An Adaptive Finite Element Method Approach to the Heat Equation and the Black-Scholes Equation

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Abstract

The abstract of the report goes here. The abstract should state the topic(s) under investigation and the main results or conclusions. Methods or approaches should be stated if this is appropriate for the topic. The abstract should be self-contained, concise and clear. The typical length is one paragraph.

Contents

1   Introduction

The origin of the Finite Element Method (FEM) is generally agreed to be a paper by Courant [4] in 1943. Though initially obscure, it gained widespread usage in engineering as computing power became more cheaply available. Since then it has become increasingly more common in the natural sciences and more recently in financial industry [11]. Though the Finite Difference Method is still overwhelming used to price options the FEM is now sometimes being instead.

Though more technical than the Finite Difference Method, under certain circumstance the FEM has stark advantages. Two notable advantages are that the FEM is simpler to use for Partial Differential Equations (PDEs) with irregular shaped domains and that there is a very well understood theory of aposteriori errors. This theory of errors, which only requires knowledge of the estimated solution, allows for the FEM to be adapted during implementation. This adaptive methodology leads to a solution where errors are guaranteed to be within certain tolerances which allows the user to analyse the effects of changing parameters. Also, adaptivity leads to a solution that should be in some sense efficient as ideally maximum accuracy achieved for the minimum degrees of freedom. This concept of efficiency is what we look to investigate further in this paper.

The pioneering work of Babuska et al [2] in the 1980s showed the first examples of how an aposteriori error estimate could be used to implement adaptive FEM. The research moved quickly and attempts at adaptive mesh refinement for parabolic PDEs began towards the end of the same decade see [5], [6] and others. Despite the research into these methods and the adoption of adaptive FEM in science and engineering there are still some theoretical results outstanding. Convergence and optimal complexity have only been shown for linear elliptical PDEs and only quite recently [7], [9]. Even these recent results only show that there is convergence to a solution and do not imply an order of convergence for the adaptive methods.

The rest of this dissertation comprises:

...

As show by Bergam [3] in their article.

The end of the introductory section would typically outline the structure of the report.

In this template, section **??** gives the background of the topic, sections **??** and 8 contain the bulk of the work and section 9 summarises and discusses what has been achieved. Appendix A displays the raw data, and certain technical calculations for section **??** are deferred to appendix B.

2　The Linear Heat Equation

The Heat Equation provides a simple situation within which we can understand and demonstrate the implementation of the Finite Element Method and adaptive algorithms. Conversely it is fundamental enough that it would allow someone reading this dissertation to easily build on our findings to access a different but connected problem for example the Black-Scholes equation. It can be thought of as a prototypical parabolic equation and this was our motivation for studying it.

In this section we will describe the derivation of the Heat Equation in one dimension. We will first do this in the steady state i.e. where $\frac{du}{dt} = 0$ and then extend this to the time dependent equation. We do this just to refresh the reader's knowledge of the properties of the equation.

2.1　The Heat Equation in Steady State

The steady state equation in one dimension can be thought of as describing a thin rod of uniform material on the interval $I = [0, L]$. As we are only considering the one dimensional problem heat is only conducted in the x direction. The rod is heated by a source f which we assume to have been acting continuously for long enough to reach the steady state.

Let q be the heat flux i.e flow of energy per unit of area per unit of time and let S be the cross section of the rod. As flux is a vector quantity it needs a direction and we take this as the direction of x increasing. The first law of thermodynamics on conservation of energy tells us that the flow out of the rod must equal the flow in from the heat source. Hence we have:

$$q(L)S(L) - q(0)S(0) = \int_I f \, dx \tag{2.1}$$

We now divide both sides of (2.1) by L and take L $\rightarrow$ 0. Hence we have the differential equation:

$$(Sq)' = f \tag{2.2}$$

Employing Fouriers law which in this context can be understood as the flux being

negatively proportional to the temperature gradient:

$$q = kT' \tag{2.3}$$

here k represents the heat conductivity of the rod.

Together (2.2) and (2.3) give us the Heat Equation:

$$-(SkT')' = f \tag{2.4}$$

2.2 The Time Dependent Problem

The time dependent problem is very similar to the steady state form. We introduce a function e which is the energy per unit length within the rod. We use the the conservation of energy principle again but this time we use the fact that the sum of the resultant heat flux is equal to the rate of change of internal energy. Here rate of change of e is denoted by $\dot{e}$

$$\int_I \dot{e}\,dx = q(0)S(0) - q(L)S(L) \int_I f\,dx \tag{2.5}$$

Once again dividing by L and letting L $\rightarrow$ 0.

$$\dot{e} + (Sq)' = f \tag{2.6}$$

Finally assuming that the energy depends linearly on the the Temperature

$$e = mT \tag{2.7}$$

And using (2.3) and combining (2.5) and (2.6) we have the transient Heat Equation which we will refer through out the rest of this paper simply as the Heat Equation:

$$m\dot{T} - (SkT')' = f \tag{2.8}$$
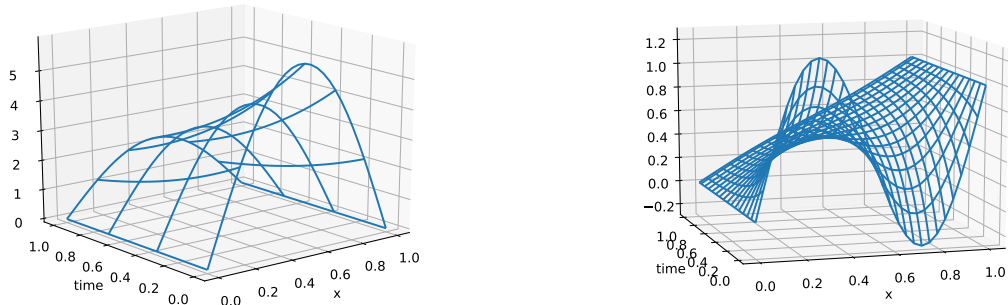
2.3   Boundary Conditions

Figure 1: $f = 0$, $m = 1$, $S = 1$, $k = \pi^- 2$

3 The Finite Element Method

This section is concerned with the discretisation of the Heat Equation by Finite Elements. Though not uncommon, the FEM is not necessarily widely known by the average postgraduate mathematician. As such we will provide a fairly detailed description of the method and full derivation of the scheme. A very nice introduction to the subject if more detail is needed is [**?**].

The underlying practical steps to the method are:

1. Finding a variational form of the equation we are looking to solve. This will allow us to look for a solution which satisfies the equation in some weak sense. This step requires the introduction of the function spaces in which we can study our equation and look for its solution. We will see that these spaces are Sobolev spaces.

2. Creating the subdivision of our domain Ω.

3. Taking the variational form, which we have at first defined in an infinite dimensional function space V, and approximating it on a finite dimensional sub-space V_h. In our case the subspace will be piecewise polynomial functions defined on our subdivision.

4. Projecting the boundary conditions onto the finite dimensional space.

Once the mathematical context of the Finite Element has been outlined we will move on to finding the weak formulation of the Heat Equation. With this done we will consider the finite dimensional subspace and the elements that will be used in our discretisation. We will close the chapter will a discussion of other possible choices for elements.

3.1 Sobolev Spaces

We begin by stating the notation to be used in this dissertation and also stating some of the relevant definitions and results from functional analysis.

Definition 3.1. Continuous Functions Let $\Omega \subset \mathbb{R}^n, n \geq 1$ be an open bounded set.

- We define $C(\Omega)$ the set of all real-valued and continuous functions defined on Ω.

- For $m \geq 1$ we write $C^m(\Omega)$ to denote the set of m times continuously differentiable functions i.e. that $C^m(\Omega) = \{f \in C(\Omega) : f^{(k)} \in C(\Omega) \quad \forall k \leq m\}$.

Definition 3.2. Function Support Let $\Omega \subset \mathbb{R}^n, n \geq 1$ be an open bounded set.

Definition 3.3. L^p Space
$$\|u\|_{L_2(\Omega)} := \Big(\int_\Omega |u|^p dx \Big)^{\frac{1}{p}} < \infty$$

Definition 3.4. Multi Index

$\alpha = (\alpha_1, \alpha_2, ...\alpha_n) \in \mathbb{N}^n$

$|\alpha| = \alpha_1 + ...\alpha_n$

Define the Multi Index

$D^\alpha = \Big(\frac{\partial}{\partial x_1} \Big)^{\alpha_1}$
$...\Big(\frac{\partial}{\partial x_n} \Big)^{\alpha_n} =$
$\frac{\partial^{|\alpha|}}{\partial x_1^\alpha ... \partial x_n^\alpha}$

We can now generalise our earlier definition of continuous differentiable functions to an analogous idea with the multi index which will be relevant to partial differentiation.

$C^m(\Omega) = \{f \in C(\Omega) : D^\alpha f \in C(\Omega) \quad \forall |a| \leq\leq m\}$

Definition 3.5. Weak Derivative

Say ψ_α is a locally integrable
$$\int_\Omega \psi_\alpha(x) \cdot v(x) dx = (-1)^\alpha \int_\Omega u(x) \cdot D^\alpha v(x) dx \forall v \in C_0^\infty(\Omega)$$

Definition 3.6. Weak Derivative

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Definition 3.7. Sobolev Space

$H^m(\Omega) = \{u \in L_2(\Omega) : D^\alpha u \in L_2(\Omega), |\alpha| \leq m\}$

$$\|u\|_{H^m(\Omega)} := \Big(\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L_2(\Omega)}^2 \Big)^{\frac{1}{2}}$$

The Lax Milgram theorem shows shows the existence and uniqueness of a solution to the PDE.

Theorem 3.1 (Lax Milgram). *Let V be a real Hilbert Space with associated norm $\|\cdot\|$. Also let $a(,)$ be a bilinear functional on V x V with:*

(a) $a(\cdot, \cdot)$ is coercive, i.e. there exists a positive constant $c_0 \forall v \in V |a(v,v)| \geq c_0 \|v\|_V^2$

(b) $a(\cdot, \cdot)$ is continuous i.e. there exists a positive constant $c_1 \forall v, w \in V |a(v,w) \leq c_1 \|v\|_V \|w\|_V$

and for $\ell(\cdot)$ a bilinear functional also on V such that

(c) $\ell(\cdot)$ is continuous i.e. there exists a positive constant $c_2 \forall v \in V |\ell(\cdot)| \leq c_2 \|v\|_V$

Then there $\exists u \in V : a(u,v) = \ell(v) \forall v \exists V$ and u is unique.

This version of the theorem is Weaker than the refinement proved by Babuska but it sufficient for our purposes. The refined theorem is available f in most textbooks on the FEM.

3.2 The Weak Formulation

Now we have defined the mathematical context we can consider our first problem.

Take a simple Heat Equation:

$$\dot{u} - u'' = f \quad x \in [0, L], \quad t \in (0, T] \tag{3.1a}$$

$$u(0, t) = u(L, t) = 0 \tag{3.1b}$$

$$u(x, 0) = u_0(x) \tag{3.1c}$$

Multplying $\dot{u} - u'' = f$ by a test function $v \in H_0^1(0, L)$ on both sides gives:

$$\int_0^L \dot{u} v dx - \int_0^L u'' v dx = \int_0^L f v dx \tag{3.2}$$

Integrating by parts yields:

$$\int_0^L \dot{u} v dx + \int_0^L u'' v + u(0)v(0) - u(L)v(L) dx = \int_0^L f v dx \tag{3.3}$$

$$\int_0^L \dot{u} v dx + \int_0^L u' v' dx = \int_0^L f v dx \tag{3.4}$$

The weak formulation of our problem is now given by:

Problem 3.1. *Weak Formulation*

Find $\quad u(\cdot, t) \in H_0^1(0, L)$:

$$\int_0^L \dot{u} v dx + \int_0^L u' v' dx = \int_0^L f v dx$$

$$\int_0^L u(x, 0) v dx = \int_0^L u_0(x) v dx$$

$\forall v \in H_0^1(0, L)$

3.3 Spatial Discretization

Subsections may be used. Use a clear structure in your report.

We denote the set of real numbers by $\mathbb{R}$, the set of integers by $\mathbb{Z}$ and the set of complex numbers by $\mathbb{C}$. Our analysis is based on the equation $e^{\pi i} = -1$ and the relation

$$\frac{2}{4} = \frac{1}{2} \tag{3.5}$$

which we verify in the appendix B. Useful consequences are

$$\frac{4}{8} = \frac{1}{2} \tag{3.6}$$

$$\frac{4}{12} + \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt = \frac{1}{3} + \sum_{n=1}^\infty \frac{1}{n^s} \tag{3.7}$$

$$\frac{2}{10} = \frac{1}{5} \tag{3.8}$$

For any $0 \neq a \in \mathbb{Z}$, the equality

$$\frac{2a}{4a} = \frac{1}{2}$$

follows from equation (3.5).

3.4 Another subsection

3.4.1 A subsubsection

Sometimes subsubsections may be appropriate.

3.4.2 Another subsubsection

This could contain a table of interesting numbers

n	1	2	3	4	5	6
F_n	1	1	2	3	5	8
B_n	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$
p_n	2	3	5	7	11	13

4 Fully Discrete Problem

4.0.1 Implicit Euler

4.0.2 Impementation

5 A Posteriori Error Analysis

When a continuous problem such as the Heat Equation is discretised and solved numerically an error is incurred. For a convergent scheme this error can be bounded in general by a priori error i.e. the estimates that are calculated before a specific approximated solution is found. A priori estimates can also be used to show stability of numerical schemes. Although useful in these applications it is rare that an a priori error estimate accurately quantifies the error present in a numerical solution. As such in a practical context the a priori error cannot be used to indicate the quality of a solution.

A posteriori error analysis uses the computed numerical solution itself to analyses the error. Where a posteriori error estimates are available and accurate this has the obvious advantage that it describes the error in the approximation without the need for an analytical solution. This replaces the heuristics and guess work that had previously guided the error analysis of the FEM approximation in complex settings.

Since the initial pioneering methods of Babuska [?] [2] in this field many a posteriori estimates have been presented. Many of these have been shown to be effective in context. The emphasis in the literature has moved now from the discovery of new a posteriori estimates to the testing of the limitations and robustness of existing a posteriori estimates all this is done primarily with the goal of validating the extensively used self-adapting FEM packages.

5.0.1 Gradient Recovery

While the overall goal of the FEM may be the numerical solution it may also be the case that the gradient is of particular interest also. The gradient in the approximation is discontinuous over element boundaries but can be smoothed out by post-processing. In fact under certain circumstances the smoothed gradient is closer to the gradient of the true solution [?]. This leads fairly directly one of the simpler a posteriori error estimates which is measuring the gradient before and after this post-processing.

We will now try to describe this process more formally. Consider the steady state one dimensional Heat Equation with conductivity co-efficient of one.

$$-u'' = f \qquad x \in [0, 1] \tag{5.1}$$

We now seek to define the energy norm. To do this we will first define the weak formulation of 5.1.

Problem 5.1. *Weak Formulation*

Find $\quad u(\cdot, t) \in H_0^1(0, 1)$:

$$\int_0^L u'v'dx = \int_0^L fvdx \qquad dx \forall v \in H_0^1(0, L)$$

$\forall v \in H_0^1(0, L)$

Write

$$\int_0^L u'v'dx = a(u, v)$$

It is clear that the bilinear form is symmetric i.e. a(w, v) = a(v, w) and it can be shown that it is coercive as described in the Lax Milgram theorem 3.1. Hence, A(w, v) is an inner product on $H_0^1(0, L)(w, v)_a$.

We can then define the associated energy norm.

$$\|w\|_a = |(w, w)_a|^{\frac{1}{2}} \tag{5.2}$$

If we measure the true error in the energy norm we get.

$$\|u - u_h\|_a^2 = \int_0^L u' - u_h' dx \tag{5.3}$$

Which is the L_2 norm of of the difference between the gradient .

If we had the true gradient we would now be able to measure the true error in the energy norm. As that is unavailable we will use an estimate of the true gradient which will be obtained by some suitable post- processing of the FEM approximation to the solution. Let the post processed FEM approximated gradient be denoted by $G_h(u_h)$ then the error indicator for 5.1 is given by:

$$\eta^2 = \int_0^L |G_h(u_h) - u_h|^2 dx \tag{5.4}$$

Clearly there are as many different error indicators as there are ways of processing the FEM approximation to the gradient.

6 Design of Adaptive Algorithms

7 Some Numerical Examples

8   Object oriented design patterns

Graphics can be included. Figure 2 shows an example. Learn about floats and pictures in the LaTeX wikibook to place the figures at the right place in the end.

Figure 2: Oh look, something happens here !

9 Conclusions

Further help on LaTeX can be found easily on the internet. The LaTeX wikibook[1] contains a lot. For instance you would find there how to type theorems and proofs nicely. Or how to include source code written in some programming language like matlab. There are long lists available with all sorts of common mathematical symbols like ξ, ∇, ∞, $\log$, $\iff$, etc.

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[1] http://en.wikibooks.org/wiki/LaTeX

A   Raw data

Material that needs to be included but would distract from the main line of presentation can be put in appendices. Examples of such material are raw data, computing codes and details of calculations.

B   Calculations for section ??

In this appendix we verify equation (3.5).

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