An Adaptive Finite Element Method Approach to the Heat Equation and the Black-Scholes Equation

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Abstract

The abstract of the report goes here. The abstract should state the topic(s) under investigation and the main results or conclusions. Methods or approaches should be stated if this is appropriate for the topic. The abstract should be self-contained, concise and clear. The typical length is one paragraph.

Contents

1 Introduction

The introductory section goes here. And remember the introduction is the last thing you write.

The origin of the Finite Element Method (FEM) is generally agreed to be a paper by Courant [4] in 1943. Though initially obscure, it gained widespread usage in engineering as computing power became more cheaply available. Since then it has become increasingly more common in the natural sciences and more recently in financial industry [11]. Though the Finite Difference Method is still overwhelming used to price options the FEM is now sometimes being instead.

Though more technical than the Finite Difference Method, under certain circumstance the FEM has stark advantages. Two notable advantages are that the FEM is simpler to use for Partial Differential Equations (PDEs) with irregular shaped domains and that there is a very well understood theory of aposteriori errors. This theory of errors, which only requires knowledge of the estimated solution, allows for the FEM to be adapted during implementation. This adaptive methodology leads to a solution where errors are guaranteed to be within certain tolerances which allows the user to analyse the effects of changing parameters. Also, adaptivity leads to a solution that should be in some sense efficient as ideally maximum accuracy achieved for the minimum degrees of freedom. This concept of efficiency is what we look to investigate further in this paper.

The pioneering work of Babuska et al [2] in the 1980s showed the first examples of how an aposteriori error estimate could be used to implement adaptive FEM. The research moved quickly and attempts at adaptive mesh refinement for parabolic PDEs began towards the end of the same decade see [5], [6] and others. Despite the research into these methods and the adoption of adaptive FEM in science and engineering there are still some theoretical results outstanding. Convergence and optimal complexity have only been shown for linear elliptical PDEs and only quite recently [7], [9]. Even these recent results only show that there is convergence to a solution and do not imply an order of convergence for the adaptive methods.

The rest of this dissertation comprises:

...

As show by Bergam [3] in their article.

The end of the introductory section would typically outline the structure of the report. In this template, section **??** gives the background of the topic, sections **??** and 4 contain the bulk of the work and section 5 summarises and discusses what has been achieved. Appendix A displays the raw data, and certain technical calculations for section **??** are deferred to appendix B.

2 The Linear Heat Equation

References can be for example textbooks [?, ?, ?, ?, ?], conventional journal articles [?, ?], conventional journal articles that are also available at an e-print server [?, ?], electronic journal articles [?], articles in conference proceedings [?], PhD theses [?, ?] or websites [?]. This template orders the references by their first citation, cites them by their number and keeps any footnotes[1] separate from the references. Other citation practices exist: Your supervisor can advise as to what is appropriate for your topic.

[1]Such as this.

3 The Finite Element Method for the Heat Equation

This section is concerned with the discretisation of the Heat Equation by Finite Elements. Though not uncommon the FEM is not necessarily widely known by the average postgraduate mathematician. As such we will provide a fairly detailed description of the method and full derivation of the scheme. A very nice introduction to the subject is [?].

The underlying practical steps to the method are:

1. Finding a variational form of the equation under consideration. This will allow us to look for a solution which satisfies the equation in some weak sense. This step requires the introduction of the function spaces where we will set our equation and look for its solution. We will see that these spaces are Sobolev spaces.

2. Dividing our domain Ω, in our case just an interval, into elements.

3. Taking the variational form which we have at first defined in an infinite dimensional function space and approximate it on a finite dimensional sub-space. In our case the subspace will be piecewise polynomial functions defined on our subdivision.

4. Projecting the boundary conditions onto the finite dimensional space.

3.1 A subsection

Subsections may be used. Use a clear structure in your report.

We denote the set of real numbers by $\mathbb{R}$, the set of integers by $\mathbb{Z}$ and the set of complex numbers by $\mathbb{C}$. Our analysis is based on the equation $e^{\pi i} = -1$ and the relation

$$\frac{2}{4} = \frac{1}{2} \tag{3.1}$$

which we verify in the appendix B. Useful consequences are

$$\frac{4}{8} = \frac{1}{2} \tag{3.2}$$

$$\frac{4}{12} + \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{3.3}$$

$$\frac{2}{10} = \frac{1}{5} \tag{3.4}$$

For any $0 \neq a \in \mathbb{Z}$, the equality

$$\frac{2a}{4a} = \frac{1}{2}$$

follows from equation (3.1).

3.2 Another subsection

3.2.1 A subsubsection

Sometimes subsubsections may be appropriate.

3.2.2 Another subsubsection

This could contain a table of interesting numbers

n	1	2	3	4	5	6
F_n	1	1	2	3	5	8
B_n	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$
p_n	2	3	5	7	11	13

Figure 1: Oh look, something happens here !

4 Implementation

Graphics can be included. Figure 1 shows an example. Learn about floats and pictures in the LaTeX wikibook to place the figures at the right place in the end.

5 Conclusions

Further help on LaTeX can be found easily on the internet. The LaTeX wikibook[2] contains a lot. For instance you would find there how to type theorems and proofs nicely. Or how to include source code written in some programming language like matlab. There are long lists available with all sorts of common mathematical symbols like ξ, ∇, ∞, $\log$, $\iff$, etc.

[2] `http://en.wikibooks.org/wiki/LaTeX`

A Raw data

Material that needs to be included but would distract from the main line of presentation can be put in appendices. Examples of such material are raw data, computing codes and details of calculations.

B Calculations for section ??

In this appendix we verify equation (3.1).

References

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