3. The Wiener Filter

3.1 The Wiener-Hopf Equation

The Wiener filter theory is characterized by:

1. The assumption that both signal and noise are random processed with known spectral characteristics or, equivalently, known auto- and cross-correlation functions.
2. The criterion for best performance is minimum mean-square error. (This is partially to make the problem mathematically tractable, but it is also a good physical criterion in many applications.)
3. A solution based on scalar methods that leads to the optimal filter weighting function (or transfer function in the stationary case).

$$\mathrm{z}(n) = \mathrm{s}(n) + \mathrm{v}(n) \longrightarrow \boxed{h(i) \text{ or } H(z)} \longrightarrow \hat{\mathrm{s}}(n)$$

JWSS LTI WSS

R_s, R_v, R_z, R_{sz} or $S_s(z), S_v(z), S_z(z), S_{sz}(z)$ known

$$J = E\left\{\left[s(n) - \hat{s}(n)\right]^2\right\}$$

Fig. 3.1-1 Wiener Filter Problem

We now consider the filter optimization problem that Wiener first solved in the 1940s. Referring to Fig. 3.1-1, we assume the following:

1. The filter input is an additive combination of signal and noise, both of which are jointly wide-sense stationary (JWSS) with known auto- and cross-correlation functions (or corresponding spectral functions).
2. The filter is linear and time-invariant. No further assumption is made as to its form.
3. The output is also wide-sense stationary.
4. The performance criterion is minimum mean-square error.

The estimate $\hat{\mathrm{s}}(n)$ of a signal s(n) is given by the convolution representation

$$\hat{\mathrm{s}}(n) = h(n) * \mathrm{z}(n) = \sum_{i=-\infty}^{\infty} h(n-i)\mathrm{z}(i) = \sum_{i=-\infty}^{\infty} h(i)\mathrm{z}(n-i), \qquad (3.1\text{-}1)$$

where z(i) is the measurement and $h(n)$ is the impulse response of the estimator. Let $\mathcal{H}$ denote the region of support of $h(n)$, defined by

$$\mathcal{H} = \{n : h(n) \neq 0\}.$$

Then, Eq. (3.1-1) can be rewritten as

$$\hat{s}(n) = \sum_{i \in \mathcal{H}} h(i) z(n-i) . \tag{3.1-2}$$

Let the mean-square estimation error (MSE) J be

$$\begin{aligned} J &= E\left\{\left[s(n) - \hat{s}(n)\right]^2\right\} \\ &= E\left\{\left[s(n) - \sum_{i \in \mathcal{H}} h(i) z(n-i)\right]\left[s(n) - \sum_{j \in \mathcal{H}} h(j) z(n-j)\right]\right\} \\ &= E\left\{s^2(n)\right\} - 2\sum_{i \in \mathcal{H}} h(i) E\left\{s(n) z(n-i)\right\} + \sum_{i \in \mathcal{H}} \sum_{j \in \mathcal{H}} h(i) h(j) E\left\{z(n-i) z(n-j)\right\}. \end{aligned} \tag{3.1-3}$$

To minimize the MSE, take the partial derivatives of J with respect to $h(i)$, for each $h(i) \neq 0$. Then, set the result equal to zero

$$\begin{aligned} \frac{\partial J}{\partial h(i)} &= -2E\left\{s(n) z(n-i)\right\} + 2\sum_{j \in \mathcal{H}} h(j) E\left\{z(n-i) z(n-j)\right\} \\ &= 0. \end{aligned} \tag{3.1-4}$$

Solving Eq. (3.1-4), we find

$$\sum_{j \in \mathcal{H}} h(j) E\left\{z(n-i) z(n-j)\right\} = E\left\{s(n) z(n-i)\right\}, \quad i \in \mathcal{H} \tag{3.1-5}$$

which we may express in the form of

$$\sum_{j \in \mathcal{H}} h(j) R_{\mathrm{z}}(j-i) = R_{\mathrm{sz}}(i), \quad i \in \mathcal{H}\,, \tag{3.1-6}$$

where $R_{\mathrm{z}}(k)$ is the autocorrelation function of z(n) and $R_{\mathrm{sz}}(k)$ is the crosscorrelation function of s(n) and z(n).

Eq. (3.1-6) is the discrete-time ***Wiener-Hopf equation***. It is the basis for the derivation of the Wiener filter.

3.2 The FIR Wiener Filter

Suppose the filter has an impulse response $h(n)$ with support $\mathcal{H}$, where $\mathcal{H}$ is a finite set, e.g., $\mathcal{H}$ = {0,1,2,…,N-1}. Then the impulse response $h(n)$ has a finite duration, and this type of filter is called a *finite impulse response* (FIR) filter.

For the FIR filter, the Wiener-Hopf equation is written

$$\sum_{j=0}^{N-1} h(j) R_{\mathrm{z}}(j-i) = R_{\mathrm{sz}}(i), \quad 0 \le i \le N-1\,. \tag{3.2-1}$$

This may be written in matrix form

$$\begin{bmatrix} R_{\mathrm{z}}(0) & R_{\mathrm{z}}(1) & \cdots & R_{\mathrm{z}}(N-1) \\ R_{\mathrm{z}}(1) & R_{\mathrm{z}}(0) & \cdots & R_{\mathrm{z}}(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_{\mathrm{z}}(N-1) & R_{\mathrm{z}}(N-2) & \cdots & R_{\mathrm{z}}(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(N-1) \end{bmatrix} = \begin{bmatrix} R_{\mathrm{sz}}(0) \\ R_{\mathrm{sz}}(1) \\ \vdots \\ R_{\mathrm{sz}}(N-1) \end{bmatrix}, \tag{3.2-2}$$

where we have used the fact that $R_{\mathrm{z}}(k) = R_{\mathrm{z}}(-k)$. Denote Eq. (3.2-2) in a convenient form

$$R_{\mathrm{z}}\underline{h} = \underline{r}_{\mathrm{sz}} \tag{3.2-3}$$

where the autocorrelation function R_{z} is symmetric and positive-definite.

Eq. (3.2-3) is solved for $\underline{h}$,

$$\underline{h} = R_{\mathrm{z}}^{-1}\underline{r}_{\mathrm{sz}}. \tag{3.2-4}$$

This is the FIR Wiener filter of order $N-1$. Note that R_{z} is a Toeplitz matrix and Eq. (3.2-4) may be solved using a computationally-efficient method such as the Levinson-Durbin algorithm.

Mean-Square Error (MSE)

We may write the MSE of the FIR Wiener filter

$$\begin{aligned}\mathrm{MSE} &= E\left\{\tilde{\mathrm{s}}(n)\left[\mathrm{s}(n)-\sum_{i=0}^{N-1}h(i)\mathrm{z}(n-i)\right]\right\}\\ &= E\left\{\tilde{\mathrm{s}}(n)\mathrm{s}(n)\right\}-\sum_{i=0}^{N-1}h(i)E\left\{\tilde{\mathrm{s}}(n)\mathrm{z}(n-i)\right\}.\end{aligned} \tag{3.2-5}$$

To simplify Eq. (3.2-5), rewrite Eq. (3.1-5) in the following form

$$\begin{aligned}E\left\{\mathrm{z}(n-i)\left[\mathrm{s}(n)-\sum_{j\in\mathcal{H}}h(j)\mathrm{z}(n-j)\right]\right\}&=0\\ E\left\{\mathrm{z}(n-i)\left[\mathrm{s}(n)-\hat{\mathrm{s}}(n)\right]\right\}&=0\\ E\left\{\mathrm{z}(n-i)\tilde{\mathrm{s}}(n)\right\}&=0,\quad i\in\mathcal{H}.\end{aligned} \tag{3.2-6}$$

This relation is known as the orthogonality principle for LTI estimators. Applying this relation to Eq. (3.2-5) gives

$$
\begin{aligned}
\mathrm{MSE} &= E\left\{\tilde{\mathrm{s}}(n)\mathrm{s}(n)\right\} \\
&= E\left\{\mathrm{s}^2(n)\right\} - \sum_{i=0}^{N-1} h(i) E\left\{\mathrm{s}(n)\mathrm{z}(n-i)\right\} \\
&= R_{\mathrm{s}}(0) - \sum_{i=0}^{N-1} h(i) R_{\mathrm{sz}}(i) \\
&= R_{\mathrm{s}}(0) - h^T \underline{r}_{\mathrm{sz}}.
\end{aligned}
\tag{3.2-7}
$$

Observe that if no filtering is performed and we simply use $\hat{\mathrm{s}}(n) = z(n)$, the MSE is

$$
\mathrm{MSE}_{\text{no filter}} = E\left\{\left[\mathrm{s}(n) - z(n)\right]^2\right\} = R_{\mathrm{s}}(0) - 2R_{\mathrm{sz}}(0) + R_{\mathrm{z}}(0). \tag{3.2-8}
$$

We can use Eq. (3.2-8) to measure the effectiveness of the Wiener filter. The reduction in MSE due to Wiener filtering is given by

$$
\text{reduction in MSE} = 10\log_{10}\left(\frac{\mathrm{MSE}_{\text{no filter}}}{\mathrm{MSE}_{\text{filter}}}\right) dB. \tag{3.2-9}
$$

We can inspect the MSE to determine whether the filter performance is acceptable. If the MSE is too high, we may wish to use a larger value of *N*.

Example 3.2-1

Suppose we have a signal $\mathrm{s}(n)$ with autocorrelation function $R_{\mathrm{s}}(k) = 0.95^{|k|}$. The signal is observed in the presence of additive white noise with variance $\sigma_{\mathrm{v}}{}^{2} = 2$. Hence $R_{\mathrm{v}}(k) = 2\delta(k)$. $\mathrm{s}(n)$ and $\mathrm{v}(n)$ are uncorrelated, zero-mean, JWSS random processes. We would like to design the optimum LTI second-order filter h.

Since the filter is the second-order, we set $N = 3$. The matrix equation Eq. (3.2-2) to be solved is

$$\begin{bmatrix} R_{\mathrm{z}}(0) & R_{\mathrm{z}}(1) & R_{\mathrm{z}}(2) \\ R_{\mathrm{z}}(1) & R_{\mathrm{z}}(0) & R_{\mathrm{z}}(1) \\ R_{\mathrm{z}}(2) & R_{\mathrm{z}}(1) & R_{\mathrm{z}}(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix} = \begin{bmatrix} R_{\mathrm{sz}}(0) \\ R_{\mathrm{sz}}(1) \\ R_{\mathrm{sz}}(2) \end{bmatrix}. \qquad (3.2\text{-}10)$$

For an additive-noise measurement model

$$\mathrm{z}(n) = \mathrm{s}(n) + \mathrm{v}(n),$$

the autocorrelation functions $R_{\mathrm{z}}(k)$ and $R_{\mathrm{sz}}(k)$ may be given as follows

$$
\begin{aligned}
R_{\mathrm{z}}(k) &= E\left\{\mathrm{z}(n)\mathrm{z}(n-k)\right\} \\
&= E\left\{\left[\mathrm{s}(n)+\mathrm{v}(n)\right]\left[\mathrm{s}(n-k)+\mathrm{v}(n-k)\right]\right\} \\
&= E\left\{\left[\mathrm{s}(n)\mathrm{s}(n-k)\right]\right\} + E\left\{\left[\mathrm{v}(n)\mathrm{v}(n-k)\right]\right\} \\
&= R_{\mathrm{s}}(k) + R_{\mathrm{v}}(k),
\end{aligned}
\tag{3.2-11}
$$

and

$$
\begin{aligned}
R_{\mathrm{sz}}(k) &= E\left\{\mathrm{s}(n)\left[\mathrm{s}(n-k)+\mathrm{v}(n-k)\right]\right\} \\
&= E\left\{\left[\mathrm{s}(n)\mathrm{s}(n-k)\right]\right\} + E\left\{\left[\mathrm{s}(n)\mathrm{v}(n-k)\right]\right\} \\
&= R_{\mathrm{s}}(k).
\end{aligned}
\tag{3.2-12}
$$

According to Eqs. (3.2-11) and (3.2-12), we have

$$
\begin{aligned}
R_{\mathrm{z}}(k) &= 0.95^{|k|} + 2\delta(k) \\
R_{\mathrm{sz}}(k) &= 0.95^{|k|}.
\end{aligned}
$$

Then Eq. (3.2-10) becomes

$$\begin{bmatrix} 3 & 0.95 & 0.9025 \\ 0.95 & 3 & 0.95 \\ 0.9025 & 0.95 & 3 \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0.95 \\ 0.9025 \end{bmatrix}. \qquad (3.2\text{-}13)$$

The solution of Eq. (3.2-13) is

$$\underline{h} = \left[0.2203 \;\; 0.1919 \;\; 0.1738\right]^T.$$

The MSE of this estimator can be computed via Eq. (3.2-7),

$$\text{MSE} = R_{\text{s}}(0) - h^T \underline{r}_{\text{sz}} = 0.4405. \qquad (3.2\text{-}13)$$

For comparison, we remark that without any filtering,

$$\text{MSE}_{\text{no filter}} = R_{\text{s}}(0) - 2R_{\text{sz}}(0) + R_{\text{z}}(0) = R_{\text{v}}(0) = 2. \qquad (3.2\text{-}14)$$

The FIR Wiener filter has reduced the MSE by

$$\begin{aligned} \text{reduction in MSE} &= 10\log_{10}\left(\frac{\text{MSE}_{\text{no filter}}}{\text{MSE}_{\text{filter}}}\right) dB \\ &= 10\log_{10}\left(\frac{2}{0.4405}\right) \qquad (3.2\text{-}15) \\ &\approx 6.5 dB. \end{aligned}$$

3.3 The Noncausal Wiener Filter

Rewrite the Wiener-Hopf equation, Eq. (3.1-6),

$$\sum_{j\in\mathcal{H}} h(j)R_{\mathrm{z}}(j-i) = R_{\mathrm{sz}}(i), \quad i\in\mathcal{H}$$

and let $\mathcal{H}=\mathbb{Z}$, where $\mathbb{Z}$ is the set of integers: $\mathbb{Z}=\{\cdots,-2,-1,0,1,2,\cdots\}$. Then, the Wiener-Hopf equation becomes

$$\sum_{j=-\infty}^{\infty} h(j)R_{\mathrm{z}}(j-i) = R_{\mathrm{sz}}(i), \quad \text{for all } i\in\mathbb{Z}. \tag{3.3-1}$$

Take the z-transform of Eq. (3.3-1)

$$H(z)S_{\mathrm{z}}(z) = S_{\mathrm{sz}}(z)$$

so that

$$H(z) = \frac{S_{\mathrm{sz}}(z)}{S_{\mathrm{z}}(z)}. \tag{3.3-2}$$

Eq. (3.3-2) is the solution for the noncausal Wiener filter.

Mean-Square Error (MSE)

As with the FIR Wiener filter of Section 3.2, we can derive an expression for the MSE. The MSE similar to Eq. (3.2-7) can be written as

$$\mathrm{MSE} = R_{\mathrm{s}}(0) - \sum_{i=-\infty}^{\infty} h(i) R_{\mathrm{sz}}(i) . \qquad (3.3\text{-}3)$$

Example 3.3-1

Let us apply a noncausal Wiener filter to the filtering problem of Example 3.2-1. In that example we found that

$$R_{\mathrm{z}}(k) = 0.95^{|k|} + 2\delta(k), \text{ and } \; R_{\mathrm{sz}}(k) = 0.95^{|k|} . \qquad (3.3\text{-}4)$$

Take z-transforms of Eq. (3.3-4)

$$\begin{aligned} S_{\mathrm{z}}(z) &= \frac{1-(0.95)^2}{(1-0.95z^{-1})(1-0.95z)} + 2 \\ &= \frac{2.3955(1-0.7931z^{-1})(1-0.7931z)}{(1-0.95z^{-1})(1-0.95z)}, \quad 0.7931 < |z| < \frac{1}{0.7931} \end{aligned}$$

and

$$S_{sz}(z) = \frac{1-(0.95)^2}{(1-0.95z^{-1})(1-0.95z)}, \quad 0.95 < |z| < \frac{1}{0.95}.$$

The noncausal Wiener filter, Eq. (3.3-2), is given by

$$\begin{aligned} H(z) = \frac{S_{sz}(z)}{S_z(z)} &= \frac{0.0975}{2.3955(1-0.7931z^{-1})(1-0.7931z)} \\ &= \frac{0.1097(1-(0.7931)^2)}{(1-0.7931z^{-1})(1-0.7931z)}, \quad 0.95 < |z| < \frac{1}{0.95}. \end{aligned}$$

The impulse response of the filter is

$$h(n) = 0.1097(0.7931)^{|n|}.$$

The mean-square errors associated with the noncausal Wiener filter is according to Eq. (3.3-3),

$$
\begin{aligned}
\mathrm{MSE} &= R_s(0) - \sum_{n=-\infty}^{\infty} h(n) R_{sz}(n) \\
&= (0.95)^0 - \sum_{n=-\infty}^{\infty} 0.1097(0.7931)^{|n|}(0.95)^{|n|} \\
&= 1 - 0.1097\left[\sum_{n=0}^{\infty}(0.7931)^n(0.95)^n + \sum_{n=-1}^{-\infty}(0.7931)^{-n}(0.95)^{-n}\right] \\
&= 1 - 0.1097\left[2\sum_{n=0}^{\infty}(0.7931)^n(0.95)^n - 1\right] \\
&= 1 - 0.1097\left[\frac{2}{1-(0.7931)(0.95)} - 1\right] = 0.2195.
\end{aligned}
$$

Since the MSE with no filter is 2 according to Eq. (3.2-14), the improvement of the noncausal filter over no filtering is

$$
\begin{aligned}
\text{reduction in MSE} &= 10\log_{10}\left(\frac{\mathrm{MSE}_{\text{no filter}}}{\mathrm{MSE}_{\text{filter}}}\right) dB \\
&= 10\log_{10}\left(\frac{2}{0.2195}\right) \\
&\approx 9.6 dB.
\end{aligned}
$$

Compared to the second-order filter in Eq. (3.2-15) in Ex. (3.2-1), the noncausal Wiener filter reduces the estimation error by an additional 3 dB.

3.4 The Causal Wiener Filter

Rewrite the Wiener-Hopf equation, Eq. (3.1-6),

$$\sum_{j\in\mathcal{H}} h(j)R_{\mathrm{z}}(j-i) = R_{\mathrm{sz}}(i), \quad i \in \mathcal{H}$$

and let $\mathcal{H} = \mathbb{N}$, where $\mathbb{N}$ is the set of integers: $\mathbb{N} = \{0,1,2,\cdots\}$. Then, the Wiener-Hopf equation becomes

$$\sum_{j=0}^{\infty} h(j)R_{\mathrm{z}}(j-i) = R_{\mathrm{sz}}(i), \quad \text{for all } i \in \mathbb{N}. \tag{3.4-1}$$

Eq. (3.4-1) can not be simplified by taking the z-transform because of its causality restriction. To solve the Wiener-Hopf equation, the spectral factorization and the causal-part extraction are necessary.

Theorem 3.4-1 (Spectral Factorization)

Let $\mathrm{x}(n)$ be a real-valued, zero-mean, WSS random process with power density spectrum $S_{\mathrm{x}}(z)$, where $S_x(z)$ is rational in z and has no poles on the unit circle. Then $S_x(z)$ can be factored into the product

$$S_x(z) = S_x^+(z) S_x^-(z), \tag{3.4-2}$$

where

$S_x^+(z)$ and $S_x^-(z)$ are rational in z,
if z_i is a pole of $S_x^+(z)$, then $|z_i| < 1$,
if z_i is a zero of $S_x^+(z)$, then $|z_i| \le 1$,
if z_i is a pole of $S_x^-(z)$, then $|z_i| > 1$,
if z_i is a zero of $S_x^-(z)$, then $|z_i| \ge 1$, and
$S_x^+(z) = S_x^-(z^{-1})$.

Example 3.4-1

Let $\mathrm{s}(n)$ be a random process described by the difference equation

$$\mathrm{s}(n) = 1.1\mathrm{s}(n-1) - 0.24\mathrm{s}(n-2) + 2\mathrm{w}(n) + 3\mathrm{w}(n-1), \tag{3.4-3}$$

where $\mathrm{w}(n)$ is a zero-mean, WSS random process with autocorrelation function $R_w(n) = 5(0.6)^{|n|}$.

We want to find the spectral factorization of $S_s(z)$.

Take the z-transform of Eq. (3.4-3)

$$S(z) = 1.1z^{-1}S(z) - 0.24z^{-2}S(z) + 2W(z) + 3z^{-1}W(z).$$

The system transfer function is

$$H(z) = \frac{S(z)}{W(z)} = \frac{2+3z^{-1}}{1-1.1z^{-1}+0.24z^{-2}} = \frac{2(1+1.5z^{-1})}{(1-0.3z^{-1})(1-0.8z^{-1})}.$$

Using the input-output power spectrum relation, the power spectrum of $s(n)$ is given by

$$\begin{aligned} S_s(z) &= H(z)H(z^{-1})S_w(z) \\ &= \frac{2(1+1.5z^{-1})}{(1-0.3z^{-1})(1-0.8z^{-1})}\frac{2(1+1.5z)}{(1-0.3z)(1-0.8z)}\frac{3.2}{(1-0.6z^{-1})(1-0.6z)}, \end{aligned} \tag{3.4-4}$$

where it is used that

$$S_w(z) = 5\frac{1-(0.6)^2}{(1-0.6z^{-1})(1-0.6z)}.$$

Collect the terms in Eq. (3.4-4) that have poles or zeros inside the unit circle to form $S_s^+(z)$,

$$S_s^+(z) = \frac{2\sqrt{3.2}(1+1.5z)}{(1-0.3z^{-1})(1-0.8z^{-1})(1-0.6z^{-1})} \quad . \tag{3.4-5}$$

Collect the terms in Eq. (3.4-4) that have poles or zeros outside the unit circle to form $S_s^-(z)$,

$$S_s^-(z) = \frac{2\sqrt{3.2}(1+1.5z^{-1})}{(1-0.3z)(1-0.8z)(1-0.6z)} \quad . \tag{3.4-6}$$

From Eqs. (3.4-5) and (3.4-6), it is noticed that

$$S_s^-(z) = S_s^+(z^{-1}) \, .$$

The Causal-Part Extraction

Consider the impulse response $h(n)$ of a real-valued LTI system with rational z-transform $H(z)$. In the time domain we can split $h(n)$ into its causal and anticausal parts such that

$$h(n) = \text{causal part of } h(n) + \text{anticausal part of } h(n) \, , \tag{3.4-7}$$

where

$$\text{causal part of } h(n) \equiv \left[h(n)\right]_+ = h(n)1(n),$$
$$\text{anticausal part of } h(n) \equiv \left[h(n)\right]_- = h(n)1(-n-1).$$

By the linearity of the z-transform, Eq. (3.4-7) becomes,

$$H(z) = \left[H(z)\right]_+ + \left[H(z)\right]_-, \tag{3.4-8}$$

where

$$\begin{aligned} \left[H(z)\right]_+ &= \mathcal{Z}\left\{h(n)1(n)\right\}, \\ \left[H(z)\right]_- &= \mathcal{Z}\left\{h(n)1(-n-1)\right\}. \end{aligned} \tag{3.4-9}$$

We now develop a method for determining $\left[H(z)\right]_+$ and $\left[H(z)\right]_-$. By long division in z, z^{-1}, or both for a rational $H(z)$, we can always convert it into the form

$$H(z) = \underbrace{\sum_{n=1}^{L} c_{-n} z^{n}}_{P_A(z)} + \underbrace{\sum_{n=0}^{M-N} c_n z^{-n}}_{P_C(z)} + \underbrace{\left(\frac{\sum_{n=0}^{N-1} b_n z^{-n}}{\sum_{n=0}^{N} a_n z^{-n}} \right)}_{Q(z)} \tag{3.4-10}$$

$H(z)$ consists of two polynomial terms, $P_A(z)$ and $P_C(z)$, and a proper fraction in z^{-1}, $Q(z)$. $P_A(z)$ corresponds to a purely anticausal sequence, and $P_C(z)$ to a purely causal sequence.

Now we examine $Q(z)$. Let $K \le N$ be the number of distinct poles of $Q(z)$, and denote these poles and their associated degrees by p_k and m_k, respectively, $k = 1, \cdots, K$. Then $Q(z)$ may be written as

$$Q(z) = \left(\frac{1}{a_0}\right) \frac{\sum_{n=0}^{N-1} b_n z^{-n}}{\prod_{k=1}^{K} (1 - p_k z^{-1})^{m_k}}.$$

Via partial fractions, we can write this expression as

$$Q(z) = \sum_{k=1}^{K} \sum_{m=1}^{m_k} \frac{q_{k,m}}{(1 - p_k z^{-1})^m} \equiv \sum_{k=1}^{K} Q_k(z). \qquad (3.4\text{-}11)$$

Suppose that $H(z)$ be stable. Then its ROC includes the unit circle and the ROC of each $Q_k(z)$ must include the unit circle. Let N_A be the total order of the poles of $H(z)$ outside the unit circle, and N_C be the total order of the poles of $H(z)$ inside the unit circle with the exception of the origin. Then $N_A + N_C = N$ and we may write

$$Q_A(z) = \sum_{\substack{k=1 \\ |p_k|>1}}^{K} Q_k(z) = \frac{\sum_{n=0}^{N_A-1} \lambda_n z^{-n}}{\sum_{\substack{k=1 \\ |p_k|>1}}^{K} (1 - p_k z^{-1})^{m_k}},$$

and

$$Q_C(z) = \sum_{\substack{k=1 \\ 0<|p_k|<1}}^{K} Q_k(z) = \frac{\sum_{n=0}^{N_C-1} \mu_n z^{-n}}{\sum_{\substack{k=1 \\ 0<|p_k|<1}}^{K} (1 - p_k z^{-1})^{m_k}}.$$

$Q_A(z)$ is the anticausal, proper, rational portion of $H(z)$, and $Q_C(z)$ is the causal, proper, rational portion of $H(z)$. Then for a stable rational $H(z)$, we have

$$\left[H(z)\right]_+ = P_C(z) + Q_C(z),$$

which is the sum of a purely causal sequence and all terms in the partial fraction expansion of

$H(z)$ that have poles inside the unit circle.

Example 3.4-2

We want the causal part of

$$H(z) = \frac{6z^2 - 51z + 128 - 109z^{-1} + 197z^{-2} - 232z^{-3} + 80z^{-4}}{2 - 17z^{-1} + 40z^{-2} - 16z^{-3}},$$

with $\mathrm{ROC} = \left\{ z : \frac{1}{2} < |z| < 4 \right\}$.

Long division in *z* gives

$$
\begin{array}{r|l}
 & 3z^2 \qquad +4 \\
\hline
2-17z^{-1}+40z^{-2}-16z^{-3} & 6z^2-51z+128-109z^{-1}+197z^{-2}-232z^{-3}+80z^{-4} \\
 & 6z^2-51z+120-\ 48z^{-1} \\
\cline{2-2}
 & \qquad 8-\ 61z^{-1}+197z^{-2}-232z^{-3}+80z^{-4} \\
 & \qquad 8-\ 68z^{-1}+160z^{-2}-\ \ 64z^{-3} \\
\cline{2-2}
 & \qquad\quad 7z^{-1}+\ 37z^{-2}-168z^{-3}+80z^{-4}
\end{array}
$$

$$H(z)=3z^2+4+\frac{7z^{-1}+\ 37z^{-2}-168z^{-3}+80z^{-4}}{2-17z^{-1}+40z^{-2}-16z^{-3}}.$$

Then long division in z^{-1} produces

$$\begin{array}{rl}
 & -5z^{-1} \quad -2 \\ \cline{2-2}
-16z^{-3}+40z^{-2}-17z^{-1}+2 \big) & 80z^{-4}-168z^{-3}+37z^{-2}+\ 7z^{-1} \\
 & 80z^{-4}-200z^{-3}+85z^{-2}-10z^{-1} \\ \cline{2-2}
 & \qquad\quad 32z^{-3}-48z^{-2}+17z^{-1} \\
 & \qquad\quad 32z^{-3}-80z^{-2}+34z^{-1}-4 \\ \cline{2-2}
 & \qquad\qquad\qquad 32z^{-2}-17z^{-1}+4
\end{array}$$

$$H(z)=3z^2+4+(-2)-5z^{-1}+\frac{4-17z^{-1}+\ 32z^{-2}}{2-17z^{-1}+40z^{-2}-16z^{-3}}.$$

Let $Q(z)$ be the rational term and find its partial fractions

$$Q(z)=\frac{4-17z^{-1}+\ 32z^{-2}}{2-17z^{-1}+40z^{-2}-16z^{-3}}=\frac{2-\frac{17}{2}z^{-1}+16z^{-2}}{(1-\frac{1}{2}z^{-1})(1-4z^{-1})^2}$$

$$=\frac{1}{(1-4z^{-1})^2}+\frac{1}{1-\frac{1}{2}z^{-1}}.$$

Hence,

$$H(z) = 3z^2 + 2 - 5z^{-1} + \frac{1}{(1-4z^{-1})^2} + \frac{1}{1-\frac{1}{2}z^{-1}}.$$

So the causal part of $H(z)$ is

$$\left[H(z)\right]_+ = 2 - 5z^{-1} + \frac{1}{1-\frac{1}{2}z^{-1}},$$

which corresponds to the sequence $2\delta(n) - 5\delta(n-1) + \left(\frac{1}{2}\right)^n 1(n)$. Also the anticausal part of $H(z)$ is

$$\left[H(z)\right]_- = 3z^2 + \frac{1}{(1-4z^{-1})^2},$$

which corresponds to $3\delta(n+2) + \left[\left(-4\right)^n 1(-n-1)\right] * \left[\left(-4\right)^n 1(-n-1)\right]$.

The Causal Wiener Filter

Assume that $S_{\mathrm{z}}(z)$ is rational in *z* and has no poles or zeros on the unit circle. The equation that we must solve to obtain the causal Wiener filter is as is given in Eq. (3.4-1)

$$\sum_{j=0}^{\infty} h(j) R_{\mathrm{z}}(j-i) = R_{\mathrm{sz}}(i), \quad i \geq 0.$$

To solve the Wiener-Hopf equation, define

$$h'(i) = R_{\mathrm{sz}}(i) - \sum_{j=0}^{\infty} h(j) R_{\mathrm{z}}(j-i), \quad \text{for all } i \in \mathbb{Z}. \tag{3.4-12}$$

Since *h* is the causal Wiener filter, $h'(i)$ can be written

$$h'(i) = \begin{cases} 0, & i \geq 0 \\ R_{\mathrm{sz}}(i) - \sum_{j=-\infty}^{\infty} h(j) R_{\mathrm{z}}(j-i), & i < 0. \end{cases} \tag{3.4-13}$$

Now we can take the *z*-transform of both sides of Eq. (3.4-12) and obtain

$$
\begin{aligned}
H'(z) &= S_{\text{sz}}(z) - H(z)S_{\text{z}}(z) \\
&= S_{\text{sz}}(z) - H(z)S_{\text{z}}^{+}(z)S_{\text{z}}^{-}(z).
\end{aligned}
$$

Dividing by $S_{\text{z}}^{-}(z)$, we have

$$
\frac{H'(z)}{S_{\text{z}}^{-}(z)} = \frac{S_{\text{sz}}(z)}{S_{\text{z}}^{-}(z)} - H(z)S_{\text{z}}^{+}(z).
$$

Extract the causal part of both sides of this equation and find

$$
\left[\frac{H'(z)}{S_{\text{z}}^{-}(z)}\right]_{+} = \left[\frac{S_{\text{sz}}(z)}{S_{\text{z}}^{-}(z)}\right]_{+} - \left[H(z)S_{\text{z}}^{+}(z)\right]_{+}. \tag{3.4-14}
$$

From Eq. (3.4-13), $h'(i)$ is purely anticausal. Therefore $H'(z)$ contains only poles outside the unit circle. Since the zeros of $S_{\text{z}}^{-}(z)$ lie outside the unit circle, the poles of $\dfrac{1}{S_{\text{z}}^{-}(z)}$ also lie outside the unit circle. We conclude that all the poles of $\dfrac{H'(z)}{S_{\text{z}}^{-}(z)}$ are outside the unit circle, and thus

$$\left[\frac{H'(z)}{S_z^-(z)}\right]_+ = 0.$$

$H(z)$ is causal by definition, so its poles lie within the unit circle. The poles of $S_z^+(z)$ are also within the unit circle. Hence all the poles of $H(z)S_z^+(z)$ are inside the unit circle, and

$$\left[H(z)S_z^+(z)\right]_+ = H(z)S_z^+(z).$$

We can not conclude anything about $\dfrac{S_{sz}(z)}{S_z^-(z)}$. Therefore, Eq. (3.4-14) becomes

$$0 = \left[\frac{S_{sz}(z)}{S_z^-(z)}\right]_+ - H(z)S_z^+(z),$$

which produces the causal Wiener filter, described by its system function $H(z)$

$$H(z) = \frac{1}{S_z^+(z)}\left[\frac{S_{sz}(z)}{S_z^-(z)}\right]_+. \tag{3.4-15}$$

<u>Example 3.4-3</u>

Consider the same situation of Examples 3.2-1 and 3.3-1 in which $\mathrm{s}(n)$ and $\mathrm{v}(n)$ are uncorrelated, zero-mean, JWSS random processes with

$$S_{\mathrm{s}}(z) = \frac{1-(0.95)^2}{(1-0.95z^{-1})(1-0.95z)},$$

and

$$S_{\mathrm{v}}(z) = 2.$$

We observe $\mathrm{z}(n) = \mathrm{s}(n) + \mathrm{v}(n)$ and estimate $\mathrm{s}(n)$.

From Example 3.3-1, we have

$$S_{\mathrm{z}}(z) = \frac{2.3955(1-0.7931z^{-1})(1-0.7931z)}{(1-0.95z^{-1})(1-0.95z)}.$$

Perform a spectral factorization on $S_{\mathrm{z}}(z)$ and obtain

$$S_{\mathrm{z}}^{+}(z) = 1.5477\frac{(1-0.7931z^{-1})}{(1-0.95z^{-1})},$$

and

$$S_z^{-}(z) = 1.5477\frac{(1-0.7931z)}{(1-0.95z)}.$$

$S_{sz}(z) = S_s(z)$ because s(n) and v(n) are uncorrelated. Therefore, we have

$$\begin{aligned}\frac{S_{sz}(z)}{S_z^{-}(z)} &= \frac{0.0630}{(1-0.95z^{-1})(1-0.7931z)} \\ &= \frac{-0.0794z^{-1}}{(1-0.95z^{-1})(1-1.2608z^{-1})} \\ &= \frac{0.2555}{1-0.95z^{-1}} - \frac{0.2555}{1-1.2608z^{-1}}\end{aligned}$$

and

$$\left[\frac{S_{sz}(z)}{S_z^{-}(z)}\right]_{+} = \frac{0.2555}{1-0.95z^{-1}}.$$

The causal Wiener filter is then

$$H(z) = \frac{1}{S_{\mathrm{z}}^{+}(z)}\left[\frac{S_{\mathrm{sz}}(z)}{S_{\mathrm{z}}^{-}(z)}\right]_{+}$$

$$= \frac{1-0.95z^{-1}}{1.5477(1-0.7931z^{-1})} \bullet \frac{0.2555}{1-0.95z^{-1}} = \frac{0.1651}{1-0.7931z^{-1}},$$

and

$$h(n) = 0.1651(0.7931)^{n}1(n).$$

The MSE associated with this filter is computed using Eq. (3.3-3)

$$\mathrm{MSE}_{\mathrm{C}} = 1-0.1651\sum_{i=0}^{\infty}(0.7931)^{i}(0.95)^{i} = 0.3302.$$

Compared to no filtering (MSE = 2), the causal Wiener filter reduces the MSE by 7.8 dB. (6.5 dB in FIR filter and 9.6 dB in noncausal filter.)