**Academic Integrity Checklist**

Please read the checklist below. Once you have verified these points, sign the checklist and submit with your assignment or test.

- I understand that I am responsible for being honest and ethical in this assessment as per Policy 71

- I have included in-text citations or footnotes when referencing words, ideas, or other intellectual property from other sources in the completion of this assessment, if applicable

- I have included a proper bibliography or works cited, which includes acknowledgement of all sources used to complete this assessment, if applicable

- The assessment was completed by my own efforts and I did not collaborate with any other person for ideas or answers

- This is the first time I have submitted this assessment (either partially or entirely) for academic evaluation

______________________________
Student Name (by signing or typing my name here I affirm my agreement to the foregoing statements)

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Student I.D. Number

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Date

UNIVERSITY OF WATERLOO

# AMATH 342 Winter 2021: Assignment 4

**Due Date: Wednesday 14 April, 2021.**
**Total points: 32.**

- Assignments below are either theory or computational. For the computational/programming assignments you may use any programming language you want **except** Maple or Mathematica. Besides your written/typed solutions you must also submit your code.
- This assignment should be submitted to Crowdmark. Please make sure your upload has the correct orientation.
- Office hrs: Tuesday's and Thursday's 10:00–10:50am (local time Waterloo, ON). A link will be emailed to you before each office hour.
- If you want to use your one time 3-day extension, the due date is Monday 19 April, 2021. Please email the TA if you plan to use your extension.
- **Important:** Please note that the final exam takes place on 22 April. Therefore, if you use your 3-day extension, please take into account that the TA might has much less time to grade your assignment before the final.

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**1.** **(0 points)** Please sign the Academic Integrity Checklist. If you do not sign the Academic Integrity Checklist you will receive a 0 for this assignment.

**2.** Consider the advection equation:

$$\partial_t u + a\partial_x u = 0, \qquad -\infty < x < \infty,\ t > 0 \quad \text{and} \quad u(x,0) = u_0(x), \tag{1}$$

with $a > 0$ a constant. Discretize the $x$-$t$ plane by choosing a mesh width $\Delta x$ and time step $\Delta t$ and define the discrete mesh points $(x_j, t_n)$ by

$$x_j = j\Delta x, \qquad j = ..., -1, 0, 1, 2, ... \quad \text{and} \quad t_n = n\Delta t, \qquad n = 0, 1, 2, ...$$

(a) **(2 points)** Show that the following difference scheme is not stable:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\left(\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}\right) = 0. \tag{2}$$

We will now derive a stable scheme.

(b) **(1 point)** From (1) it is clear that $\partial_t u = -a\partial_x u$. Show that $\partial_{tt} u = a^2 \partial_{xx} u$.

(c) **(2 points)** Using the Taylor series expansion of $u(x, t + \Delta t)$ around $t$ and question (b), show that

$$u(x, t + \Delta t) = u(x,t) - \Delta t a \partial_x u(x,t) + \tfrac{1}{2}(\Delta t)^2 a^2 \partial_{xx} u(x,t) + \text{h.o.t.} \tag{3}$$

(h.o.t = higher order terms).

(d) **(2 points)** Approximating the $x$-derivatives in (3) by central differences, show that we obtain the following finite difference scheme:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{2\Delta x} a \left(U_{j+1}^n - U_{j-1}^n\right) + \frac{\Delta t^2}{2\Delta x^2} a^2 \left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right). \tag{4}$$

(e) **(3 points)** Define $\nu = a\Delta t/\Delta x$. Using Fourier analysis, find the amplification factor. Under what condition on $\nu$ is the finite difference scheme (4) stable?

(over)

**3**. Given is the following problem:

$$\partial_t u = \partial_x \left(a(x)\partial_x u\right), \qquad u(0,t) = 0,\ u(1,t) = 0,\ u(x,0) = u_0(x), \quad \text{where } a(x) > 0. \tag{5}$$

(a) **(2 points)** Find the finite difference scheme for (5) using the $\theta$-method in time. You may use that central differences in space for the $\partial_x \left(a(x)\partial_x u\right)$ term is given by

$$\partial_x \left(a(x)\partial_x u\right) \approx \frac{a_{j+1/2}(U_{j+1} - U_j) - a_{j-1/2}(U_j - U_{j-1})}{(\Delta x)^2}. \tag{6}$$

where $a_{j+1/2} = a(x_{j+1/2})$ and $x_{j+1/2} = \frac{1}{2}(x_j + x_{j+1})$.

(b) **(2 points)** Show that the maximum principle is satisfied if

$$2\Delta t(1-\theta) \max_{x\in[0,1]} a(x) \leq (\Delta x)^2.$$

(c) **(5 points)** Show that (6) is a consistent discretization of $\partial_x \left(a(x)\partial_x u\right)$.

**4. (4 points)** Let $0 < x < 1$ and $0 < t < T_F$ and consider the following mixed initial-boundary value problem:

$$\partial_t u = 2\partial_{xx} u, \qquad u(0,t) = u(1,t) = 1, \qquad u(x,0) = f(x), \tag{7}$$

where $f(x) = 2$ if $x = 0.5$ and $f(x) = 1$ otherwise. Suppose the mesh points are chosen to satisfy

$$0 = x_0 < x_1 < x_2 < ... < x_{J-1} < x_J = 1$$

where $x_j - x_{j-1} = \Delta x$ for all $j = 1, 2, ..., J$. Implement the $\theta$-method for problem (7). To solve the implicit system you need to solve a matrix system of the form $AU^{n+1} = F$. Write down the matrix $A$ and vector $F$. Now compute a solution using $\theta = \frac{1}{2}$ and $J = 10$. Make two plots: in the first plot plot the solution at all computed time levels between 0 and $T_F = 0.07$ using $\Delta t = \Delta x^2$ (so plot the solution at $t = 0, t = \Delta t, t = 2\Delta t, ..., t = T_F$); in the second plot, do the same but use $\Delta t = 0.5\Delta x^2$. Explain the difference in solution.

(over)

**5**. Being able to capture boundary layer effects in fluid dynamics is very important and the grid plays an important role in being able to capture these effects.

Let $0 < x < 1$ and $0 < t < 10$ and consider the following advection-diffusion problem:

$$\partial_t u + \partial_x u - \frac{1}{Re}\partial_{xx} u = 0, \tag{8}$$

where $Re$ is the Reynolds number. Consider the following boundary conditions $u(0,t) = 1$ and $u(1,t) = 0$ and initial condition $u(x,0) = 1 - x$. The exact steady-state solution to this problem is given by

$$u(x) = \frac{\exp(Re) - \exp(Re\, x)}{\exp(Re) - 1}. \tag{9}$$

To discretize (8), we first use a uniform grid, where

$$x_i = (i-1)\Delta x, \qquad i = 1, 2, ..., N+1, \tag{10}$$

where $\Delta x = 1/N$. As finite difference method, we use, for $i = 2, 3, ..., N$,

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(U_i^n - U_{i-1}^n\right) + \frac{1}{Re}\frac{\Delta t}{\Delta x^2}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right), \tag{11}$$

(a) **(5 points)** Take $Re = 40$. Implement the finite difference discretization (11). For this, take $N = 64$ and $\Delta t$ small enough such that your scheme is stable. In one figure, plot the exact steady solution given by (9) and the numerical approximation at time $t = 10$. At $t = 10$, compute the error $E = \max_{i=1,...,N} |U_i - u(x_i)|$. Compute this error also when $N = 128$. Based on these two errors, what would you say is the order of the finite difference method given by (11)?

We will now compute the solution on the following *non-uniform* grid:

$$x_i = \begin{cases} 2(1-c)(i-1)/N, & \text{for } i = 1, 2, ..., N/2+1, \\ 1 - c + \frac{2c(i-1-N/2)}{N}, & \text{for } i = N/2+1, N/2+2, ..., N+1, \end{cases} \tag{12}$$

where $c = (2/Re)\ln(N)$. The reason to use this non-uniform grid is that for high Reynolds numbers $Re$ the solution rapidly changes in a small neighbourhood of the point $x = 1$. This non-uniform grid captures these changes better than a uniform grid.

A finite difference approximation to (8), on the non-uniform grid (12), is given by

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x_i}\left(U_i^n - U_{i-1}^n\right) + \frac{1}{Re}\frac{\Delta t}{\Delta x_{i+1}}\left(\frac{U_{i+1}^n - U_i^n}{\Delta x_{i+1}} - \frac{U_i^n - U_{i-1}^n}{\Delta x_i}\right), \tag{13}$$

for $i = 2, 3, ..., N$, and where $\Delta x_i = x_i - x_{i-1}$.

(b) **(3 points)** Take $Re = 40$. Implement the finite difference discretization (13). For this, take $N = 64$ and $\Delta t$ small enough such that your scheme is stable. In one figure, plot the exact steady solution given in part (a) and the numerical approximation at time $t = 10$. At $t = 10$, compute the error $E = \max_{i=1,...,N} |U_i - u(x_i)|$. Compute this error also when $N = 128$. Based on these two errors, what would you say is the order of the finite difference method given by (13)?

(c) **(1 point)** Now take $Re = 1000$. In the same figure, plot the solution computed on the uniform grid and the solution on the non-uniform grid. Use $N = 64$. Explain the difference.