Academic Integrity Checklist

Please read the checklist below. Once you have verified these points, sign the checklist and submit with your assignment or test.

- I understand that I am responsible for being honest and ethical in this assessment as per Policy 71
- I have included in-text citations or footnotes when referencing words, ideas, or other intellectual property from other sources in the completion of this assessment, if applicable
- I have included a proper bibliography or works cited, which includes acknowledgement of all sources used to complete this assessment, if applicable
- The assessment was completed by my own efforts and I did not collaborate with any other person for ideas or answers
- This is the first time I have submitted this assessment (either partially or entirely) for academic evaluation

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Student Name (by signing or typing my name here I affirm my agreement to the foregoing statements)

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Student I.D. Number

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Date

UNIVERSITY OF WATERLOO

AMATH 342 Winter 2021: Assignment 1

Due Date: Friday 29 January, 2021. Total points: 35.

- Assignments below are either theory or computational. For the computational assignments you may use any programming language you want **except** Maple or Mathematica. Besides your written/typed solutions you must also submit your code.
- This assignment should be submitted to Crowdmark. Please make sure your upload has the correct orientation.
- Office hrs: Tuesday's and Thursday's 10:00–11:00am (local time Waterloo, ON). A link will be emailed to you before each office hour.
- If you want to use your one time 3-day extension, the due date is Wednesday 3 February, 2021. Please email the TA if you plan to use your extension.

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1. **(0 points)** Please sign the Academic Integrity Checklist. If you do not sign the Academic Integrity Checklist you will receive a 0 for this assignment.
2. **(5 points)** Exercise 1.1 of the book by Arieh Iserles. Only prove convergence of the implicit midpoint rule (1.12). You do not need to prove convergence of the theta method (1.13).
3. Consider the scalar linear problem $y' = f(y)$, $y(0) = 1$, $t \in [0, t^\star]$, where $f(y) = ay$ with $a < 0$.

   (a) **(1 point)** Write down Euler's method for this problem.

   (b) **(3 points)** Now take $a = -2$ and $t^\star = 100$. To prove that Euler's method is convergent, we proved that
   $$|e_{n,h}| \leq \frac{c}{\lambda}(\exp(t^\star\lambda) - 1)h,$$
   where λ is the Lipschitz constant associated with $f(y)$ and c a constant bounding the $\mathcal{O}(h^2)$ term in the Taylor series expansion of $y(t_{n+1})$ (see the proof of Theorem 1.1 of the Arieh Iserles book). A reasonable choice is $c = 2$ (see page 7 of Arieh Iserles book). Find a value for λ and therefore a bound for the norm of the error of the form $|e_{n,h}| \leq \alpha h$ with $\alpha \in \mathbb{R}^+$. Is this bound of any use in practice?

   (c) **(4 points)** An improved error bound is given by
   $$|e_{n,h}| \leq \tfrac{1}{2}t^\star a^2 h.$$
   Prove this error bound. With $a = -2$ and $t^\star = 100$, compare this error bound to that found in (b).
   Hint: Show that
   $$|e_{n,h}| = |(1 + ah)^n - \exp(anh)|.$$
   Then use that for $-1 \ll x \leq 0$, and $n = 0, 1, 2, ...$ that
   $$\exp(nx) - \tfrac{1}{2}nx^2\exp((n-1)x) \leq (1 + x)^n \leq \exp(nx).$$

4. We are given the following ODE:

$$y'' + by' + cy = 0, \qquad t \in [0,1], \qquad y(0) = 1,\ y'(0) = 0,$$

where $b^2 = 4c$.

(a) **(1 point)** Find the exact solution to the above problem.

(b) **(1 point)** Write the above second-order ODE as a system of first order ODEs.

(c) **(5 points)** Implement Euler's method to solve the system of first order ODEs. Take $b = 10$. In 3 separate figures plot both the numerical solution and the exact solution. In Figure 1 use $h = 0.5$, in Figure 2 use $h = 0.05$ and in Figure 3 use $h = 0.005$.

(d) **(2 points)** Let the error on a grid with time step h be defined as:

$$E_h = \max_{0 \le nh \le 1} |y_{n,h} - y(nh)|$$

For $h = 0.5$, $h = 0.05$ and $h = 0.005$ compute E_h. You will see that the error satisfies $E_h = \mathcal{O}(h^p)$. What is p? Explain your answer.

(e) **(5 point)** Write down the theta-method for the system of first order ODEs of question (b). Repeat questions (c) and (d) but instead of the Euler method, use the θ-method with $\theta = 0.5$. Which of the two methods discussed here converges faster to the exact solution? Explain your answer.

5. The Lorenz equations are a simplified model of convection in the earths atmosphere, and is given by

$$\begin{aligned}
\frac{dx}{dt} &= \sigma(y - x) \\
\frac{dy}{dt} &= x(\rho - z) - y \\
\frac{dz}{dt} &= xy - \beta z
\end{aligned}$$

where σ, ρ and β are system parameters.

(a) **(5 points)** Take $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$. Implement Euler's method to solve the Lorenz equations. Let $t \in [0, 50]$ and take $h = 0.002$. As initial condition take $(x_0, y_0, z_0) = (1, 0, 0)$. Plot the solution where the horizontal axis is x and the vertical axis is z.

(b) **(1 point)** Repeat question (a) but with $\rho = 14$.

(c) **(2 points)** Search online or in a textbook for the definitions of 'chaos', 'strange attractor', 'Lorentz attractor', etc. to explain the plots found in questions (a) and (b). (A short two or three sentence explanation is sufficient.)