**AMATH 455/655**

# Lab 3: Control of Water Height in a Tank

**Winter 2021**

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The report is due no later than 11:59pm Monday, March 22, and should be submitted through Crowdmark. Questions marked with a (*) are required for AMath655 students; optional bonus questions for Amath455. You may choose to collaborate with one classmate for the Simulation Part. However, the submitted work should reflect your own understanding and be in your own words. Indicate the name of your collaborator at the top of your report if applicable.

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Consider a single tank of water filled by a pump and draining through a hole at the base. The inflow to the tank is proportional to the voltage $V$ applied to the pump:

$$F_{\text{in}}(t) = K_m V(t) \text{ cm}^3/\text{sec}.$$

From Bernouilli's Law, the velocity of the flow out of the tank is

$$v_{out}(t) = \sqrt{2gL(t)} \text{ cm/sec},$$

where $g$ is the gravitational acceleration in cm/sec$^2$, and $L(t)$ is the height of water in the tank in cm. The volume flow rate out is thus

$$F_{\text{out}}(t) = a\sqrt{2gL(t)} \text{ cm}^3/\text{sec},$$

where $a$ is the area of the outflow orifice in cm$^2$. Since the density of the water doesn't change, conservation of mass can be replaced by conservation of volume. Letting $A$ be the cross-sectional area ($cm^2$ ) of the tank,

$$A\dot{L}(t) = K_m V(t) - a\sqrt{2gL(t)}. \tag{1}$$

1. Determine the equilibrium level $L_0$ as a function of a constant applied voltage $V_0$.
2. Linearize (1) about the equilibrium $L_0$ determined in Question 1 with voltage $V$ as the control input.
3. Is the equilibrium point of the original model (1) with constant voltage $V(t) = V_0$ locally asymptotically stable?
4. Does external stability of the controlled linearized system imply local asymptotic stability? Assume that the implementation of the controller is controllable and observable. Explain your answer.
5. Define a new state variable $x(t) = L(t) - L_0$, input $u(t) = V(t) - V_0$ and write the linearized system in state space form. The measured output is the water height $x(t)$.
6. Compute the transfer function, $P(s)$, of the linearized system determined in Question 5.

| | | |
|---|---|---|
| Tank 1 Outflow orifice diameter | $d$ | 0.4763 cm |
| Tank 1 Area of the outflow orifice | $a$ | $a = \pi(d/2)^2 = 0.1781$ cm$^2$ |
| Tank 1 Cross-sectional Area | $A$ | 15.5179 cm$^2$ |
| gravitational constant | $g$ | 980 cm/sec$^2$ |
| Flow Constant | $K_m$ | 3.88 cm$^3$/(V·sec) |

Table 1:

7. Find the reference voltage $V_0$ associated with water height $L_0 = 15$cm. Use the parameter values in Table 1.

8. Consider the system linearized about $L_0$ with the values of $L_0$, $V_0$ in Question 7, and a controller

$$H(s) = K_p + K_i/s, \tag{2}$$

where $K_p$ is the constant proportional gain, and $K_i$ is the constant integral gain. Such controllers are known as PI controllers.

(a) Compute the various transfer functions $\frac{1}{1+PH}$, $\frac{H}{1+PH}$ , $\frac{-P}{1+PH}$ for the closed loop system.

(b) Write $P$ and $H$ as ratios of coprime polynomials:

$$P = \frac{n_P}{d_P}, \qquad H = \frac{n_H}{d_H}.$$

(That is, $n_P$ and $d_P$ have no common zeros; and similarly for $n_H, d_H$.) Define the *characteristic polynomial*

$$\kappa = n_P n_H + d_P d_H .$$

Since $\lim_{s\to\infty} P(s) = 0$, the closed loop is stable if and only if $\kappa$ has no (closed) right-half plane zeros. (This is shown in the text.)

i. Consider $K_i = 0$ (i.e., with only proportional control). For what range of values of $K_p$ is the closed-loop system externally stable? All physical parameters are positive.

ii. Consider $K_i \neq 0$. For what range of values of $K_p$ and $K_i$ the closed-loop system externally stable?

9. (a) The denominator in the transfer functions for the closed loop system with PI controller determined in Question 8a is a second-order polynomial and can be written in the form

$$s^2 + 2\zeta\omega_n s + \omega_n^2.$$

Determine the natural frequency, $\omega_n$, and the damping constant, $\zeta$, in terms of the feedback gains $K_p$, $K_i$.

(b) Determine values of $K_p$ and $K_i$ so that the system exhibits a 98% settling time $t_s$ of less than 5 seconds and a less than one percent overshoot $PO$. (Note: $t_s = \frac{4}{\zeta\omega_n}$ and $PO = 100\exp(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}})$. How does the system response change if a smaller value of $K_p$ is used?

10. Construct a Simulink model of the **nonlinear** model of the tank as follows:

   (i) Create a new simulink model file.

   (ii) Add a **Subsystem** block from the **Commonly Used Blocks** library. Rename "Subsystem" as "Tank system".

   (iii) Double-click on the subsystem block to open it in a new window. Rename "in" as $V$ and "out" as $L$. Delete the existing link between 'in' and 'out'.

   (iv) Implement equation (1) between 'in' and 'out'. [Hint: You may need the following blocks: gain, sum, integrator. You may also use **square-root function** available in the pull-down menu of **Math function** block, which can be found in **Math operations**].

   (v) Close the subsystem window. Connect the output of the 'tank system' to a scope block.

   (vi) Add a **Constant** block, which can be found in **Commonly Used Blocks** library. Rename it as $L_0$. Set the value to 15. Add a **square-root function**, followed by a **gain** block to obtain the value of desired voltage $V_0$. Connect it to the input of the 'tank system'.

   (a) *Include screen shot of the Simulink model* (that is, the main system plus the subsystem) that was created based on steps (i) to (vi).

   (b) *Submit a plot of the open loop system's response for* 100 *seconds.*

   (c) Now, change the input $L_0$ to $L_0 \pm 1$ by adding a block with a square wave generator of magnitude $\pm 1$ and frequency 0.1 Hz. (Since the magnitude of the square wave changes more slowly than the system settling time, this will yield the response to a series of "constant" inputs. *Submit a plot of the system's response for* 100 *seconds. What happens to the water height in the tank?*

11. Add a PI closed loop control to the Simulink model in Question 10.

   (a) *Include screenshot of the Simulink model* and ensure that both the main system and the subsystem are included.

   (b) Choose values for $K_p$ and $K_i$ based on your analysis in Question 8 . *Submit a plot of the system's response with constant reference input* 15.

   (c) Add a square wave input (pulse) of magnitude 1 and frequency 0.1Hz so the input is either 14 or 16. *Submit a plot of the system response over* 100 *seconds.*

   (d) *Compare the response to the open loop system.*

(e) Keeping the same value of $K_i$, choose $K_p > 0$ half that of the value used in question 11b. *Submit a plot of the system's response with the same input used in question 11c for* 100 *seconds. How does the system's response change? Is this similar to that predicted by linear analysis?*

(f) Use the same value of $K_p$ as in question 11b and change the value of $K_i$ to half of the original value. *Submit a plot of the system's response with the same input used in question 11c for* 100 *seconds. How does the system's response change? Is this similar to that predicted by linear analysis?*

(g) What is the steady-state error to a step input, in terms of $K_p$ if $K_i = 0$?

(h) What is the steady-state error to a step input if $K_i \neq 0$? Compare the steady-state error with a proportional controller ($K_i = 0$) and with a PI controller.