AMATH 455/655

Lab 2: Frequency Response of a Second-Order System

Winter 2021

The report is due no later than 11:59pm Monday, February 22, and should be submitted through Crowdmark. Questions marked with a (*) are required for AMath655 students; optional bonus questions for Amath455. You may choose to collaborate with one classmate for the Simulation Part. However, the submitted work should reflect your own understanding and be in your own words. Indicate the name of your collaborator at the top of your report if applicable.

Part 1

The dynamics of a mass-spring damper system, with applied force $u(t)$ are modelled by

$$m_l\ddot{x}(t) + d\dot{x}(t) + kx(t) = bu(t), \tag{1}$$

where m_l is the mass, d is the system damping, k is the spring constant and b is a proportionality constant for the input force.

1. The position is measured. Find the Laplace transform of the map from input u to x in equation (1) with zero initial conditions. Verify that the transfer function from the input $u(s)$ to $x(s)$ is of the form

$$\frac{\eta}{s^2 + 2\xi\omega_n s + \omega_n^2}. \tag{2}$$

   What are η, ξ and ω_n in terms of the system parameters?

2. Generate a Bode plot of the transfer function described in equation (2) using the parameters in Table 1.

m_l	0.52 kg
k	142 N/m
d	2 Ns/m
b	k

Table 1: Parameter Values for cart-spring system

   Use the Bode plot to fill in Table 2. (The Matlab commands ‘tf’ and ‘Bode’ will be helpful in generating the Bode plot and determining the values of the gain and phase at each given frequency. Type ‘help tf’ and ‘help bode’ for more information. Also, when creating Bode plots, the ‘grid on’ option is a useful feature to have.)

   Include as part of your report the m-file that generates the transfer function and the Bode plot.

Table 2: Gain and phase of second-order system using transfer function

ω	gain (dB)	phase (degrees)
5		
10		
20		
80		
ω_n		

3. Change the natural frequency ω_n to 30 and also 5 $/s$. Keep the other parameters the same. Plot the frequency response of the three systems on the same plot. (The MATLAB command 'hold on' maybe helpful for achieving this, or you can plot 3 Bode plots at once. Be sure to include a legend clearly indicating which Bode plots corresponds to which values of ω_n.)

4. Explain how the natural frequency, ω_n, can be estimated from the frequency response.

Part 2

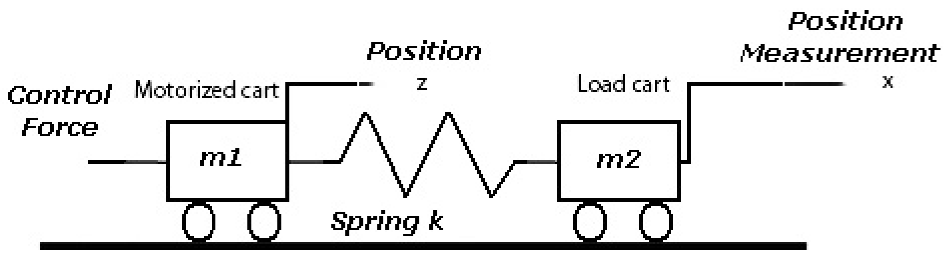

Figure 1: Lab apparatus

The system involves 2 carts, a load cart and a motorized cart, connected through a spring. (See Figure 1.) In the figure z is the position of the motorized cart and x is the position of the load cart. The force on the load cart is applied indirectly by a motorized cart as shown in Figure 1. The load cart dynamics are modelled here by

$$m_l\ddot{x}(t) + d\dot{x}(t) + k(x(t) - z(t)) = 0.$$

which is the same as equation (1), except with $z(t)$ instead of $u(t)$.

1. Recall from Lab 1 that the motor cart with no spring attached can be modelled by

$$m_m\ddot{z}(t) + \nu\dot{z}(t) = \alpha V_m(t) + \beta\dot{z}(t) \tag{3}$$

   where $V_m(t)$ is the voltage applied to the motor. The values of the parameters are shown in Table 3.

   The addition of the spring and load cart alters the force on the cart. The motion of the motorized cart is modelled by

$$m_m\ddot{z}(t) + \nu\dot{z}(t) + k(z(t) - x(t)) = \alpha V_m(t) + \beta\dot{z}(t). \tag{4}$$

2. Write the system in state-space form with states $\begin{bmatrix} x(t) & z(t) & \dot{x}(t) & \dot{z}(t) \end{bmatrix}$. Is the system controllable with the voltage as control?

3. Let $\hat{z}(s)$, $\hat{x}(s)$ and $\hat{V}_m(s)$ indicate the Laplace transforms of $z(t)$, $x(t)$ and $V_m(t)$, respectively. Take Laplace transforms, with zero initial conditions, of (3) and (4).

4. What is the transfer function from $\hat{z}$ to $\hat{x}$? Compare to the transfer function in part 1.

m_m	0.94 kg
ν	5.4 kg/s
α	1.5167
β	-6.7967

Table 3: Parameter values for motorized cart

5. Show that the transfer function from the motor voltage, $\hat{V}_m(s)$, to the motorized cart position, $\hat{z}(s)$, has the form

$$\frac{\alpha(m_l s^2 + ds + k)}{\gamma_4 s^4 + \gamma_3 s^3 + \gamma_2 s^2 + \gamma_1 s + \gamma_0}.$$

What are γ_i, $i = 0, 1 \cdots 4$, in terms of the system parameters m_m, ν, etc.? What are the numerical values of γ_i?

6. The motor cart position $z(t)$ will be considered as the the input to the system. To determine the frequency response of $x(t)$, the position $z(t)$ needs to be a sine wave. This part is concerned with design of a controller to accomplish this.

   Construct a Simulink model of this system as follows.

   (a) Use a **Sine Wave** as the system input (found in the **Sources** library).

   (b) Add a **Subsystem** block from the **Commonly Used Blocks** library.

   (c) Double-click on the subsystem block to open it in a new window. Add a second **Out1** block from the **Commonly Used Blocks** library and delete the existing connection between the original input and output blocks. Label one of the outputs as $\hat{z}(s)$, the motorized cart position, and the other as $\hat{x}(s)$, the load cart position.

   (d) You have already calculated the transfer functions from $\hat{V}_m(s)$ to $\hat{z}(s)$ and from $\hat{z}(s)$ to $\hat{x}(s)$. Use two **Transfer Fcn** blocks from the **Continuous** library to generate the outputs $\hat{z}(s)$ and $\hat{x}(s)$ from the input $\hat{v}_m(s)$.

   (e) Close the subsystem window. Connect both the outputs of the subsystem to a scope via a **Mux** block, which can be found in the **Commonly Used Blocks** library.

   (f) Add a **Saturation** block directly before the subsystem. Set the upper and lower limits to +2.5 and -2.5 respectively. This limits the control input that can be applied.

   (g) Add a **To File** block from the **Sinks** library to save the data as a ".mat" file for easier manipulation.

   (h) Add feedback to the system by splitting the "$\hat{z}$" output of the subsystem and subtracting it from the input sine wave. This requires adding a **Summation block** with two input ports (one positive and the other negative). Put a feedback **Gain block** after the summation block, and before the saturation. Use $K = 12$ in the feedback.

   As part of your report, include screenshots of Simulink model (that is, the main system plus the subsystem) that was created based on steps (a) to (h).

7. Using an input sine wave of amplitude 0.4 and frequency 5 rad/s, run the simulation for 10 seconds. Submit a plot of the input, $z(t)$, and the output, $x(t)$, positions on the *same plot*.

8. Repeat the previous step with frequencies 10, 20, 80 rad/s and the natural frequency. Create a separate plot for each frequency as for the previous step.

9. In this step you calculate the gain and phase of the system at various frequencies using the plots from the previous steps. Recall that the asymptotic response to input $u(t) = e^{i\omega t}$ is output $y(t) = G(i\omega)e^{i\omega}$. The response to $\sin(\omega t)$ is therefore $\mathrm{Im}(G(i\omega)e^{i\omega})$. Writing $G(i\omega) = M(\omega)e^{i\phi(\omega)}$, the asymptotic response to $A\sin(\omega t)$, where A is a constant, is $AM(\omega)\sin(\omega t + \phi(\omega))$. The gain at a particular frequency, ω, is $M(\omega)$, which is a ratio of the amplitude of the output to the amplitude of the input. That is, the gain is

$$\text{Gain} = 20\log_{10} M(\omega) = 20\log_{10}\left(\frac{\text{amplitude of output}}{\text{amplitude of input}}\right).$$

The phase of the system at a particular frequency $\phi(\omega)$ is the phase difference between the input and output.

The first method to calculate gain and phase requires looking at the graphs closely by zooming in, and then estimating the appropriate values. The second method is creating a program in MATLAB that will give a more precise value of the gain and phase shift. For the second method, the MATLAB code must be included as part of the solution.

Method 1 : To determine the phase difference between two sinusoids, two equivalent points on each sinusoid need to be compared: it is convenient to consider the point where each sinusoid is zero. From your plot, measure the time difference Δt between when the input sinusoid is zero and when the output sinusoid is zero. Knowing the time difference, and the frequency ω of the sinusoids, you can use the fact that $\omega = \frac{\Delta\theta}{\Delta t}$ to calculate the phase difference as $\phi = \Delta\theta = \omega\Delta t$, and then convert to degrees. Fill in the various values in the correct columns of Table 4.

Method 2 (via MATLAB CODE): Create a function file in MATLAB that will give the gain and phase for a given frequency of the sine wave. Include the MATLAB code as part of your solution.

Table 4: Gain and phase data for simulated system

$\omega(rad/s)$	input mag.	output mag.	Gain(dB)	Δt	Phase (rad)	Phase (degrees)
5						
10						
20						
80						
ω_n						

10. (*) Compare the values in Table 2 with the values obtained in Table 4. Are these values similar? Suggest reasons for any discrepancies.

11. Apply signals with higher frequency than the ones given in the table and discuss how the gain and phase changes in response to high frequency input.

12. According to the results obtained in Table 4 with different frequencies for the given system, suggest a technique to obtain the natural frequency of any unknown system.