AMATH 455/655

Lab 4: Control of an Inverted Pendulum

Winter 2021

The report is due no later than 11:59pm Monday, April 5, and should be submitted through Crowdmark. Questions marked with a (*) are required for AMath655 students; optional bonus questions for Amath455. You may choose to collaborate with one classmate for the Simulation Part. However, the submitted work should reflect your own understanding and be in your own words. Indicate the name of your collaborator at the top of your report if applicable.

Consider an inverted pendulum balanced on a moving cart. (Figure 1). The cart can move in the direction d, the pendulum can rotate in the plane.

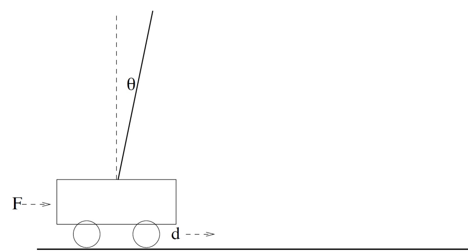

Figure 1: Inverted pendulum and cart

Let $f(t)$ be force on the cart (N), $d(t)$ cart position (m) and $\theta(t)$ the angle of the pendulum (rad). The angle is measured from the vertical up position, with clockwise angles positive. Hamilton's Principle can be used to derive the mathematical model, two second-order differential equations:

$$\begin{aligned} (M+m)\ddot{d}(t) + \epsilon\dot{d}(t) + mI_p\ddot{\theta}(t)\cos(\theta(t)) - m\dot{\theta}^2(t)I_p\sin(\theta(t)) &= f(t), \\ mI_p\cos(\theta(t))\ddot{d}(t) + \frac{4}{3}m\ddot{\theta}(t)I_p^2 - mgI_p\sin(\theta(t)) &= 0. \end{aligned}$$

Note that viscous damping $\epsilon\dot{d}(t)$ has been included in the model. The cart is moved to balance the pendulum about the vertical. The force f is applied to the cart through voltage u to a motor so

$$f(t) = \alpha u(t) - \beta\dot{d}(t)$$

where the second term represents electrical resistance in the motor.

1. Define the state $\mathbf{x} = (d, \theta, \dot{d}, \dot{\theta})$ and write the equations in first-order form

   $$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}, u).$$

   Linearize the equations about the upright equilibrium $(0,0,0,0)$, and write the linearized equations in state-space form:

   $$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t).$$

2. Find the equilibrium points of the original model and determine whether they are locally asymptotically stable or unstable.

3. Substitute in the parameter values in Table 1.

4. Show that the linearized system is controllable.

5. Is a short or long pendulum easier to balance? Explain why this would be the case, with reference to the eigenvalues of the linearized system.

6. Where should you look to make it easier to balance? Speculate briefly on reasons for this.

7. A linear-quadratic (LQ) controller will be used to stabilize the pendulum. Since there is only one control, the control weight can be set to $R = 1$ and the relative weighting of state and control adjusted by changing Q. The resulting controller should stabilize the system with little overshoot so that the system does not leave the region where the linearization is valid. Also, the control signal should not exceed 5 volts.

   Trial and error shows that state weight $Q = Q_1$ where $Q_1 = diag\begin{bmatrix}5000 & 3000 & 20 & 20\end{bmatrix}$, which penalizes positions heavily with little cost on the velocities, works well. (Other choices of Q are of course possible.) Use a Matlab routine such as *care* or *are* to solve the algebraic Riccati equation and hence the LQ-optimal feedback control K_1 for the linear system.

8. Produce plots of the response (d, θ, u) of the controlled linearized system with control K_1 to the initial condition $(0, 0.06\text{rad}, 0, 0)$. (Useful Matlab routines are *initial* , *lsim*. The command *subfigure* may also be useful.)

9. For state weight $\mathbf{Q}_2 = diag\begin{bmatrix}5000 & 15000 & 20 & 20\end{bmatrix}$, find the corresponding feedback controller K_2. Plot the response (d, θ, u) with the control K_2 and the same initial condition as in step 8. Compare the response to that in step 8.

10. Also design a controller K_3, with the state weight Q_1, but using a different estimate for the friction coefficient, $\epsilon = 2.0\text{kg/s}$. Produce plots, using the original linearized plant model, of the response (d, θ, u) of the controlled system with initial condition $(0, 0.06\text{rad}, 0, 0)$. Compare the different plots.

11. For what range of ϵ is the linearized plant and state feedback controller $\mathbf{K}_3$ stable?

12. The position $d(t)$ and angle $\theta(t)$ are measured. Define the appropriate matrix C defining the vector $y(t)$. (The matrix C will be 2×4.).

    Show that the system is observable.

13. Since the velocities aren't measured, an estimate of the full state needs to be designed. This will have the form

$$\frac{d\hat{\mathbf{x}}}{dt}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{L}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t)) + \mathbf{B}u(t). \tag{1}$$

    If $\mathbf{A} - \mathbf{LC}$ is Hurwitz, $\lim_{t\to\infty} \hat{\mathbf{x}}(t) = \mathbf{x}(t)$.

Parameter	Description	Value
M	cart mass (with weight)	0.815 kg
m	pendulum mass	0.210 kg
I_p	distance from pivot to pendulum centre of mass	0.305 m
g	gravitational constant	9.8 m/s^2
α	voltage to force conversion factor	1.719 N/V
β	electrical resistance	7.682 Ns/m
ϵ	coefficient of friction	3.86 kg/s

Table 1: Parameter Values

(a) Show that the eigenvalues of $\mathbf{A} - \mathbf{LC}$ can be placed arbitrarily by choice of $\mathbf{L}$.

(b) Previous experience has shown that chosing $\mathbf{L}$ so that the eigenvalues of $\mathbf{A} - \mathbf{LC}$ are at $-60, -90, -120, -150$ works well. Use the Matlab routine *place* to find $\mathbf{L}$.

14. Make a simulink diagram to simulate the controlled linear system with measurements d, θ. The controller will be
$$u(t) = \mathbf{K}_1 \hat{\mathbf{x}}(t)$$
with estimator (1).

    Include a screen shot of your Simulink diagram.

15. Check the diagram by seeing if you obtain the same plots as in step 8 if the initial condition on the pendulum and the estimator is the same; that is $\hat{\mathbf{x}}(0) = \mathbf{x}(0)$.

16. What happens if you put a different initial condition on the estimator than on the plant?

17. Add an external input to the pendulum and set it to a step. Compare the response of the plant and the estimated plant. Use the initial conditions from step 16.

18. (*) Repeat steps 14 and 15 with the linear pendulum model replaced by the nonlinear model. Compare the response with that of the linear system.

19. (*) With the nonlinear model, simulate the system with the controller K_3 designed using the smaller friction coefficient. Discuss the response.