Appendix C: Complex Numbers

- $z = a + bi$
- $i = \sqrt{-1}$
- $Im(z) = b$
- $Re(z) = a$

$i^0 = 1$

$i = \sqrt{-1}$

- $i^2 = -1$
- $i^3 = i \cdot i^2 = -\sqrt{-1}$

- $|z|$ = modulus of $z = \sqrt{a^2+b^2}$
- $\overline{z}$ = complex conjugate $= a - bi$
- $z\overline{z} = |z|^2$

↳ <u>polar form</u>

$z = r(\cos\theta + i\sin\theta)$ $\quad z = re^{i\theta}$

- principal argument:

if $-\pi < \theta < \pi$

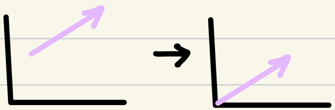
to find, add $2\pi + \theta$ until θ is in range

- $z^n = r(\cos n\theta + i\sin n\theta)^n$ or $r^n e^{in\theta}$

Chapter 1 Vectors

Standard position

has an origin at 0,0

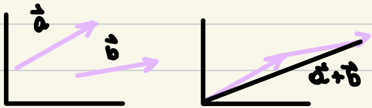


dot product (•)

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

vector addition



cross product

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\vec{u} \times \vec{v} = \begin{pmatrix} u_2v_3 - v_2u_3 \\ -(u_1v_3 - v_1u_3) \\ u_1v_2 - v_1u_2 \end{pmatrix}$$

length / norm ($\|\vec{v}\|$)

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1v_1 + v_2v_2 + v_3v_3}$$

↳ $\|c\vec{v}\| = c\,\|\vec{v}\|$

normalizing vectors

$$\frac{1}{\|\vec{v}\|}\vec{v}$$

projection of $\vec{v}$ onto $\vec{u}$

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\,\vec{u}$$

the Cauchy inequality

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \cdot \|\vec{v}\|$$

the triangle inequality

$$|\vec{u} + \vec{v}| \leq \|\vec{u}\| + \|\vec{v}\|$$

distance

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

Chapter 2: systems of linear equations

Solving linear equations

$$\begin{aligned} x - y - z &= 2 \\ 3x - 3y + 2z &= 16 \\ 2x - y + z &= 9 \end{aligned}$$

↳ augmented coefficient matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array}\right] \xrightarrow[R_2 - 3R_1]{R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & -1 & 5 \end{array}\right] \xrightarrow{\frac{R_2}{5} \Leftrightarrow R_3}$$

row echelon form

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_2 - R_1} \xrightarrow{R_1 + R_2} \xrightarrow{R_1 + R_3}$$

reduced row echelon

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{aligned} x &= 3 \\ y &= -1 \\ z &= 2 \end{aligned}$$

homogenous system

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

↳ homogenous systems always have a solution (either the trivial solution or infinite many solutions)

matrix solutions

pivot: leading entry

free variable: a column with no pivots

consistent systems: a system w solutions

unique solution: matrix with no free var.

linear combination

when a vector $\vec{w}$ is written in terms of other vectors (eg $\vec{w} = 2\vec{v} + 3\vec{y}$)

Spanning set

Let S be the set of vectors $\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_n\}$

Span (S) is the set of all linear combinations of S

If Span $(S) = \mathbb{R}^n$ then S is a spanning set of $\mathbb{R}^n$

linearly dependent

a set of vectors $\{\vec{v}_1 \dots \vec{v}_n\}$ is linearly dependent if there is at least one non trivial scalar such that $c_1\vec{v}_1 \dots c_n\vec{v}_n = 0$

Chapter 3 Matrices

matrix multiplication

$A \times B = AB$

$m\times n \quad n\times m \quad m\times m$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} = \begin{bmatrix} au+bw+cy & av+bx+cz \\ du+ew+fy & dv+ex+fz \end{bmatrix}$$

transpose

if A is an $m\times n$ matrix

A^T will be the $n\times m$ matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Symmetric matrices

square matrix is symmetric if $A^T = A$

the inverse of a matrix

A matrix A is invertible if and only if there exists a matrix B such that

$AB = I_n$ and $BA = I_n$

↳ $[A \mid I] \to [I \mid A^{-1}]$

elementary matrix

any matrix that can be obtained by performing elementary row operations to the Identity matrix

↳ how to express a matrix A as a product of elementary matrices

if a reduces to I_n, let each row operation be 1 to n. Perform row operations 1 to n to the I_n and name it $E_1, \dots E_n$. $A = ((E_1)^{-1} \cdots (E_n)^{-1})$

row equivalent matrices

A and B are R.E. when A + row oper = B

Subspace

any collection of vectors W is a Subspace of $\mathbb{R}^n$ if

W has the $\vec{0}$,

W is closed under addition

W is closed under scalar multiplication

↳ spans are always a subspace of $\mathbb{R}^n$

basis

a basis for a subspace W is a subspace that spans W and is linearly indep.

possible basis for a matrix A

- row (A)
- col (A)

Dimension

the # of vectors in any basis for a subspace

rank

number of pivots - $\text{rank}(A^T) = \text{rank}(A)$

nullity

the dimention of the nullspace of A

$\text{rank}(A) + \text{nullity}(A) = 0$

Linear transformation

a transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$ is linear (=)

1. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$ & $\vec{v}$ in $\mathbb{R}^n$
2. $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v}$ in $\mathbb{R}^n$ and all scalars c

Chapter 4 Eigenvalues and Eigenvectors

Determinants

- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

- $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-1)^{i+j}\left(a\begin{vmatrix} e & f \\ h & i \end{vmatrix}\right) + (-1)^{i+j}\left(b\begin{vmatrix} d & f \\ g & i \end{vmatrix}\right) + (-1)^{i+j}\left(c\begin{vmatrix} d & e \\ g & h \end{vmatrix}\right)$

- $A = \left[\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array}\right] \begin{array}{cc} a & b \\ d & e \\ g & h \end{array}$

  $-u \quad -v \quad -w$

  $+x \quad +y \quad +z$

$\det A = x + y + z - u - v - w$

- a square matrix is invertible if and only if $\det A \neq 0$

$\det A = \det A^T$

$\det(AB) = (\det A)(\det B)$

- Let A be an $n \times n$ matrix, if $A\vec{x} = \lambda\vec{x}$ then the scalar λ is an eigenvalue the vector $\vec{x}$ is called an eignvector

eigenspace

A collection of all eig-envectors and $\vec{0}$ that correspond to an eigenvalue (E_λ). span[eignvector]

how to find eignvalues for matrix A

$\det(A - \lambda I) = 0$; characteristic polynomial

- the solutions of the polynomial are the eigenvalues of A

how to find eigenvectors for λ

Solve $[A \mid \lambda - I]$, the set of vectors will be the eigenvectors

Algebraic multiplicity

how many times λ shows as a root

geometric multiplicity

the dim of an eigenspace

Simmilar Matrices

A is similar to B ($A \sim B$ if there is an invertible matrix P such that $P^{-1}AP = B$

↳ if $A \sim B$ then

- $\det A = \det B$
- $A^m \sim B^m$
- A and B have same charastisteric poly

Diagonalization

A matrix is diagonalizable if there is a diagonal matrix D and an invertible matrix P such that

$$P^{-1}AP = D$$

- $A^n = PD^nP^{-1}$

↳ to find P:

find eignvalues of A and eignvectors for each eigenvalue.

$P = [\vec{v}_1, \ldots, \vec{v}_n]$

Chapter 5 Orthogonality

orthogonal set

a set is orthogonal if all pairs of vectors in the set are orthogonal
aka $v_i \cdot v_i = 0$ for $i \neq j$ $i,j = 1,2,\dots k$

orthogonal basis

a basis of subspace W that orthogonal

theorem 5.2

if β is an orthogonal basis for a subspace W. When $\vec{w}$ is a vector in W, then there are unique scalars such that

$\vec{w} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$, then

$$c_i = \frac{\vec{w}\cdot\vec{v}_i}{\vec{v}_i\cdot\vec{v}_i} \quad \text{for } i = 1\dots k$$

then $[\vec{w}]_\beta = \begin{bmatrix} c_i \\ \vdots \\ c_k \end{bmatrix}$

Orthonormal set

a set is orthonormal if it is orthogonal and its vectors are unit
this is: $v_i \cdot v_j = 0$ and $v_i \cdot v_i = 1$
for $i \neq j$ and $i = 1,\dots,k$
↳ normalize the vectors of an orthogonal set

orthogonal matrices

a matrix whose columns form an orthonormal set

- if a matrix is orthogonal then

$$Q^{-1} = Q^T$$

orthogonal complement

the set of all vectors that are orthogonal to subspace W ($W^\perp$)
$W^\perp = \{\vec{v} \text{ in } \mathbb{R}^n : \vec{v}\cdot\vec{w} = 0 \text{ forall } \vec{w} \text{ in } W\}$

- $(\text{col}(A))^\perp = \text{null}(A^T)$

Orthogonal projections

Let $\{\vec{u}_1, \dots, \vec{u}_k\}$ be a basis for subspace W then the proj of $\vec{v}$ onto W

$$\text{proj}_W(\vec{v}) = \frac{\vec{u}_1\cdot\vec{v}}{\vec{u}_1\cdot\vec{u}_1} + \dots + \frac{\vec{u}_k\cdot\vec{v}}{\vec{u}_k\cdot\vec{v}}$$

and $\text{perp}_W(\vec{v}) = \vec{v} - \text{proj}_W(\vec{v})$

the orthogonal decomposition theorem

there are unique vectors $\vec{w}$ in subspace W and $\vec{w}^\perp$ in $W^\perp$ such that

$$\vec{v} = \vec{w} + \vec{w}^\perp$$

$\vec{w} = \text{proj}_W(\vec{v})$
$\vec{w}^\perp = \text{perp}_W(\vec{v})$

- $\dim W + \dim W^\perp = n$

Gram - Schmidt Process

a process to orthogonalize a basis for subspace W.

Let $\{\vec{x}_1, \ldots, \vec{x}_k\}$ be a basis for subspace W

- $\vec{v}_1 = \vec{x}_1$
- $\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2)$
- $\vec{v}_3 = \vec{x}_3 - \text{proj}_{\vec{v}_1}(\vec{x}_3) - \text{proj}_{\vec{v}_2}(\vec{x}_3)$

  ...

then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3 \ldots, \vec{v}_n\}$ will be an orthogonal basis for W

QR Factorization

Let A be an m×n matrix with linearly dependent columns Then A can be factored as $A = QR$ where Q is an m×n matrix with orthonormal column and R is an invertible upper triangular matrix

1. find a basis for A
2. use GS to find an orth. basis
3. normalize basis
4. $R = Q^T A$

orthogonally diagonize matrices

A square matrix A is orthogonally diagonalizable if there exists an orthogonal matrix Q and a diagonal matrix D such that $Q^T A Q = D$

- A is symmetric if and only if it is orthogonally diagonalizable
- if A is a symmetric matrix then any two eigenvectors corresponding to distinct eigenval. of A are orthogonal

1. find char. polynomial and λs
2. find corresponding eigenvectors
3. normalize eigenvectors
4. Q = matrix of normalized eig.vec.