Problem Set 1

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Problem 1

1a and 1b

Table 1: **Estimated coefficients for AR(1)**

Model	Lags	Coefficients	Std. Errors	P-value
AR(1)	AR(1)	0.29	0.09	0.00
AR(1)	Intercept	4.54	0.54	0.00

Table 2: **Estimated coefficients for AR(2)**

Model	Lags	Coefficients	Std. Errors	P-value
AR(2)	AR(1)	0.25	0.09	0.01
AR(2)	AR(2)	0.12	0.09	0.20
AR(2)	Intercept	4.53	0.61	0.00

Table 3: **Estimated coefficients for MA(1)**

Model	Lags	Coefficients	Std. Errors	P-value
MA(1)	MA(1)	0.24	0.09	0.01
MA(1)	Intercept	4.54	0.48	0.00

Table 4: **Estimated coefficients for MA(2)**

Model	Lags	Coefficients	Std. Errors	P-value
MA(2)	MA(1)	0.22	0.09	0.02
MA(2)	MA(2)	0.16	0.10	0.13
MA(2)	Intercept	4.53	0.53	0.00

Table 5: **Estimated coefficients for ARMA(1,1)**

Model	Lags	Coefficients	Std. Errors	P-value
ARMA(1,1)	AR(1)	0.74	0.28	0.01
ARMA(1,1)	MA(1)	-0.52	0.37	0.17
ARMA(1,1)	Intercept	4.49	0.71	0.00

Table 6: **Estimated coefficients for ARMA(1,2)**

Model	Lags	Coefficients	Std. Errors	P-value
ARMA(1,2)	AR(1)	0.94	0.07	0.00
ARMA(1,2)	MA(1)	-0.72	0.12	0.00
ARMA(1,2)	MA(2)	-0.12	0.10	0.22
ARMA(1,2)	Intercept	4.26	1.07	0.00

Table 7: **Estimated coefficients for ARMA(2,1)**

Model	Lags	Coefficients	Std. Errors	P-value
ARMA(2,1)	AR(1)	1.10	0.14	0.00
ARMA(2,1)	AR(2)	-0.15	0.11	0.18
ARMA(2,1)	MA(1)	-0.87	0.10	0.00
ARMA(2,1)	Intercept	4.26	1.07	0.00

Table 8: **Estimated coefficients for ARMA(2,2)**

Model	Lags	Coefficients	Std. Errors	P-value
ARMA(2,2)	AR(1)	1.29	0.48	0.01
ARMA(2,2)	AR(2)	-0.32	0.45	0.47
ARMA(2,2)	MA(1)	-1.06	0.49	0.03
ARMA(2,2)	MA(2)	0.16	0.41	0.70
ARMA(2,2)	Intercept	4.27	1.06	0.00

Table 9: **Estimated BIC and AIC for AR models**

	AR(1)	AR(2)
BIC	700.35	703.50
AIC	691.99	692.35

Table 10: **Estimated BIC and AIC for MA models**

	MA(1)	MA(2)
BIC	702.34	704.87
AIC	693.98	693.72

Table 11: **Estimated BIC and AIC for ARMA models**

	ARMA (1,1)	ARMA (2,1)	ARMA (1,2)	ARMA (2,2)
BIC	702.96	706.67	706.48	711.15
AIC	691.81	692.73	692.55	694.42

1c

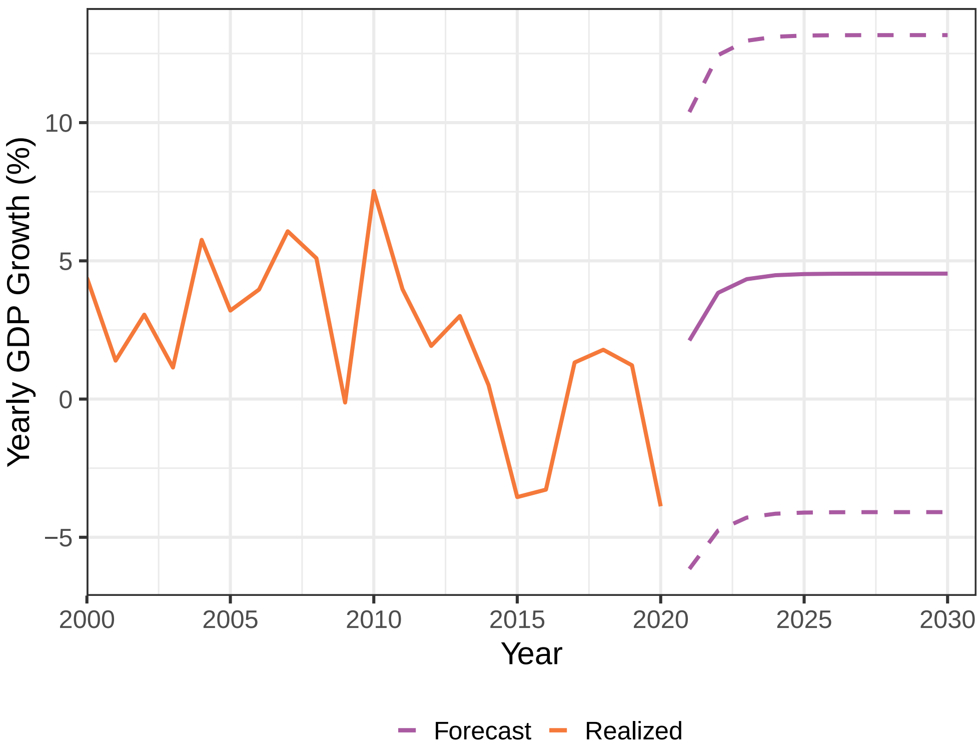


Figure 1: GDP Growth and 10-Year-Ahead Forecast using an AR(1) Model.

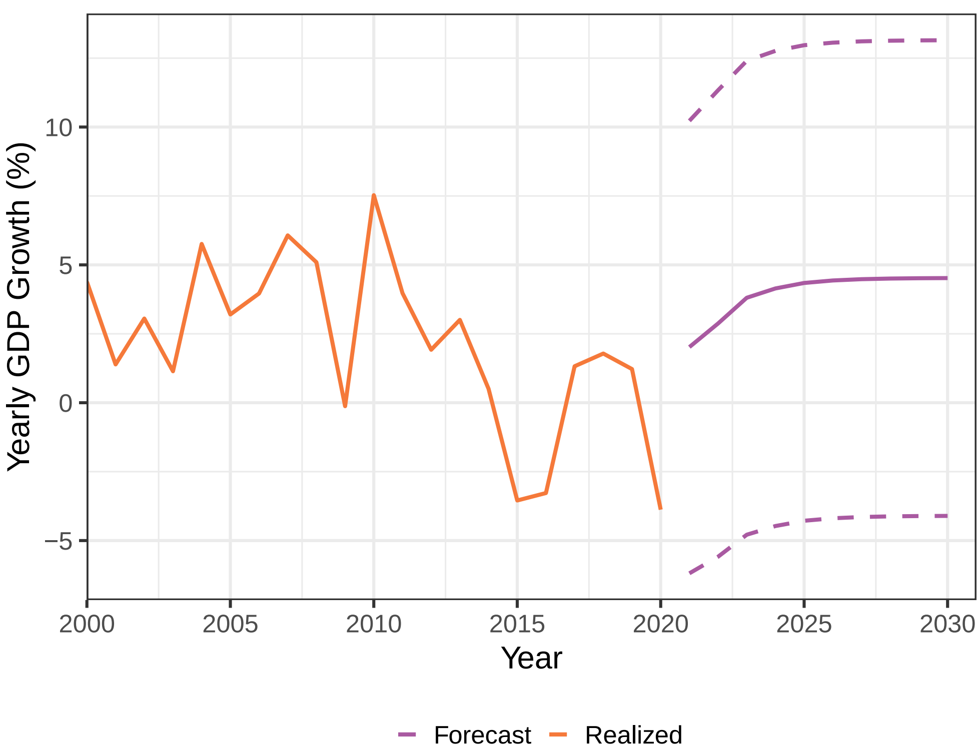


Figure 2: GDP Growth and 10-Year-Ahead Forecast using an AR(2) Model.

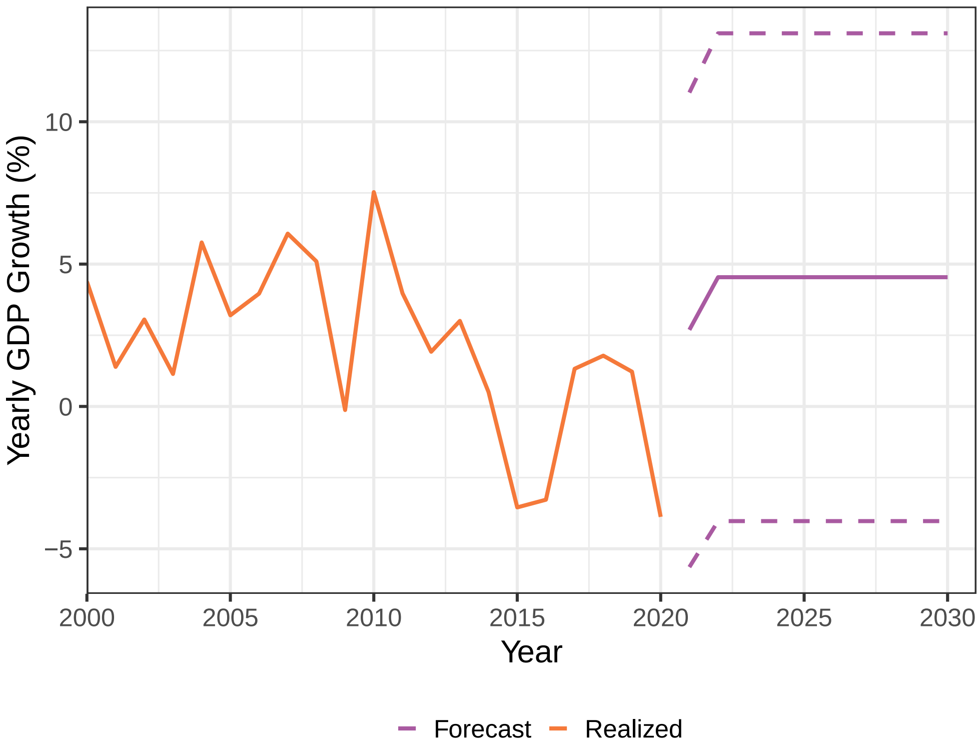


Figure 3: GDP Growth and 10-Year-Ahead Forecast using an MA(1) Model.

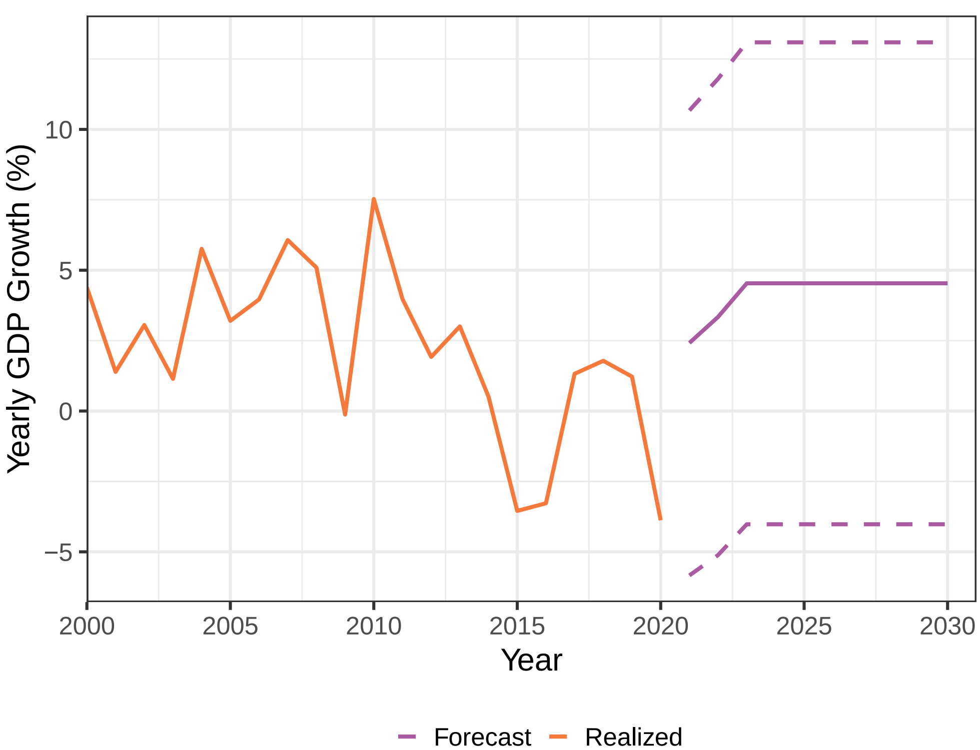


Figure 4: GDP Growth and 10-Year-Ahead Forecast using an MA(2) Model.

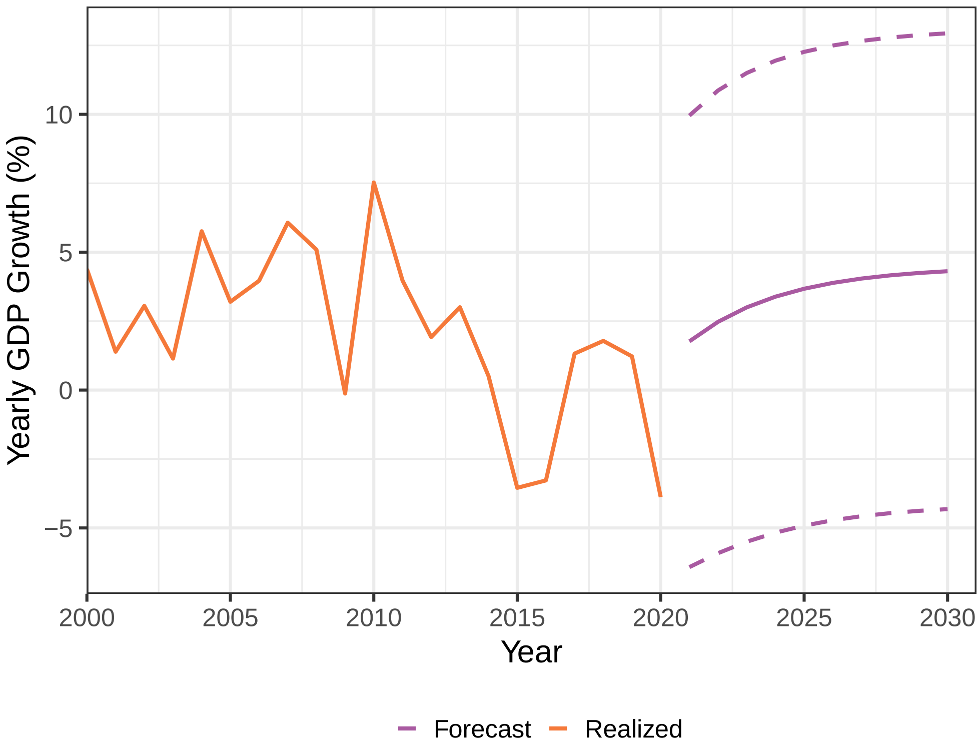


Figure 5: GDP Growth and 10-Year-Ahead Forecast using an ARMA(1,1) Model.

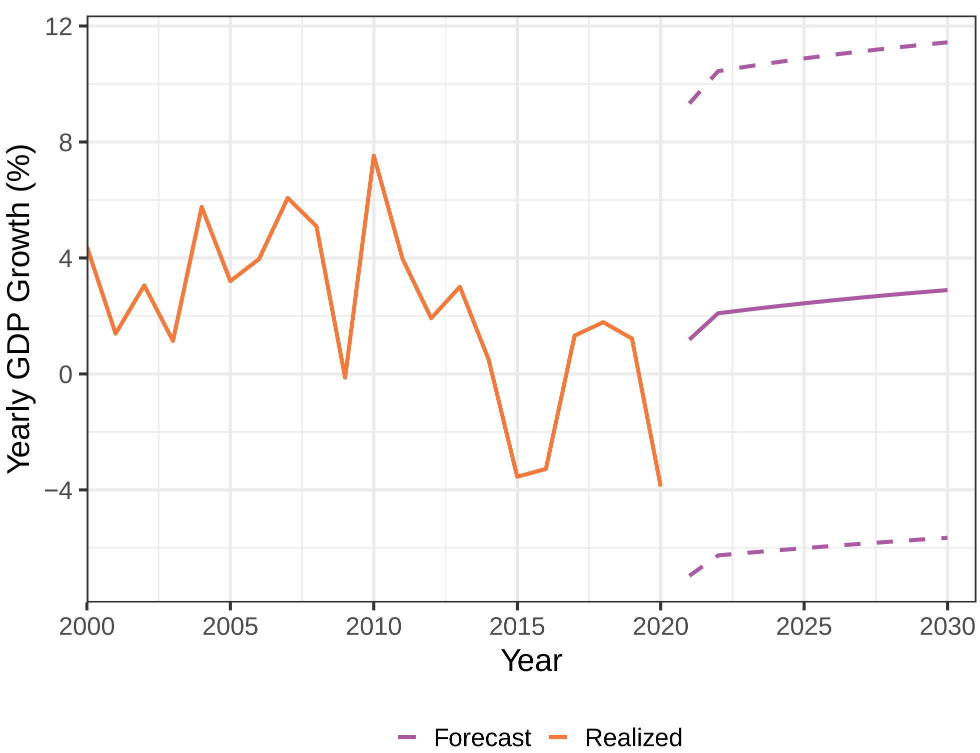


Figure 6: GDP Growth and 10-Year-Ahead Forecast using an ARMA(1,2) Model.

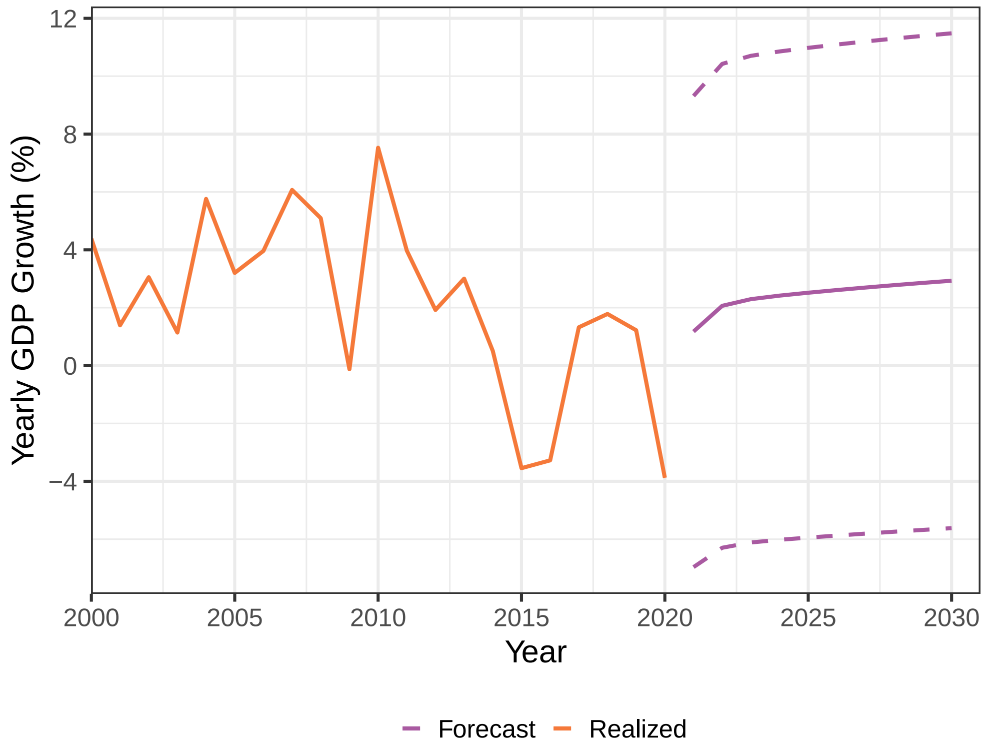


Figure 7: GDP Growth and 10-Year-Ahead Forecast using an ARMA(2,1) Model.

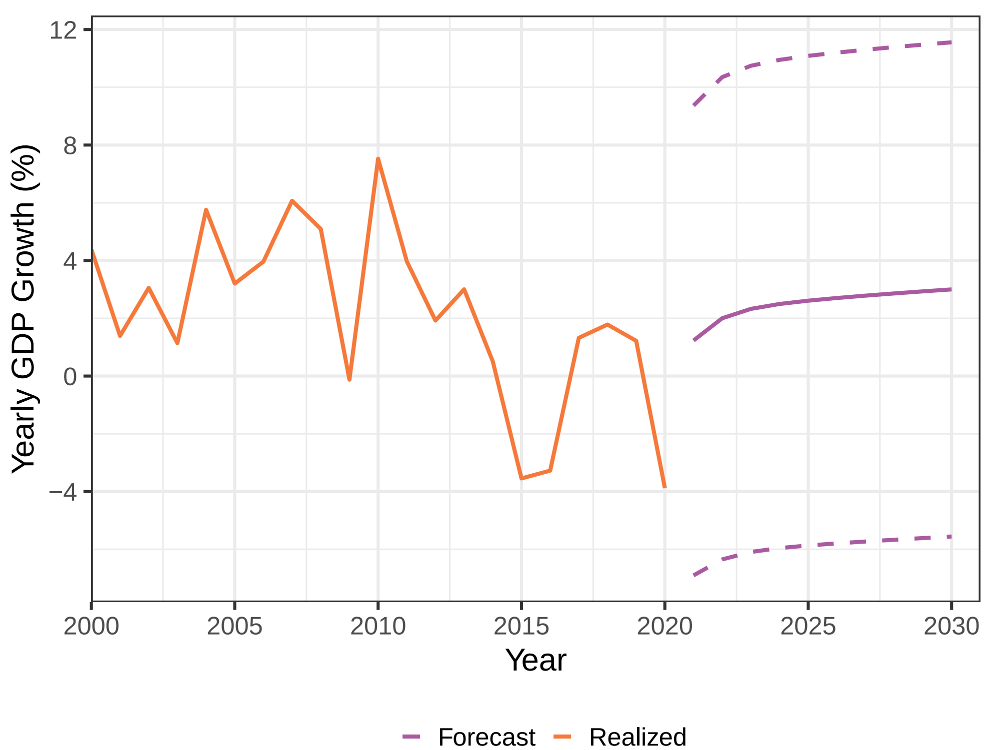


Figure 8: GDP Growth and 10-Year-Ahead Forecast using an ARMA(2,2) Model.

Moreover, answer the following question (30 points): "Among these eight models, which one would you choose as the best model? Explain your choice."

After analyzing the AIC and BIC criteria, we conclude that the ARMA(1,1) and AR(1) models are the top-performing models. While the ARMA(1,1) model minimizes the AIC criterion, the AR(1) model gives the lowest BIC criterion value. Although the AR(1) and ARMA(1,1) models are quite similar, we suggest selecting the AR(1) model as the preferred choice.

ARMA(1,1) minimiza AIC
AR(1) minimiza AR(1)

Problem 2

2.1

Given the MA(1) model

$$Y_t = c + \theta \cdot \epsilon_{t-1} + \epsilon_t,$$

where $c = 0$, $\theta = 0.5$ and ϵ_t is i.i.d $N(0, 1)$. To analyze the **intercept estimator** of this model, we can examine its convergence in probability, convergence in distribution (illustrated as well by a GIF[1] for the convergence in distribution), and test size control. These figures provide insights into the behavior of the estimator and its performance.

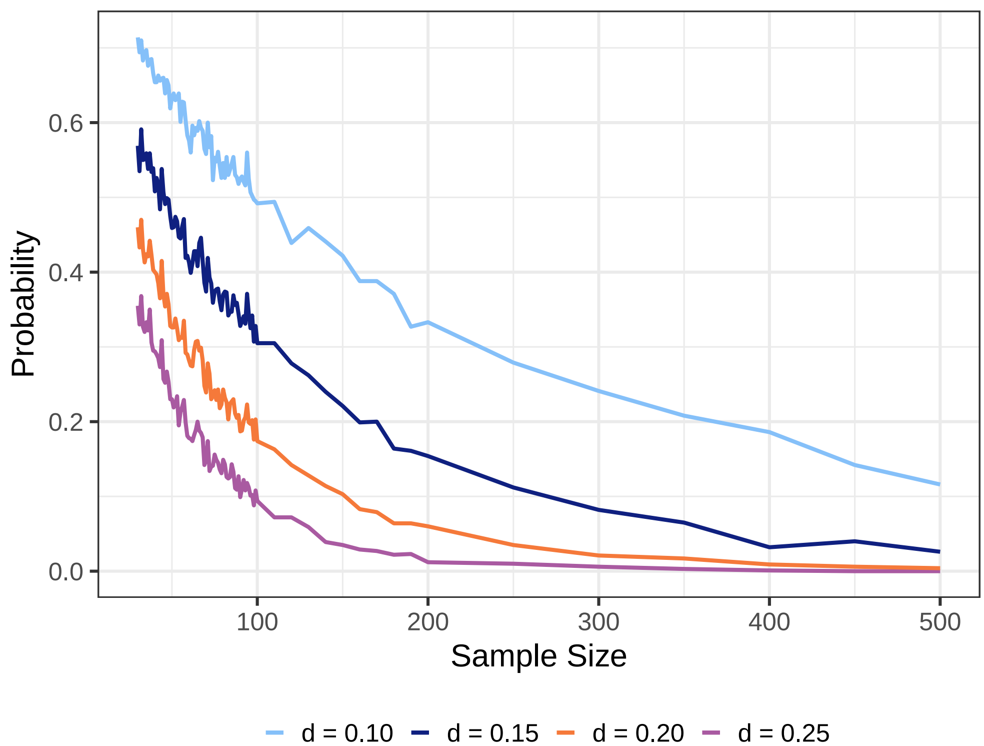


Figure 9: Convergence in probability.

[1] To see the GIF is recommended to open the PDF file in Adobe Acrobat Reader

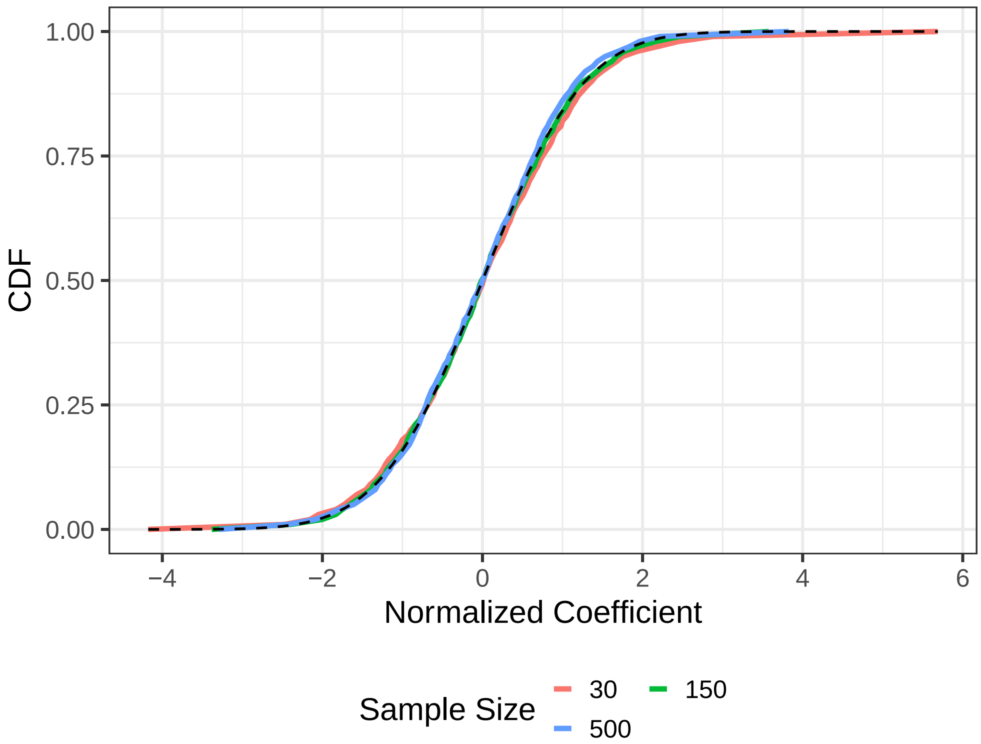
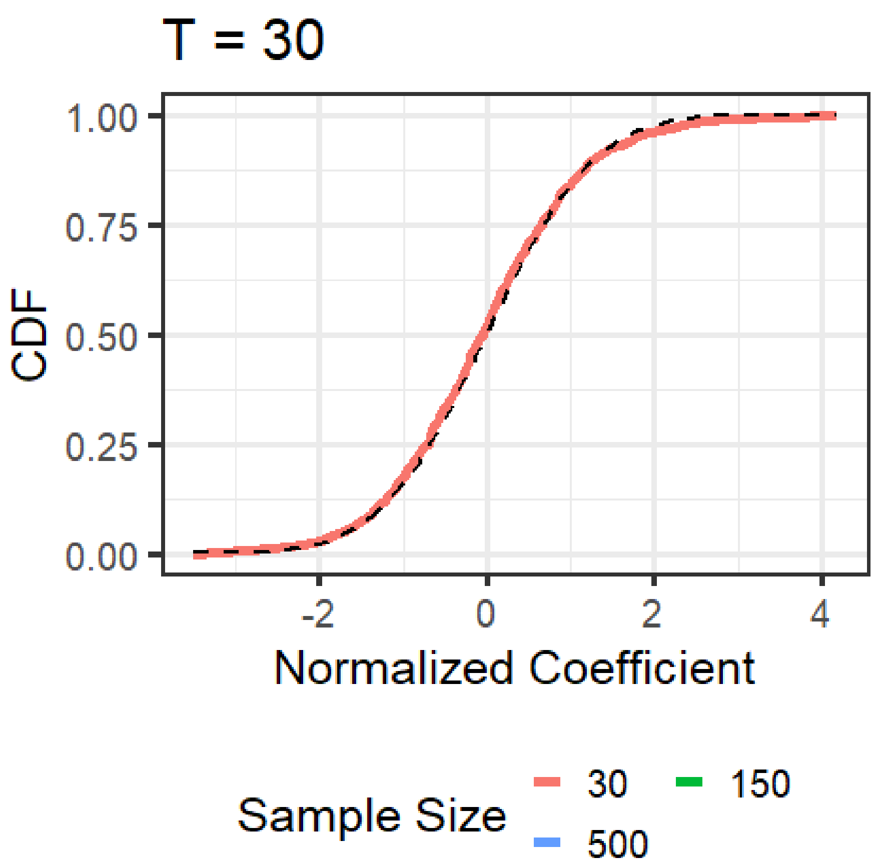


Figure 10: Convergence in distribution.



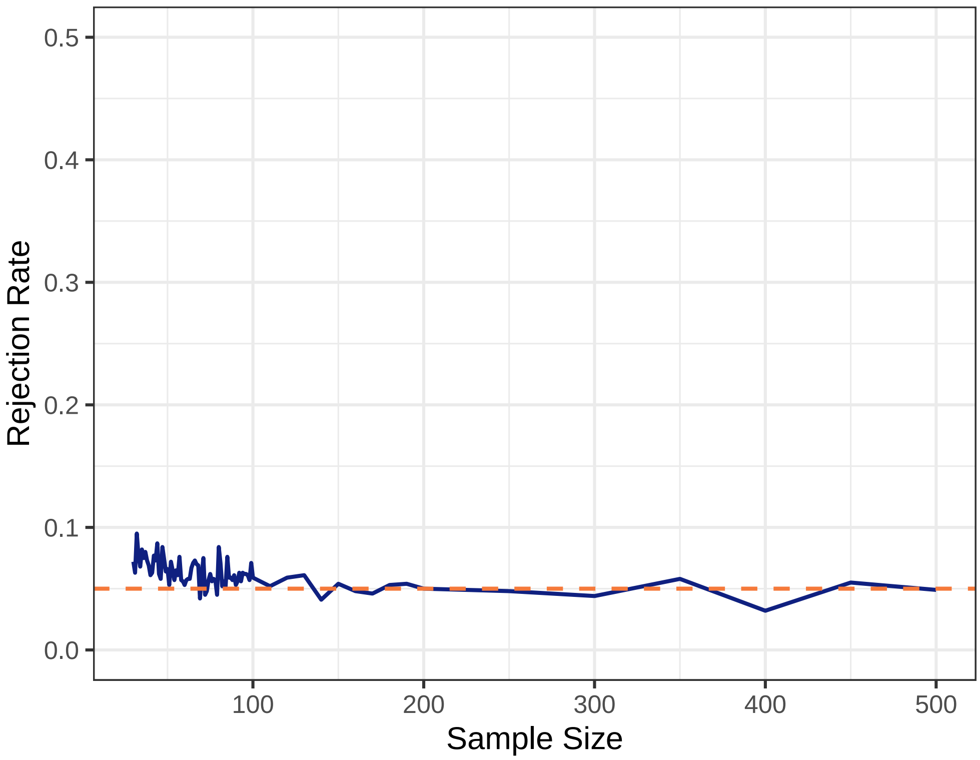


Figure 11: Test size control

For the MA(1) model above, if ϵ_t is i.i.d $exp(1)$ we have the following figures:

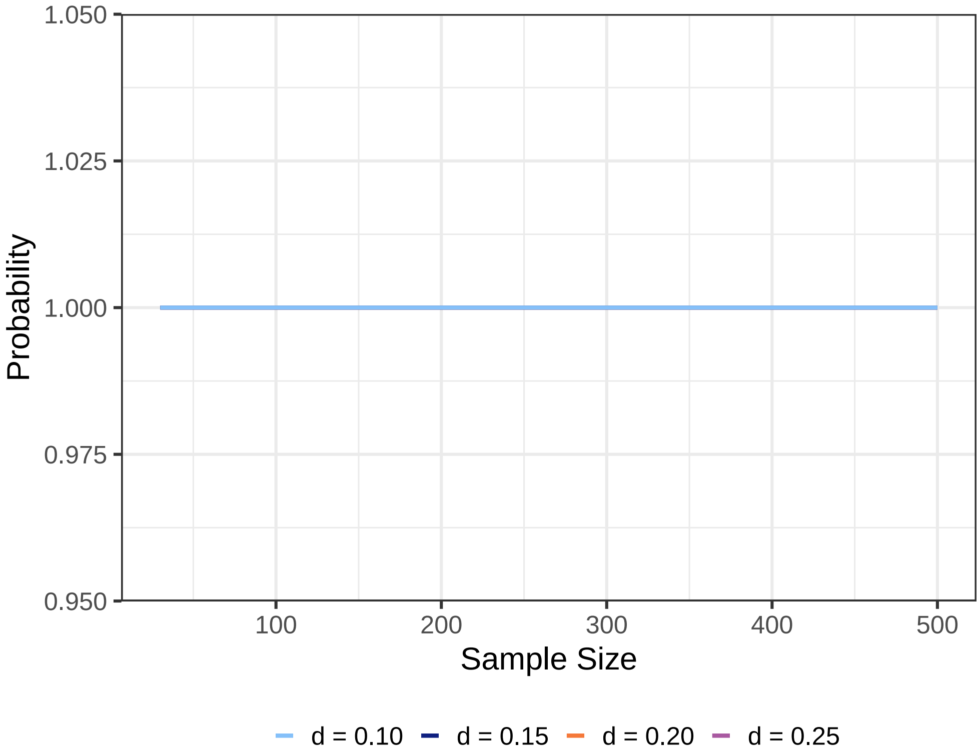


Figure 12: Convergence in probability.

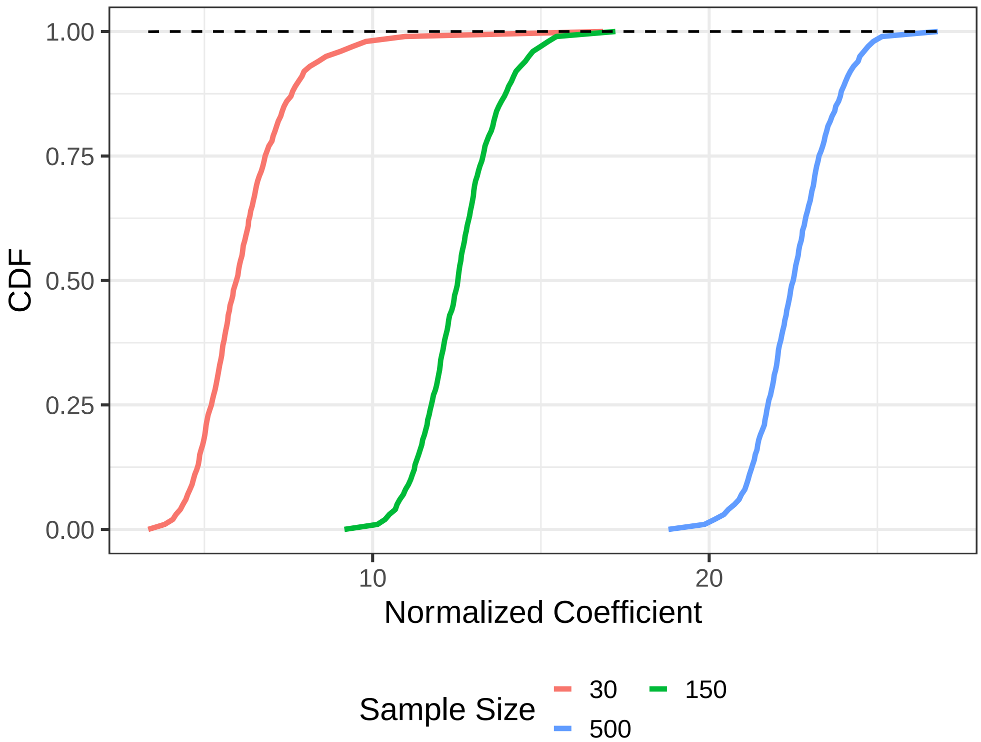
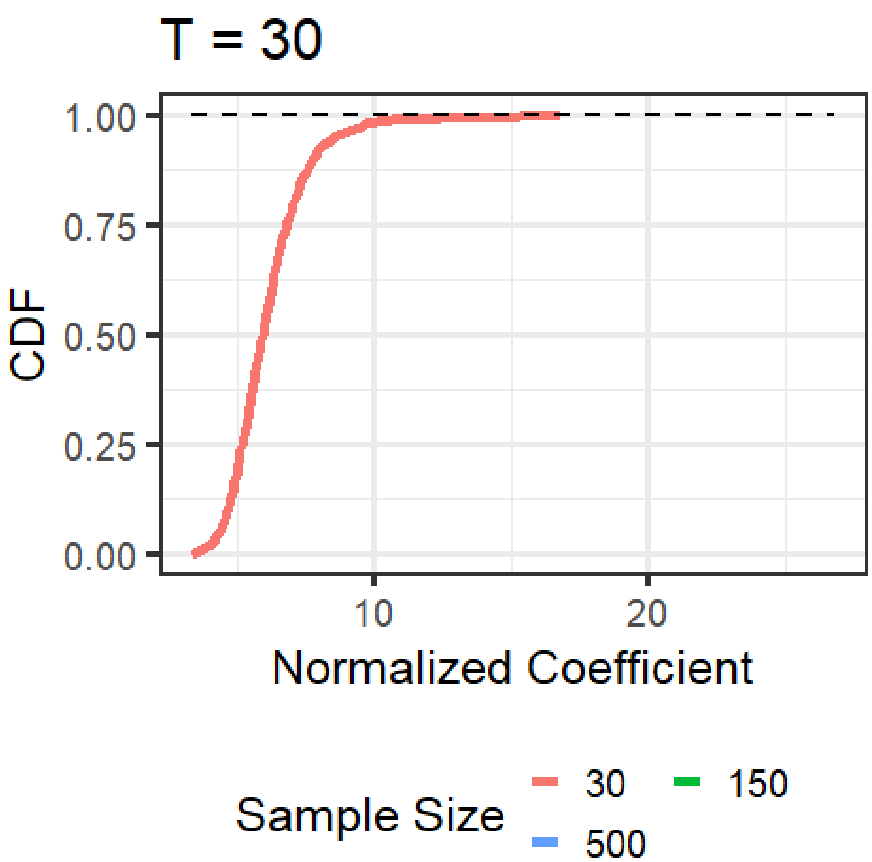


Figure 13: Convergence in distribution.



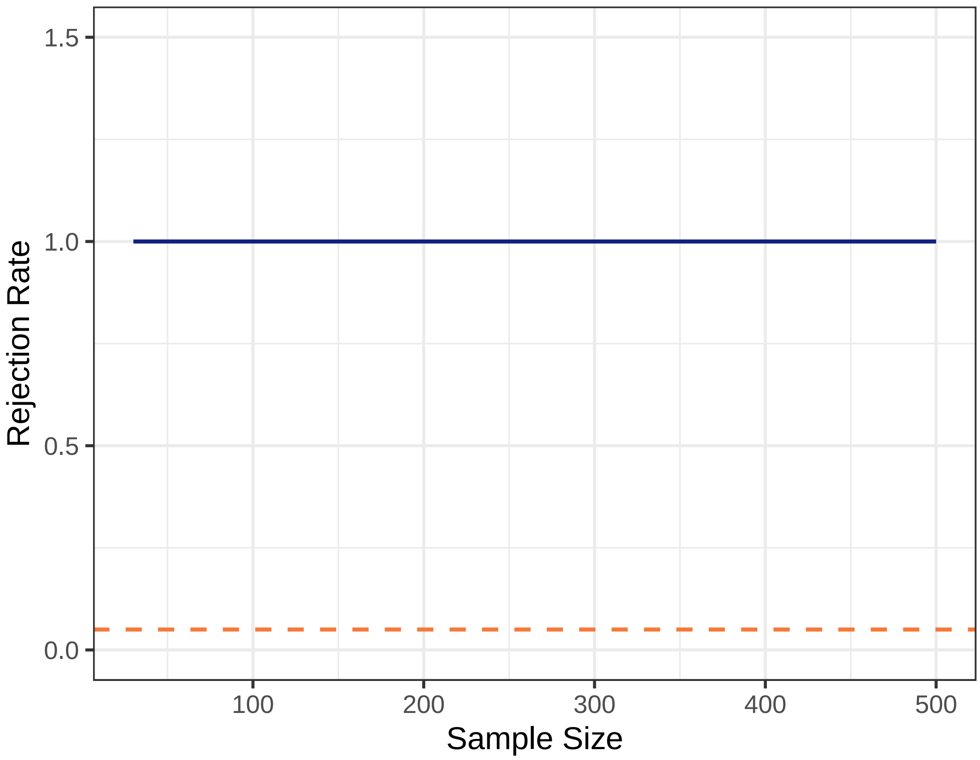


Figure 14: Test size control

2.2

Given the AR(1) model

$$Y_t = c + \phi \cdot Y_{t-1} + \epsilon_t,$$

where $c = 0$, $\phi = 0.3$ and ϵ_t is i.i.d $N(0, 1)$. To analyze the **intercept estimator** of this model, we can examine its convergence in probability, convergence in distribution (illustrated as well by a GIF[2] for the convergence in distribution), and test size control. These figures provide insights into the behavior of the estimator and its performance.

[2]To see the GIF is recommended to open the PDF file in Adobe Acrobat Reader

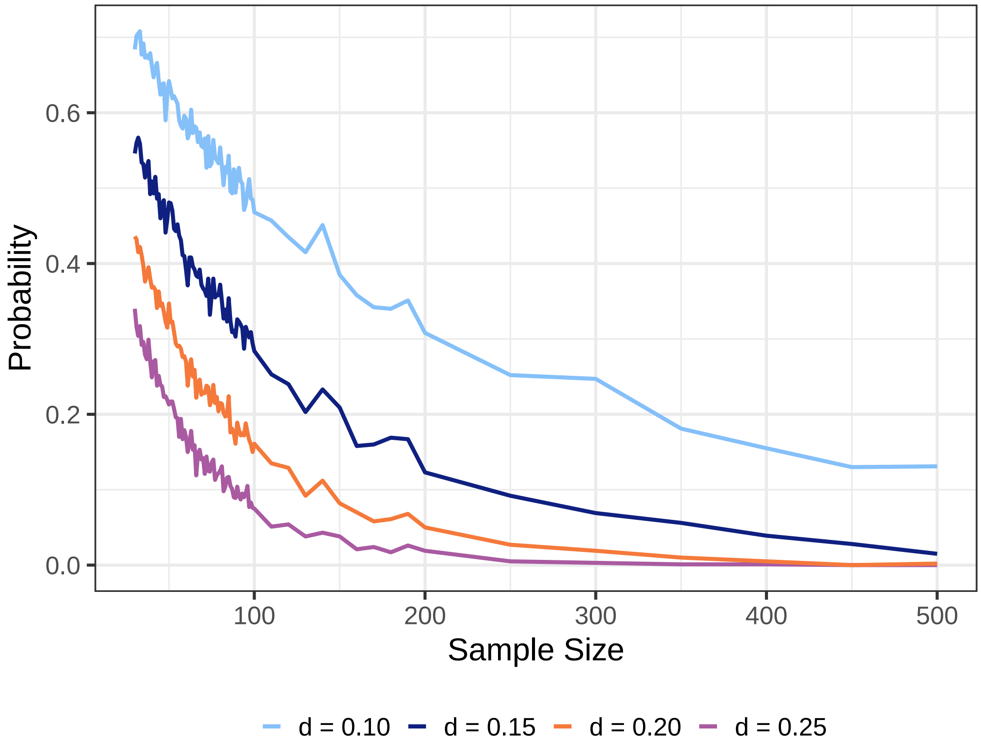


Figure 15: Convergence in probability.

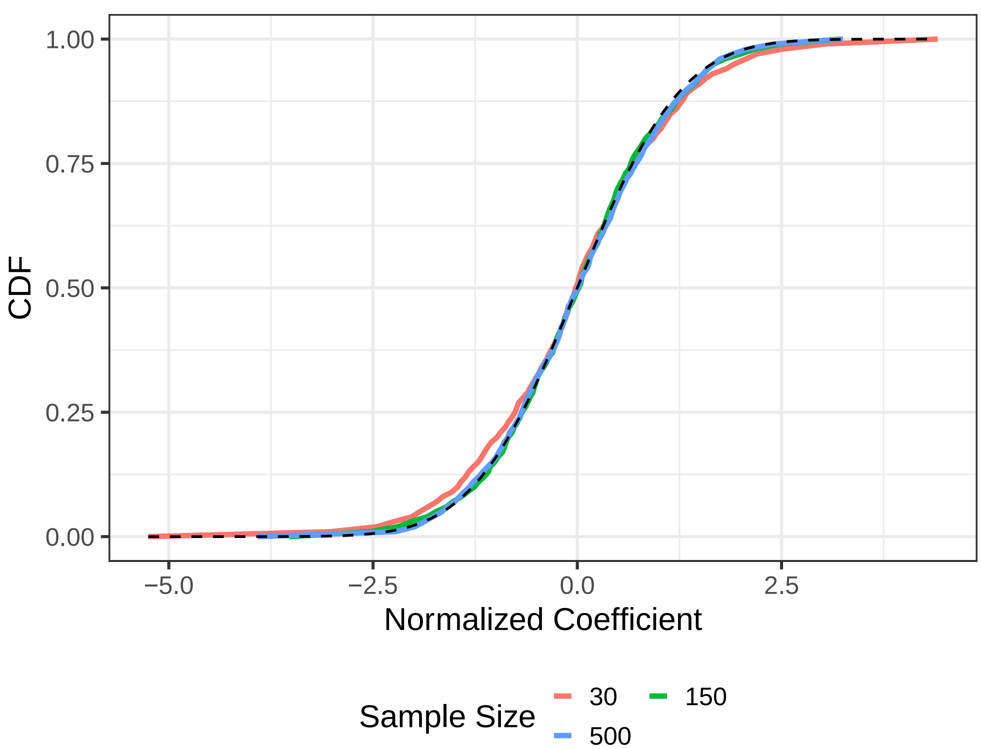
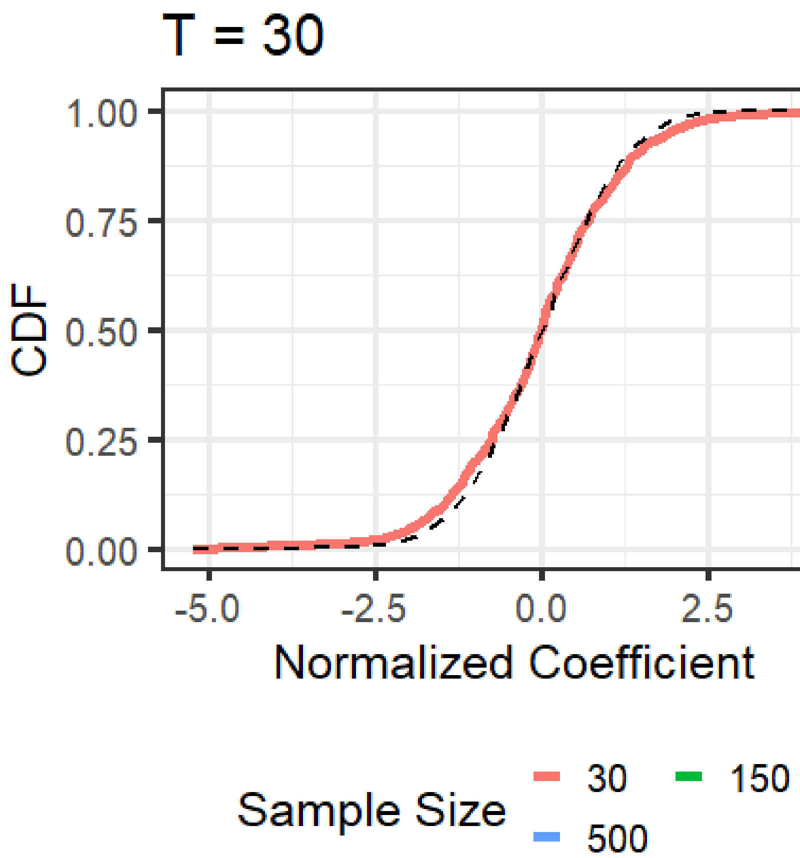


Figure 16: Convergence in distribution.



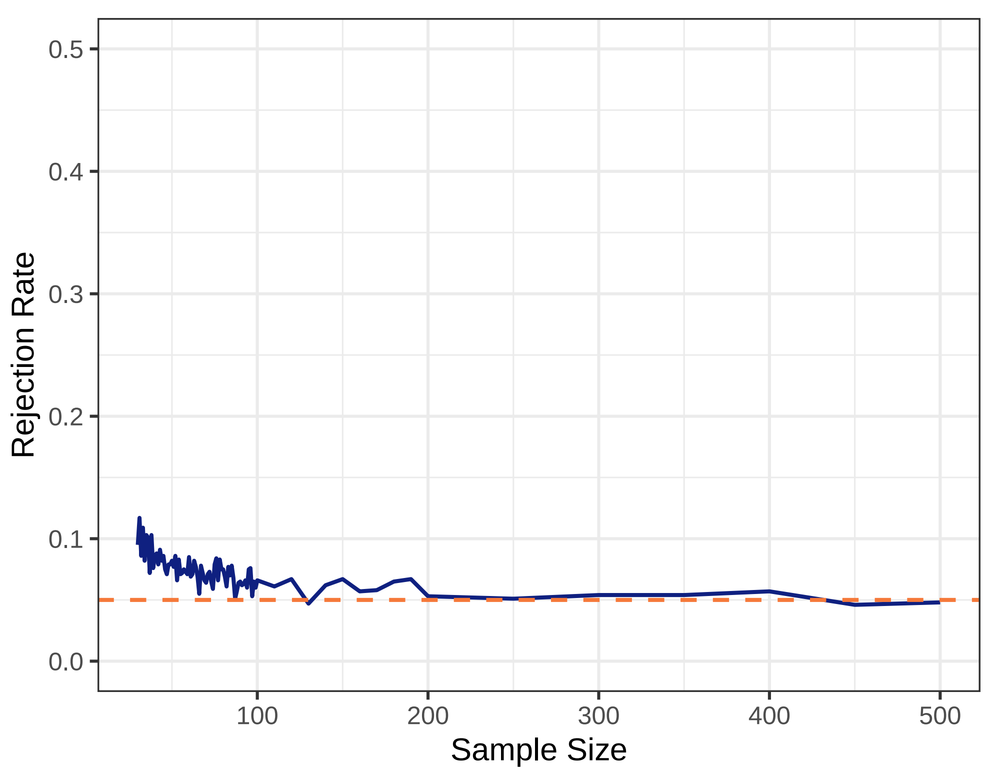


Figure 17: Test size control

For the AR(1) model above, if ϵ_t is i.i.d $exp(1)$ we have the following figures:

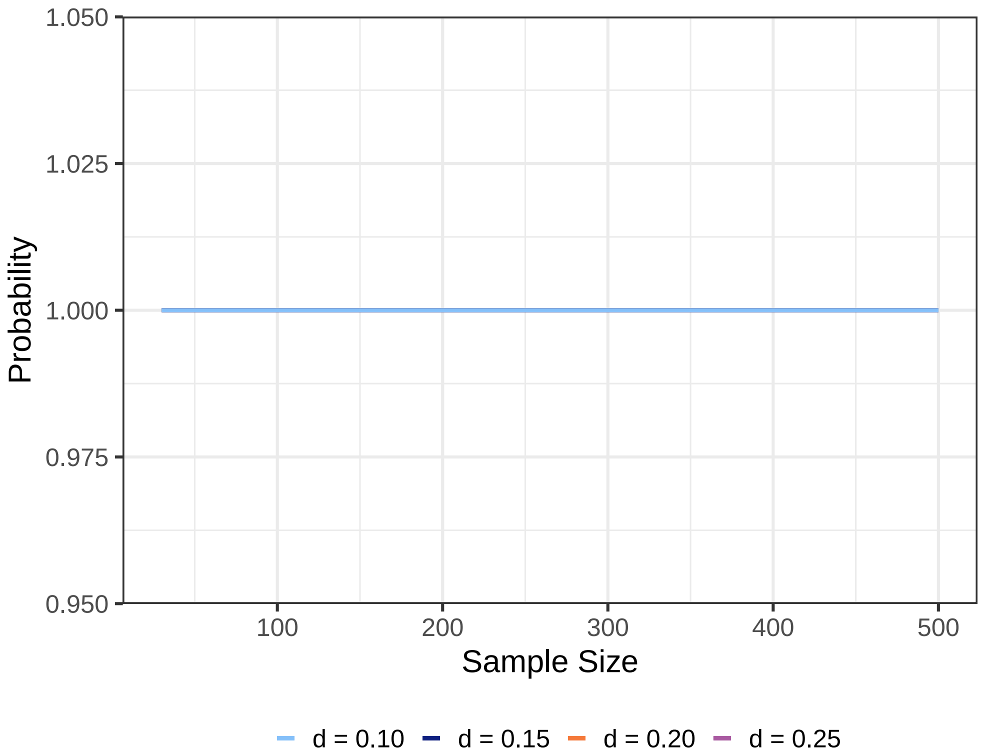


Figure 18: Convergence in probability.

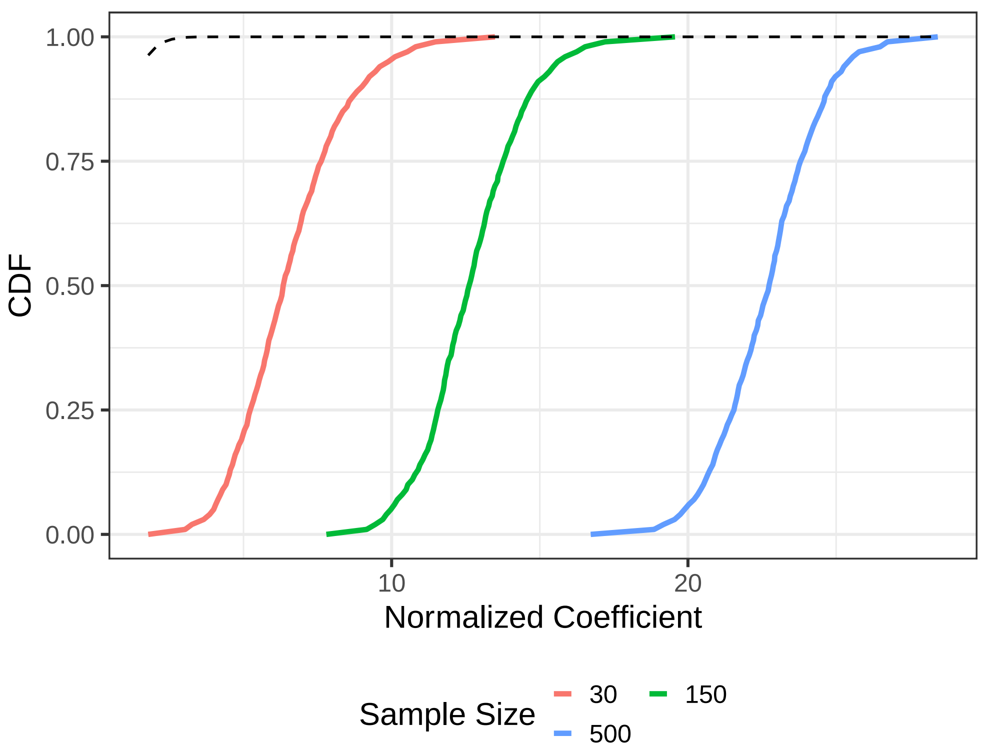
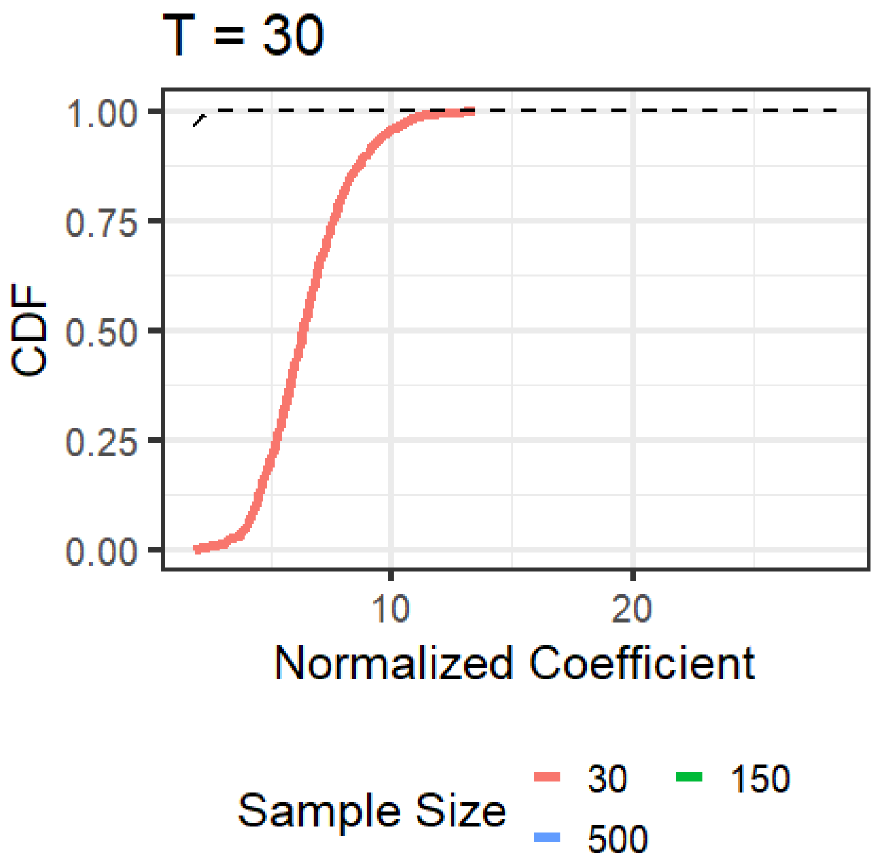


Figure 19: Convergence in probability.



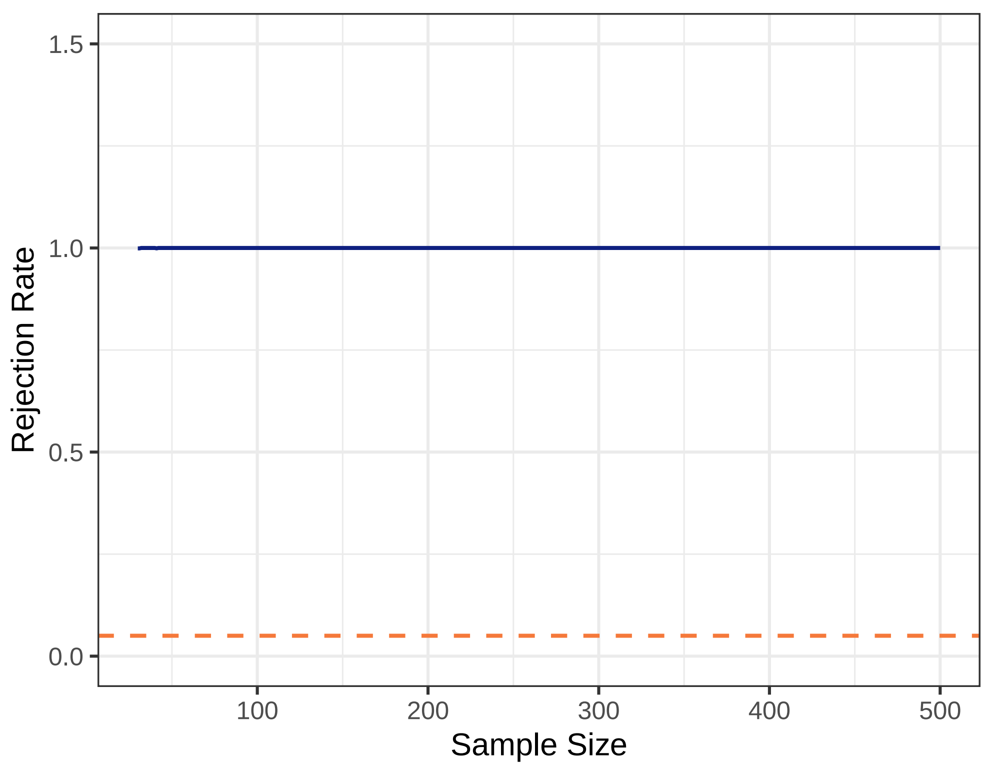


Figure 20: Convergence in distribution.

2.3

Given the AR(1) model

$$Y_t = c + \phi \cdot Y_{t-1} + \epsilon_t,$$

where $c = 0$, $\phi = 0.3$ and ϵ_t is i.i.d $N(0, 1)$. To analyze the **the estimator for the first autoregressive coefficient** of this model, we can examine its convergence in probability, convergence in distribution (illustrated as well by a GIF[3] for the convergence in distribution), and test size control. These figures provide insights into the behavior of the estimator and its performance.

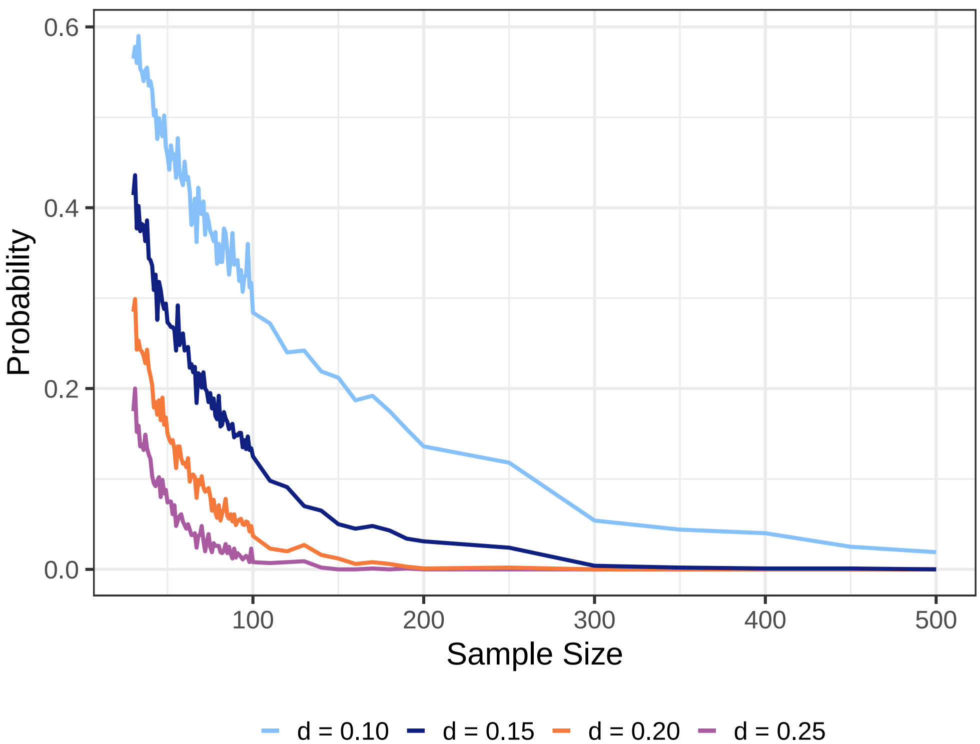


Figure 21: Convergence in probability.

[3]To see the GIF is recommended to open the PDF file in Adobe Acrobat Reader

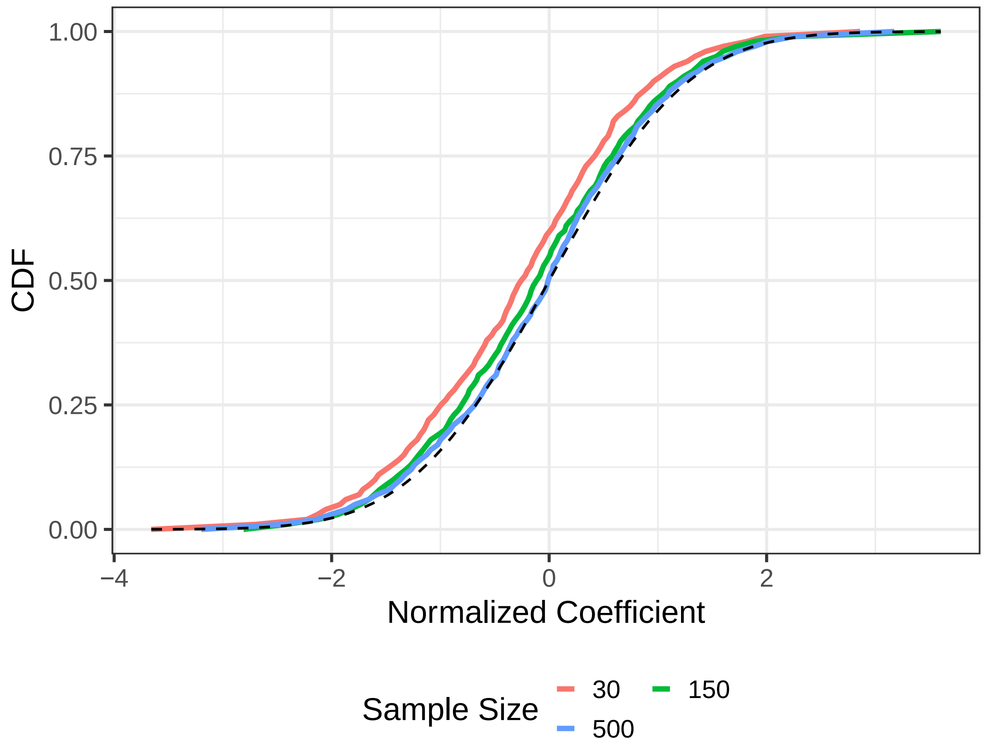
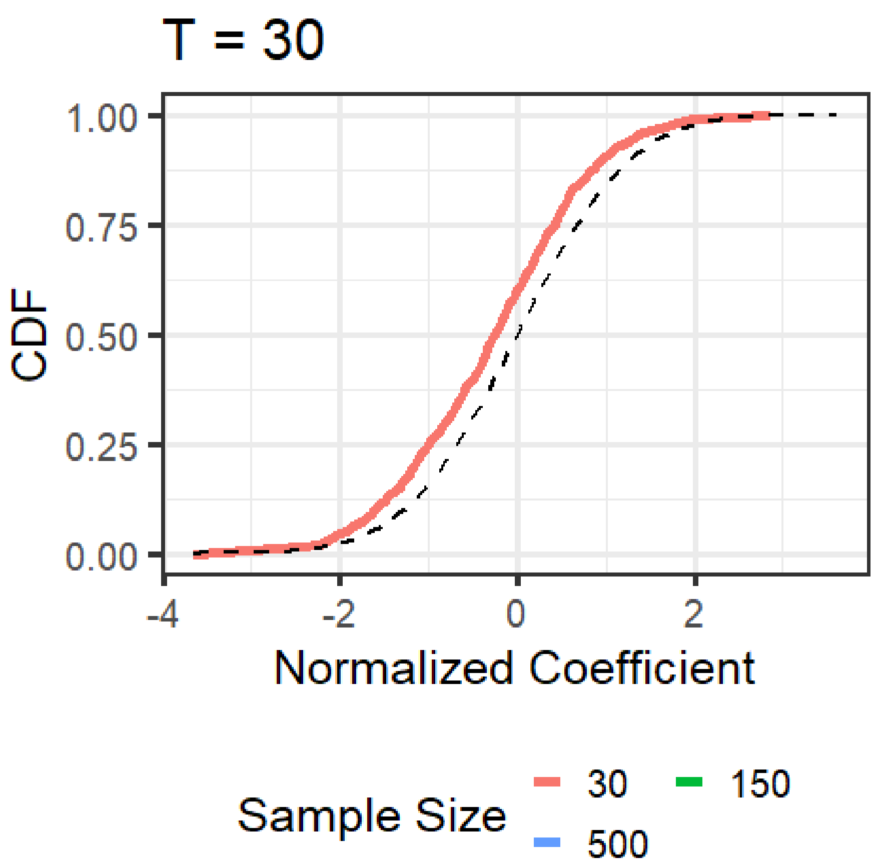


Figure 22: Convergence in distribution.



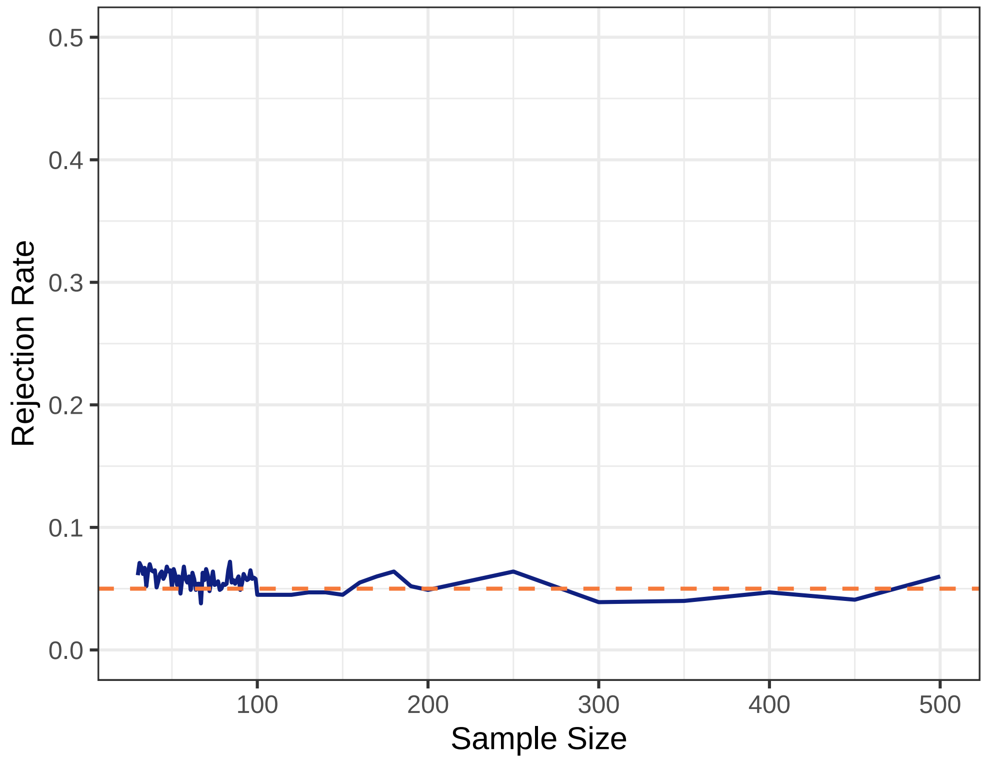


Figure 23: Test size control

For the AR(1) model above, if ϵ_t is i.i.d $exp(1)$ we have the following figures:

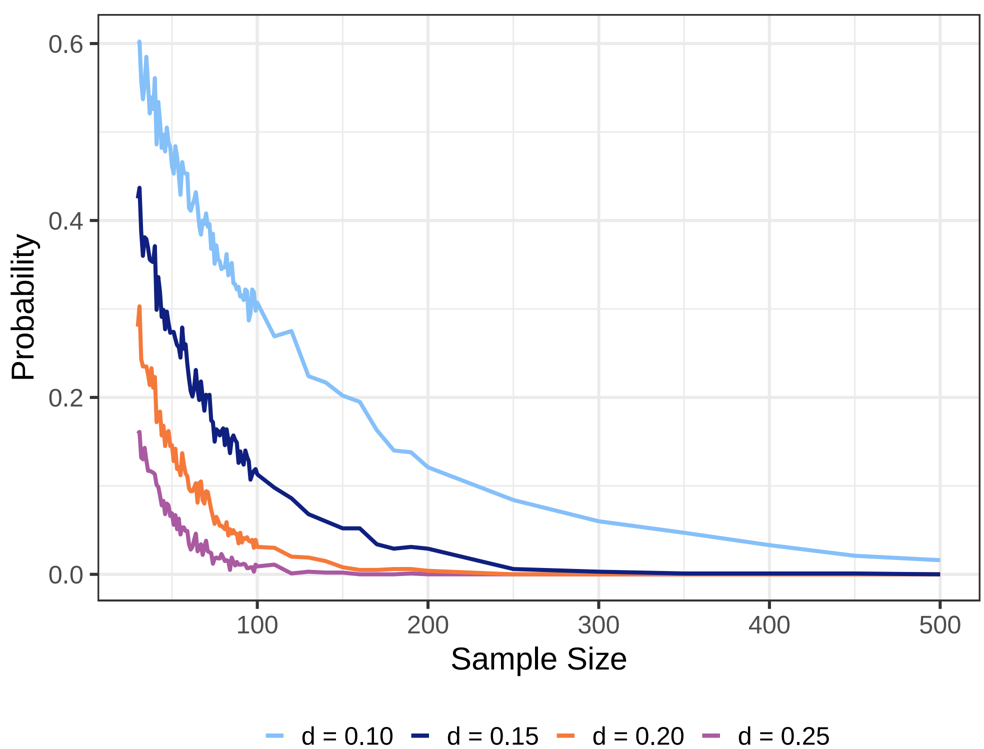


Figure 24: Convergence in probability.

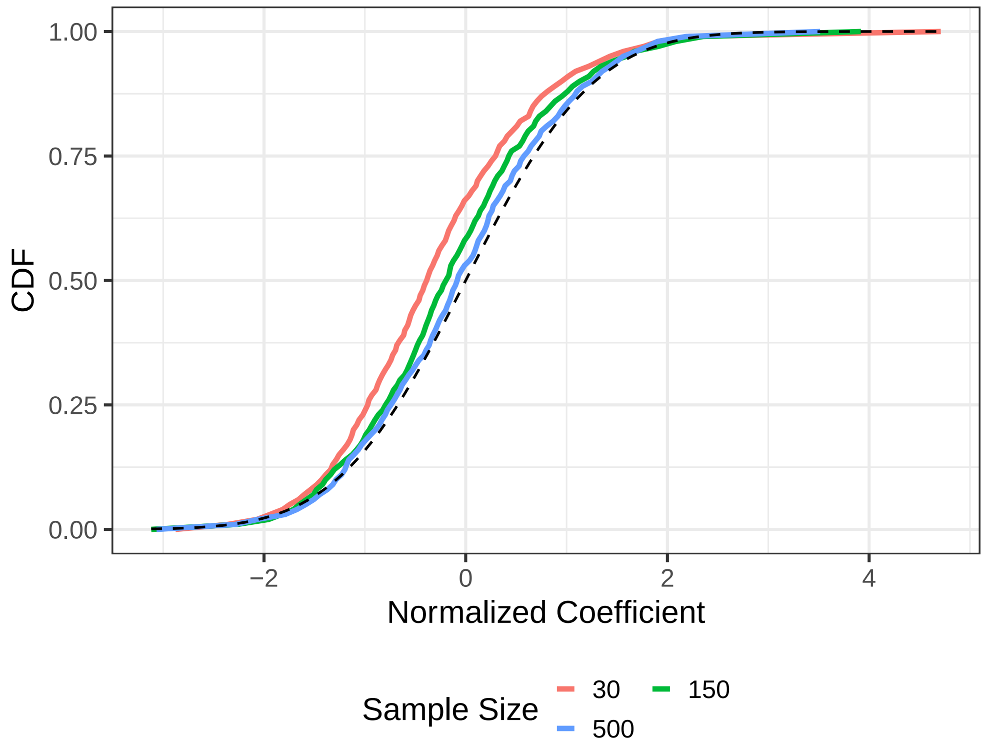
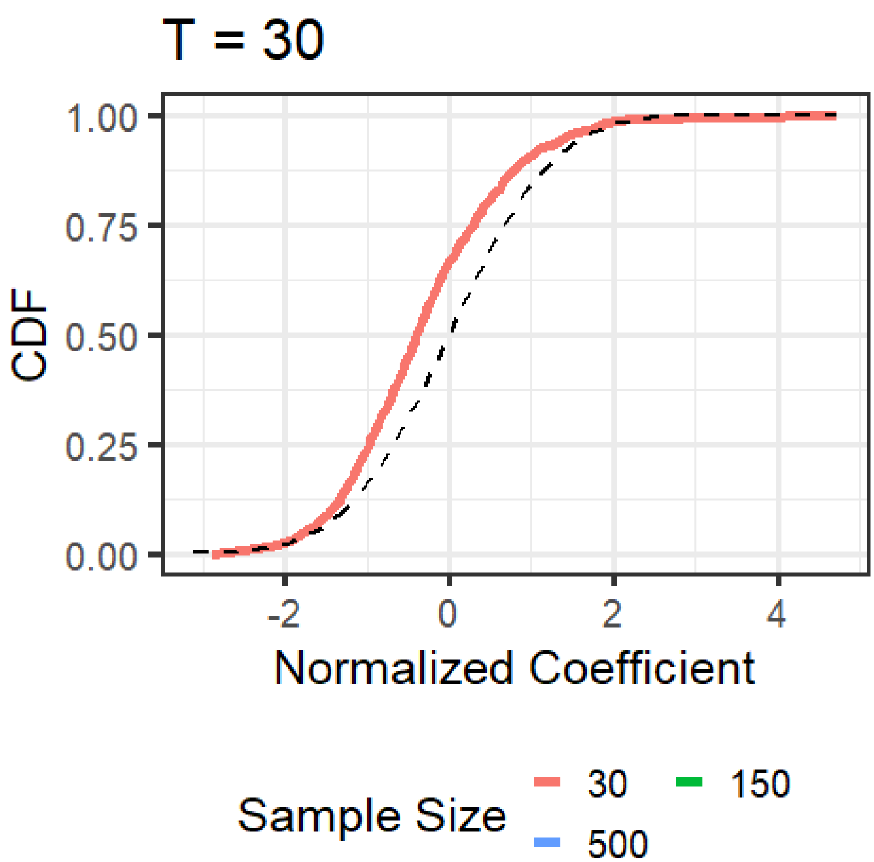


Figure 25: Convergence in probability.



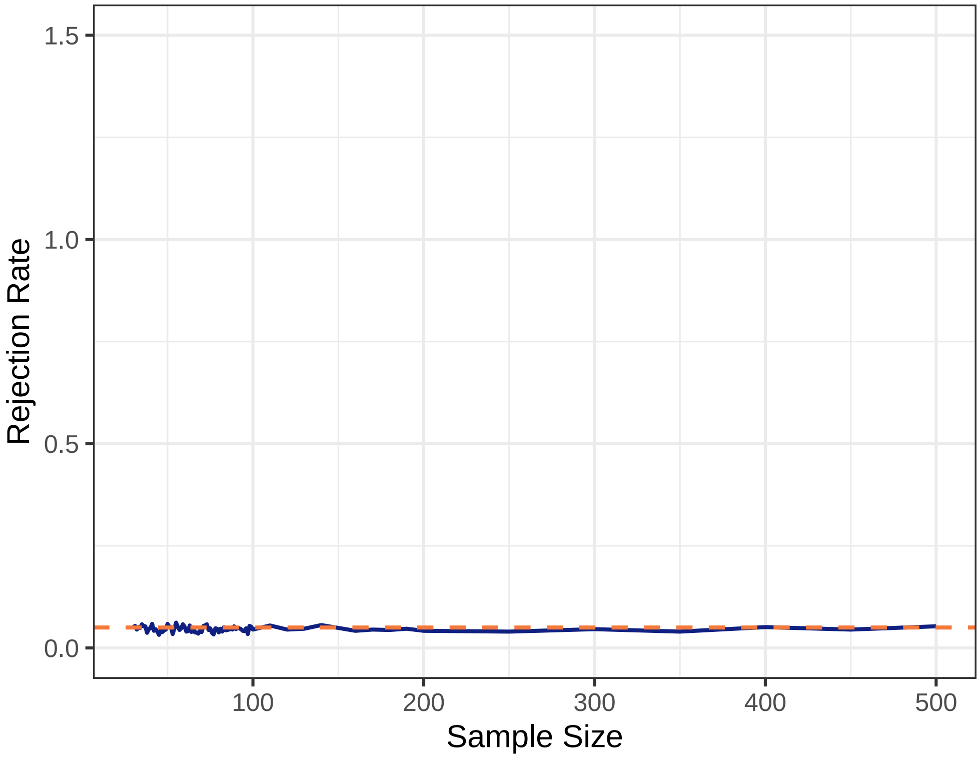


Figure 26: Convergence in distribution.

2.4

Given the ARMA(1,1) model

$$Y_t = c + \phi \cdot Y_{t-1} + \theta \cdot \epsilon_{t-1} + \epsilon_t,$$

where $c = 0$, $\phi = 0.3$, $\theta = 0.5$ ϵ_t is i.i.d $N(0, 1)$.

To analyze the **intercept estimator** of this model, we can examine its convergence in probability, convergence in distribution (illustrated as well by a GIF[4] for the convergence in distribution), and test size control. These figures provide insights into the behavior of the estimator and its performance.

[4]To see the GIF is recommended to open the PDF file in Adobe Acrobat Reader

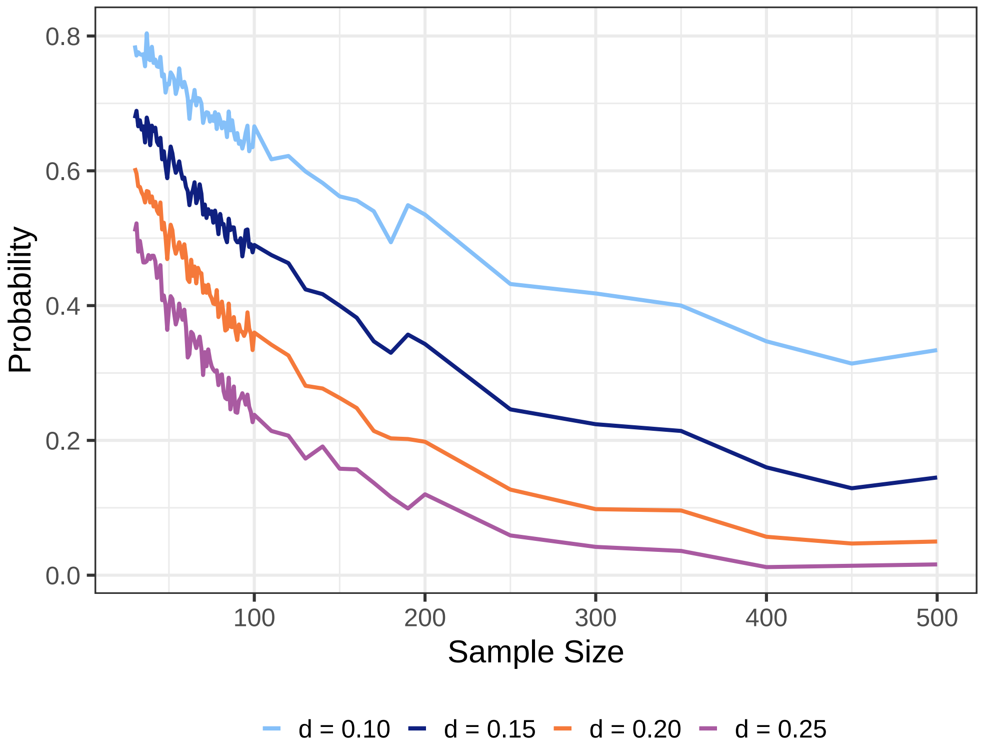


Figure 27: Convergence in probability.

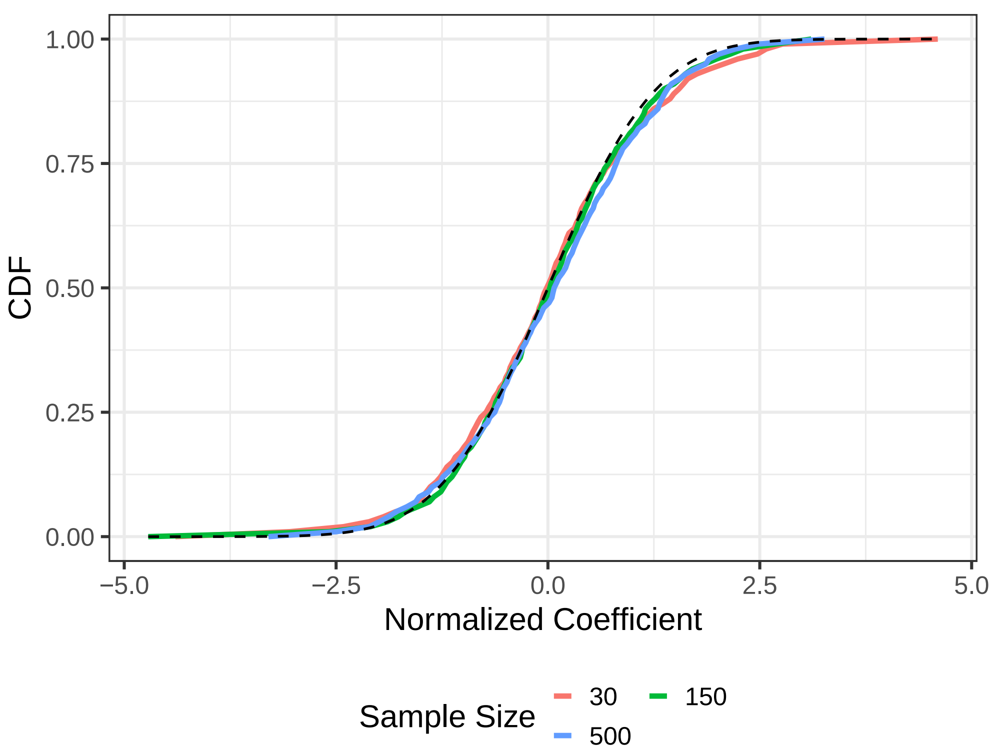
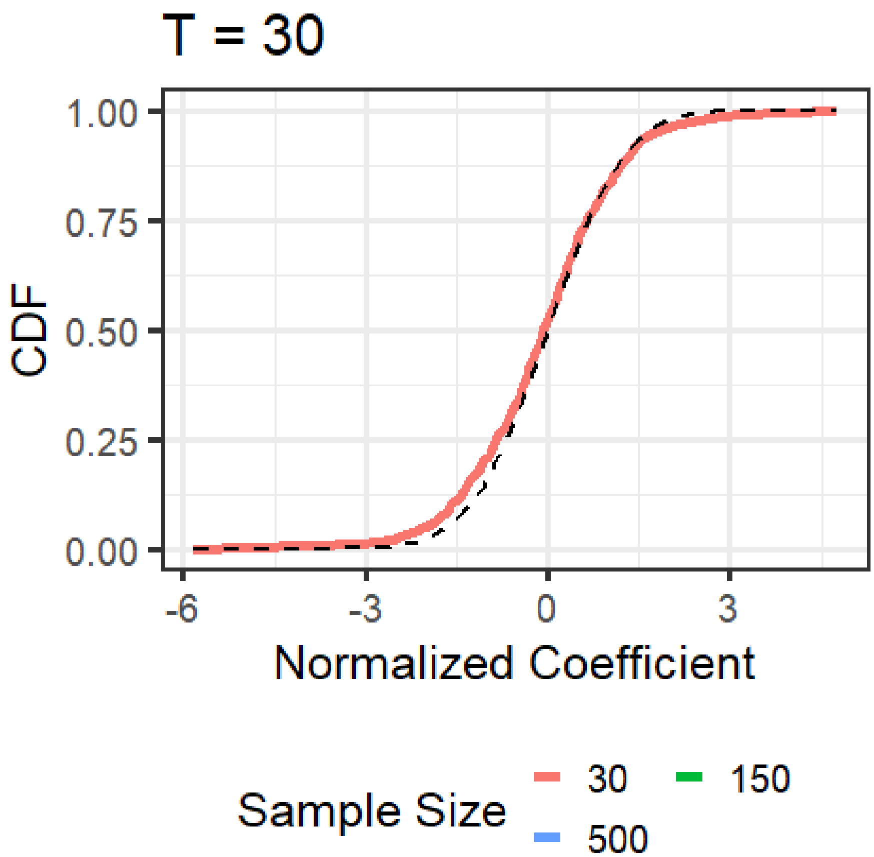


Figure 28: Convergence in distribution.



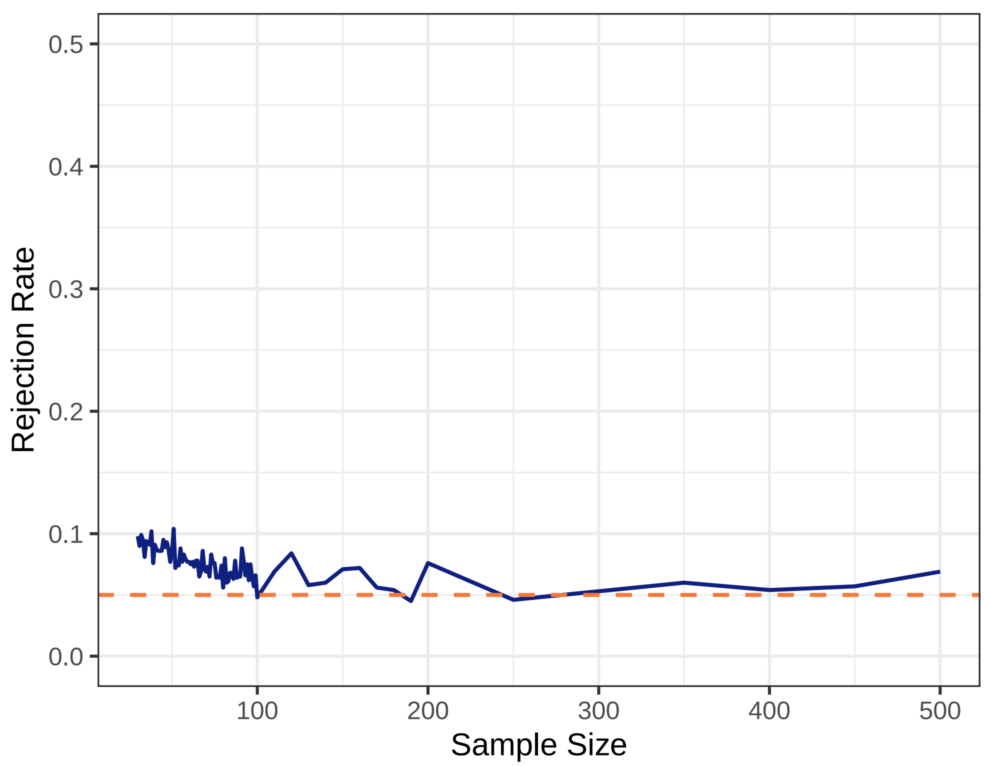


Figure 29: Test size control

For the ARMA(1,1) model above, if ϵ_t is i.i.d $exp(1)$ we have the following figures:

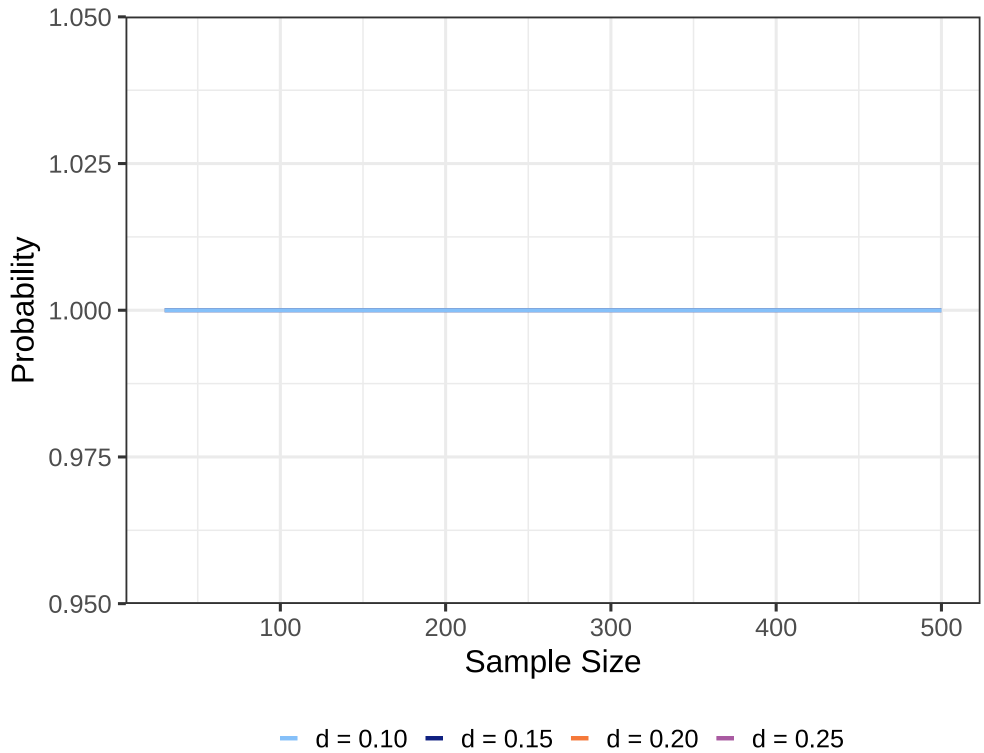


Figure 30: Convergence in probability.

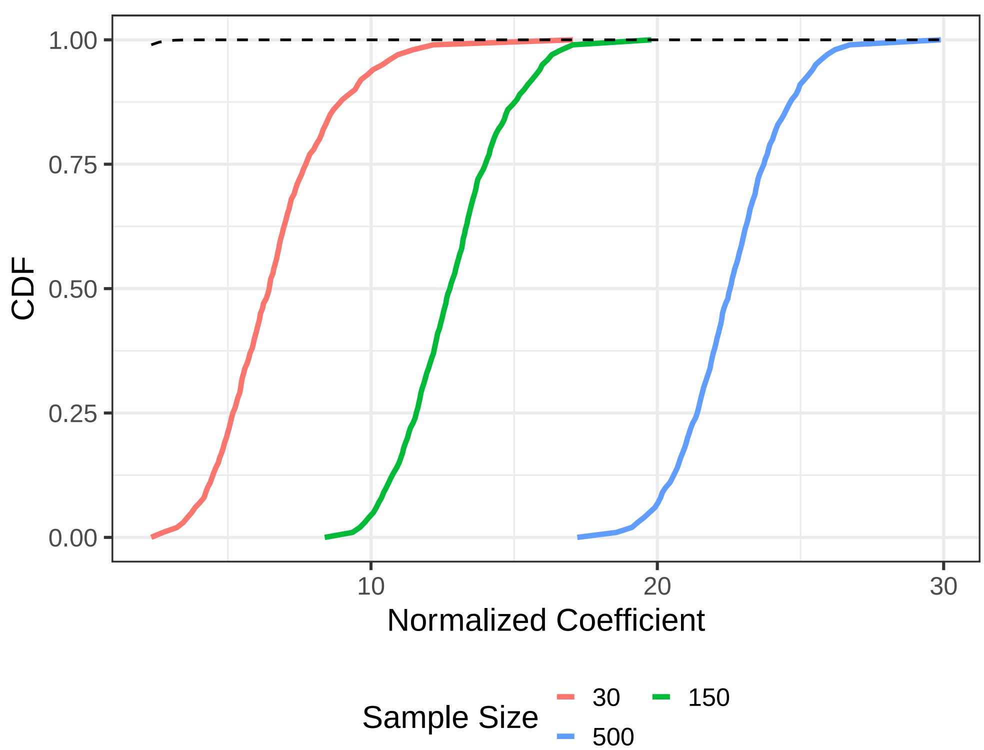
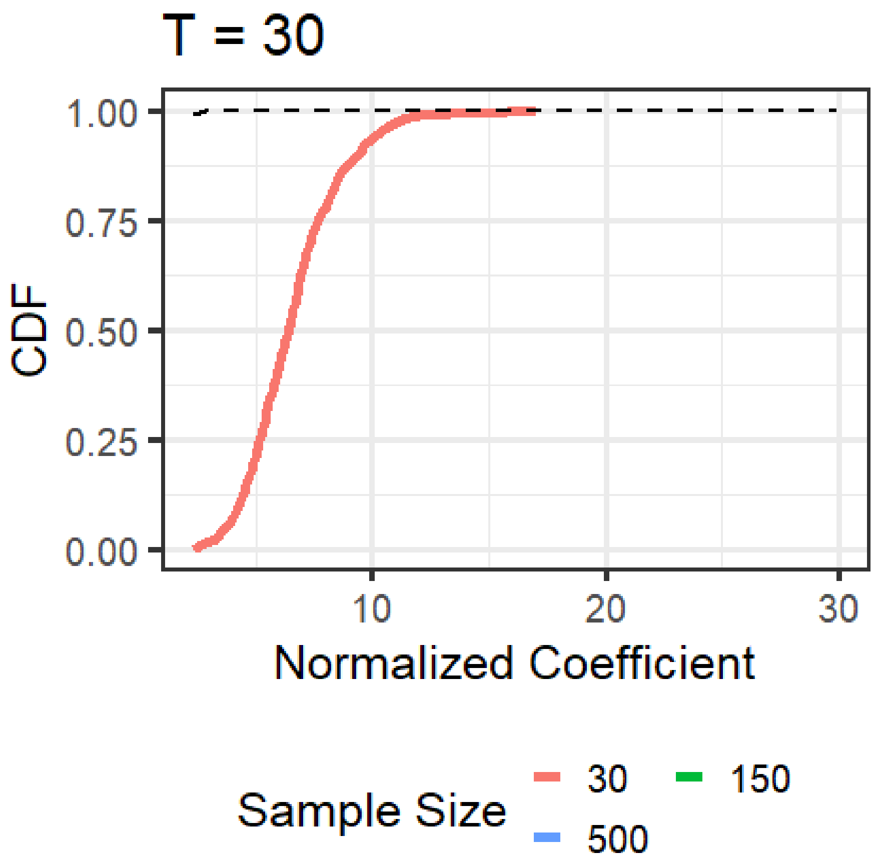


Figure 31: Convergence in probability.



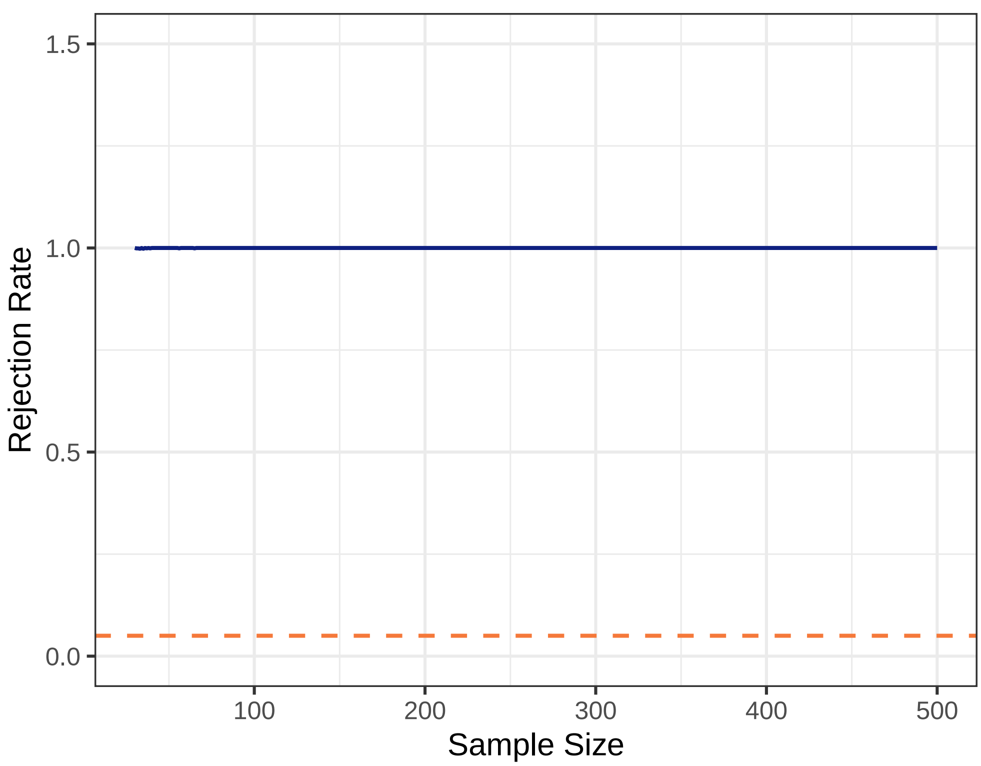


Figure 32: Convergence in distribution.

2.5

Given the ARMA(1,1) model

$$Y_t = c + \phi \cdot Y_{t-1} + \theta \cdot \epsilon_{t-1} + \epsilon_t,$$

where $c = 0$, $\phi = 0.3$, $\theta = 0.5$ ϵ_t is i.i.d $N(0, 1)$.

To analyze the **estimator for the first auto-regressive coefficient** of this model, we can examine its convergence in probability, convergence in distribution (illustrated as well by a GIF[5] for the convergence in distribution), and test size control. These figures provide insights into the behavior of the estimator and its performance.

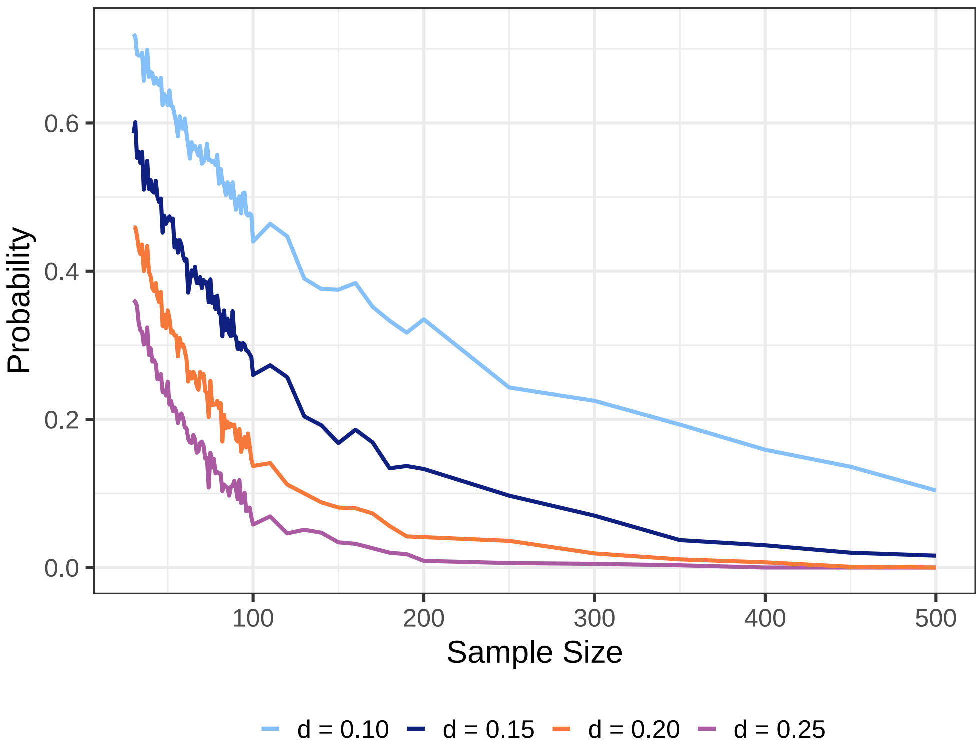


Figure 33: Convergence in probability.

[5]To see the GIF is recommended to open the PDF file in Adobe Acrobat Reader

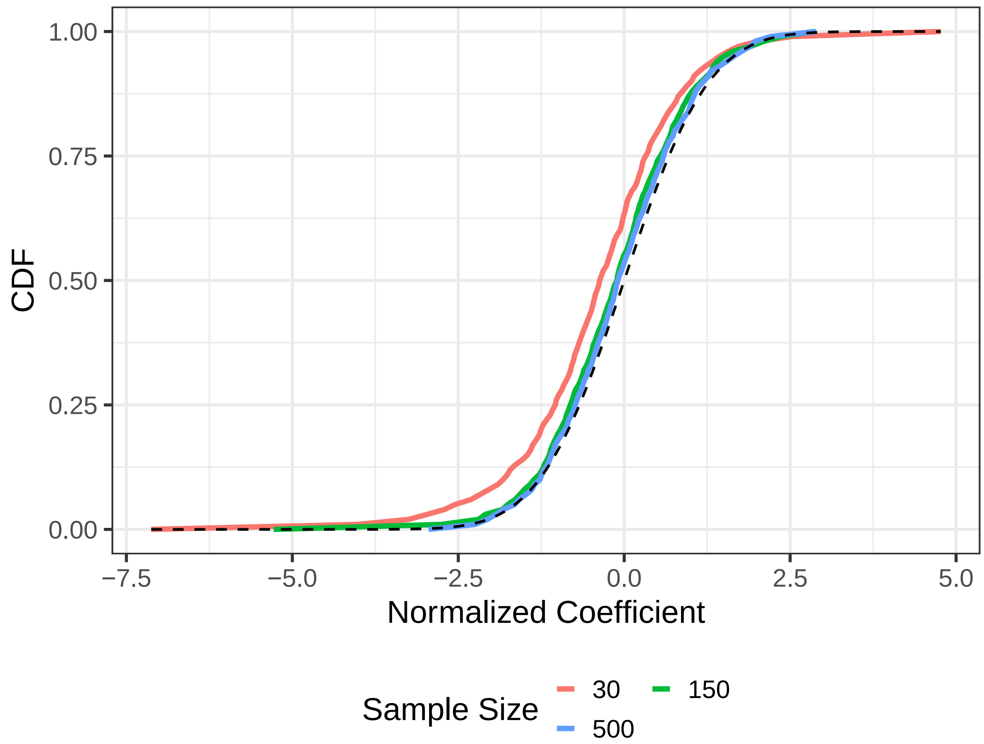
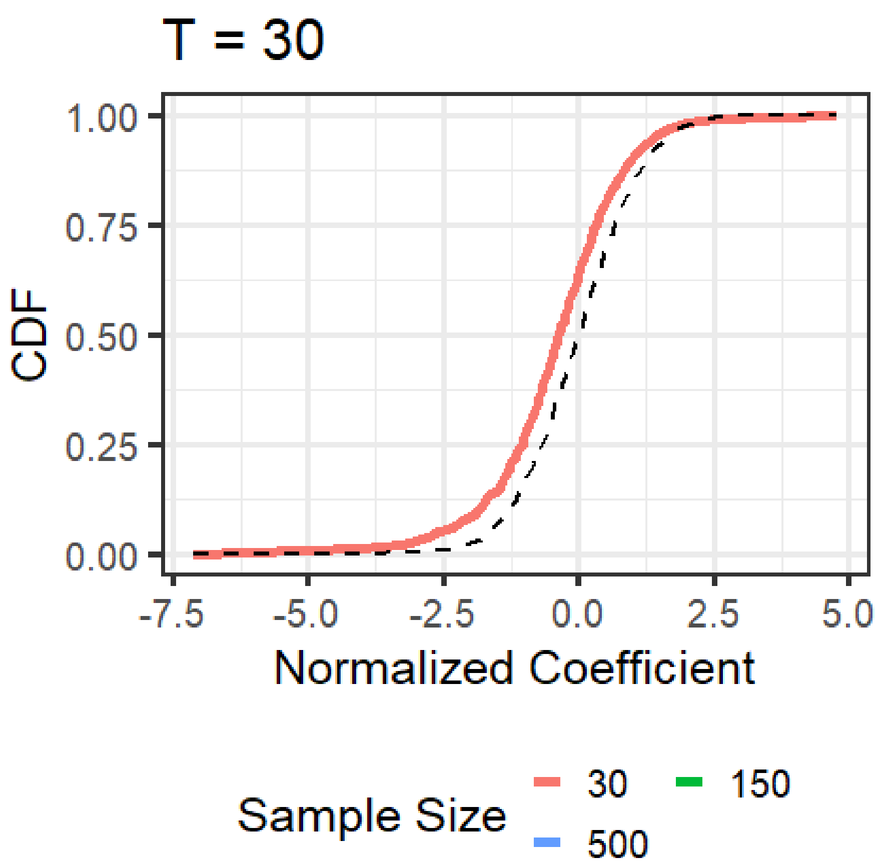


Figure 34: Convergence in distribution.



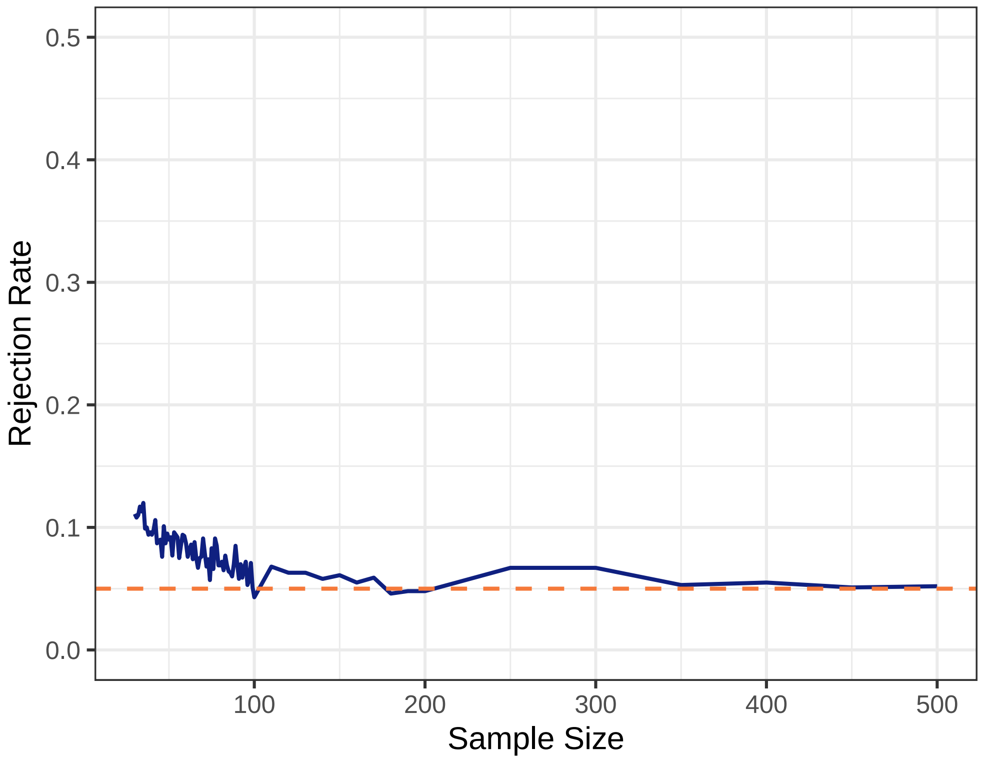


Figure 35: Test size control.

For the ARMA(1,1) model above, if ϵ_t is i.i.d $exp(1)$ we have the following figures:

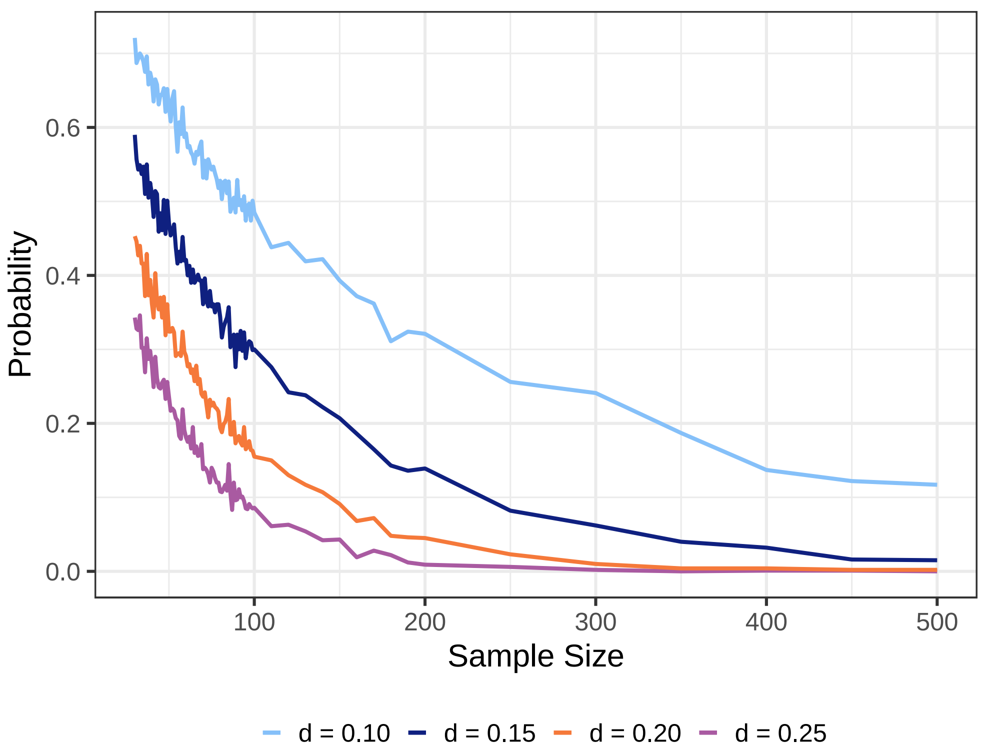


Figure 36: Convergence in probability.

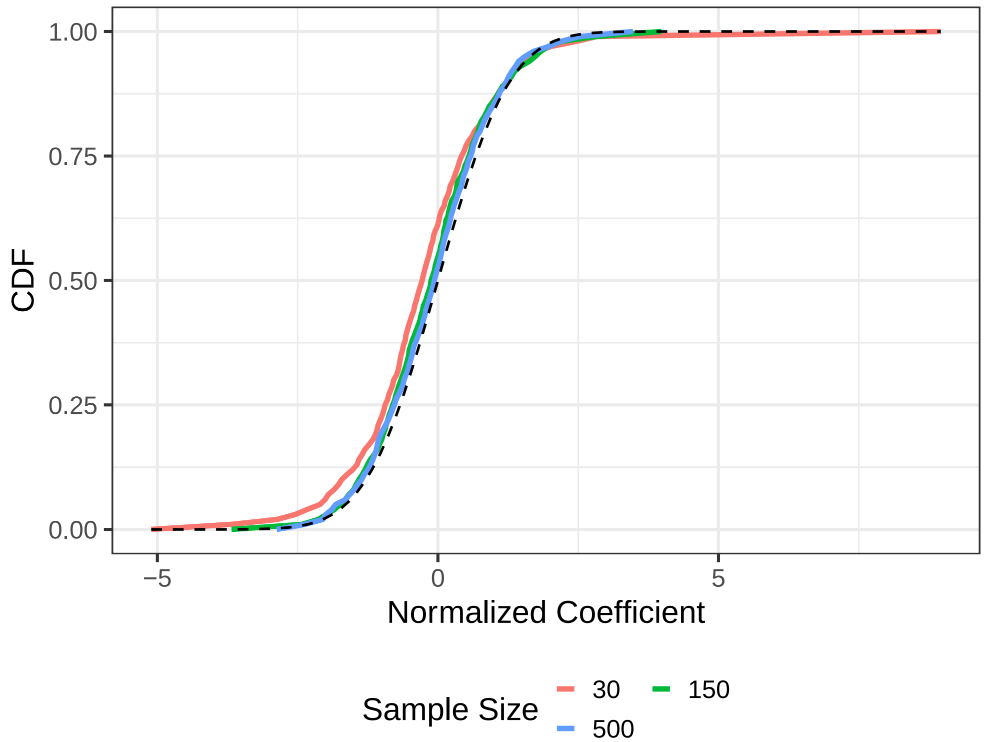
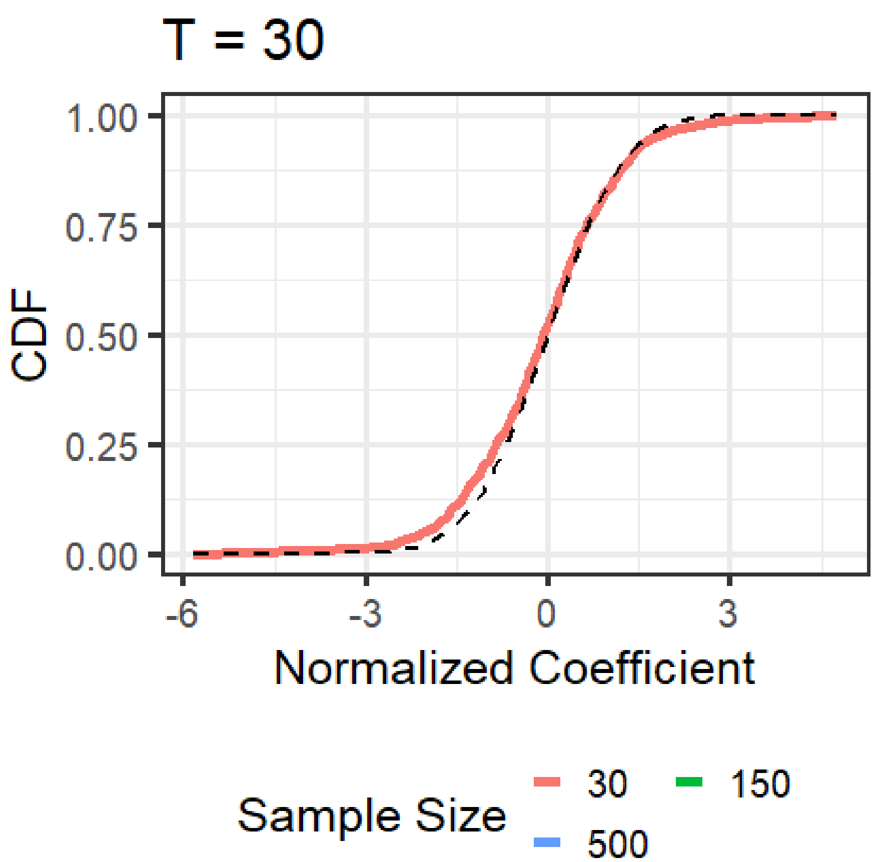


Figure 37: Convergence in probability.



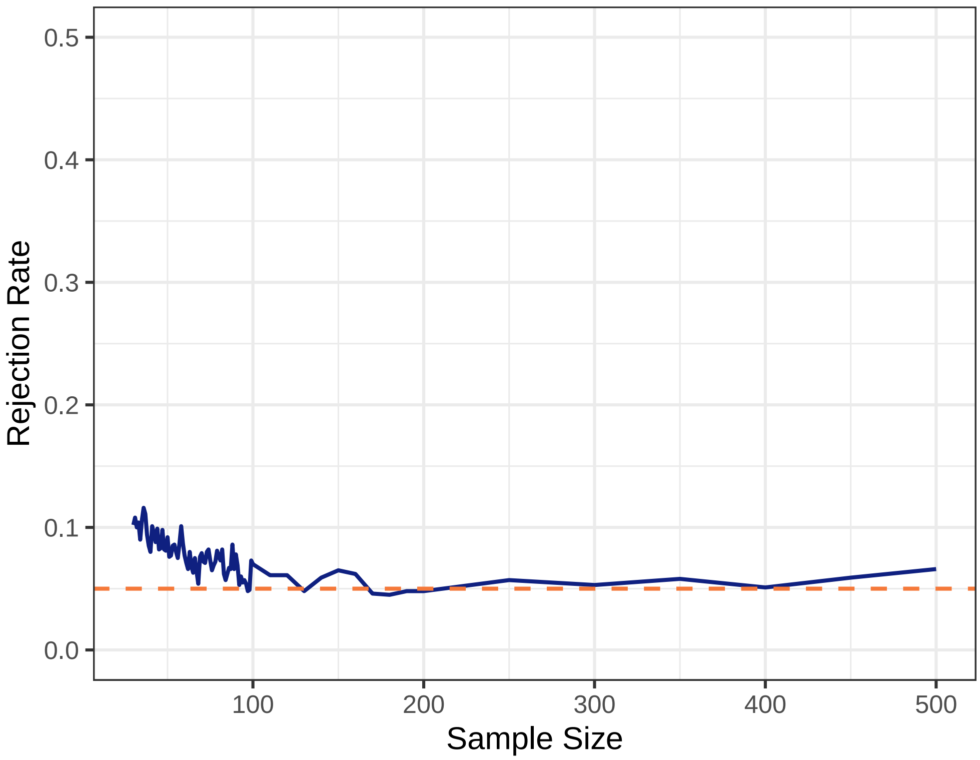


Figure 38: Convergence in distribution.

2.6

Given the ARMA(1,1) model

$$Y_t = c + \phi \cdot Y_{t-1} + \theta \cdot \epsilon_{t-1} + \epsilon_t,$$

where $c = 0$, $\phi = 0.3$, $\theta = 0.5$ ϵ_t is i.i.d $N(0, 1)$.

To analyze the **estimator for the first moving average coefficient** of this model, we can examine its convergence in probability, convergence in distribution (illustrated as well by a GIF[6] for the convergence in distribution), and test size control. These figures provide insights into the behavior of the estimator and its performance.

[6]To see the GIF is recommended to open the PDF file in Adobe Acrobat Reader

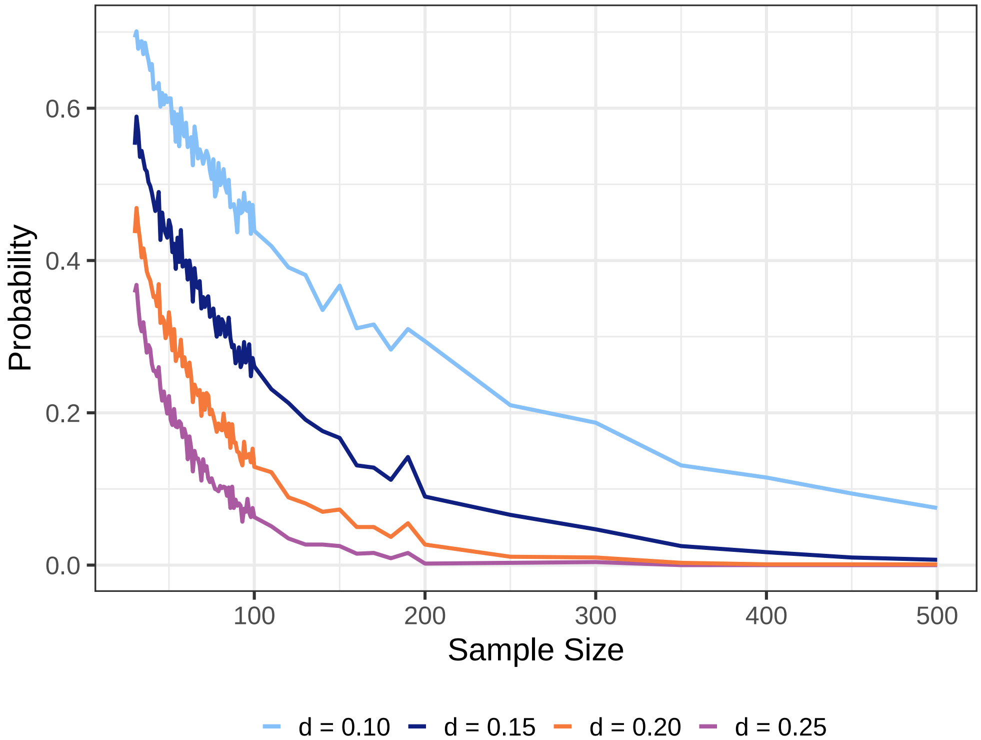


Figure 39: Convergence in probability.

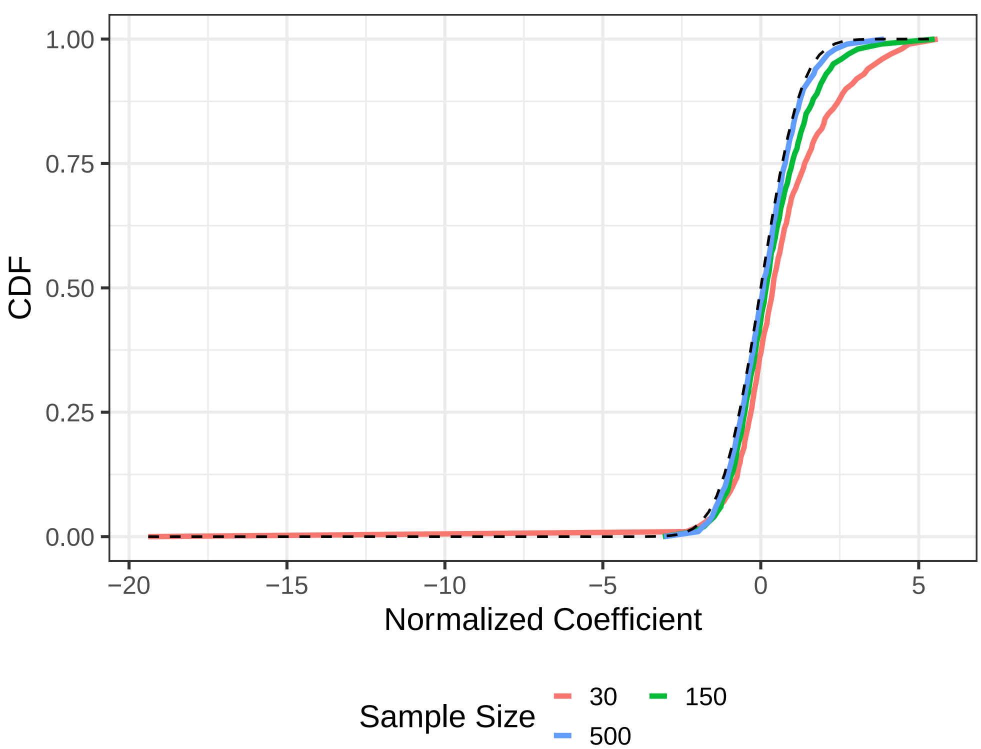


Figure 40: Convergence in distribution.

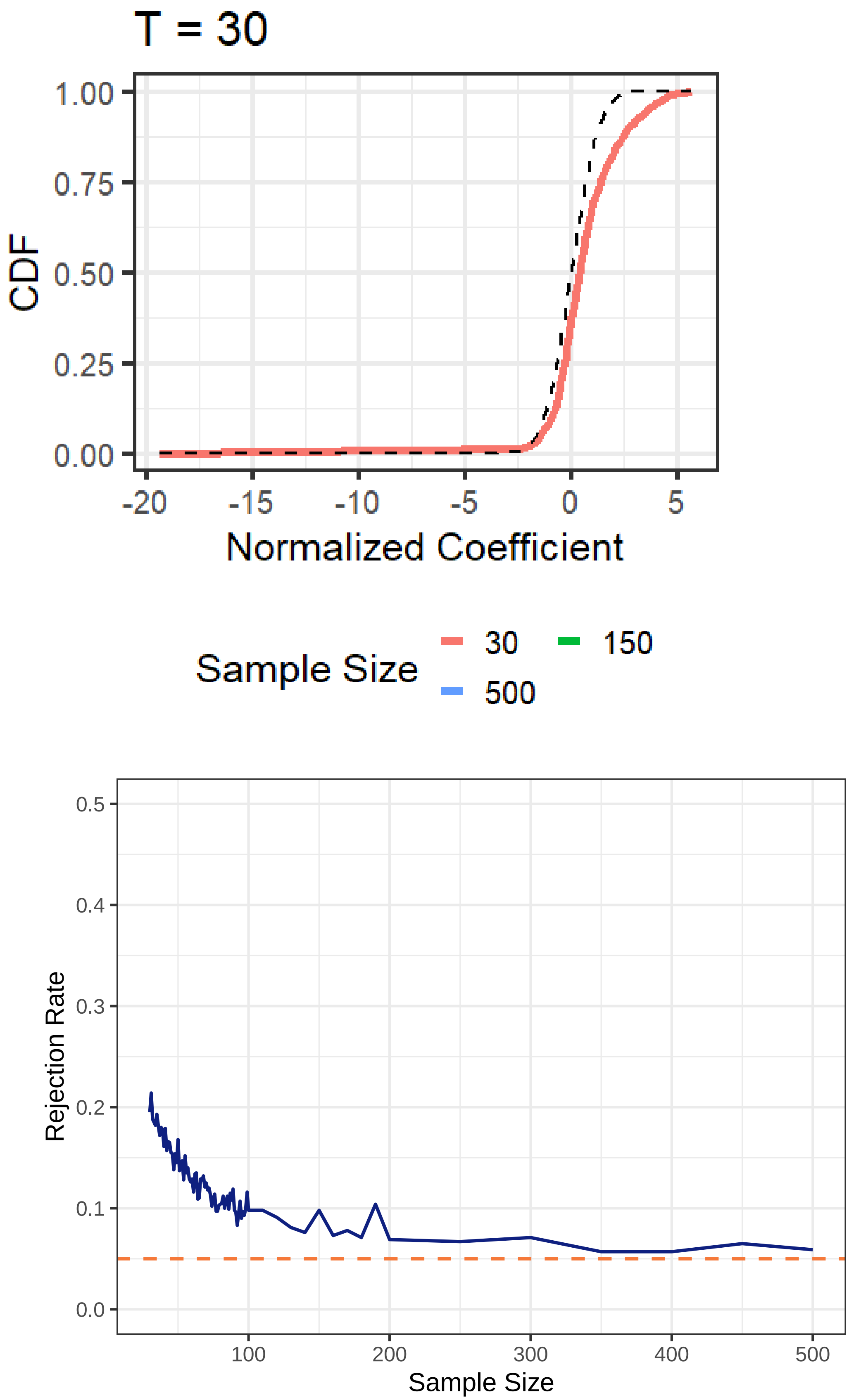


Figure 41: Test size control.

For the ARMA(1,1) model above, if ϵ_t is i.i.d $exp(1)$ we have the following figures:

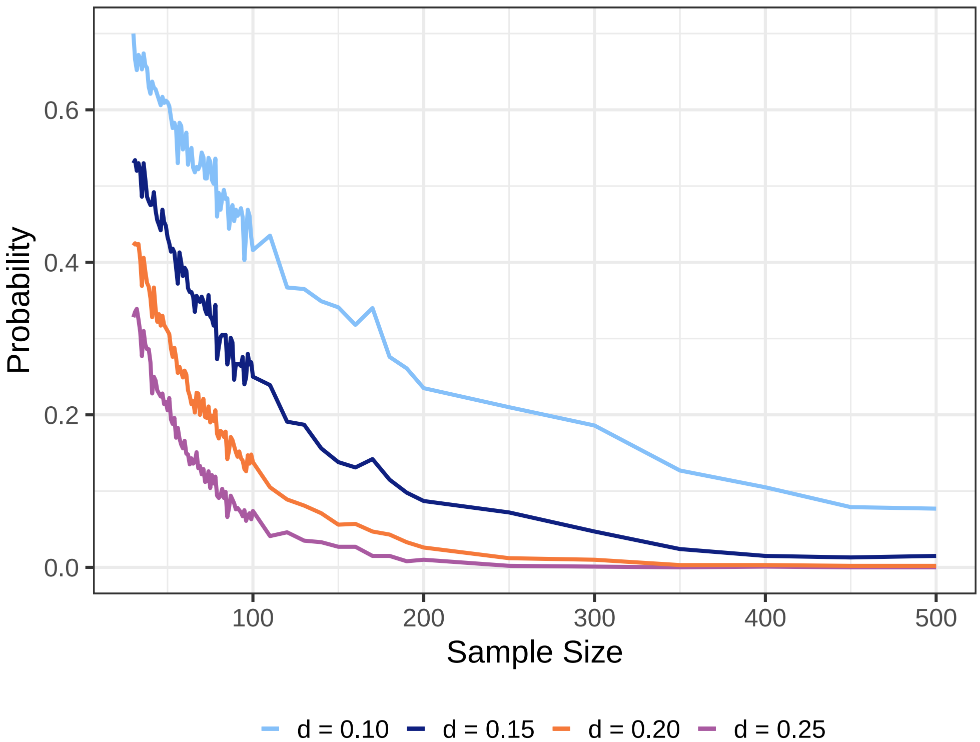


Figure 42: Convergence in probability.

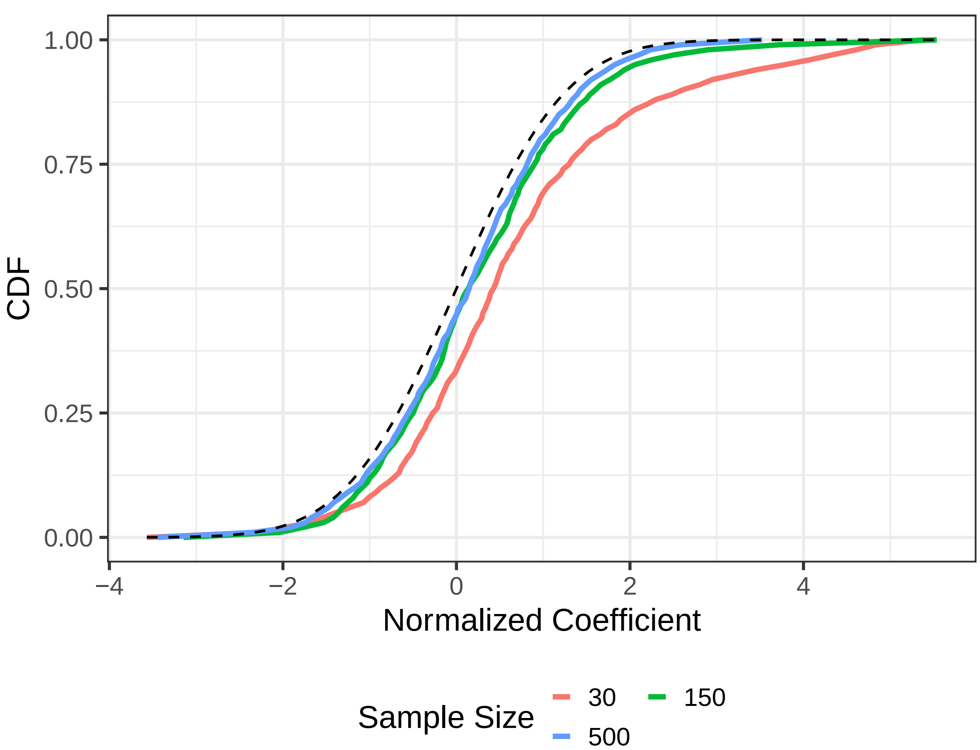


Figure 43: Convergence in probability.

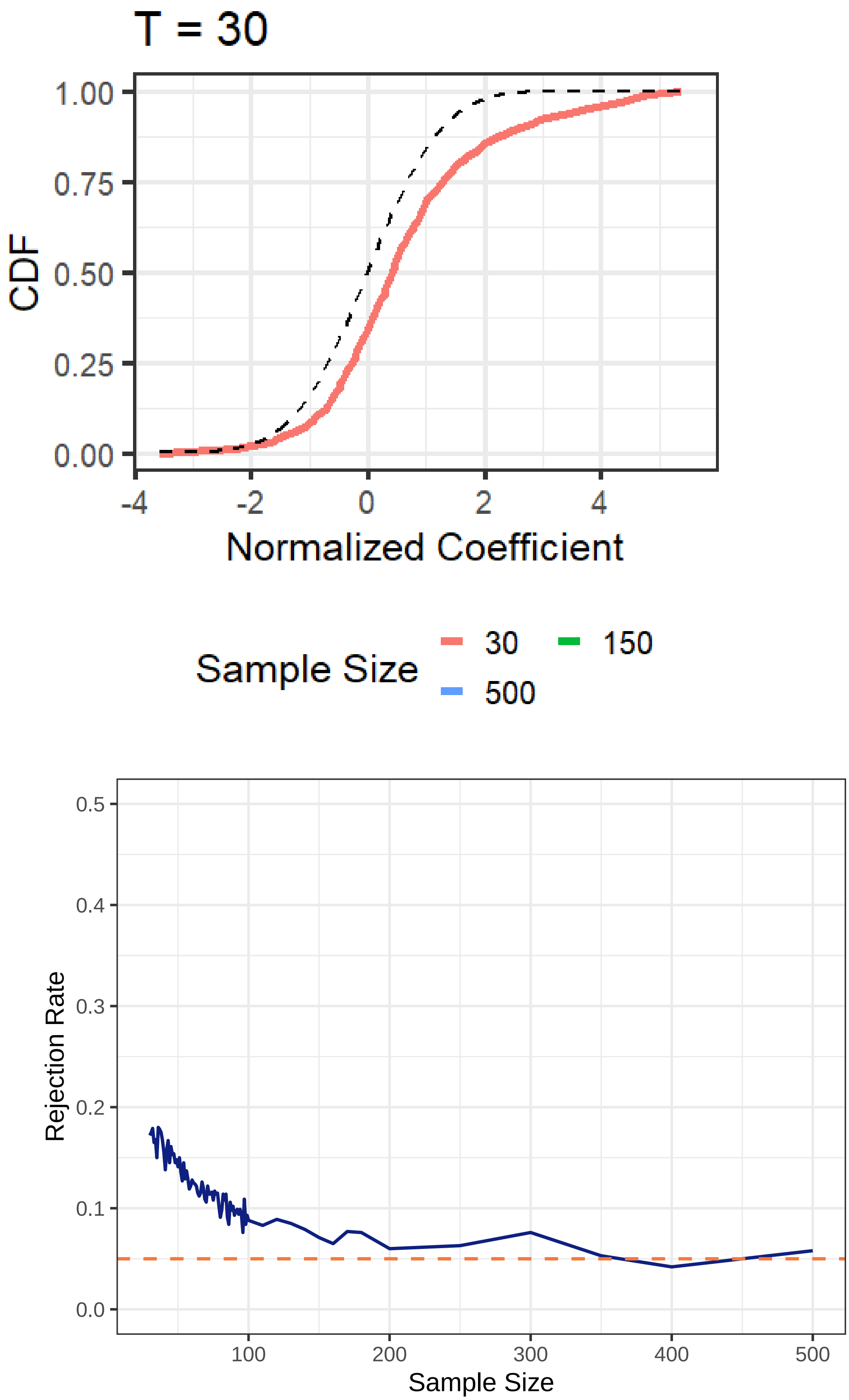


Figure 44: Convergence in distribution.

Moreover, answer the following question (20 points): Based on the simulations above, do you feel comfortable with a sample size of 500 periods for any stochastic process? Explain your answer.

When assuming **i.i.d. Gaussian errors** the sample size of 500 periods has a significant effect in terms of convergence in probability and convergence in distribution compared to smaller sample sizes. This can be attributed to the i.i.d. errors satisfying the Central Limit Theorem and Law of Large Numbers in this scenario. Additionally, as the sample size increases to 500 periods for any stochastic process, the rejection rate of the hypothesis test approaches the 5% significance level. Moreover, even for small samples, our estimators tend to follow a normal distribution.

erros gaussianos
· CLT
· LLN
· p/ n pequena, o estimador ~ Normal

Under the assumption of **i.i.d. exponential errors**, the analysis of the **intercept estimator** reveals that convergence in probability is consistently 100% regardless of the sample size, due to the misspecification of the ARMA model. In other words, the underlying model assumptions are not met, leading to biased estimates that do not converge to the expected value and may yield misleading results. Convergence in distribution shows that, as more data points are included in the estimation, the distribution of the normalized coefficient approaches the expected behavior under the assumption of i.i.d. exponential errors, albeit with a slower convergence. As for test size control, the coefficient of the model is rejected 100% of the time.

erros exp
· ARMA misspecified
· conv prob ok
· conv distr
↳ slower convergence
· test size
↳ rejeita 100% das vezes.

Furthermore, under the assumption of **i.i.d. exponential errors**, when analyzing the **coefficient estimator**, if the error tolerance is small (e.g., $d = 0.1$), a large sample size is required to claim that the probability of having an error greater than 0.1 is small. However, with $d = 0.25$, the probability decreases rapidly, even with exponential errors. Despite the model misspecification, the model exhibits well-behaved convergence in probability.

Regarding the convergence in distribution of the estimated coefficient of the model, each Monte Carlo repetition involves comparing the cumulative distribution function (CDF) of the evaluated model with the CDF of the normal distribution, which represents the asymptotic expectation. Furthermore, convergence in distribution refers to the point-wise convergence of the CDF, even for models with exponential errors.

In the rejection rate graph, we perform a t-test to determine if the estimated coefficient significantly differs from the true parameter. As the sample size increases, the rejection rate approaches 5%. Conversely, with a small sample size, the rejection rate naturally tends to be higher.

Overall, it is important to note that ML estimation tends to yield reasonable results even when the error distribution is non-Normal.

Problem 3

3.1

A weakly stationary stochastic process can be represented as an MA(q) process given by:

$$Y_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q} \tag{1}$$

Furthermore, this MA(q) process is ergodic for all moments. Where

$$E[Y_t] = \mu$$

$$E[(Y_t - \mu)^2] = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2) \cdot \sigma^2$$

$$\gamma_j = \begin{cases} 0 & \text{for } j > q \\ \sigma^2(\theta_j + \theta_{j+1} \cdot \theta_1 + \theta_{j+2} \cdot \theta_{\cancel{1}\,2} + \cdots + \theta_q \cdot \theta_{q-j}) & \text{for } j \in \{1, \ldots, q\} \end{cases}$$

3.2

A stochastic process $\{Y_t\}$, i.e., an AR(1) process

$$Y_t = c + \phi \cdot Y_{t-1} + \epsilon_t,$$

is not weakly stationary when $|\phi| \geq 1$. Where $\{\epsilon_t\}$ is white noise and c and ϕ are constants.

Problem 4

To show that $\gamma_j = \gamma_{-j}$ we have to remember that the auto-covariance is given by:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (y_t - \mu_t)(y_{t-j} - \mu_{t-j}) f_{Y_{t-j},\ldots,Y_t}(y_{t-j},\ldots,y_t) dy_{t-j} \cdots dy_t$$

Furthermore:

$$\gamma_{j,t} := E[(Y_t - \mu_t) \cdot (Y_{t-j} - \mu_{t-j})] \tag{2}$$

According to Hamilton (p.45) "The autocovariance $\gamma_{j,t}$ can be viewed as the (1,j+1) element of the variance-covariance matrix of the vector $\mathbf{x}_t$. For this reason, the autocovariances are described as the second moments of the process for Y_t".

Since the process $\{Y_t\}$ is weakly stationary (a.k.a. covariance-stationary), i.e., if neither the mean μ_t nor the autocovariances $\gamma_{j,t}$ depend on the date t:

$$E[Y_t] = \mu \quad \text{for all } t \tag{3}$$

$$E[(Y_t - \mu_t) \cdot (Y_{t-j} - \mu_{t-j})] = \gamma_j \quad \text{for all } t \text{ and } j. \tag{4}$$

i.e., the covariance between Y_t and Y_{t-j} depends only on j, the length of time separating the observations, and not on t, the date of the observation. Under these conditions $\gamma_j = \gamma_{-j}$ represent the same magnitude. Then,

$$E[(Y_t - \mu)(Y_{t-j} - \mu)] = \gamma_{-j} \quad \text{for all t} \tag{5}$$

And,

$$E[(Y_{t+j} - \mu)(Y_{t+j-j} - \mu)] = \gamma_j \tag{6}$$

Therefore, we can replace t with t+j , remaining

$$E[(Y_{t+j} - \mu)(Y_{t+j-j} - \mu)] = E[(Y_t - \mu)(Y_{t+j} - \mu)] \tag{7}$$

$$\gamma_j = \gamma_{-j} \quad \text{for all integers j} \tag{8}$$