

Sets

DEFINITION 1 A *set* is an unordered collection of objects.

DEFINITION 2 The objects in a set are called the *elements*, or *members*, of the set. A set is said to *contain* its elements.

We will now introduce notation used to describe membership in sets. We write $a \in A$ to denote that a is an element of the set A . The notation $a \notin A$ denotes that a is not an element of the set A . Note that lowercase letters are usually used to denote elements of sets.

There are several ways to describe a set. One way is to list all the members of a set, when this is possible. We use a notation where all members of the set are listed between braces. For example, the notation $\{a, b, c, d\}$ represents the set with the four elements a, b, c , and d .

EXAMPLE

The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$.

EXAMPLE

The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$.

EXAMPLE

The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \dots, 99\}$.

Another way to describe a set is to use **set builder** notation. We characterize all those elements in the set by stating the property or properties they must have to be members. For instance, the set O of all odd positive integers less than 10 can be written as



$$O = \{x \mid x \text{ is an odd positive integer less than } 10\},$$

or, specifying the universe as the set of positive integers, as

$$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}.$$

We often use this type of notation to describe sets when it is impossible to list all the elements of the set. For instance, the set \mathbf{Q}^+ of all positive rational numbers can be written as

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p \text{ and } q\}.$$

These sets, each denoted using a boldface letter, play an important role in discrete mathematics:

$\mathbf{N} = \{0, 1, 2, 3, \dots\}$, the set of **natural numbers**

$\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, the set of **integers**

$\mathbf{Z}^+ = \{1, 2, 3, \dots\}$, the set of **positive integers**

$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**

\mathbf{R} , the set of **real numbers**

The set $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$ is a set containing four elements, each of which is a set. The four elements of this set are \mathbf{N} , the set of natural numbers; \mathbf{Z} , the set of integers; \mathbf{Q} , the set of rational numbers; and \mathbf{R} , the set of real numbers.

DEFINITION 3 Two sets are *equal* if and only if they have the same elements. That is, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

EXAMPLE

The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so $\{1, 3, 3, 3, 5, 5, 5, 5\}$ is the same as the set $\{1, 3, 5\}$ because they have the same elements.

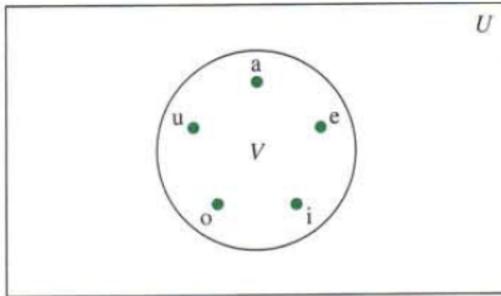


FIGURE 1 Venn Diagram for the Set of Vowels.

DEFINITION 4 The set A is said to be a *subset* of B if and only if every element of A is also an element of B . We use the notation $A \subseteq B$ to indicate that A is a subset of the set B .

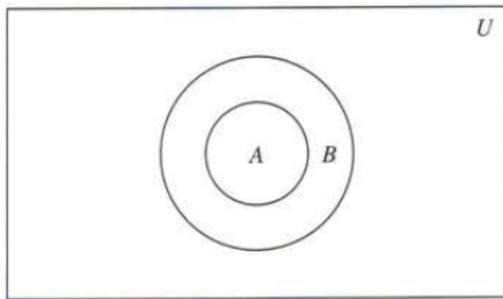


FIGURE 2 Venn Diagram Showing that A Is a Subset of B .

THEOREM 1 For every set S ,
 (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.

When we wish to emphasize that a set A is a subset of the set B but that $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B . For $A \subset B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A .

$$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

DEFINITION 5 Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S . The cardinality of S is denoted by $|S|$.

EXAMPLE

Let A be the set of odd positive integers less than 10. Then $|A| = 5$.

EXAMPLE

Let S be the set of letters in the English alphabet. Then $|S| = 26$.

EXAMPLE

Because the null set has no elements, it follows that $|\emptyset| = 0$.

The Power Set

DEFINITION 7 Given a set S , the *power set* of S is the set of all subsets of the set S . The power set of S is denoted by $P(S)$.

EXAMPLE

What is the power set of the set $\{0, 1, 2\}$?

Solution: The power set $P(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence,

$$P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

EXAMPLE

What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

Solution: The empty set has exactly one subset, namely, itself. Consequently,

$$P(\emptyset) = \{\emptyset\}.$$

The set $\{\emptyset\}$ has exactly two subsets, namely, \emptyset and the set $\{\emptyset\}$ itself. Therefore,

$$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$

If a set has n elements, then its power set has 2^n .

Cartesian Products

The order of elements in a collection is often important. Because sets are unordered, a different structure is needed to represent ordered collections. This is provided by **ordered n -tuples**.

DEFINITION 8 The *ordered n -tuple* (a_1, a_2, \dots, a_n) is the ordered collection that has a_1 as its first element, a_2 as its second element, \dots , and a_n as its n th element.

DEFINITION 9 Let A and B be sets. The *Cartesian product* of A and B , denoted by $A \times B$, is the set of all ordered pairs (a, b) , where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

EXAMPLE

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Solution: The Cartesian product $A \times B$ is

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

EXAMPLE

Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$, where A and B are as in Example 16.

Solution: The Cartesian product $B \times A$ is

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

This is not equal to $A \times B$, which was found in Example 16. 

The Cartesian product of more than two sets can also be defined.

DEFINITION 10 The *Cartesian product* of the sets A_1, A_2, \dots, A_n , denoted by $A_1 \times A_2 \times \dots \times A_n$, is the set of ordered n -tuples (a_1, a_2, \dots, a_n) , where a_i belongs to A_i for $i = 1, 2, \dots, n$. In other words,

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n\}.$$

EXAMPLE

What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

Solution: The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c) , where $a \in A$, $b \in B$, and $c \in C$. Hence,

$$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}. \quad \blacktriangleleft$$

Exercises

1. List the members of these sets.
 - a) $\{x \mid x \text{ is a real number such that } x^2 = 1\}$
 - b) $\{x \mid x \text{ is a positive integer less than } 12\}$
 - c) $\{x \mid x \text{ is the square of an integer and } x < 100\}$
 - d) $\{x \mid x \text{ is an integer such that } x^2 = 2\}$
4. Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$, and $D = \{4, 6, 8\}$. Determine which of these sets are subsets of which other of these sets.
7. Determine whether each of these statements is true or false.

a) $0 \in \emptyset$	b) $\emptyset \in \{0\}$
c) $\{0\} \subset \emptyset$	d) $\emptyset \subset \{0\}$
e) $\{0\} \in \{0\}$	f) $\{0\} \subset \{0\}$
g) $\{\emptyset\} \subseteq \{\emptyset\}$	

- 17.** What is the cardinality of each of these sets?
- a) $\{a\}$ b) $\{\{a\}\}$
c) $\{a, \{a\}\}$ d) $\{a, \{a\}, \{a, \{a\}\}\}$
-
- 19.** Find the power set of each of these sets, where a and b are distinct elements.
- a) $\{a\}$ b) $\{a, b\}$ c) $\{\emptyset, \{\emptyset\}\}$
- 28.** Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find
- a) $A \times B \times C$ b) $C \times B \times A$
c) $C \times A \times B$ d) $B \times B \times B$

Set Operations

DEFINITION 1 Let A and B be sets. The *union* of the sets A and B , denoted by $A \cup B$, is the set that contains those elements that are either in A or in B , or in both.

An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B . This tells us that

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

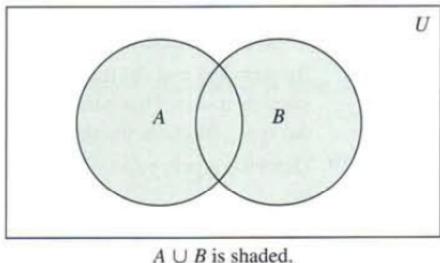
EXAMPLE

The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$; that is, $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$.

DEFINITION 2 Let A and B be sets. The *intersection* of the sets A and B , denoted by $A \cap B$, is the set containing those elements in both A and B .

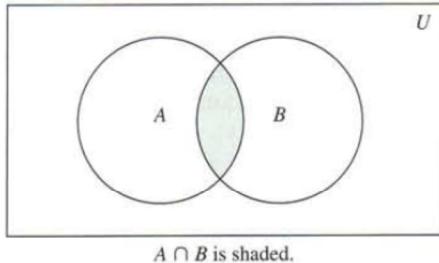
An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B . This tells us that

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$



$A \cup B$ is shaded.

FIGURE 1 Venn Diagram Representing the Union of *A* and *B*.



$A \cap B$ is shaded.

FIGURE 2 Venn Diagram Representing the Intersection of *A* and *B*.

EXAMPLE

The intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 3\}$; that is, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$. ◀

DEFINITION 3 Two sets are called *disjoint* if their intersection is the empty set.

EXAMPLE

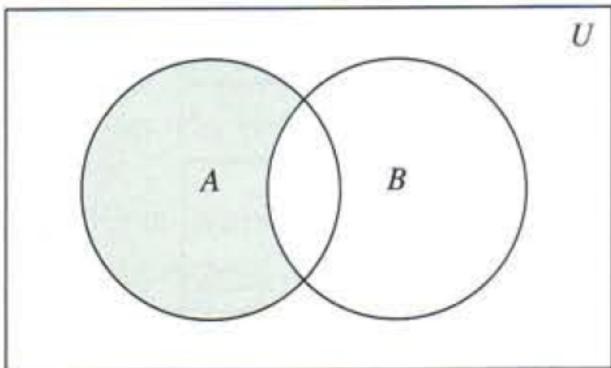
Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$, A and B are disjoint.

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

DEFINITION 4 Let A and B be sets. The *difference* of A and B , denoted by $A - B$, is the set containing those elements that are in A but not in B . The difference of A and B is also called the *complement of B with respect to A* .

An element x belongs to the difference of A and B if and only if $x \in A$ and $x \notin B$. This tells us that

$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$



$A - B$ is shaded.

FIGURE 3 Venn Diagram for the Difference of A and B .

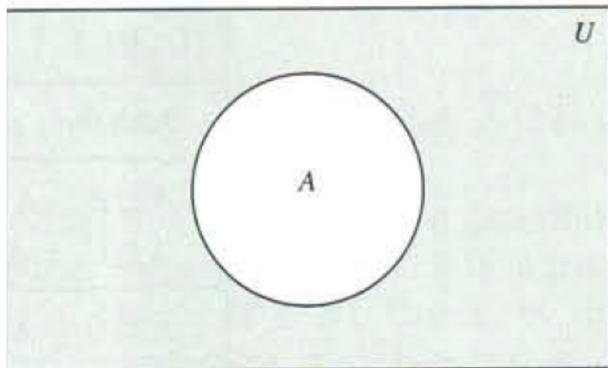
EXAMPLE

The difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{5\}$; that is, $\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$. This is different from the difference of $\{1, 2, 3\}$ and $\{1, 3, 5\}$, which is the set $\{2\}$. ◀

DEFINITION 5 Let U be the universal set. The *complement* of the set A , denoted by \bar{A} , is the complement of A with respect to U . In other words, the complement of the set A is $U - A$.

An element belongs to \bar{A} if and only if $x \notin A$. This tells us that

$$\bar{A} = \{x \mid x \notin A\}.$$



\bar{A} is shaded.

FIGURE 4 Venn Diagram for the Complement of the Set A .

DEFINITION 6 The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

We use the notation

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^n A_i$$

to denote the union of the sets A_1, A_2, \dots, A_n .

DEFINITION 7 The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

We use the notation

$$A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{i=1}^n A_i$$

to denote the intersection of the sets A_1, A_2, \dots, A_n .

EXAMPLE

Let $A_i = \{i, i + 1, i + 2, \dots\}$. Then,

$$\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n \{i, i + 1, i + 2, \dots\} = \{1, 2, 3, \dots\},$$

and

$$\bigcap_{i=1}^n A_i = \bigcap_{i=1}^n \{i, i + 1, i + 2, \dots\} = \{n, n + 1, n + 2, \dots\}.$$

TABLE 1 Set Identities.

<i>Identity</i>	<i>Name</i>
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(A)} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Computer Representation of Sets

There are various ways to represent sets using a computer. One method is to store the elements of the set in an unordered fashion. However, if this is done, the operations of computing the union, intersection, or difference of two sets would be time-consuming, because each of these operations would require a large amount of searching for elements. We will present a method for storing elements using an arbitrary ordering of the elements of the universal set. This method of representing sets makes computing combinations of sets easy.

Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used). First, specify an arbitrary ordering of the elements of U , for instance a_1, a_2, \dots, a_n . Represent a subset A of U with the bit string of length n , where the i th bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A . Example 18 illustrates this technique.

EXAMPLE

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U , the subset of all even integers in U , and the subset of integers not exceeding 5 in U ?

Solution: The bit string that represents the set of odd integers in U , namely, $\{1, 3, 5, 7, 9\}$, has a one bit in the first, third, fifth, seventh, and ninth positions, and a zero elsewhere. It is

10 1010 1010.

(We have split this bit string of length ten into blocks of length four for easy reading because long bit strings are difficult to read.) Similarly, we represent the subset of all even integers in U , namely, $\{2, 4, 6, 8, 10\}$, by the string

01 0101 0101.

The set of all integers in U that do not exceed 5, namely, $\{1, 2, 3, 4, 5\}$, is represented by the string

11 1110 0000. 

Using bit strings to represent sets, it is easy to find complements of sets and unions, intersections, and differences of sets. To find the bit string for the complement of a set from the bit string for that set, we simply change each 1 to a 0 and each 0 to 1, because $x \in A$ if and only if $x \notin A$. Note that this operation corresponds to taking the negation of each bit when we associate a bit with a truth value—with 1 representing true and 0 representing false.

EXAMPLE

We have seen that the bit string for the set $\{1, 3, 5, 7, 9\}$ (with universal set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$) is

10 1010 1010.

What is the bit string for the complement of this set

Solution: The bit string for the complement of this set is obtained by replacing 0s with 1s and vice versa. This yields the string

01 0101 0101,

which corresponds to the set $\{2, 4, 6, 8, 10\}$. 

EXAMPLE

The bit strings for the sets $\{1, 2, 3, 4, 5\}$ and $\{1, 3, 5, 7, 9\}$ are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

Solution: The bit string for the union of these sets is

$$11 1110 0000 \vee 10 1010 1010 = 11 1110 1010,$$

which corresponds to the set $\{1, 2, 3, 4, 5, 7, 9\}$. The bit string for the intersection of these sets is

$$11\ 1110\ 0000 \wedge 10\ 1010\ 1010 = 10\ 1010\ 0000,$$

which corresponds to the set $\{1, 3, 5\}$.



Problems:

4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find
 - a) $A \cup B$.
 - b) $A \cap B$.
 - c) $A - B$.
 - d) $B - A$.
14. Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.
25. Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$, and $C = \{4, 5, 6, 7, 8, 9, 10\}$. Find
 - a) $A \cap B \cap C$.
 - b) $A \cup B \cup C$.
 - c) $(A \cup B) \cap C$.
 - d) $(A \cap B) \cup C$.
50. Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of these sets with bit strings where the i th bit in the string is 1 if i is in the set and 0 otherwise.
 - a) $\{3, 4, 5\}$
 - b) $\{1, 3, 6, 10\}$
 - c) $\{2, 3, 4, 7, 8, 9\}$
51. Using the same universal set as in the last problem, find the set specified by each of these bit strings.
 - a) 11 1100 1111
 - b) 01 0111 1000
 - c) 10 0000 0001
52. What subsets of a finite universal set do these bit strings represent?
 - a) the string with all zeros
 - b) the string with all ones

Relations

Relations and Their Properties

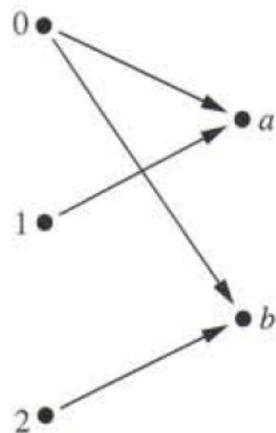
Introduction

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called binary relations. In this section we introduce the basic terminology used to describe binary relations. Later in this chapter we will use relations to solve problems involving communications networks, project scheduling, and identifying elements in sets with common properties.

DEFINITION 1 Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B . We use the notation $a R b$ to denote that $(a, b) \in R$ and $a \not R b$ to denote that $(a, b) \notin R$. Moreover, when (a, b) belongs to R , a is said to be **related to b by R** .

EXAMPLE



R	a	b
0	×	×
1	×	
2		×

EXAMPLE

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B . This means, for instance, that $0 R a$, but that $1 \not R b$. Relations can be represented graphically,

Relations on a Set

DEFINITION 2 A *relation on the set A* is a relation from A to A .

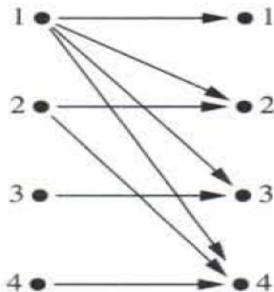
In other words, a relation on a set A is a subset of $A \times A$.

EXAMPLE

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b , we see that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

EXAMPLE

How many relations are there on a set with n elements?

Solution: A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of $A \times A$. Thus, there are 2^{n^2} relations on a set with n elements. For example, there are $2^{3^2} = 2^9 = 512$ relations on the set $\{a, b, c\}$.

Properties of Relations

DEFINITION 3 A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

EXAMPLE

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$

$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$

$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$

$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$

$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$

$$R_6 = \{(3, 4)\}.$$

Which of these relations are reflexive?

Solution: The relations R_3 and R_5 are reflexive because they both contain all pairs of the form (a, a) , namely, $(1, 1), (2, 2), (3, 3)$, and $(4, 4)$. The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, R_1, R_2, R_4 , and R_6 are not reflexive because $(3, 3)$ is not in any of these relations. 

EXAMPLE

Is the “divides” relation on the set of positive integers reflexive?

Solution: Because $a | a$ whenever a is a positive integer, the “divides” relation is reflexive. (Note that if we replace the set of positive integers with the set of all integers the relation is not reflexive because 0 does not divide 0.) 

DEFINITION 4 A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

EXAMPLE

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$

$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$

$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$

$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$

$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$

$$R_6 = \{(3, 4)\}.$$

Which of the relations are symmetric?

Solution: The relations R_2 and R_3 are symmetric, because in each case (b, a) belongs to the relation whenever (a, b) does. For R_2 , the only thing to check is that both $(2, 1)$ and $(1, 2)$ are in the relation. For R_3 , it is necessary to check that both $(1, 2)$ and $(2, 1)$ belong to the relation, and $(1, 4)$ and $(4, 1)$ belong to the relation. The reader should verify that none of the other relations is symmetric. This is done by finding a pair (a, b) such that it is in the relation but (b, a) is not.

EXAMPLE

Is the “divides” relation on the set of positive integers symmetric?

Solution: This relation is not symmetric because $1 \mid 2$, but $2 \nmid 1$.

DEFINITION 5 A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

EXAMPLE

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$

$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$

$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$

$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$

$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$

$$R_6 = \{(3, 4)\}.$$

Which of the relations are transitive?

Solution: R_4 , R_5 , and R_6 are transitive. For each of these relations, we can show that it is transitive by verifying that if (a, b) and (b, c) belong to this relation, then (a, c) also does. For instance, R_4 is transitive, because $(3, 2)$ and $(2, 1)$, $(4, 2)$ and $(2, 1)$, $(4, 3)$ and $(3, 1)$, and $(4, 3)$ and $(3, 2)$ are the only such sets of pairs, and $(3, 1)$, $(4, 1)$, and $(4, 2)$ belong to R_4 . The reader should verify that R_5 and R_6 are transitive.

R_1 is not transitive because $(3, 4)$ and $(4, 1)$ belong to R_1 , but $(3, 1)$ does not. R_2 is not transitive because $(2, 1)$ and $(1, 2)$ belong to R_2 , but $(2, 2)$ does not. R_3 is not transitive because $(4, 1)$ and $(1, 2)$ belong to R_3 , but $(4, 2)$ does not. 

Exercises

2. a) List all the ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 5, 6\}$.
3. For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is transitive.
 - a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
 - b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
 - c) $\{(2, 4), (4, 2)\}$
 - d) $\{(1, 2), (2, 3), (3, 4)\}$
 - e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
 - f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$