# Inverses and Rules of Matrix Arithmetic:

- Properties of Matrix Operations
- Inverse of 2 by 2 Matrix and its properties
- Powers of a Matrix
- Matrix polynomial

### Properties of Matrix Operations

For matrices A and B, however, $AB$ and $BA$ need not be equal. Equality can fail to hold for three reasons:

1- It can happen that the product $AB$ is defined but $BA$ is undefined. For example, this is the case if $A$ is $2\times3$ a matrix and $B$ is $3\times4$ a matrix.

2- Also, it can happen that $AB$ and $BA$ are both defined but have different sizes. This is the situation if $A$ is $2\times3$ a matrix and $B$ is $3\times2$ a matrix.

3- Finally, it is possible to have $AB \neq BA$ even if both $AB$ and $BA$ are defined and have the same size.

**Example** ($AB$ and $BA$ need not be equal)

$$A = [1, 2] \quad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad AB = [1]$$

$$BA = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} B = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \qquad AB = \begin{bmatrix} -1 & 3 \\ -3 & 7 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$$

*Thus,* $AB \neq BA$.

### Properties of Matrix Operations

*Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.*

(1) $A+B = B+A$ *Commutative low for addition*

(2) $A + (B + C) = (A + B) + C$ *Associative low for addition*

(3) $A(BC)=(AB)C$ *Associative low for multiplication*

(4) $A(B \pm C) = AB \pm AC$ *left distributive low*

(5) $(A \pm B)C = AC \pm BC$ *Right distributive low*

(6) $\alpha(A \pm B) = \alpha A \pm \alpha B$

(7) $(\alpha \pm \beta)A = \alpha A \pm \beta A$

(8) $\alpha(\beta A) = \alpha\beta A$

(9) $\alpha(BA) = (\alpha B)A = B(\alpha A)$

## Properties of matrix multiplication:

If $A$, $B$, and $C$ are matrices (with sizes such that the given matrix products are defined) and $c$ is a scalar, then the following properties are true.

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $c(AB) = (cA)B = A(cB)$

## Problem 1: Consider the matrices

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$$

Prove that $A(BC) = (AB)C$

Proof:

$$(AB)C = \left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix}\right)\begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 13 & 14 \end{bmatrix}.$$

$$A(BC) = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 13 & 14 \end{bmatrix}$$

## Problem 1: Prove that $AB \neq BA$ where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$

Proof:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}$$

**Example:** Associativity of Matrix Multiplication

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad BC = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}$$

Thus

$$(AB)C = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

And

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

So $(BA)C \neq A(BC)$.

# Unit Matrix:

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

For instance, if $n = 1, 2,$ or $3$, we have

$$\underset{1\times 1}{I_1 = [1]}, \qquad \underset{2\times 2}{I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \qquad \underset{3\times 3}{I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$


For instance, if $n = 1, 2,$ or $3$, we have

$$\underset{1\times 1}{I_1 = [1]}, \qquad \underset{2\times 2}{I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \qquad \underset{3\times 3}{I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$


# Example:

$$\text{(a)}\ \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\text{(b)}\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

**Zero Matrices**

A matrix, all of whose entries are zero, such as

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [0]$$

is called a ***zero matrix***. A zero matrix will be denoted by *0*; if it is important to emphasize the size, we shall write $0_{m \times n}$ for the $m \times n$ zero matrix.

**Properties of Zero Matrices**

*Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.*

(1) $A + 0 = 0 + A = A$

(2) $A - A = 0$

(3) $0 - A = -A$

(4) $A0 = 0;\ 0A = 0$

Some of the rules of arithmetic for real numbers do not carry over to matrix arithmetic, for example consider the following two standard results in the arithmetic of real numbers.

If $ab = ac$ and $a \neq 0$, then $b = c$. (This is called the *cancellation law*.)

If $ab = 0$, then at least one of the factors on the left is 0.

## Example

Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$$

We get

$$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \quad \text{and} \quad AD = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, although $A \neq 0$, it is *incorrect* to cancel the $A$ from both sides of the equation $AB = AC$ and write $B = C$. Also, $AD = 0$, yet $A \neq 0$ and $D \neq 0$. Thus, the cancellation law is not valid for matrix multiplication, and it is possible for a product of matrices to be zero without either factor being zero.

# Matrix power:

$$A^k = \underbrace{AA \cdot \ \cdot \ \cdot A}_{k \text{ factors}}.$$

$A^0 = I_n$ (where $A$ is a square matrix of order $n$)

Example:

Find $A^3$ for the matrix $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$.

SOLUTION $A^3 = \left( \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} \right) \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -6 \end{bmatrix}$

## *The Transpose of a Matrix*

The **transpose** of a matrix is formed by writing its columns as rows. For instance, if $A$ the $m \times n$ matrix shown by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix},$$

Size: $m \times n$

Size: $m \times n$

then the transpose, denoted by $A^T$, is the $n \times m$ matrix below

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}.$$

Size: $n \times m$

Example:

Find the transpose of each matrix.

(a) $A = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $D = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$

SOLUTION (a) $A^T = \begin{bmatrix} 2 & 8 \end{bmatrix}$ (b) $B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ (c) $C^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $D^T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}$

# Properties of Transpose of a matrix:

If $A$ and $B$ are matrices (with sizes such that the given matrix operations are defined) and $c$ is a scalar, then the following properties are true.

1. $(A^T)^T = A$ Transpose of a transpose
2. $(A + B)^T = A^T + B^T$ Transpose of a sum
3. $(cA)^T = c(A^T)$ Transpose of a scalar multiple
4. $(AB)^T = B^T A^T$ Transpose of a product

# Show that

$(AB)^T = B^T A^T$

$A = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$

# Solution:

$$AB = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & -1 \\ -1 & 2 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

## Definition

The ***trace of an n×n matrix A***, denoted by Tr($A$), is defined to be the sum of the entries on the main diagonal of $A$. That is

$$\mathrm{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

**Example:** The trace of the matrix

$$B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\mathbf{Tr(B) = -1 + 5 + 7 + 0 = 11}$$

The square matrix $A$ is said to be invertible if there is a matrix $B$ of same size such that

$$AB = BA = I$$

In this case $B$ is called the inverse of $A$, and $A$ is said to be ***invertible*** matrix. The notation for the inverse is $A^{-1}$. If no such matrix $B$ can be found, then $A$ is said to be ***singular***.

**Example**

The matrix

$$B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \quad \text{is an inverse of} \quad A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

Since

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Example:

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA=\begin{bmatrix}1 & -2\\ 1 & -1\end{bmatrix}\begin{bmatrix}-1 & 2\\ -1 & 1\end{bmatrix}=\begin{bmatrix}-1+2 & 2-2\\ -1+1 & 2-1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

**Theorem**

The matrix

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

is invertible if $ad-bc\neq 0$, in which case the inverse is given by the formula

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}=\begin{bmatrix}\frac{d}{ad-bc} & -\frac{b}{ad-bc}\\ -\frac{c}{ad-bc} & \frac{a}{ad-bc}\end{bmatrix}$$

Example:

(a) $A=\begin{bmatrix}3 & -1\\ -2 & 2\end{bmatrix}$ (b) $B=\begin{bmatrix}3 & -1\\ -6 & 2\end{bmatrix}$

Example:

$$A=\begin{bmatrix}1 & 4\\ -1 & -3\end{bmatrix}.$$

## Theorem

If $A$ and $B$ are invertible matrices of the same size, then $AB$ is invertible and

$$(AB)^{-1}=B^{-1}A^{-1}$$

## Example

Consider the matrices

$$A=\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix},\quad B=\begin{bmatrix}3 & 2\\ 2 & 2\end{bmatrix},\quad AB=\begin{bmatrix}7 & 6\\ 9 & 8\end{bmatrix}$$

Applying the formula in Theorem, we obtain

$$A^{-1}=\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix},\quad B^{-1}=\begin{bmatrix}1 & -1\\ -1 & \frac{3}{2}\end{bmatrix},\quad (AB)^{-1}=\begin{bmatrix}4 & -3\\ -\frac{9}{2} & \frac{7}{2}\end{bmatrix}$$

Also

$$A^{-1}=\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix},\quad B^{-1}=\begin{bmatrix}1 & -1\\ -1 & \frac{3}{2}\end{bmatrix},\quad (AB)^{-1}=\begin{bmatrix}4 & -3\\ -\frac{9}{2} & \frac{7}{2}\end{bmatrix}$$

Therefore, $(AB)^{-1}=B^{-1}A^{-1}$

1. $(A^{-1})^{-1} = A$
2. $(A^{k})^{-1} = A^{-1}A^{-1} \cdot \cdot \cdot A^{-1} = (A^{-1})^{k}$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}, c \neq 0$
4. $(A^{T})^{-1} = (A^{-1})^{T}$

**Example** (**Properties of Exponents**)
Let the matrices $A$ and $A^{-1}$ be

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

Also,

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

so,

$$(A^3)^{-1} = \frac{1}{(11)(41) - (30)(15)}\begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = (A^{-1})^3$$

## Matrix Polynomials

If $A$ is a square matrix, say $n \times n$, and if

$$p(x) = a_0 + a_1x + a_2x^2 + \cdot \cdot \cdot + a_mx^m$$

is any polynomial, then we define the $n \times n$ matrix $p(A)$ to be

$$p(A) = a_0I + a_1A + a_2A^2 + \cdot \cdot \cdot + a_mA^m$$

where $I$ is the $n \times n$ identity matrix; that is, $p(A)$ is obtained by substituting $A$ for $x$ and replacing the constant term $a0$ by the matrix $a0I$. Last expression is called a ***matrix polynomial in A***.

**Example** (A Matrix Polynomial)
Find $p(A)$ for

$$p(x) = x^2 - 2x - 3 \quad \text{and} \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

**Solution**

$$\begin{aligned} p(A) &= A^2 - 2A - 3I \\ &= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

or more briefly, $p(A) = 0$.

**Example** (Inverse of a Transpose)

Consider a general $2 \times 2$ invertible matrix and its transpose:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Since $A$ is invertible, its determinant $ad - bc$ is nonzero. But the determinant of $A^T$ is also $ad - bc$, so $A^T$ is also invertible. It follows

$$(A^T)^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{c}{ad-bc} \\ -\frac{b}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

which is the same matrix that results if $A-1$ is transposed (verify). Thus,

$$(A^T)^{-1} = (A^{-1})^T$$

**Exercises**

1- Let

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix}, \quad a = 4, \quad b = -7$$

Show that

(a) $(A + B) + C = A + (B + C)$

(b) $(AB)C = A(BC)$

(c) $(a + b)C = aC + bC$

(d) $a(B - C) = aB - aC$

2- Using the matrices and scalars in Exercise 1, verify that

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(aC)^T = aC^T$

(d) $(AB)^T = B^T A^T$

3- Compute the inverses of the following matrices.

(a) $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$

(c) $C = \begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix}$ (d) $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

4- Use the matrices $A$, $B$, and $C$ in Exercise 3 to verify that

(a) $(AB)^{-1} = B^{-1}A^{-1}$

(b) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

5.

Let $A$ be the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

In each part, find $p(A)$.

(a) $p(x) = x - 2$

(b) $p(x) = 2x^2 - x + 1$