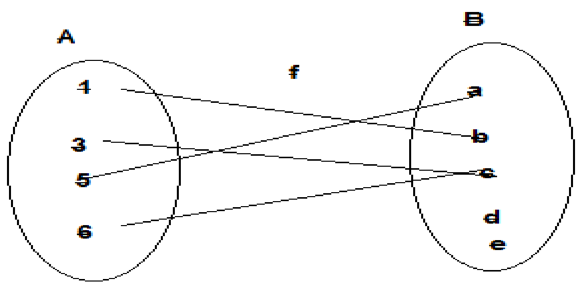
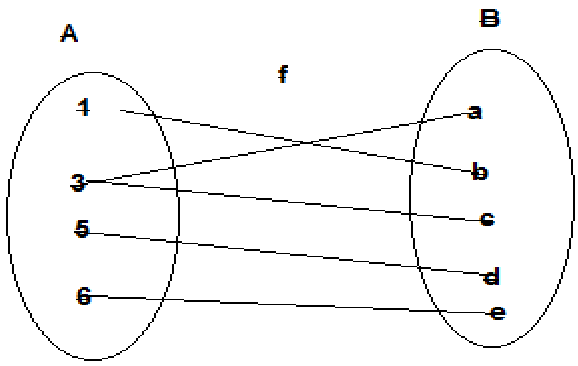
Functions

This assignment is an example of a function. The concept of a function is extremely important in mathematics and computer science. For example, in discrete mathematics functions are used in the definition of such discrete structures as sequences and strings. Functions are also used to represent how long it takes a computer to solve problems of a given size. Many computer programs and subroutines are designed to calculate values of functions. Recursive functions, which are functions defined in terms of themselves, are used throughout computer science; they will be studied in Chapter 4. This section reviews the basic concepts involving functions needed in discrete mathematics.

DEFINITION 1 Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.



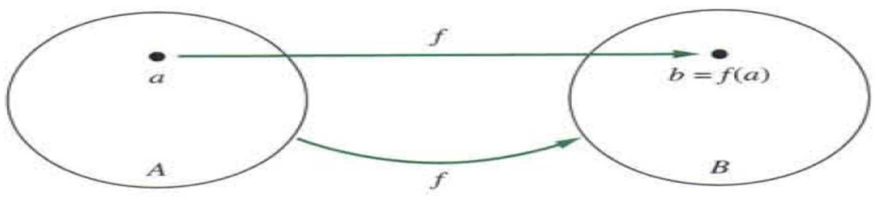
So, $f: A \to B$ is a function



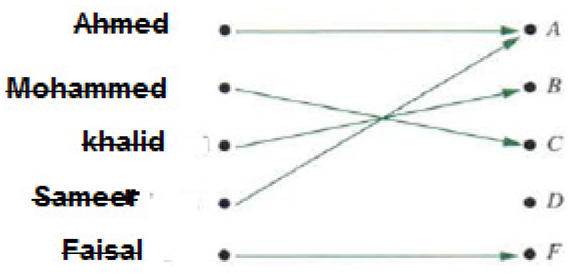
***f* is not a function from A to B, because f(3) is not unique.**

$$f(3) = a \text{ and } f(3) = c.$$

Remark: Functions are sometimes also called **mappings** or **transformations.**



Functions are specified in many different ways , Sometimes we explicitly state the assignments , as Figures. Often we give a formula , such as $f(x) = x + 1$ to define a function other times we use a computer program to specify a function.



Assignment of Grades in a discrete Mathematics class.

Definition: If f is a function from A to B ,we say that A is the domain of $f$and B is the co-domain of f, if $f(a) = b$, we say that b is the image of a. The range of f is the set of all images of elements of A.

EXAMPLE

Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example, $f(11010) = 10$. Then, the domain of f is the set of all bit strings of length 2 or greater, and both the codomain and range are the set {00, 01, 10, 11}.

EXAMPLE

Let $f: \mathbf{Z} \to \mathbf{Z}$ assign the square of an integer to this integer. Then, $f(x) = x^2$, where the domain of f is the set of all integers, we take the the codomain of f to be the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, \ldots\}$.

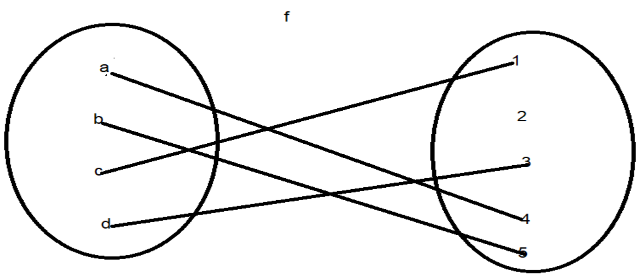
One –to One Functions

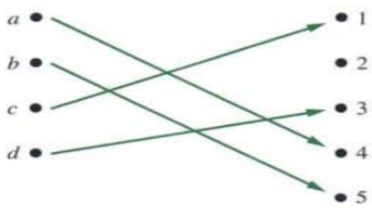
A function f is said to be *one-to-one,* or *injective,* if and only if $f(a) = f(b)$ implie that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if i is one-to-one.

Note that a function f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$. This way of expressing that f is one-to-one is obtained by taking the contrapositive of the implication in the definition.

EXAMPLE

Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.





Note that the function $f(x) = x^2$ with its domain restricted to $\mathbf{Z}^+$ is one-to-one. (Technically, when we restrict the domain of a function, we obtain a new function whose values agree with those of the original function for the elements of the restricted domain. The restricted function is not defined for elements of the original domain outside of the restricted domain.) ◄

Example:

Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.

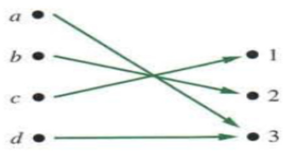
Solution: The function $f(x) = x + 1$ is a one-to-one function. To demonstrate this, note that $x + 1 \neq y + 1$ when $x \neq y$. ◄

DEFINITION

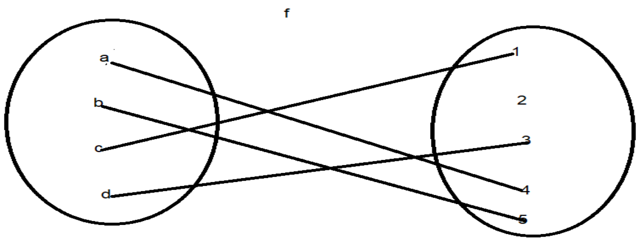
A function f from A to B is called *onto,* or *surjective,* if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a *surjection* if it is onto.

EXAMPLE

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?



Example: The given function is not onto



EXAMPLE

Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: The function f is not onto because there is no integer x with $x^2 = -1$, for instance.

EXAMPLE

Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?

Solution: This function is onto, because for every integer y there is an integer x such that $f(x) = y$. To see this, note that $f(x) = y$ if and only if $x + 1 = y$, which holds if and only if $x = y - 1$.

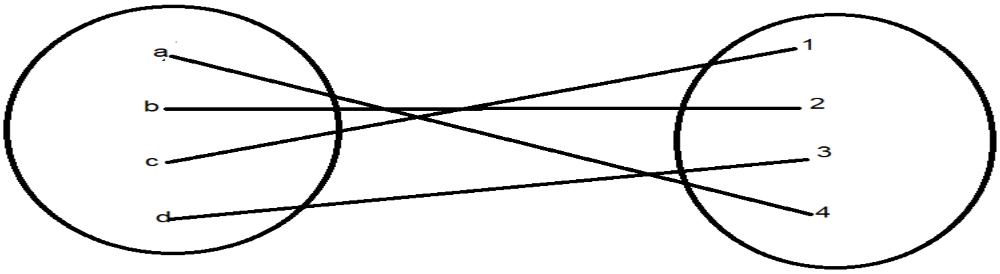
DEFINITION

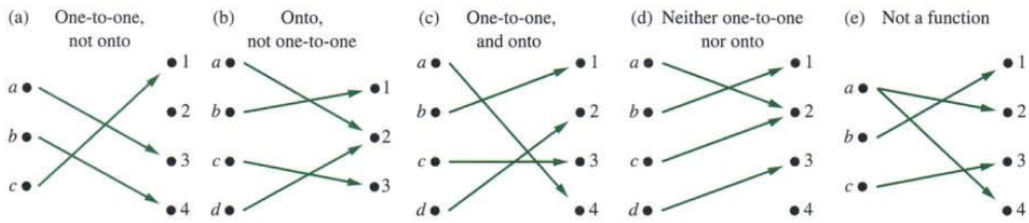
The function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto.

EXAMPLE

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f a bijection?

Solution: The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the codomain are images of elements in the domain. Hence, f is a bijection. ◀

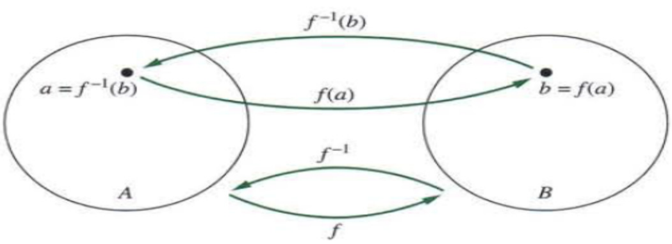




DEFINITION

Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

The Function f^{-1} Is the Inverse of Function f.



EXAMPLE

Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

EXAMPLE

Let $f : \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f has an inverse because it is a one-to-one correspondence, as we have shown. To reverse the correspondence, suppose that y is the image of x, so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of $\mathbf{Z}$ that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$.

EXAMPLE

Let f be the function from $\mathbf{R}$ to $\mathbf{R}$ with $f(x) = x^2$. Is f invertible?

Solution: Because $f(-2) = f(2) = 4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible.

Some Important Functions

There are two important functions in discrete mathematics, namely **floor function** and **ceiling function.**

1. Floor function ($\lfloor\rfloor$):
The floor function assigns to the real number x the largest integer that is less than or equal to x. the value of the floor x is denoted by $\lfloor x \rfloor$. Thus $\lfloor x \rfloor = n \leftrightarrow n \le x < n+1$.

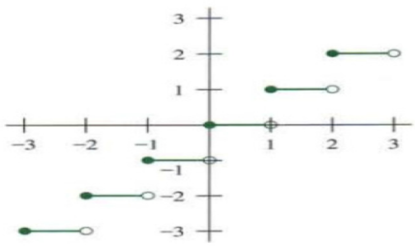
2. Ceiling function ($\lceil\rceil$):
The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x. the value of the ceiling x is denoted by $\lceil x \rceil$. Thus $\lceil x \rceil = n \leftrightarrow n-1 < x \le n$

Examples: $\left\lfloor \frac{1}{2} \right\rfloor = 0$, $\left\lceil \frac{1}{2} \right\rceil = 1$, $\left\lfloor -\frac{1}{2} \right\rfloor = -1$, $\left\lceil -\frac{1}{2} \right\rceil = 0$, $\lfloor 3.1 \rfloor = 3$, $\lceil 3.1 \rceil = 4$, $\lfloor 7 \rfloor = 7$, $\lceil 7 \rceil = 7$

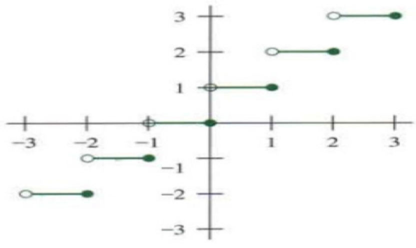
Example: Find the values of

1) $\left\lfloor \frac{1}{2} + \left\lceil \frac{3}{2} \right\rceil \right\rfloor$ 2) $\left\lceil \frac{1}{2} + \left\lfloor \frac{1}{2} \right\rfloor + \left\lceil \frac{1}{2} \right\rceil \right\rceil$ 3) $\left\lfloor \frac{1}{2} . \left\lceil \frac{5}{2} \right\rceil \right\rfloor$

The graphs of floor and ceiling functions are as follows:



$y = \lfloor x \rfloor$



$y = \lceil x \rceil$

Example: Find the values of

$$1)\ \left\lfloor \frac{1}{2} + \left\lceil \frac{3}{2} \right\rceil \right\rfloor \quad 2)\ \left\lceil \frac{1}{2} + \left\lfloor \frac{1}{2} \right\rfloor + \left\lceil \frac{1}{2} \right\rceil \right\rceil \quad 3)\ \left\lfloor \frac{1}{2} . \left\lceil \frac{5}{2} \right\rceil \right\rfloor$$

Solution:

$$1)\ \left\lfloor \frac{1}{2} + \left\lceil \frac{3}{2} \right\rceil \right\rfloor = \lfloor 0.5 + \lceil 1.5 \rceil \rfloor = \lfloor 0.5 + 2 \rfloor = \lfloor 2.5 \rfloor = 2$$

$$2)\ \left\lceil \frac{1}{2} + \left\lfloor \frac{1}{2} \right\rfloor + \left\lceil \frac{1}{2} \right\rceil \right\rceil = \lceil 0.5 + \lfloor 0.5 \rfloor + \lceil 0.5 \rceil \rceil = \lceil 0.5 + 0 + 1 \rceil = \lceil 1.5 \rceil = 2$$

$$3)\ \left\lfloor \frac{1}{2} . \left\lfloor \frac{5}{2} \right\rfloor \right\rfloor = \lfloor 0.5 . \lfloor 2.5 \rfloor \rfloor = \lfloor 0.5 . 2 \rfloor = \lfloor 1 \rfloor = 1$$

Example: Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$ for all real numbers x and y.

Solution: For x= $\frac{1}{2}$ and y= $\frac{1}{2}$

LHS: $\lceil x + y \rceil = \left\lceil \frac{1}{2} + \frac{1}{2} \right\rceil = \lceil 1 \rceil = 1$

RHS: $\lceil x \rceil + \lceil y \rceil = \left\lceil \frac{1}{2} \right\rceil + \left\lceil \frac{1}{2} \right\rceil = 1 + 1 = 2$

LHS $\neq$ RHS. Hence it is disproved.

Theorem: For all real numbers x and all integers m,prove that

$$\lfloor x + m \rfloor = \lfloor x \rfloor + m$$

Solution: By definition of $\lfloor x \rfloor$, $\lfloor x \rfloor = n \iff n \le x < n+1$, for some $n \in \mathbb{Z}$

Adding $m \in \mathbb{Z}$, we get

$$m + n \le x + m < m + n + 1$$

This can be written as

$$\lfloor x + m \rfloor = m + n$$

$$\lfloor x + m \rfloor = \lfloor x \rfloor + m \quad (\lfloor x \rfloor = n).$$

Note: If x is a real number and m is an integer, then $\lceil x + m \rceil = \lceil x \rceil + m$.

TABLE 1 Useful Properties of the Floor and Ceiling Functions. (n is an integer)
(1a) $\lfloor x \rfloor = n$ if and only if $n \le x < n+1$ (1b) $\lceil x \rceil = n$ if and only if $n-1 < x \le n$ (1c) $\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$ (1d) $\lceil x \rceil = n$ if and only if $x \le n < x+1$
(2) $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a) $\lfloor -x \rfloor = -\lceil x \rceil$ (3b) $\lceil -x \rceil = -\lfloor x \rfloor$
(4a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ (4b) $\lceil x + n \rceil = \lceil x \rceil + n$

Exercises

1. Find these Values of the following:
   a) $\lfloor 1.1 \rfloor$
   b) $\lceil 1.1 \rceil$
   c) $\lfloor -0.1 \rfloor$
   d) $\lfloor -0.1 \rfloor$
   e) $\lceil 2.99 \rceil$
   f) $\lceil -2.99 \rceil$
   g) $\left\lfloor \frac{1}{2} + \left\lceil \frac{1}{2} \right\rceil \right\rfloor$
   h) $\left\lceil \left\lfloor \frac{1}{2} \right\rfloor + \left\lceil \frac{1}{2} \right\rceil + \frac{1}{2} \right\rceil$

Recurrence Relation:

A Recurrence relation for the sequence $\{a_k\}$ is an equation that expresses an in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, a_2, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a non-negative integer.

Solution of a Recurrence Relation: A sequence is called a solution of recurrence relation if it term satisfy the recurrence relation.

Example: let $\{a_k\}$ be a sequence that satisfies that recurrence relation
$a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$ and suppose $a_0 = 3,\ a_1 = 5$.
What are a_2 *and* a_3?

Solution: Given recurrence relation $a_n = a_{n-1} - a_{n-2}\ ;n \geq 2$

At $n = 2$ $\quad a_2 = a_{2-1} - a_{2-2}$

$a_2 = a_1 - a_0 = 5 - 3 = 2 \qquad \Rightarrow \quad a_2 = 2$

At $n = 3$ $\quad a_3 = a_{3-1} - a_{3-2}$

$a_3 = a_2 - a_1 = 2 - 5 = -3 \qquad \Rightarrow \quad a_2 = -3$

Example: let $\{a_k\}$ be a sequence that satisfies that recurrence relation
$a_n = 3a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$ and suppose $a_0 = 3,\ a_1 = 5$.
What are a_2 *and* a_3?

Solution: Given the recurrence relation $a_n = a_{n-1} - a_{n-2}; n \geq 2$.

At $n = 2$ $\quad a_2 = 3a_{2-1} - a_{2-2}$

$a_2 = 3a_1 - a_0 = 15 - 3 = 12 \qquad \Rightarrow \quad a_2 = 12$

At $n = 3$ $\quad a_3 = 3a_{3-1} - a_{3-2}$

$a_3 = 3a_2 - a_1 = 36 - 5 = 31 \qquad \Rightarrow \quad a_3 = 31$

Example: Find the first-five terms of the sequence defined by each of these recurrence relation 1) $a_n = 6a_{n-1} \quad ; \quad a_0 = 2$

2) $a_n = a_{n-1} + a_{n-3} \quad ; \quad a_0 = 1,\ a_1 = 2,\ a_2 = 0$

Solution:

1) Given recurrence relation $a_n = 6a_{n-1} \quad ; \quad a_0 = 2\ ; n \geq 1$

At $n = 1$ $\quad a_1 = 6a_{1-1}$

$a_1 = 6a_0 = 6(2) = 12 \qquad \Rightarrow \quad a_1 = 12$

At $n = 2$ $\quad a_2 = 6a_{2-1}$

$a_2 = 6a_1 = 6(12) = 72 \qquad \Rightarrow \quad a_2 = 72$

At $n = 3$ $\quad a_3 = 6a_{3-1}$

$a_3 = 6a_2 = 6(72) = 432 \qquad \Rightarrow \quad a_3 = 432$

At $n = 4$ $\quad a_4 = 6a_{4-1}$

$a_4 = 6a_3 = 6(432) = 2592 \qquad \Rightarrow \quad a_4 = 2592$

2) Given recurrence relation $a_n = a_{n-1} + a_{n-3}$; $a_0 = 1,\ a_1 = 2,\ a_2 = 0$

At $n = 3$ $\quad a_3 = a_{3-1} + a_{3-3}$

$a_3 = a_2 + a_0 = 0 + 1 = 1 \quad \Longrightarrow \quad a_3 = 1$

At $n = 4$ $\quad a_4 = a_{4-1} + a_{4-3}$

$a_4 = a_3 + a_1 = 1 + 2 = 3 \quad \Longrightarrow \quad a_3 = 3$

<u>A Homogeneous Linear Recurrence Relation:</u>

A linear homogeneous recurrence relation with constant coefficients is of the form: $\quad a_n = c_1 a_{n-1} + c_2\, a_{n-2} + c_3 a_{n-3} + \ ... + c_k\, a_{n-k}$

Where $c_1, c_2, c_3, \dots, c_k$ are real numbers and $c_k \neq 0$

<u>Solving Linear Homogeneous Recurrence Relation with constant coefficients:</u>

a) <u>A linear homogeneous Recurrence Relation with constant coefficients for degree 1</u>

$a_n = c_1 a_{n-1}$ $\qquad$ [degree 1]

$a_n - c_1 a_{n-1} = 0$ $\qquad$ ..... (1)

$$\boxed{\begin{aligned} a_n &= r \\ a_{n-1} &= r^{1-1} = r^0 = 1 \end{aligned}}$$

The Characteristic equation is $\quad r = c_1$

The General Solution is $\quad a_n = A(r)^n$

Example: What is the solution of the recurrence relation

$a_n = 6a_{n-1}$ $\quad$ with $\ a_0 = 2,$

Solution: Given recurrence relation $a_n = 6a_{n-1}$; $a_0 = 2$

$a_n = 6a_{n-1}$ $\quad$ ... (1) $\quad$ [degree 1]

The Characteristic equation is $\Longrightarrow\Longrightarrow\Longrightarrow$ $r = 6$ $\ $ the General solution is $a_n = A(6)^n$

Using $a_0 = 2$ $\ ,\Longrightarrow\Longrightarrow a_0 = A(6)^0 = 2 \quad \Longrightarrow\Longrightarrow A = 2$

Then General solution is $\ a_n = 2(6)^n$

b) <u>A linear homogeneous Recurrence Relation with constant coefficients for degree 2</u>

$a_n = c_1 a_{n-1} + c_2\, a_{n-2}$ $\qquad$ [degree 2]

$a_n - c_1 a_{n-1} - c_2\, a_{n-2} = 0$ $\qquad$ ..... (1)

$$\boxed{\begin{aligned} a_n &= r^2 \\ a_{n-1} &= r^{2-1} = r \\ a_{n-2} &= r^{2-2} = r^0 = 1 \end{aligned}}$$

The Characteristic equation is

$r^2 - c_1 r - c_2 = 0$ $\qquad$ .... $\qquad$ (2)

$$r = \frac{-(-c1) \pm \sqrt{(-c_1)^2 - 4.1.(-c_2)}}{2}$$

The formula of quadratic equation is $\ ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note equation (2) is of degree 2 in 'r' solving it for ' r ' we have following three cases

s. num.	Root	General Solution
1	Distinct Root $r_1,\ r_2$	$a_n = A(r_1)^n + B(r_2)^n$
2	Equal Root $r_1,\ r_1$	$a_n = (A + nB)(r_1)^n$

Example: What is the solution of the recurrence relation

$a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2,\ a_1 = 7$?

Solution: Given recurrence relation $a_n = a_{n-1} + 2a_{n-2}$; $a_0 = 2,\ a_1 = 7$

$$a_n - a_{n-1} - 2a_{n-2} = 0 \quad \dots \quad (1) \quad [\text{ degree } 2\ ]$$

The Characteristic equation is $\Longrightarrow\Longrightarrow\Longrightarrow$

$a_n = r^2$
$a_{n-1} = r^{2-1} = r$
$a_{n-2} = r^{2-2} = r^0 = 1$

$$r^2 - r - 2 = 0$$

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2}$$

$r_1 = \frac{1+3}{2} = \frac{4}{2} = 2$ or $r_2 = \frac{1-3}{2} = \frac{-2}{2} = -1 \implies r_1 = 2\ ,\ r_2 = -1$

the Solution is

$$a_n = A(2)^n + B(-1)^n \quad \dots \ (2)$$

Using $a_0 = 2$, we get from (2) $\implies a_0 = A(2)^0 + B(-1)^0$

$\implies 2 = A + B \quad \dots.(3)$

Using $a_1 = 7$, we get from (2) $\implies a_1 = A(2)^1 + B(-1)^1$

$\implies 7 = 2A - B \quad \dots.(4)$

Solving (3) and (4) *then* $A = 3\ \&\ B = -1$

Using A *and* B in (2)

The General Solution $a_n = 3(2)^n - (-1)^n$

(3)+(4)
$2 = A + B$ ...(3)
$\underline{7 = 2A - B}$ ...(4)
$9 = 3A$
$\frac{3A}{3} = \frac{9}{3} \implies A = 3$
$Using\ A = 3\ in(3)$
$2 = 3 + B$
$\implies B = 2 - 3 = -1$

Example: Solving the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1,\ a_1 = 6$?

Solution: Given recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$; $a_0 = 1,\ a_1 = 6$

$$a_n - 6a_{n-1} + 9a_{n-2} = 0 \quad \dots \quad (1) \quad [\text{ degree } 2\ ]$$

The Characteristic equation is

$$r^2 - 6r + 9 = 0$$

$$r = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(9)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36-36}}{2} = \frac{6 \pm \sqrt{0}}{2} = \frac{6 \pm 0}{2}$$

$r_1 = \frac{6+0}{2} = \frac{6}{2} = 3$ or $r_2 = \frac{6-0}{2} = \frac{6}{2} = 3 \implies r_1 = r_2 = 3$

the Solution is $a_n = (A + nB)(3)^n \quad \dots \ (2)$

Using $a_0 = 1$, we get from (2) $\implies a_0 = (A + 0.B)(3)^0$

$\implies 1 = A \quad \dots . (3)$

Using $a_1 = 6$, we get from (2) $\implies a_1 = (1 + 1.B)(3)^1$

$\implies 6 = 3(1 + B)$

$\implies 2 = 1 + B \implies B = 1$

from (3) and (4) $then\ A = 1\ \&\ B = 1$

Using $A\ \ and\ \ B$ in (2)

The General Solution $a_n = (1 + n)(3)^n$

Example: Solving the following recurrence relation

1. $a_n = a_{n-1} + 6a_{n-2}$ with $a_0 = 3,\ a_1 = 6$?
2. $a_n = 6a_{n-1} - 8a_{n-2}$ with $a_0 = 4,\ a_1 = 10$?
3. $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 4,\ a_1 = 1$?