Matrices and Matrix Operations

Definitions and its Properties

A ***matrix*** is a rectangular array of numbers. The numbers in the array are called the ***entries*** in the matrix.

Examples

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \;\; 1 \;\; 0 \;\; -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

The size of the array is – written as m × n, where

$$\underset{\text{number of rows}}{m} \times \underset{\text{number of columns}}{n}$$

Notation

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{bmatrix} \quad \text{rows}$$

$$\text{columns}$$

A ≡ uppercase denotes a matrix

a ≡ lower case denotes an entry

Some points about matrices

1- A matrix with only one column is called a column matrix (or a column vector), and a matrix with only one row is called a row matrix (or a row vector). Thus, in Example 1 the 2×1 matrix is a column matrix, the 1×4 matrix is a row matrix, and the 1×1 matrix is both a row matrix and a column matrix.

2- When discussing matrices, it is common to refer to numerical quantities as ***scalar***.

3- When compactness of notation is desired, the preceding matrix can be written as

$$\left[a_{ij}\right]_{m\times n} \quad \text{or} \quad \left[a_{ij}\right]$$

The first notion being used when it is important in the discussion to know the size and the second when the size need not be emphasized.

4- Usually, we shall match the letter denoting a matrix with the letter denoting its entries: thus, for a matrix B we would generally use b_{ij} for the entry in row i and j and for a matrix C we would use the notion c_{ij} .

5- The entry in row i and column j of a matrix A is also commonly denoted by the symbol $(A)_{ij}$.

$$(A)_{ij} = a_{ij}$$

And for the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 7 & 0 \end{bmatrix}$$

we have $(A)_{11} = 2, (A)_{12} = -3, (A)_{21} = 7,$ and $(A)_{22} = 0$.

Some Types of Matrices

1- If $m = n$, the matrix is called square. In this case we have
(1a) A matrix A is said to be *diagonal* if

$$a_{ij} = 0 \qquad i \neq j.$$

(1b) A diagonal matrix A may be denoted by $diag(d_1, d_2, \dots, d_n)$ where

$$a_{ii} = d_i \qquad a_{ij} = 0 \qquad j \neq i.$$

The diagonal matrix $diag(1, 1, \dots, 1)$ is called the identity matrix and is usually denoted by

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

Or simply I, when n is assumed to be known. 0 = diag(0, . . . , 0) is called the zero matrix.

(1c) A square matrix L is said to be lower triangular if

$$l_{ij} = 0 \quad i < j.$$

(1d) A square matrix U is said to be upper triangular if

$$u_{ij} = 0 \; i > j.$$

(1e) A square matrix A is called symmetric if

$$a_{ij} = a_{ji}.$$

2- A rectangular matrix A is called nonnegative if

$$a_{ij} \geq 0 \qquad all\ i, j.$$

It is called positive if

$$a_{ij} > 0 \qquad all\ i, j.$$

Examples:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ 5 & 6 & 1 \end{bmatrix}$$

The matrix A is a lower triangular matrix.

The matrix $B = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$

is an upper triangular matrix.

The matrix $C = \begin{bmatrix} 2 & 0 & 5 \\ 2 & 5 & 2 \\ 1 & 4 & 7 \end{bmatrix}$

is a nonnegative matrix. While the matrix

$$H = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 1 & 4 \\ 7 & 8 & 9 \end{bmatrix}$$

is a positive matrix.

The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ is a symmetric matrix.

Operations on matrices

Definition (Definition of equality of matrices) Tow matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if they have the same size and $(A)_{ij} = (B)_{ij}$, or equivalently, $a_{ij} = b_{ij}$ for all i and j.

Example: Consider the four matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad C = [1 \quad 3], \text{ and } D = \begin{bmatrix} 1 & 2 \\ x & 4 \end{bmatrix}.$$

Matrices A and B are ***not*** equal because they are of different size. Similarly, B and C are not equal. Matrices A and D are equal if and only if $x = 3$.

Definition :(Definition of addition matrices) If $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then

$$(A \pm B)_{ij} = (A)_{ij} \pm (B)_{ij} = a_{ij} \pm b_{ij}$$

Example.

(a) $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) The sum of

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

is undefined.

Definition: (Definition of scalar multiplication by matrix)
If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, then the scalar multiple of A by c is the $m \times n$ matrix given by

$$(cA)_{ij} = c(A)_{ij} = ca_{ij}$$

Example: For the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

find **(a)** $3A$, (b) $-B$, and **(c)** $3A - B$.

Solution.

$$\textbf{(a)}\ 3A = 3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

$$\textbf{(b)}\ -B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

$$\textbf{(c)}\ 3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -13 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

Properties of Matrix addition and scalar multiplication:

If A, B, and C are $m \times n$ matrices and c and d are scalars, then the following properties are true.

1. $A + B = B + A$	Commutative property of addition
2. $A + (B + C) = (A + B) + C$	Associative property of addition
3. $(cd)A = c(dA)$	Associative property of multiplication
4. $1A = A$	Multiplicative identity
5. $c(A + B) = cA + cB$	Distributive property
6. $(c + d)A = cA + dA$	Distributive property

Definition: (Definition of linear combination) If $A_1, A_2, \ldots, A_n$ are matrices of the same size and $c_1, c_2, \ldots, c_n$ are scalars, then an expression of the form

$$c_1A_1 + c_2A_2 + \cdots + c_nA_n$$

is called a linear combination of $A_1, A_2, \ldots, A_n$ with coefficients $c_1, c_2, \ldots, c_n$.

Example. If

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

Then

$$2A - B + \tfrac{1}{3}C = 2A + (-1)B + \tfrac{1}{3}C$$

$$= \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 & 2 \\ 4 & 3 & 11 \end{bmatrix}$$

Definition . (Definition of multiplication matrices) If

$$A = [a_{ij}] = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{vmatrix}$$

is an $m \times r$ matrix and

$$B = [b_{ij}] = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{ij} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & & & & & \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

is an $r \times n$ matrix, then the product AB is $m \times n$ matrix

$$AB = [c_{ij}]$$

where

$$c_{ij} = (AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{12}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}.$$

Example

Consider the matrices

$$A=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}, \qquad B=\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}$$

Since A is a 2×3 matrix and B is a 3×4 matrix, the product AB is a 2×4 matrix. To determine, for example, the entry in row 2 and column 3 of AB, we single out row 2 from A and column 3 from B. Then, as illustrated below, we multiply corresponding entries together and add up these products.

$$\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}=\begin{bmatrix}\boxed{} & \boxed{} & \boxed{} & \boxed{}\\ \boxed{} & \boxed{} & \boxed{26} & \boxed{}\end{bmatrix}$$

$$(2\cdot4)+(6\cdot3)+(0\cdot5)=26$$

The entry in row 1 and column 4 of AB is computed as follows:

$$\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}=\begin{bmatrix}\boxed{} & \boxed{} & \boxed{} & \boxed{13}\\ \boxed{} & \boxed{} & \boxed{} & \boxed{}\end{bmatrix}$$

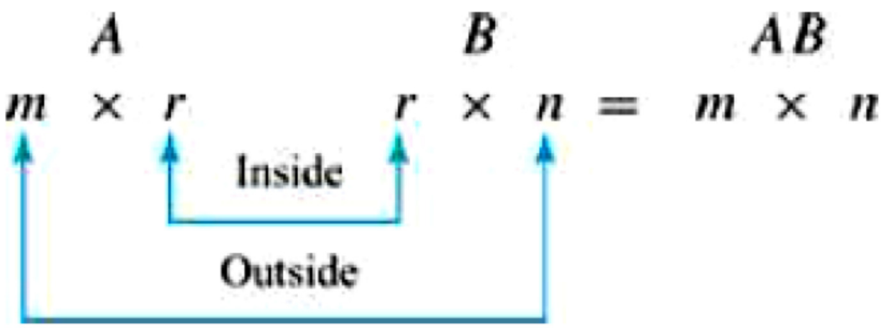
$$(1\cdot3)+(2\cdot1)+(4\cdot2)=13$$

The computations for the remaining entries are

$$(1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) = 12$$
$$(1 \cdot 1) - (2 \cdot 1) + (4 \cdot 7) = 27$$
$$(1 \cdot 4) + (2 \cdot 3) + (4 \cdot 5) = 30$$
$$(2 \cdot 4) + (6 \cdot 0) + (0 \cdot 2) = 8$$
$$(2 \cdot 1) - (6 \cdot 1) + (0 \cdot 7) = -4$$
$$(2 \cdot 3) + (6 \cdot 1) + (0 \cdot 2) = 12$$

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

The definition of matrix multiplication requires that the number of columns of the first factor A be the same as the number of rows of the second factor B in order to form the product AB. If this condition is not satisfied, the product is undefined. A convenient way to determine whether a product of two matrices is defined is to write down the size of the first factor and, to the right of it, write down the size of the second factor. If, as in 3, the inside numbers are the same, then the product is defined. The outside numbers then give the size of the product.



$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

Examples:

(a) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}$

$2 \times 3 \qquad 3 \times 3 \qquad 2 \times 3$

(b) $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$

$2 \times 2 \qquad 2 \times 2 \qquad 2 \times 2$

(c) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$2 \times 2 \qquad 2 \times 2 \qquad 2 \times 2$

(d) $\begin{bmatrix} 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$

$1 \times 3 \qquad 3 \times 1 \qquad 1 \times 1$

Properties of matrix multiplication:

If A, B, and C are matrices (with sizes such that the given matrix products are defined) and c is a scalar, then the following properties are true.

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $c(AB) = (cA)B = A(cB)$

Problem 1: Consider the matrices

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$$

Prove that $A(BC) = (AB)C$

Proof:

$$(AB)C = \left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix} \right) \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 13 & 14 \end{bmatrix}.$$

$$A(BC) = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 8 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 13 & 14 \end{bmatrix}$$

Problem 1: Prove that $AB \neq BA$ where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$

Proof:

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}$$

Unit Matrix:

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

For instance, if $n = 1, 2,$ or 3, we have

$$\underset{1 \times 1}{I_1 = [1]}, \qquad \underset{2 \times 2}{I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \qquad \underset{3 \times 3}{I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

Properties of Unit matrix:

If A is a matrix of size $m \times n$, then the following properties are true.

1. $AI_n = A$
2. $I_mA = A$

Example:

(a) $\begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

Matrix power:

$$A^k = \underbrace{AA \cdot \ \cdot \ \cdot A}_{k \text{ factors}}.$$

$A^0 = I_n$ (where A is a square matrix of order n)

Example:

Find A^3 for the matrix $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$.

SOLUTION $A^3 = \left(\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\right)\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -6 \end{bmatrix}$

The Transpose of a Matrix

The **transpose** of a matrix is formed by writing its columns as rows. For instance, if A the $m \times n$ matrix shown by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix},$$

Size: $m \times n$

Size: $m \times n$

then the transpose, denoted by A^T, is the $n \times m$ matrix below

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}.$$

Size: $n \times m$

Example:

Find the transpose of each matrix.

(a) $A = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $D = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$

SOLUTION (a) $A^T = \begin{bmatrix} 2 & 8 \end{bmatrix}$ (b) $B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ (c) $C^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $D^T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}$

Properties of Transpose of a matrix:

If A and B are matrices (with sizes such that the given matrix operations are defined) and c is a scalar, then the following properties are true.

1. $(A^T)^T = A$ Transpose of a transpose
2. $(A + B)^T = A^T + B^T$ Transpose of a sum
3. $(cA)^T = c(A^T)$ Transpose of a scalar multiple
4. $(AB)^T = B^TA^T$ Transpose of a product

Show that

$(AB)^T = B^TA^T$

if

$$A=\begin{bmatrix}2&1&-2\\-1&0&3\\0&-2&1\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}3&1\\2&-1\\3&0\end{bmatrix}$$

Solution:

$$AB=\begin{bmatrix}2&1&-2\\-1&0&3\\0&-2&1\end{bmatrix}\begin{bmatrix}3&1\\2&-1\\3&0\end{bmatrix}=\begin{bmatrix}2&1\\6&-1\\-1&2\end{bmatrix}$$

$$(AB)^T=\begin{bmatrix}2&6&-1\\1&-1&2\end{bmatrix}$$

$$B^TA^T=\begin{bmatrix}3&2&3\\1&-1&0\end{bmatrix}\begin{bmatrix}2&-1&0\\1&0&-2\\-2&3&1\end{bmatrix}=\begin{bmatrix}2&6&-1\\1&-1&2\end{bmatrix}$$

Definition

The ***trace of an nxn matrix A***, denoted by Tr(A), is defined to be the sum of the entries on the main diagonal of A. That is

$$\mathrm{Tr}(A)=a_{11}+a_{22}+\cdots+a_{nn}.$$

Example: The trace of the matrix

$$B=\begin{bmatrix}-1&2&7&0\\3&5&-8&4\\1&2&7&-3\\4&-2&1&0\end{bmatrix}$$

Tr(B) = -1 + 5 + 7 + 0 = 11

<u>Exercise</u>

1- Suppose that A, B, C, D, and E are matrices with the following sizes:

A	B	C	D	E
(4×5)	(4×5)	(5×2)	(4×2)	(5×4)

Determine which of the following matrix expressions are defined. For those that are defined, give the size of the resulting matrix.

(a) BA *(b)* $AC + D$ *(c)* $AE + B$ *(d)* $AB + B$
(e) $E(A + B)$ *(f)* $E(AC)$ *(g)* E^TA *(h)* $(A^T + E)D$

2- Consider the matrices

$$A=\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B=\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C=\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},$$

$$D=\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E=\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Compute the following (where possible).

(a) $D + E$ *(b)* $D - E$ *(c)* $5A$ *(d)* $-7C$ *(e)* $2B - C$
(f) $4E - 2D$ *(g)* $-3(D + 2E)$ *(h)* $A - A$ *(i)* $tr(D)$ *(j)* $tr(D - 3E)$
(k) $4\ tr(7B)$ *(l)* $tr(A)$

3- Let

$$A=\begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$

Find

(a) the first row of AB (b) the third row of AB
(c) the second column of AB (d) the first column of BA
(e) the third row of AA (f) the third column of AA