Ch 2 Transformations (function or mapping)

Def^n A transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$, $T: \mathbb{R}^n \to \mathbb{R}^m$ is a rule that assigns each vector $x \in \mathbb{R}^n$ to a vector $T(x) \in \mathbb{R}^m$. $\mathbb{R}^n$ is domain of T and $\mathbb{R}^m$ is co-domain of T.

Def^n Let $T: \mathbb{R}^n \to \mathbb{R}^m$ is a transformation. Then the set $\mathbb{R}^n$ is domain and $\mathbb{R}^m$ is co-domain of T. And for $x \in \mathbb{R}^n$, the value $T(x) \in \mathbb{R}^m$ is called image of x under the transformation T. The set of all such images of x under T (i.e. $T(x)$) is called the range of T.

e.g. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ & $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x) = Ax$

Find the images under T of $u = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$

$T(u) = Au = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

$T(v) = Av = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix} = 2\begin{bmatrix} a \\ b \end{bmatrix}$

Def^n: matrix transformation: for each $x \in \mathbb{R}^n$, $T(x)$ is computed as $Ax \in \mathbb{R}^m$, where A is $m \times n$ matrix that behaves as a transformation operator.

e.g. $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

Define $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x) = Ax$

(a) find $T(u)$ (b) find x in $\mathbb{R}^2$ whose image under T is b

(c) Is there more than one $x \in \mathbb{R}^2$ whose image under T is b?

(d) Determine if c is in the range of T.

(a) $T(u) = Au = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 6-5 \\ -2-7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$ (check the dimension of A & x.)

(b) let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then $T(x) = b \Rightarrow Ax = b$

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \Rightarrow \text{Am} \begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix}$$

After several EROs (do yourself), we get following reduced echolon matrix.

$\sim \begin{bmatrix} \boxed{1} & 0 & 1.5 \\ 0 & \boxed{1} & -0.5 \\ 0 & 0 & 0 \end{bmatrix}$ The last row implies the system is consistent and hence the solⁿ exists. Since there is no columns without pivot, we get the unique solⁿ as $x_1 = 1.5$, & $x_2 = 0.5$

$\Rightarrow x = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix}$ in $\mathbb{R}^2$ whose image under T is b.

(c) In the solⁿ of (b), x has no free variable and the solⁿ is unique. Hence, there is no other solution of the system.

(d) From (c), there is exactly one range, b, of T. So, c is not a range of T.

Alternatively, we could proceed as $Ax = c$

$\Rightarrow \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and solve this as in b. This should give the inconsistent system, implying c is not a range of T.

Contraction & Dilation

A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x) = rx$ for some scalar r. Then T is called contraction when $0 \leq r \leq 1$ and T is called dilation when $r > 1$.

The matrix of a LT

A transformation T is linear iff

$T(u+v) = T(u) + T(v)$ for any u, v in domain of T.

$T(cu) = c\,T(u)$ for any u in domain T & for any scalar c.

A single equivalent condition for linearity of T is for all $\alpha, \beta \in F$ u, v domain of T,

$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$.

Note: If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear then $T(0) = 0$, If $T(0) \neq 0$ then T is not linear

Theorem 1 Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear transformation. Then there exists a unique matrix A such that $T(x) = Ax$ for all $x \in \mathbb{R}^n$

In fact, A is $m \times n$ matrix whose j^{th} column is the vector $T(e_j)$ where e_j is the jth column of the identity matrix in $\mathbb{R}^n$. $\underbrace{A}_{\text{Standard matrix}} = [T(e_1) \; \cdots \; T(e_n)]$. (3)

ex. prove that contraction map is linear transformation. (LT)

proof: we know that map $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x) = rx$, where $0 \le r \le 1$ is called contraction map.

Let $u, v \in \mathbb{R}^2$ and c & d are scalars. Then using first condition for LT, $(T(u+v) = T(u) + T(v))$

$$T(cu + dv) = r(cu + dv)$$
$$= c\,ru + d\,rv$$
$$= c\,T(u) + d\,T(v)$$

$\therefore$ T is linear transformation.

ex Show that the transformation T defined by $T(x_1, x_2) = (2x_1 - 3x_2,\ x_1 + 4,\ 5x_2)$ is not linear.

proof. $T(u+v) = T(u_1+v_1,\ u_2+v_2)$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$u+v = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix}$$

$$= \begin{bmatrix} 2(u_1+v_1) - 3(u_2+v_2) \\ u_1+v_1+4 \\ 5(u_2+v_2) \end{bmatrix}$$

$$= \begin{bmatrix} 2u_1 + 2v_1 - 3u_2 - 3v_2 \\ u_1 + v_1 + 4 \\ 5u_2 + 5v_2 \end{bmatrix}$$ This is from LHS

Now, RHS, $T(u) + T(v) = T(u_1, u_2) + T(v_1, v_2)$

$$= \begin{bmatrix} 2u_1 - 3u_2 \\ u_1 + 4 \\ 5u_2 \end{bmatrix} + \begin{bmatrix} 2v_1 - 3v_2 \\ v_1 + 4 \\ 5v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2u_1 + 2v_1 - 3u_2 - 3v_2 \\ u_1 + v_1 + 8 \\ 5u_2 + 5v_2 \end{bmatrix}$$ The second component of LHS & HRS vectors is not the same.

So, $T(u+v) \ne T(u) + T(v)$. Hence, it is not a LT.

Alternatively, for $T(x_1, x_2) = (2x_1 - 3x_2,\ x_1 + 4,\ 5x_2)$

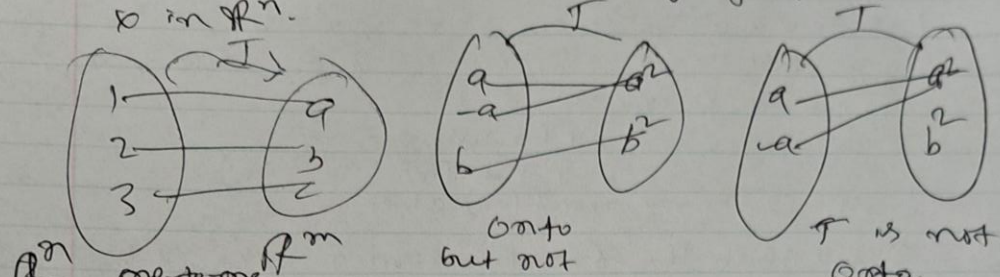
$$T(0,0) = (2\times0 - 3\times0,\ 0+4,\ 5\times0) = (0,4,0) \ne (0,0,0)$$
$\Rightarrow T(0) \ne 0$. $\therefore$ T is not LT.

Def^n (onto)

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each element in $\mathbb{R}^m$ is the image of at least one x in $\mathbb{R}^n$.

one-to-one

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each element in $\mathbb{R}^m$ is the image of at most one x in $\mathbb{R}^n$.



one-to-one and onto

Onto but not one-to-one

T is not Onto.

Theorem Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one iff the equation $T(x) = 0$ has only the trivial sol^n.

Theorem Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a LT and let A be the standard matrix for T. Then,

(a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$.

(b) T is one-to-one iff the columns of A are linearly independent.

Ex Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$

(a) show that T is one-to-one LT

(b) Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$. (See the class sol^n or apply the above theorem to solve those questions.)