Lec 3

Elementary row operations (ERO)

1) Replacement — $R_k \leftarrow R_k \pm cR_m$
2) Interchange - Interchang two rows $R_k \leftrightarrow R_m$
3) Scaling $R_k \leftarrow cR_k$

1.2 Echelon form or Reduced Echelon form

A rectangular matrix is in echelon form if A has the following three properties:

1) All non-zero rows are above any rows of all zeros.
2) Each leading entry (pivot) of a row is in col^n to the right of the leading entry of the row above it.
3) All entries in a column below a leading entry are zeros

Reduced echelon form →
4) The leading entry in each non zero row is 1
5) Each leading 1 is the only nonzero entry in its col^n.

Def^n: Any matrix in echelon form is called echelon matrix.

□ - any non zero
* - any value including 0.

$$\begin{bmatrix} \square & * & * & * \\ 0 & \square & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \square & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \square & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \square & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \square & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \square & * \end{bmatrix}$$

pivot position

pivot col^{ns}

Note: ① Different sequence of row operations gives different echelon matrix of any non-zero matrix.

② A non-zero matrix have a unique reduced echelon matrix.

e.g.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \sim \begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

pivot

After some EROs, we get the following echelon matrix.

$$\sim \begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & \boxed{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivots

pivot col^ns

Ex 9

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

discuss the forward phase & backward phase row-reduction

(In forward phase, we begin with R1, R2, ... Rn.)

Do yourself the EROs to get the following echelon matrix

$$\text{echelon} \sim \begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

up to this steps It's forward phase.

pivot col^ns.

row reduced

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

For backward phase, we make the echelon matrix as (row) reduced echelon matrix by starting with the last row having pivot element (R_3 in this case) and move upward (R_2 & then R_1).

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Sol^n of LS

The row reduced echelon ~~form~~ matrix leads to the sol^n of a linear system (LS) when the row reduction algorithm is applied to the augmented matrix of LS.

e.g. Augmented matrix

$$\begin{bmatrix} \boxed{1} & 0 & -2 & 3 & 0 & : -24 \\ 0 & \boxed{1} & -2 & 2 & 0 & : -7 \\ 0 & 0 & 0 & 0 & \boxed{1} & : 4 \end{bmatrix} \Rightarrow 6 \text{ col}^{ns} \Rightarrow 5 \text{ variables}$$

The LS produced is

$$\left[\begin{aligned} x_1 - 2x_3 + 3x_4 &= -24 \\ x_2 - 2x_3 + 2x_4 &= -7 \\ x_5 &= 4 \end{aligned}\right.$$

x_1, x_2 & x_5 are basic variable because col 1st, 2nd & 5th columns are pivot column and x_3 and x_4 are free variables because there are no pivot columns in 3rd & 4th col^{ns}.

$x_5 = 4$ (fixed

$$\left.\begin{aligned} \Rightarrow x_1 &= -24 + 2x_3 - 3x_4 \\ x_2 &= -7 + 2x_3 - 2x_4 \\ x_3 &= \text{free} \\ x_4 &= \text{free} \\ x_5 &= \text{fixed} \end{aligned}\right\} \text{general sol}^{ns} \text{ of the } \underline{LS}$$

Theorem 2 Existence and Uniqueness Th

A LS is consistent iff the right most col^n of the augmented matrix is not a pivot col^n (A LS has sol^n iff an echelon form of the augmented matrix has no row of the form [0 0 . . . 0 b] with b ≠ 0. If the system is consistent and no free variable ⇒ unique sol^n and if free variable infinitely many sol^{ns} exist.

(4)

e.g. show that LS has no sol^n or exactly one sol^n

$x + y = 5$

$x + y = 3$

Ans
$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

The LS is inconsistent as the last row is of a form of $[0 \quad 0 \quad b]$ with $b \neq 0$

$$\begin{aligned} x + y &= 3 \\ x - y &= 1 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} \boxed{1} & 1 & 3 \\ 0 & \boxed{-2} & -2 \end{bmatrix}$$

~~$R_1 \to R_1 + \frac{1}{2} R_2$~~

$-2x_2 = -2$

$x_2 = +1$

$x_1 + x_2 = 3$

$x_1 = 3 - 1 = 2$

unique sol^n

Q8 Solve the LS if consistent.

$2x - 3y + 7z = 5$

$3x + y - 3z = 13$

$2x + 19y - 47z = 32$

$1 + \frac{3}{2} \times 3$

$11/2$

Ans
$$\begin{bmatrix} 2 & -3 & 7 & 8 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{bmatrix}$$

$R_2 \to R_2 - \frac{3}{2} R_1$ & $R_3 \to R_3 - R_1$

$-\frac{27}{2} \times 4 \quad 2$

$\frac{11}{2} \times 4 \quad 2$

$$\begin{bmatrix} 2 & -3 & 7 & 8 \\ 0 & 11/2 & -27/2 & 11/2 \\ 0 & 22 & -54 & 27 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 4R_2} \begin{pmatrix} 2 & -3 & 7 & 8 \\ 0 & 11/2 & -27/2 & 11/2 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

The last row is in form $[0 \quad 0 \quad 0 \quad 0 \quad b]$ with $b \neq 0$

and hence inconsistent as per the theorem.

e.g. $x_1 + 3x_2 = 2$ } find h & k s.t. show this LS has
$3x_1 + hx_2 = k$ } many soln

Ans $\begin{bmatrix} \boxed{1} & 3 & 2 \\ 3 & h & k \end{bmatrix}$ $\xrightarrow{R_2 \Rightarrow R_2 - 3R_1}$ $\begin{bmatrix} \boxed{1} & 3 & 2 \\ 0 & h-9 & k-6 \end{bmatrix}$

echelon form

For infinite solution at least one free variable and soln to exist $\Rightarrow$ $h-9=0$, $k-6=0$

$h=9$ & $k=6$

Vector eqn — row vector & col^n vectors

Two vectors in $\mathbb{R}^2$ are equal iff their corresponding entries are equal.

$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $\vec{u} = \vec{v}$ iff $u_1 = v_1$, $u_2 = v_2$

$u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ & $v = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\Rightarrow u \neq v$.

$u + v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$

Algebraic properties of $\mathbb{R}^n$

i) $u + v = v + u$ ii) $c(u+v) = cu + cv$

iii) $(u+v)+w = u+(v+w)$ iv) $(c+d)u = cu + du$

v) $u + 0 = 0 + u = u$ (vi) $c(du) = d(cu) = (cd)u$

vii) $u + (-u) = u - u = 0$ (viii) $1.u = u.1 = u$.

Linear combination

Given $u_1, u_2, u_3, \ldots, u_n$ in $\mathbb{R}^n$ and scalars $c_1, c_2, c_3, \ldots c_n$ and if y in $\mathbb{R}^n$ is defined by $y = c_1u_1 + c_2u_2 + \cdots + c_nu_n$ then vector $y \in \mathbb{R}^n$ is called a LC of $u_1, u_2 \ldots$ with $c_1, c_2 \ldots$

(6)

Note: $y \in \mathbb{R}^n$ is linear combination of vectors $u_1, u_2, u_3, \ldots, u_n \in \mathbb{R}^n$ is consistence if $x_1u_1 + x_2u_2 + \ldots + x_nu_n = y$

Def^n Subset of $\mathbb{R}^n$ spanned by vectors

If $u_1, u_2, \ldots, u_n$ are in $\mathbb{R}^n$ then the set of all linear combinations of vectors is denoted by span $\{u_1, u_2, \ldots, u_n\}$ and is called subset of $\mathbb{R}^n$ spanned (or generated) by vectors $u_1, u_2, \ldots, u_n$. The span $\{u_1, u_2, \ldots, u_n\}$ with weights $c_1, c_2, \ldots, c_n$ can be written as $c_1u_1 + c_2u_2 + \ldots + c_nu_n$

Note: A vector b is in span $\{u_1, u_2, \ldots, u_n\}$ if linear system with augmented matrix $[u_1 \; u_2 \ldots u_n \; b]$ represent a consistence system.

e.g. ~~show that~~ Let $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in the span $\{u, v\}$ for all h & k.

The augmented matrix form of u, v & b is

$$[u \; v \; b] = \begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix}$$ with row operations

$$\sim \begin{bmatrix} 1 & 0 & (h-2k)/4 \\ 0 & 1 & (h+2k)/4 \end{bmatrix}$$ sol^n exists

This is a consistence system for all values of h & k.

If not consistent then vector b does not span the given vectors.

e.g. Let $a_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $a_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$

For what value of h is b in the plane spanned by a_1 & a_2?

If $a_1x_1 + a_2x_2 = b$ has sol^n then b is in the plane spanned by a_1 & a_2.

Augment matrix form of a_1, a_2 & b

$$[a_1 \; a_2 \; b] = \begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix}$$

$R_2 \to R_2 - 4R_1, \; R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & h+8 \end{bmatrix}$$

$R_3 \to R_3 - \frac{3}{5}R_2$ $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 0 & (h+8)+\frac{3}{5}\times 15 \end{bmatrix}$

$(h+8) - \frac{3}{5}(-15)$

$h \neq -17$ $h + 17$

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$R_3 \Rightarrow R_3 - \frac{3}{5}R_2$ $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 0 & h+17 \end{bmatrix}$ This is consistent only when $h+17=0 \Rightarrow h=-17$ //

Thus for $h=-17$ the vector b is in the plane spanned by a_1 & a_2.

The matrix eqn $\underset{m\times n}{A}x_n = b_m = [a_1 \; a_2 \; \cdots \; a_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$= x_1a_1 + x_2a_2 + \cdots x_na_n$

$x_1 + 2x_2 - x_3 = 4$

$-5x_2 + 3x_3 = 1$

$$\overset{A}{\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{b}{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}$$

Three ways to represent the LS

i) Matrix eqn ii) vector equation iii) System of linear eqn.

Theorem 4 Let A be an $m \times n$ matrix. Then the following statements are equivalent.

a) For each b in $\mathbb{R}^m$, the equation $Ax=b$ has a soln

b) Each b in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row. (A only)

ex Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$
- $Ax=b$ has soln for each $b \in \mathbb{R}^4$
- coln of A span $\mathbb{R}^4$
- is each $b \in \mathbb{R}^4$ a LC

Just check whether A has pivot positions in each row. elementary row operations we get

$\sim \begin{bmatrix} \boxed{1} & 3 & 0 & 3 \\ 0 & \boxed{2} & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{5} \end{bmatrix}$ no pivot on the third row, so all are not correct!

Homogeneous Linear System (HLS)

Def^n A linear system is called HLS if it can be written as $Ax = 0$, where A is $m \times n$ matrix, x is vector of $n \times 1$ and 0 be a null matrix of order $m \times 1$.

Def^n Trivial and non-trivial sol^n of HLSs)
Let $Ax = 0$ be a HLS. The equation $Ax = 0$ always has one sol^n $x = 0$, where 0 is null vector, such sol^n is called trivial sol^n. And the non-zero sol^n of $Ax = 0$ is called non-trivial sol^n.

Note: The HLS have non-trivial sol^ns iff the HLS has at least one free variable.

e.g. Determine if the HLS has a trivial sol^n.

$$x_1 - 3x_2 + 7x_3 = 0$$
$$-2x_1 + x_2 - 4x_3 = 0$$
$$x_1 + 2x_2 + 9x_3 = 0$$

$\Rightarrow$ The AM of the HLS is

$$\begin{bmatrix} \boxed{1} & -3 & 7 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & 2 & 9 & 0 \end{bmatrix}$$

$\Rightarrow$ check for the free variable by converting the AM into echelon matrix.

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} \boxed{1} & -3 & 7 & 0 \\ 0 & \boxed{-5} & 10 & 0 \\ 0 & 5 & 2 & 0 \end{bmatrix} \sim R_3 \to R_3 + R_2 \begin{bmatrix} \boxed{1} & -3 & 7 & 0 \\ 0 & \boxed{-5} & 10 & 0 \\ 0 & 0 & \boxed{12} & 0 \end{bmatrix}$$

Here, we have pivots in 1^st, 2^nd & 3^rd columns, so no free variable exists. x_1, x_2 & x_3 are basic variables and the sol^ns are $12x_3 = 0 \Rightarrow x_3 = 0$, $-5x_2 + 10x_3 = 0 \Rightarrow x_2 = 0$, similarly $x_1 = 0$

$x = 0$. Thus the system has trivial solution.

e.g.
$$\left.\begin{aligned} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_1 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{aligned}\right\} \Rightarrow \text{AM} \begin{bmatrix} \boxed{2} & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix}$$

$$\underset{\substack{R_2 \to R_2+R_1 \\ R_3 \to R_3 - 2R_1}}{\sim} \begin{bmatrix} \boxed{2} & -5 & 8 & 0 \\ 0 & \boxed{-12} & 9 & 0 \\ 0 & 12 & -9 & 0 \end{bmatrix} \underset{R_3 \to R_3+R_2}{\sim} \begin{bmatrix} \boxed{2} & -5 & 8 & 0 \\ 0 & \boxed{-12} & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{R_2 \to \frac{R_2}{-12}}{\sim} \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & 1 & -\cancel{\tfrac{9}{12}}\tfrac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_1 \to R_1+5R_2}{\sim} \begin{bmatrix} \boxed{2} & 0 & \frac{17}{4} & 0 \\ 0 & \boxed{1} & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{R_1 \to \frac{R_1}{2}}{\sim} \begin{bmatrix} \boxed{1} & 0 & \frac{17}{8} & 0 \\ 0 & \boxed{1} & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we 1st and 2nd as pivot col^ns, so, x_1 & x_2 are basic variables and 3rd col^n has no pivot, so x_3 is a free variable. This implies this HLS has non-trivial sol^n as

$x_2 - \frac{3}{4}x_3 = 0 \Rightarrow x_2 = \frac{3}{4}x_3$

$x_1 \quad + \frac{17}{8}x_3 = 0 \Rightarrow x_1 = -\frac{17}{8}x_3$

$0x_3 = 0 \Rightarrow 0 = 0 \Rightarrow x_3$ is free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{17}{8}x_3 \\ \frac{3}{4}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -17/8 \\ 3/4 \\ 1 \end{bmatrix} = x_3 v \Rightarrow x = tv \quad t \in \mathbb{R}$$

parametric / vector form

The sol^n is in Span{v}

Note
If the equation has only one free variable, then the sol^n set is in a line through origin. If it has two free variables then the sol^n is a plane containing origin. If $Ax = 0$ has $x = 0$ as sol^n then the sol^n set is in Span{0}.

The sol^n of non-homogenous LS ($Ax = b$)

e.g.
$$\left.\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned}\right\} \Rightarrow A_m \begin{bmatrix} \boxed{1} & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix}$$

$$\underset{R_2 \to R_2+4R_1}{\sim} \begin{bmatrix} \boxed{1} & 3 & 1 & 1 \\ 0 & \boxed{3} & 6 & 3 \\ 0 & -3 & -6 & -3 \end{bmatrix} \underset{R_3 \to R_3+R_2}{\sim} \begin{bmatrix} \boxed{1} & 3 & 1 & 1 \\ 0 & \boxed{3} & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{R_2 \to R_2/3}{\sim} \begin{bmatrix} \boxed{1} & 3 & 1 & 1 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_1 \to R_1 - 3R_2}{\sim} \begin{bmatrix} \boxed{1} & 0 & -5 & -2 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consistent, the last row is of form $[0 \quad 0 \quad 0 \quad b]$ with $b = 0$. Basic variables $= x_1, x_2$ & x_3 is free variable.

The general sol^n is:

$x_1 - 5x_3 = -2 \Rightarrow x_1 = -2 + 5x_3$

$x_2 + 2x_3 = 1 \Rightarrow x_1 = 1 - 2x_3$

x_3 is free.

Parametric form

$$\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2+5x_3 \\ 1-2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

particular sol^n

$\Rightarrow X = p + x_3 v \Rightarrow X = p + t v$ for $t \in \mathbb{R}$.

sol^n for homogenous part

Linear dependence

The set of vectors $\{v_1, v_2, \ldots v_n\}$ in $\mathbb{R}^n$ is called linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0 \Rightarrow x_1 = 0 = x_2 = \cdots = x_n.$$

The set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^n$ is called linearly dependent if the vector equation

$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$ implies not all x_j are zero for $j = 1, \ldots, n$.

How the check for linear dependence & independence?

i) Write the vector eq^n $x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$, x_i are scalars

ii) Write the augmented matrix $[v_1 \; v_2 \; \cdots \; v_k \; 0]$

iii) Change the AM to echelon form.

→ If there is at least one free variable then set of vectors are linearly dependent - (non-trivial sol^n)

→ If there is no free variable then the set of vectors are linearly independent. (trivial sol^n case).

Theorem If a set contains more vectors than the entries (components) in each vector, then the set of such vectors are linearly dependent.