

① Important questions to be taken care of

Ch 1 Function of one variable (we talk only real valued f)

$f: \mathbb{R} \rightarrow \mathbb{R}$ Range

domain
(function definition)

Study carefully
Exempl 1 off the book

Find domain and range of:

- i) $y = x^2$ ii) $y = \frac{1}{x}$ iii) $y = \sqrt{4-x}$ iv) $y = \sqrt{x}$
See all the solutions in book.

Here are some more from exercise:

③ Find domain and range of

i) $h(x) = \sqrt{4-x^2}$

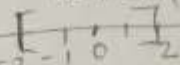
For domain (all the valid values of x that when plugged into $h(x) = \sqrt{4-x^2}$ should give a real number):

$\Rightarrow (4-x^2) \geq 0$ \because If $(4-x^2) < 0$, then the square root of -ve number is not real number

changing sign, we get

$x^2 - 4 \leq 0$
 $\Rightarrow (x-2)(x+2) \leq 0$
 $\Rightarrow \text{Domain } -2 \leq x \leq 2$

$x-2 \leq 0 \quad \& \quad (x+2) \leq 0$
 $\Rightarrow x \leq 2 \quad \Rightarrow x \leq -2$
domain $\quad \text{i.e. } x \geq 2$



To find range, we have

$h(x) = y = \sqrt{4-x^2}$ (squaring both sides)

$y^2 = 4 - x^2 \Rightarrow x^2 = 4 - y^2$

$\Rightarrow x = \pm \sqrt{4-y^2}$, Again, for all values of $(4-y^2) \geq 0$, we get corresponding real values for x .

$\Rightarrow y^2 - y \leq 0 \Rightarrow (y+2)(y-2) \leq 0 \Rightarrow -2 \leq y \leq 2$

(Imp. to check with question)

But in our question, the only possible values of y given $x \in [-2, 2]$, is positive number because square root of -ve number is not allowed $\Rightarrow \text{range } = 0 \leq y \leq 2$

③ (ii) Given $g(x) = \frac{2x+1}{x-3}$, the valid domain (2)

In this case in $x-3 \neq 0$, this is to avoid division by zero issue

$\Rightarrow x \neq 3$ not included

So, domain = $(-\infty, 3) \cup (3, \infty)$

For range, let's inverse the function:

$$g(x) = y = \frac{2x+1}{x-3} \Rightarrow xy - 3y = 2x+1$$

$$\Rightarrow xy - 2x = 3y+1$$

$$x(y-2) = 3y+1$$

$$\Rightarrow x = \frac{3y+1}{y-2}$$

$$\Rightarrow y-2 \neq 0$$

The possible value of $y = (-\infty, 2) \cup (2, \infty)$

Now, checking back to the question, there is no restriction imposed by the question for this range \Rightarrow range = $(-\infty, 2) \cup (2, \infty)$

(Let's see, if we say $\frac{2}{1}$ is also in the range, then, we can say that $y=2 \Rightarrow 2 = \frac{2x+1}{x-3}$

$$\Rightarrow 2x-6 = 2x+1, -6 = 1, \text{ does not make sense.}$$

(Hence, 2 should be excluded from the range)

\Rightarrow See the Vertical Line test to check for a function. Solve the examples from book. Like $x^2 + y^2 = 9$

\Rightarrow A18: study carefully the examples in the book about Even and odd function: Symmetry one more example from exercise:

5) (iv) check for odd or even function

$$f(x) = 2|x| + 1$$

$$\Rightarrow f(-x) = 2|-x| + 1 = 2|x| + 1 = f(x) \Rightarrow \text{even f}^n$$

$$v) f(x) = 3 \Rightarrow f(-x) = 3 = f(x) \Rightarrow \text{even f}^n$$

$$\text{iii) } h(x) = 1 + x^3 - x^5 \Rightarrow h(-x) = 1 + (-x)^3 - (-x)^5 = 1 - x^3 + x^5 \neq h(x)$$

Neither odd nor even / take -1 common $\Rightarrow -1(x^3 - 1 - x^5) \neq \pm h(x)$

Study carefully. Example 8 of the book.

$$f(x) = \begin{cases} 1-x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$
 we discussed this in detail in the class!

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Exercise 1.1 (1) Evaluation:

i) $f(x) = 4 - 3x$, $\frac{f(3+h) - f(3)}{h}$ [Just put $3+h$ in place of x]

$$\Rightarrow \frac{4 - 3(3+h) - (4 - 3 \cdot 3)}{h} = \frac{4 - 9 - 3h - 4 + 9}{h} = \frac{-3h}{h} = -3$$

1.2 Linear mathematical model

Study carefully the example 1 in the book (we also discussed this in the class).

Ex. 1.2 (2) We have: 100 chairs cost \$2200

$\Rightarrow (100, 2200)$ one point in x, y coordinate

Also given in 300 chairs cost \$4800

$\Rightarrow (300, 4800)$. Here the x variable is

Number of chairs and y is the cost. So, we can build a linear model in the form of $y = mx + b$ with those two data points given.

We know, $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4800 - 2200}{300 - 100} = \frac{2600}{200} = 13$

So, we can take any one point and this slope to make a linear model. Find the intercept b !

$2200 = 13 \cdot 100 + b \Rightarrow b = 2200 - 1300 = 900$
 $(y = mx + b)$

So, our final model is $y = 13x + 900$

\Rightarrow If we want to build one more chair, the cost

would go up by \$13, $\frac{dy}{dx} = 13$

The value of b indicates even if we don't produce a single chair ($x=0$), we incur the cost (fixed cost) of \$900.

1.3) Combination of function composite (4)

Study carefully Example 2 in the book (imp)
for composite functions and finding their domain

Exercise 1.3

(2) (vi) $f(x) = x^2$, $g(x) = 1 - \sqrt{x}$

$\Rightarrow f \circ g = f(g(x)) = f(1 - \sqrt{x}) = (1 - \sqrt{x})^2$

$\because f(x) = x^2$
we plug $(1 - \sqrt{x})$
as x

$= 1 - 2\sqrt{x} + x = 1 + x - 2\sqrt{x} = f \circ g$

Here, domain of $f \circ g$ is that we can't take $x \leq 0$ values

as we have \sqrt{x} term in the definition

And hence the domain of $f \circ g$ is $[0, \infty)$, 0 included

(try yourself to find the range)

$\Rightarrow g \circ f = g(f(x)) = g(x^2) = 1 - \sqrt{x^2} = g \circ f$

Domain is, we can't have $\sqrt{x} < 0$

First, need to find domain of $g \circ f$

$\sqrt{x} \geq 0 \Rightarrow x \geq 0$

So, domain of $g(x) = [0, \infty) = A$

Then, we need to find the domain of $g(g(x)) = 1 - \sqrt{1 - \sqrt{x}}$

Here, we must have $1 - \sqrt{x} \geq 0$

$\Rightarrow \sqrt{x} \leq 1 \Rightarrow x \leq 1$

Again, \sqrt{x} can't be negative,

i.e. $x \geq 0 \Rightarrow 0 \leq x \leq 1$, i.e. domain $= [0, 1] = B$

\therefore Domain of $g \circ f = A \cap B = [0, \infty) \cap [0, 1] = [0, 1]$

Note: We need to find the domain of inner f^n first &

give it a name say A and then the whole composite

f^n , $g(g(x))$ and give it a name, say B . AND the

domain of the composite f^n is $A \cap B$.

But, in the case of $f \circ g$ just above, we had inner

f^n $g(x)$ whose domain is only restricted by $\sqrt{x} \geq 0$,

i.e. $A = [0, \infty)$ and for B we had

$f \circ g = f(g(x)) = 1 + x - 2\sqrt{x}$, again the same condition of $x \geq 0$
 $\therefore B = [0, \infty) \Rightarrow A \cap B = [0, \infty)$

Continuing with previous question

$$g \circ f = g(f(x)) = g(x^2) = 1 - \sqrt{x^2} = 1 - x$$

in this case, domain of $g \circ f \supset A \cap B$, inner f²

$$A = \text{domain of } f(x) = x^2 = (-\infty, \infty)$$

$$B = \text{domain of } g(x) = 1 - x = (-\infty, \infty)$$

$$\Rightarrow A \cap B = (-\infty, \infty)$$

You can try for $f \circ f$, yourself!

Ch 2 Limit

1) Find the limit: $\lim_{x \rightarrow \infty} \frac{\sqrt{2x^2+1}}{3x-5}$, if we directly apply the limit $x \rightarrow \infty$ in our f², then we get $\frac{\infty}{\infty}$ indeterminate

So, to solve this type of problem, we need to bring the variable in the form of $\frac{c}{x}$, c is a constant

We can manipulate the f² as

$$\lim_{x \rightarrow \infty} \frac{\sqrt{\frac{2x^2}{x^2} + \frac{1}{x^2}}}{\frac{3x}{x} - \frac{5}{x}}, \text{ multiplying both denominator \& numerator by } \frac{1}{x}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}}, \text{ now we can apply limit}$$

$$\Rightarrow \frac{\sqrt{2+0}}{3-0} = \frac{\sqrt{2}}{3} \quad \left(\text{Since } \frac{1}{\infty} = 0 \right)$$

1) $\lim_{x \rightarrow 2} \frac{x^4 - 16}{x^2 - 4}$, here also, we can't apply limit directly as we get $\frac{0}{0}$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{(x^2 - 4)(x^2 + 4)}{(x^2 - 4)} = \lim_{x \rightarrow 2} x^2 + 4 = 4 + 4 = 8$$

We will review limit & continuity in class next week

ch3 Derivatives

(6)

3.1 Tangent (Imp)

Study carefully Example 1 to find the equation of tangent to $y = x^2$ at the point $P(1, 1)$.
We discussed this in the class!

(Imp) Exercise 3.1 Find eqⁿ of tangent to $y = f(x)$ at $P(x_1)$

1 (d) $y = \frac{2x+1}{x+2}$, $P = (1, 1)$ & tangent.
First, we need to find the slope m at $x = 1$

$$m = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{1}{x-1} \left[\frac{2x+1}{x+2} - \frac{2+1}{1+2} \right]$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1}{(x-1)} \left[\frac{2x+1}{x+2} - 1 \right] = \lim_{x \rightarrow 1} \frac{2x+1 - x-2}{(x+2)(x-1)}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{(x-1)}{(x-1)(x+2)} = \lim_{x \rightarrow 1} \frac{1}{x+2} = \frac{1}{1+2} = \frac{1}{3}$$

Now, we have m and a given point $P(1, 1)$ for a tangent line.

So, eqⁿ of this tangent line is

$$(y - y_1) = m(x - x_1)$$

$$\Rightarrow y - 1 = \frac{1}{3}(x - 1) \Rightarrow 3y - 3 = x - 1$$

$$\Rightarrow \boxed{y = \frac{x+2}{3}}$$

3.3 Derivative as a fⁿ.

(Imp) See, Example 1 & 2 and try to understand it.

Exercise 3.3 (4) $g(t) = \frac{1}{\sqrt{t}}$

We know that $g'(t) = \lim_{h \rightarrow 0} \frac{g(t+h) - g(t)}{h}$ by definition

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{t+h}} - \frac{1}{\sqrt{t}}}{h} \Rightarrow \lim_{h \rightarrow 0} \frac{\sqrt{t} - \sqrt{t+h}}{h \sqrt{t} \sqrt{t+h}} \cdot \frac{\sqrt{t} + \sqrt{t+h}}{\sqrt{t} + \sqrt{t+h}}$$

(Imp)

$$\Rightarrow \lim_{h \rightarrow 0} \frac{t - t - h}{h \sqrt{t} \sqrt{t+h} (\sqrt{t} + \sqrt{t+h})}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{-1}{\sqrt{t} \sqrt{t+h} (\sqrt{t} + \sqrt{t+h})} = \frac{-1}{\sqrt{t} \sqrt{t} (\sqrt{t} + \sqrt{t})} = \frac{-1}{2t\sqrt{t}}$$

Since the root f^2 is defined only for non-negative x values and the rational f^3 is defined only for non-zero denominator value, both $g(t)$ and $h(t)$ are defined for only $t \in (0, \infty)$ (Important point + note)
 Try to solve other problems in this exercise yourself!

3.4 Review of derivative

Remember all the rules and derivatives of trigonometric & transcendental (\lg & e^x) functions. Apply these rules and solve for the derivatives of any order for the given problem.

Exercise 3.4 (8) differentiate.

$$\begin{aligned} \textcircled{1} f(x) &= 3x^2 - 2\cos x \\ \Rightarrow \frac{dy}{dx} &= \frac{d}{dx}(3x^2 - 2\cos x) = \frac{d}{dx}(3x^2) - \frac{d}{dx}(2\cos x) \\ &= 6x + 2\sin x \end{aligned}$$

$$\textcircled{2} g(\theta) = e^{2\theta}(\tan\theta - \theta)$$

$$\frac{dg(\theta)}{d\theta} = e^{2\theta} \frac{d(\tan\theta - \theta)}{d\theta} + (\tan\theta - \theta) \frac{d(e^{2\theta})}{d\theta}$$

$$= e^{2\theta}(\sec^2\theta - 1) + (\tan\theta - \theta) \frac{d(e^{2\theta})}{d\theta}$$

$$= e^{2\theta}(\sec^2\theta - 1) + (\tan\theta - \theta) e^{2\theta} \cdot 2$$

$$= e^{2\theta}[\sec^2\theta - 1 + 2(\tan\theta - \theta)]$$

and so on ———

Try solving all problems of the exercise.

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