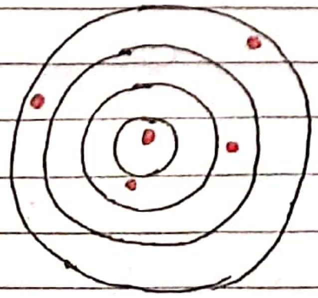
1.1 Errors in Numerical Calculations :-

Accuracy & Precision

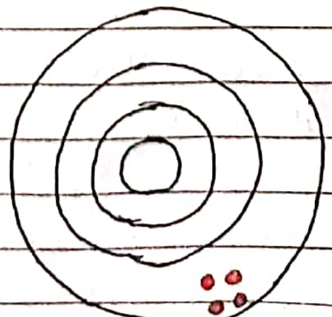
However hard we try, there is always some diff. between the true value and the measured value. We cannot avoide the presence of error in any numerical calculations. Any measurement and calculations can be characterised by accuracy and precision. These are two important measures of observational error.

Accuracy is how close or far the measured value agrees with the true value. In other words, accuracy is the proximity of result of measurement to the true value.

precision refers to how close or far the successive measurements are. In other words, precision is the degree to which repeated measurement show the same result under similar conditions.



Higher accuracy
low precision



Low accuracy
high precision

It is always recomended to have measurements with high accuracy and high precision.

Exact and Approximate Numbers:

The numbers that can be represented ~~in floating~~ point representation using finite no. of digits are the exact numbers. Eg: 1, 2, 3, 1/2, 7.25 etc.

The numbers that cannot be represented in floating point representation using finite no. of digits are called approximate numbers. Eg: 1/3, $\sqrt{2}$, 22/7 etc. These numbers need infinite no. of digits for exact representation. We can approximate their values using finite no. of digits.

~~Errors and~~ Types of Errors in Numerical calculat.

The deviation of measured/estimated/approximated value from the true value is called error.

We have the following types of errors in numerical calculations:

(a) Inherent Error: Errors which exists due to the erronous problem statement or approximation is called inherent error. The errors due to the limitation of computing aids is callso the inherent error.

Inherent errors cannot be completely elliminated. We can reduce the inherent error by taking accurate data, correct problem statement and the precise computation aids.

For example, let $x = 1/3$ and $y = 22/7$.
$= 0.333$ $= 3.14$

Any calculation using these values of x & y will suffer from inherent error.

(b) Absolute Error: The magnitude of difference betn the true value and measured value is called as absolute error.

$$\text{Absolute Error } (E_A) = |\text{True value} - \text{Measured val}|$$
$$= |\text{True Error}|$$

The absolute error may not be a good way to represent an error due to the following two reasons.

(i) It is hard to get true value.
(ii) It doesn't reflect the error size. we cannot decide the acceptance of error based on the absolute error.

Eg: let A 100 mm rod is measured to be 101 mm with an absolute error of |1mm|.
similarly let the diameter of a pin is 0.5 mm & is measured to be 1.5 mm with absolute error of |1mm|. In both cases, absolute error is |1mm|. It is acceptable in first case and is not acceptable in second case.

Date ______ Page ______

(c) Relative Error :-
The ratio of absolute error to the true value is called relative error (or fractional error). It represents the magnitude of error in comparison to the true value.

$$\text{Relative error}(E_R) = \left|\frac{\text{Absolute Error}}{\text{True Value}}\right|$$

Sometimes, relative error is also represented in % as,

$$\text{Relative Error}(E_R) = \left|\frac{\text{Absolute Error}}{\text{True Value}}\right| \times 100\%$$

Relative error is highly preferred error representation because it doesn't represent the magnitude/scale of the error only, rather it represents the error with respect to the true value.

Eg: let a 20 m rod is measured to be 20m 1cm and a 0.5 cm pin is measured to be 1.5 cm.

In first case,

$$E_R = \left|\frac{20 - 20.01}{20}\right| \times 100\%$$

$$= 0.05\% \quad \text{(Acceptable)}$$

In second case,

$$E_R = \left|\frac{0.5 - 1.5}{0.5}\right| \times 100\%$$

$$= 200\% \quad \text{(Not acceptable)}.$$

(d) Approximate Error :-

In most of the numerical problems, there is no luxury of knowing the true value. In such cases we have to depend on our own numerical results to decide the amount of error. In such situation, we can estimate the approximate error. We can use the successive approximations to calculate the approximate error.

The approximate error is given by,

$$E_A = \text{Current Approximation} - \text{Previous Approximation}$$

If we take the absolute value of approximate error, it is known as absolute approximate error.

~~Most commonly used~~
Relative approximate error is the most commonly used estimation for ~~ret~~ approximate error, Given by,

$$\epsilon_a = \frac{\text{Approximate Error}}{\text{Current Approximation}} \cancel{\times 100\%}$$

Often, we may express the approximate error in the form of percentage relative approximate error.

$$\epsilon_a = \frac{\text{Approximate Error}}{\text{Current Approximation}} \times 100\%$$

Approximate error is used in iterative methods to decide the acceptance of estimated result. It helps to accept the result based on precision of the error.

Example: Let $f(x) = 7e^{0.5x}$
Estimate the relative approximate error in calculating $f'(x)$ at $x=2$ by taking $\Delta x = 0.3$ & $\Delta x = 0.15$.
use approximation formula

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

→ Here, $f'(x) \simeq \frac{f(x+\Delta x) - f(x)}{\Delta x}$

for $\Delta x = 0.3$,

$$f'(2) \simeq \frac{f(2+0.3) - f(2)}{0.3}$$

$$= 10.2643$$

for $\Delta x = 0.15$,

$$f'(2) \simeq \frac{f(2+0.15) - f(2)}{0.15}$$

$$= 9.8747$$

Approximate error E_t = Current approximation − P.A.

$$= 9.8747 - 10.2643$$

$$= -0.3896$$

Relative approximate error (ϵ_a) = $\frac{\text{Approximate Error}}{\text{Current Approximation}}$

$$= \frac{-0.3896}{9.8747}$$

$$= -0.0395$$

percentage relative approximate error $= -3.95\%$

Absolute percentage approximate error $= 3.95\%$.

(e) Round-off Error:-

It is impossible to represent some numbers exactly. Such numbers are represented using a fixed no. of digits. Such representation of numbers suffer from round-off error. For example, representation of some irrational numbers like π, roots of some positive integers like $\sqrt{2}$ etc. has round-off error.

Round-off error occurs due to approximate representation of numbers. It is hard to control such errors.

Round-off error = Actual Number − Representation of the no.

Example: If 1/3 is represented as 0.33, ~~what~~ How much is the round-off error?

→ Round off error = Actual No. − Representation

$= \frac{1}{3} - 0.33$

$= 0.333333\ldots - 0.33$

$= 0.003333\ldots$

(f) Truncation Error:

The error occurred in numerical calculations due to approximation in mathematical process or formula is called truncation error.

For example, consider the Maclaurin series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

If we approximate the expression using only three terms,

$$e^x = 1 + x + x^2/2!$$

T. Error (E_T) = Exact Value − Represented V

In this case, truncation error is $= \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

Example: Find the truncation error in calculating the first derivative of $f(x) = 5x^3$ at $x = 7$ using step size 0.25.

→ The definition of the exact first order derivative of $f(x)$ is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

However, if we calculate the derivative numerically, h has to be finite. Such error caused by taking finite h is a truncation error in the process of differentiation.

Here,

Exact value of $f'(x)$ at $x=7$ is

$$f'(7) = 15x^2$$
$$= 15 \times 7 \times 7$$
$$= 735.$$

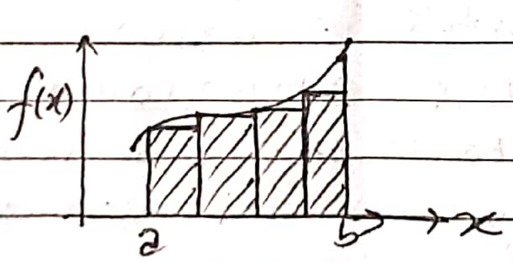
The approximate value is

$$f'(7) = \cancel{f(7)} \frac{f(7+0.25) - f(7)}{0.25}$$

$$= 761.5625$$

$\therefore$ Truncation error (TE) $= 735 - 761.5625$
$= -26.5625$

Note that, truncation error ~~may~~ also occurs in case the integration also. The integral $\int_a^b f(x)\,dx$ doesn't give exact area under the curve $f(x)$ in the range $[a, b]$



Sources of Errors

In any nemerical method, we may get errors due to the following sources.

1. Error in mathematical modeling of the problem.
2. Instrumental error (error in measuring tools).
3. Observational error.
4. Round-off error, truncation error, approximate error.

Systematic Vs Random Errors

Systematic Error	Random Error
Error that deviates from true value by a fixed amount in a particular direction is systematic error.	Error whose value varies/fluctuates in any unpredectable amount in any direction (+ve/-ve) is called random error.
It remains constant or changes in a regular fashion in repeated measurements.	It is varying in both magnitude and sign
Instrumental errors & different types of inherent errors are examples of systematic error.	Observation error, error due to incorrect calibration of measuring devices, etc. are examples of random error
Once discovered, they are easy to eliminate either in the source or in the result (mathematically).	Such errors are identified by observing inconsistent result and are hard to eliminate.

Propagation of Errors.

If we perform calculation by using approximate values, the error in representation of each value is transfered to the result, which is known as error propagation.

a. Error Propagation in addition & subtraction:

~~Let a and b be two numbers represented with errors Δa and Δb~~. During addition or subtraction of the numbers, the ~~error~~ propagated error is obtained by the sum of errors in each number.

Let a and b be two numbers represented with errors Δa and Δb. Then, the ~~sum~~ addition and subtraction of a and b will have the error $(\Delta a + \Delta b)$.

i.e. $$\text{sum} = (a+b) \pm (\Delta a + \Delta b)$$
$$\text{Difference} = (a-b) \pm (\Delta a + \Delta b)$$

Q. Two rods of lengths 100 mm and 50 mm are measured by using a mm-scale having least count of 1 mm. Calculate the addition & subtraction values of the lengths with propagated error.

→ Here, $l_1 = 100 \pm 1$ mm $\Rightarrow 99 \le l_1 \le 101$
$l_2 = 50 \pm 1$ mm $\Rightarrow 49 \le l_2 \le 51$

Adding the values,
$$\text{sum} = l_1 + l_2$$

Here, the sum is not an exact value rather, it is in a range.

At minimum, sum = $99 + 49 = 148$
At maximum, sum = $101 + 51 = 152$

$\therefore$ sum $= 150 \pm 2 \Rightarrow 148 \leq$ sum ≤ 152

Now, subtracting the values,

Difference $= l_1 - l_2$
At minimum, Diff. $= 99 - 51 = 48$
At maximum, Diff. $= 101 - 49 = 52$

$\therefore$ difference $= 50 \pm 2 \Rightarrow 48 \leq$ diff ≤ 52

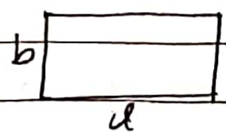
b. Error Propagation in multiplication & division

Let a and be be two numbers represented with errors Δa and Δb respectively. The error propagated after ~~addition~~ multiplication & division of the numbers is given by

$$\boxed{\text{Propagated Error} = \left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right) a.b}$$ for multiplication a.b.

$$\boxed{\text{Propagated Error} = \left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right) . \frac{a}{b}}$$ for division a/b.

Q. Determine the area of the given rectangle with proper error limits.



$l = 4 \pm 0.4$ m
$b = 2 \pm 0.2$ m

→ $A = l \times b$

$= 4 \times 2$

$= 8\ m^2$

Error in the result (ΔA) can be obtained by,

$$\Delta A = \left(\frac{\Delta l}{l} + \frac{\Delta b}{b}\right) \cancel{(l \times b)}\ A$$

$$= \left(\frac{0.4}{4} + \frac{0.2}{2}\right) \times 8$$

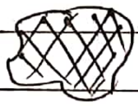
$$= (0.1 + 0.1)\, 8$$

$$= 1.6\ m^2$$

$\therefore$ Area $= A \pm \Delta A$

$= (8 \pm 1.6)\ m^2$

Q. Determine the density of the given object with proper error limits.



$M = 6 \pm 0.3\ kg$, $V = 2 \pm 0.2\ m^3$

→ $d = \frac{M}{V}$

$= 6/2 = 3\ kg/m^3$

Error in the result (Δd) can be obtained by,

$$\Delta d = \left(\frac{\Delta M}{M} + \frac{\Delta V}{V}\right) \cdot d$$

$$= \left(\frac{0.3}{6} + \frac{0.2}{2}\right) \cdot 3$$

$$= (0.05 + 0.1)\, 3$$

$$= 0.45$$

$\therefore$ density $= 3 \pm 0.45\ kg/m^3$.

Q. Find the absolute & relative error in the sum $\sqrt{6}+\sqrt{7}+\sqrt{8}$ correct upto four significant digits.

→ Here, $\sqrt{6} = 2.4495$

$\sqrt{7} = 2.6458$

$\sqrt{8} = 2.8284$

Taking four significant digits only,

$\sqrt{6} = 2.449$ with error $= 0.0005$

$\sqrt{7} = 2.645$ with error $= 0.0008$

$\sqrt{8} = 2.828$ with error $= 0.0004$

Now,

$$\text{sum} = (\sqrt{6}+\sqrt{7}+\sqrt{8}) \pm (0.0005 + 0.0008 + 0.0004)$$

$$= (2.449 + 2.645 + 2.828) \pm (0.0017)$$

$$= 7.922 \pm 0.0017$$

∴ Absolute Error $= 0.0017$

Relative Error $= \dfrac{0.0017}{7.922} = 0.000215 = 0.0215\%$

Date
Page

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Review of Taylor's Theorem

The Taylor's theorem gives an expression which can be used to determine the value of a function at some point closer to a point for which the functional value is known, and the function is continuously differentiable at the known point.

Let $f(x)$ be a function whose functional value at a point x_0 is known & $f(x)$ is continuously differe-ntiable at x_0. Then the value of the function at another point x_0+h can be approximated using the Taylor's theorem as:

$$\boxed{f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \cdots}$$

OR

$$\boxed{f(x_0+h) = \sum_{K=0}^{\infty} \frac{h^K}{K!} f^K(x_0)}$$

Applications ~~Characteristics~~ of Taylor's Theorem

(i) It helps to predict the functional value at a point based on the functional value and successive derivatives at another known point.

(ii) It helps to predict functional value at a point even without knowing function itself. (If higher order derivatives known)

(iii) It is based on the higher order derivatives of a continuously differentiable function at some known point.

(iv) we can use Taylor's series for solving the differential equations.

(v) Non standard functions can be approximated to the standard ones using Taylor's series.

(vi) It can be used to solve the definite integrals.

In a nutshell, Taylor's theorem helps to express any function in the form of series, which The series representation of a function may be useful in solving many mathematical problems.

Note: A Taylor's series expanded about the point $x = 0$ is called Maclaurin series.

Q. Find the value of $f(6)$ given that $f(4) = 125$, $f'(4) = 74$, $f''(4) = 30$, $f'''(4) = 6$ and all the higher order derivatives of $f(x)$ at $x = 4$ are zero.

Sol^n → The Taylor's series expansion of a function $f(x)$ about the point x_0 is written as,

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \dots\dots$$

Taking $x_0 = 4$ and $h = 2$, we get

$$f(4+2) = f(4) + 2\cdot f'(4) + \frac{4}{2}f''(4) + \frac{8}{6}f'''(\cancel{64}) + 0$$

$$= 125 + 2.74 + 2.30 + \frac{8}{6}.6 + 0$$

$$= 341.$$

Q. Find the Taylor's series expansion of the function $f(x) = 4x^3 + 3x - 8$. Consider $x = 3$ as base point and determine the expression for $f(x)$ at $x = 5$. Also verify that the result given by Taylor's theorem is correct.

→ Here,

$$f(x) = 4x^3 + 3x - 8$$
$$x_0 = 3$$
$$f'(x) = 12x^2 + 3 \Rightarrow f'(3) = 111$$
$$f''(x) = 24x \Rightarrow f''(3) = 72$$
$$f'''(x) = 24 \Rightarrow f'''(3) = 24$$

Using Taylor's series,

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \cdots$$

Taking $x_0 = 3$ & $h = 2$,

$$f(5) = f(3) + 2.f'(3) + \frac{4}{2}f''(3) + \frac{8}{6}f'''(3) + 0$$
$$= 109 + 222 + 144 + 32 + 0$$
$$= 507$$

Check:

$$f(x) = 4x^3 + 3x - 8$$
$$f(5) = 4 \times 5 \times 5 \times 5 + 3 \times 5 - 8$$
$$= 507.$$

Correct.

~~Solving~~ Non-Linear Equations:-

Any equations of degree two or more are called non-linear equations. Non-linear equations may be algebraic (purely polynomial) or transcendental (Containing trigonometric, logarithmic or exponential terms).

Non-linear equations can be solved using various analytical methods but as the degree goes on increasing it becomes difficult to solve by using analytical approach. There may be cases in which we cannot solve a non-linear equation by using the analytical methods. In such cases, we can use numerical methods to solve the equations.

Numerical Method

Numerical Method is an approach for solving a complex mathematical problem in terms of simple ~~mathematical algebraic~~ arithmetic operations. It gives quick solution of the problems by approximation. Numerical methods for a mathematical problems are easy to program and run faster in computer.

Numerical methods can be used to solve different problems such as:

(a) Solution of linear/non-linear equations.
(b) curve fitting

(c) Interpolation problems.
(d) Solving definite integrals
(e) Solving initial/boundary value problems (diff. equations).
(f) Solving partial differential equations.

Characteristics of Numerical Methods :-

(a) Accuracy :- The accuracy of a numerical method depends on the number of iteration performed.

(b) Convergence :- The convergence (or convergence rate) of a numerical method defines how fast the desired result ~~is~~ can be obtained.

(c) Numerical stability :- Numerical stability of a numerical method defines how likely the ~~numerical meth~~ solution gets diversified on small deviation in the procedure. Usually the numerical methods are unstable.

(d) Efficiency :- The efficiency of a numerical method defines how much effort it needs to solve the problem by using computers or by humans.

(e) Approximate Result :- By analytical method, exact solution can be obtained. However ~~by~~ by numerical methods, only approximate solution can be obtained.

Solution of Non-linear Equations :-

Different algebraic polynomials and transcendental equations can be solved by numerical method. We can solve the non-linear equations to find their approximate roots by using Numerical method

Following are the different numerical methods for solving the non-linear equations to find their approximate solution.

a. Trial and Error Method
b. Bisection (Half-Interval) Method
c. Regula Falsi (False-position) Method.
d. Newton-Raphson (Newton's) Method
e. Secant Method
f. Fixed Point Iteration Method
g. Horner's Method.

Solving Non-Linear Equation by Trial & Error Method

It is the simplest type of ~~non-linear~~ numerical method to solve a non-linear equation. This method is based on the smart guesses we can make in each iteration. The accuracy of the solution completely depends on the guesses and the number of iterations. There is no guarantee of convergence and the solution is highly unstable.

Procedure:

1. Make an initial guess.
2. Check for the correctness
3. If the error is not acceptable, repeat steps 1, 2, 3.
4. End.

Q. What are the sources of errors? Discuss various types of errors. Find the roots of the equation $x^2 + 5.6x - 14 = 0$ by trial & error method up to 4 significant digits. [1+3+4] (TU CSIT 077)

→ for last/numerical part only.

The equation is,

$$x^2 + 5.6x - 14 = 0$$

Let the initial guess be $x = 0$.

$\Rightarrow$ $0^2 + 5.6 \times 0 - 14 = 0$

or, $-14 = 0$. (LHS $= -14$)

Date
Page

let $x = 2$, then $2^2 + 5.6 \times 2 - 14 = 0$
or, $1.2 = 0$ (LHS = 1.2)

Tabulating the successive iterations,

(Internally use Bisection method)

SN	Guess(x)	LHS (error)
1	$x = 0$	-14
2	$x = 2$	1.2
3	$x = 1.5$	-4.67
4	$x = 1.75$	-1.1375
5	$x = 1.875$	0.0156
6	$x = 1.8125$	-0.5648
7	$x = 1.8438$	-0.2751
8	$x = 1.8594$	-0.12999
9	$x = 1.8672$	-0.057244
10	$x = 1.8711$	-0.0208
11	$x = 1.87305$	-0.0026
12	$x = 1.874025$	0.006509
13	$x = 18735$	0.0016
14	$x = 1.873275$	-0.0005008

$\therefore$ Approximate root $= 1.873275$ Ans.

Q. Solve the non-linear equation $f(x) = x^4 + 3x^2 - 6$ using trial & error method.

[Ans: $x = 1.171444$, -1.171444

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Limitations of trial-and-error method

There is no guarantee of convergence.
Accuracy of the solution depends on the choice of guess in each iteration and the number of iterations.
There is no guarantee of convergence.
Highly unstable method.

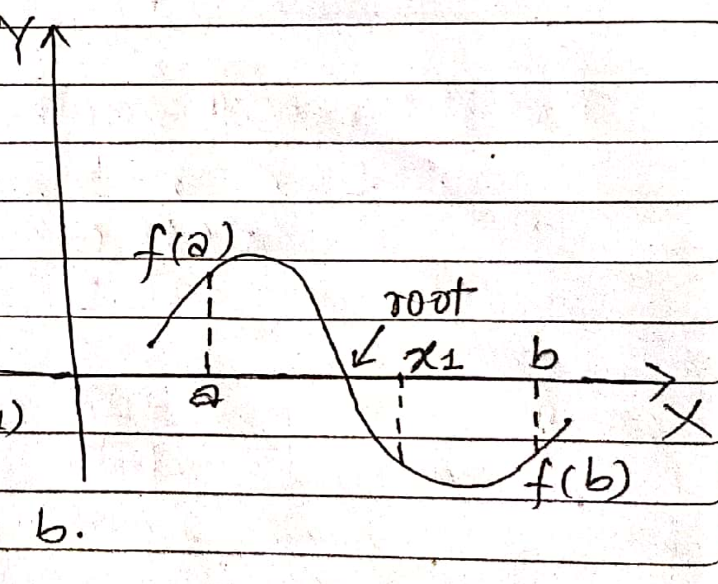
Bisection Method :-

It is one of the simplest and reliable numerical method for solving the non-linear eqns. This method is also known as binary chopping or half interval method.

Let $f(x)$ be a non-linear function which is continuous in the interval $[a, b]$ ~~and~~ such that $f(a)$ and $f(b)$ are of opposite sign. Then the function $f(x)=0$ has at least one root between $x=a$ and $x=b$.

Here,

$f(a).f(b) < 0$ is the condition for the convergence of the root.



The first approximation (x_1) can be estimated to lie at the middle of a and b.

$$x_1 = \frac{a+b}{2}$$

If $f(x_1) = 0$, x_1 is the required root. Otherwise, we will assume either $a = x_1$ or $b = x_1$ such that $f(a).f(b) < 0$.

In the successive iterations, we assume that the root lies at the mid point between a and b. The iterations stop if the new approximation is in the acceptable error range.

Algorithm :-

1. Start.
2. Define function $f(x) = 0$ and acceptable error $E > 0$.
3. Input two initial guesses for the roots as a & b.
4. Compute $f(a)$ and $f(b)$.
5. If $f(a) \cdot f(b) > 0$, display message "Root Cannot be found between a and b" and go to step .
6. Compute the first approximation of the root as

$$r_1 = (a+b)/2$$

7. If $f(r_1) = 0$, go to step
   If $f(r_1) \cdot f(b) < 0$, set $a = r_1$
   If $f(a) \cdot f(r_1) < 0$, set $b = r_1$
8. Compute the next approximation of the root as
$$r_2 = (a+b)/2$$
9. Compute the relative approximate error as:
$$Err = \left| \frac{r_2 - r_1}{r_2} \right| \text{ (or any other way).}$$
10. If $Err > E$, set $r_1 = r_2$ and go to Step 7.
11. Display r_1 as the root.
12. Stop.

Date
Page

Q. Solve $x^3 - 2x - 5 = 0$ using bisection method to find a real root of $f(x)$. Check your result up to the 0.08% accuracy.

→ Here, $f(x) = x^3 - 2x - 5$

Let the initial guesses be $a = 2$ & $b = 3$.

Then, $f(2) = -1 < 0$

$f(3) = 16 > 0$

The real root of $f(x)$ lies between 2 & 3.

Acceptable error $(E) = 0.0008$.

First approximation $(r_1) = (a+b)/2$

$= (2+3)/2$

$= 2.5$

$f(2.5) = 5.625 \neq 0$ ($x = 2.5$ is not a solution).

The root lies between $a = 2$ & $b = 2.5$.

Next approximation $(r_2) = (a+b)/2$

$= 2.25$

Err $= |2.25 - 2.5|$

$= 0.25$ (Not acceptable).

~~= 0.1111 (Not acceptable).~~

Tabulating the successive iterations,

n	a	b	$r_n = (a+b)/2$	$f(r_n)$
1.	2	3	2.5	5.625
2.	2	2.5	2.25	1.891
3.	2	2.25	2.125	0.3457
4.	2	2.125	2.0625	-0.3513
5.	2.0625	2.125	2.0938	-0.0089
6.	2.0938	2.125	2.1094	0.1671

n	a	b	$r_n = (a+b)/2$	$f(r_n)$
7	2.0938	2.1094	2.1016	0.0789
8	2.0938	2.1016	2.0977	0.0352
9	2.0938	2.0977	2.0958	0.0133
10	2.0938	2.0958	2.0948	0.0027
11	2.0938	2.0948	2.0943	−0.0028

Here, Error = |Present Approx − Previous Approx|

$= |2.0943 - 2.0948|$

$= 0.0005$ (Acceptable).

∴ Approximated root = 2.0943

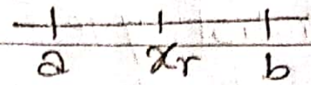
Q. Find at least one root of $f(x) = x^2 + \tan x + e^x$. Correct upto 3 decimal places using bisection method.

→

Convergence of Bisection Method:-

In bisection method, the interval of the root is reduced by half after each iteration.

After n iterations,
the interval is
reduced to $(a-b)/2^n$.



Let the width be Δx.

$$\frac{a-b}{2^n} = \frac{\Delta x}{2^n}$$

$\therefore$ Error after n^{th} iteration $|E_n| = \Delta x/2^n$

If we perform one more iteration,

$$|E_{n+1}| = \frac{\Delta x}{2^{n+1}} = \frac{\Delta x/2^n}{2} = \frac{E_n}{2}$$

Hence, in bisection method, the error is halfed after each iteration. The bisection method converges to real root in linear fashion.

Advantages:-

Since the method brackets the roots, it has guarantee of convergence.
After each iteration, the error is reduced by half.
The method has slow but steady rate of convergence.
It provides good accuracy.
It is easy to program.

Date ____
Page ____

Disadvantages:

It has slower rate of convergence.
If one of the initial guess lies closer to the root, it takes large no. of iterations.
Initial guess cannot be performed for some function (such as $f(x) = x^2$).
The convergence of the method depends on initial guess.

Number of Iterations Required:-

Let a & b be two initial guesses.
After n iterations, the error can be represented as,

$$E = \frac{b-a}{2^n} \quad \text{(Maximum possible error)}$$

or, $2^n = (b-a)/E$

or, $\log 2^n = \log\left(\frac{b-a}{E}\right)$

or, $\boxed{n = \frac{\log(b-a) - \log E}{\log 2}}$

$[\log a^b = b \log a]$
$\log a/b = \log a - \log b$
log is of base 10

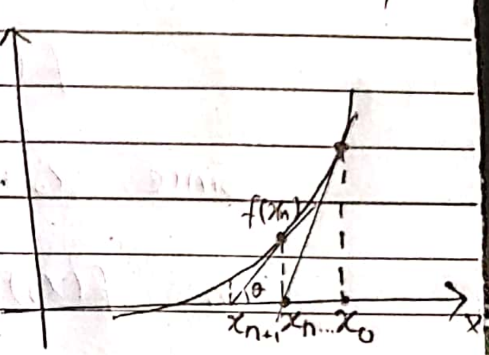
Here, n is the minimum number of iterations & E is the maximum acceptable error.

Q. Find the minimum no. of iterations needed to solve $e^x - x^2 + 3x - 2 = 0$ using bisection method with error 10^{-2}
[Ans: $n \simeq 7$]

P.N.K.

Newton - Raphson Method :-

It is one of the simplest and fastest methods to approximate the root (real or complex) of a non-linear equation. It was developed by Newton & Josheph Raphson. It is non-bracketed method and uses only one initial guess.



Let $f(x)$ be a non-linear function and x_0 be an initial guess for the root. Let us draw a tangent to $f(x)$ at the point corresponding to x_0.

The tangent cuts x-axis at x_1, which is the next approximation for the root. If we draw another tangent corresponding to x_1, we will get another approximation x_2 and so on. In each iteration, we get closer and closer to the real root.

Consider the n^{th} approximation x_n. If we draw tangent to $f(x)$ corresponding to x_n, we get new approximation x_{n+1}.

From the diagram,

$$\tan\theta = \frac{f(x_n)}{x_n - x_{n+1}}$$

$$\text{or, } \quad f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}}$$

$$\therefore \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

We can derive Newton-Raphson formula using the Taylor's series.

Let x_0 be an approximate root of $f(x) = 0$, with error h. Let $x_1 = x_0 + h$ be the exact root. Then from Taylor's series,

$$f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots\cdots$$

Since $x_0 + h$ is real root, $f(x_0 + h) = 0$.

or, $f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$

or, $f(x_0) + h f'(x_0) = 0$ [Neglecting higher powers of h]

or, $h = -\frac{f(x_0)}{f'(x_0)}$

or, $x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$

or, ∴ $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

In general

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Algorithm

1. Start
2. Define function $f(x)$, acceptable error $e \geq 0$, derivative function $f'(x)$.
3. Input an initial guess x_0.
4. Compute $f(x_0)$ & $f'(x_0)$.
5. Find the first approximation as

$$x_1 = x_0 - f(x_0)/f'(x_0)$$

6. Find the next approximation as

$$x_2 = x_1 - f(x_1)/f'(x_1)$$

7. Compute the error as.

$$error = |x_2 - x_1|$$

8. If error > e, set $x_1 = x_2$ and go to step-6.
9. Display x_2 as the root.
10. Stop.

Q. Find the root of equation $e^x - 4x^2$ using the Newton-Raphson method upto 5 decimal places.

[CSIT - 2072]

→ Here

$$f(x) = e^x - 4x^2$$
$$f'(x) = e^x - 8x$$
$$e = 0.000001$$
$$x_0 = 0 \quad \text{(initial guess)}$$

$f(0) = 1$
$f'(0) = 1$

Using Newoton-Raphson method,

$x_1 = x_0 - f(x_0)/f'(x_0) \quad \because \left[x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \right]$
$= 0 - 1/1$
$= -1$

$x_2 = x_1 - f(x_1)/f'(x_1)$
$= (-1) - (-3.63212)/8.36788$
$= -0.56595$

Tabulating the successive iterations,

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0.	0	1	1	-1
1.	-1	-3.63212	8.36788	-0.56595
2.	-0.56595	-0.713377	5.09542	-0.4259464
3.	-0.4259464	-0.072569	4.06072	-0.408075
4.	-0.408075	-0.001172	3.929529	-0.4077767
5.	-0.4077767	-0.0000000367	3.9273409	-0.4077766

$\therefore x = -0.40777$ is the required root correct to 5 decimal places.

Convergence of Newton-Raphson Method :-

Let $f(x)$ be a non-linear function & α be its real root. After n iterations, let x_n be the approximate root with error e_n.

$$x_n = \alpha + e_n$$

In next iteration

$$x_{n+1} = \alpha + e_{n+1}$$

Using Newton, raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

or, $$\cancel{\alpha} + e_{n+1} = \cancel{\alpha} + e_n - \frac{f(\alpha + e_n)}{f'(\alpha + e_n)}$$

or, $$e_{n+1} = e_n - \frac{f(\alpha+e_n)}{f'(\alpha+e_n)}$$

$$= e_n - \frac{f(\alpha) + e_n f'(\alpha) + e_n^2/2!\, f''(\alpha) + \cdots + \cdots}{f'(\alpha) + e_n f''(\alpha) + e_n^2/2!\, f'''(\alpha) + \cdots + \cdots}$$

$$\approx e_n - \frac{\cancelto{0}{f(\alpha)} + e_n f'(\alpha)}{f'(\alpha) + e_n f''(\alpha)}$$ (Neglecting higher powers of e)

$$= \frac{\cancel{e_n f'(\alpha)} + e_n^2 f''(\alpha) - \cancel{e_n f'(\alpha)}}{f'(\alpha) + e_n f''(\alpha)}$$

$$= \frac{e_n^2 f''(\alpha)}{f'(\alpha) + e_n f''(\alpha)}$$

or, $$e_{n+1} \simeq \frac{e_n^2 f''(\alpha)}{f'(\alpha) + e_n f''(\alpha)}$$

$$= \frac{e_n^2 f''(\alpha)}{f'(\alpha)\left(1 + e_n \frac{f''(\alpha)}{f'(\alpha)}\right)}$$

$$= e_n^2 \frac{f''(\alpha)}{f'(\alpha)} \cdot \left[1 + e_n \frac{f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

Using Binomial expansion $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots \infty$.

$$e_{n+1} = e_n^2 \frac{f''(\alpha)}{f'(\alpha)} \cdot \left[1 - e_n \frac{f''(\alpha)}{f'(\alpha)} + \cdots - \cdots + \cdots\right]$$

or, $$e_{n+1} \simeq e_n^2 \frac{f''(\alpha)}{f'(\alpha)}$$ (Neglecting higher power of e_n).

or, $$e_{n+1} = K e_n^2$$

$\therefore$ $\boxed{e_{n+1} \propto e_n^2}$.

Here, we see that error in next approximation is proportional to the square of the error in previous approximation. Hence Newton-Raphson method has quadratic convergence. It is very fast method. (It is non-bracketed method so there is no guarantee of convergence).

Advantages:

(i) It has quadratic convergence & is very fast method.
(ii) It needs only one initial guess.
(iii) If the initial guess is close to the real root, this method converges very quickly.
(iv) It is easy to program.

Disadvantages:

(i) It has computational overhead of finding derivatives of the function at different points.
(ii) It fails to give reasult if $f'(x_n) = 0$, so it has poor numerical stability.
(iii) Near local maxima/minima, it fails to give next approximation.
(iv) It fails to give result if tangent to $f(x)$ corresponding to x_0 cuts x-axis again at x_0.
(v) It is non-bracketed method and convergence is not guaranteed.

Q. Calculate the root of $x \sin x + \cos x = 0$ using the Newton-Raphson method correct upto 10^{-4}.

Q. Use Newton's method to find the root of $x = e^{-x}$. Correct upto six decimal digits.

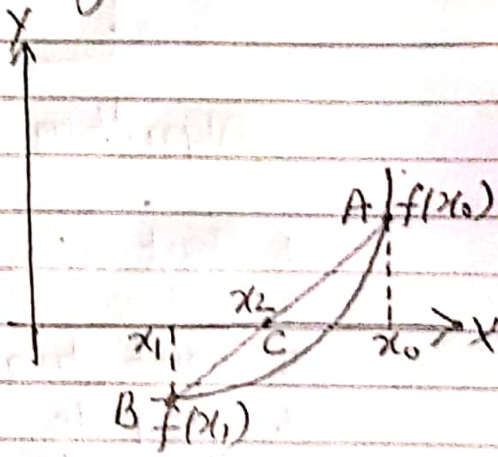
Q. Find the cube root of 30 using NR-method upto three decimal places.

Q. Find the value of $\sqrt{18}$ using NR-method.

Secant Method :

It is an open bracketed method to find the approx. root of a non-linear function. It is faster than the bisection method and slower than Newton-Raphson's method. Newton-Raphson method has an overhead of finding derivatives but this method has solved such overhead. This method starts with two initial guesses but there is no guarantee of convergence.

Let $f(x)$ be a non-linear function and x_0, x_1 be the initial guesses. Let us draw a secant line joining the points corresponding to x_0 and x_1. Here, the secant cuts X-axis at point x_2 which is our first approximation.



Slope of the secant line is,

Slope of AB = Slope of BC

or, $$\frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{0-f(x_1)}{x_2-x_1}$$

or, $$x_2 - x_1 = -\frac{f(x_1)\ (x_1-x_0)}{f(x_1)-f(x_0)}$$

or, $$\boxed{x_2 = x_1 - \frac{f(x_1)}{f(x_1)-f(x_0)}(x_1-x_0)}$$

Similarly, drawing another secant line corresponding to the point x_1 & x_2, we get x_3 as;

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

In general,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(x_{n-1})} \cdot (x_n - x_{n-1})}$$

Algorithm

1. Start.
2. Define $f(x)$ and acceptable error $e > 0$.
3. Input two initial guesses (x_0 & x_1).
4. Compute $f(x_0)$ & $f(x_1)$.
5. Find the first approximation as:

$$x_2 = x_1 - \frac{f(x_1)}{f(x_1) - f(x_0)} (x_1 - x_0)$$

6. Find the next approximation as:

$$x_3 = x_2 - \frac{f(x_2)}{f(x_2) - f(x_1)} (x_2 - x_1)$$

7. Compute the error as:
   error $= |x_3 - x_2|$
8. If error $> e$, set $x_1 = x_2$ & $x_2 = x_3$ go to step-6.
9. Display x_3 as the root. (or x_2 also).
10. Stop.

CSIT-073

Q. Estimate the root of $x^2-4x-10=0$ using secant method, correct upto four decimal places.

→ Here, $f(x) = x^2-4x-10$

let $x_0 = 5 \Rightarrow f(x_0) = f(5) = -5$
$x_1 = 6 \Rightarrow f(x_1) = f(6) = 2$ } root lies between 5 & 6.

From secant formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n)-f(x_{n-1})}(x_n - x_{n-1})$$

First approximation,

$$x_2 = x_1 - \frac{f(x_1)}{f(x_1)-f(x_0)}(x_1-x_0)$$

$$= 6 - \frac{2(6-5)}{2+5}$$

$$= 5.7143$$

$$f(x_2) = -0.204$$

Tabulating successive iterations,

n	x_{n-1}	x_n	$f(x_{n-1})$	$f(x_n)$	x_{n+1}
1.	$x_0 = 5$	$x_1 = 6$	$f(x_0) = -5$	$f(x_1) = 2$	$x_2 = 5.7143$
2.	$x_1 = 6$	$x_2 = 5.7143$	$f(x_1) = 2$	$f(x_2) = -0.204$	$x_3 = 5.7407$
3.	$x_2 = 5.7143$	$x_3 = 5.7407$	$f(x_2) = -0.204$	$f(x_3) = -0.0072$	$x_4 = 5.7417$
4.	$x_3 = 5.7407$	$x_4 = 5.7417$	$f(x_3) = -0.0072$	$f(x_4) = 0.00032$	$x_5 = 5.7417$

∴ Ans: 5.7417

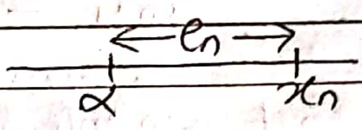
Convergence :-

from secant formula, we have,

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(x_{n-1})} \cdot (x_n - x_{n-1}) \quad \text{——— (i)}$$

Let α be the real root of $f(x)$ & e_{n+1}, e_n & e_{n-1} be the errors corresponding to the approximate values x_{n+1}, x_n & x_{n-1} respectively.

Then,
$$x_{n+1} = \alpha + e_{n+1}$$
$$x_n = \alpha + e_n$$
$$x_{n-1} = \alpha + e_{n-1}$$



With these values, eqn (i) becomes,

$$e_{n+1} = e_n - \frac{f(\alpha + e_n)}{f(\alpha + e_n) - f(\alpha + e_{n-1})} (e_n - e_{n-1})$$

$$\text{or, } e_{n+1} = \frac{e_{n-1} f(\alpha + e_n) - e_n f(\alpha + e_{n-1})}{f(\alpha + e_n) - f(\alpha + e_{n-1})}$$

Using Taylor's series: $f(x_0 + h) = f(x_0) + hf'(x_0) + \cdots$

$$e_{n+1} = \frac{e_{n-1}(\cancel{f(\alpha)} + e_n f'(\alpha) + \cdots) - e_n(\cancel{f(\alpha)} + e_{n-1} f'(\alpha) + \cdots)}{(\cancel{f(\alpha)} + e_n f'(\alpha) + \cdots) - (\cancel{f(\alpha)} + e_{n-1} f'(\alpha) + \cdots)}$$

$$\text{or, } e_{n+1} = \frac{e_{n-1}\left(e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \cdots\right) - e_n\left(e_{n-1} f'(\alpha) + \frac{e_{n-1}^2}{2!} f''(\alpha) + \cdots\right)}{e_n f'(\alpha) - e_{n-1} f'(\alpha)}$$

or, $e_{n+1} = \frac{e_n e_{n-1}}{2} \frac{(e_n f''(\alpha) - e_{n-1} f''(\alpha))}{e_n f'(\alpha) - e_{n-1} f'(\alpha)}$

or, $e_{n+1} = \frac{e_n e_{n-1}}{2} \frac{f''(\alpha)\ \cancel{(e_n - e_{n-1})}}{f'(\alpha)\ \cancel{(e_n - e_{n-1})}}$

or, $e_{n+1} = e_n \cdot e_{n-1} \cdot K$ ——— (ii)

$K = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)}$

Let p be the rate of convergence, then according to the definition of rate of convergence,

$$e_{n+1} = A e_n^p$$

$$e_n = A e_{n-1}^p \Rightarrow e_{n-1} = e_n^{1/p} A^{-1/p}$$

With all these values, eqn (ii) becomes,

$$A e_n^p = K.\ e_n .\ e_n^{1/p} .\ A^{-1/p}$$

or $A e_n^p = (K . A^{-1/p})\ e_n^{1+1/p}$

or, $p = 1 + 1/p$ $\left(\text{for } A = K \cdot A^{-1/p}\right)$

or, $p^2 - p - 1 = 0$ $\left(\text{use } p = \frac{a \pm \sqrt{b^2 - 4ac}}{2a}\right)$

Solving, we get $p = 1.618$ (Golden ratio).

Here, ~~B~~ Secant method has super-linear convergence. It is faster than Bisection method but slower than Newton-Raphson method.

Advantages:

It has faster rate of convergence than Bisection method.
It is not required to evaluate derivatives as in case of Newton-Raphson Method.
It requires only one functional evaluation per iteration.
Initial guesses can be taken freely (no Condition needed).

Disadvantages:

It needs two initial guesses.
It has no guarantee of convergence.
It is open-bracketed method and hence the initial guesses not necessarily bound the root.
Slower than Newton-Raphson method.
Since the method has no guarantee of convergence, maximum allowed iterations must be defined while programming.

Fixed Point Iteration Method :

It is a numerical method to find the approximate root of an algebraic or transcendental (non-linear) function. The method is based on fixed point theorem of algebra.

Let $f(x)$ be a non-linear function. We can rewrite the function as $x = g(x)$. Firstly we find the interval in which the root lies. Then we make an initial guess x_0.

The first approximation $(x_1) = g(x_0)$
Second approximation $(x_2) = g(x_1)$
. . . .

Generalizing, we get the iteration scheme as:

$$\boxed{x_{n+1} = g(x_n)}$$

The sequence of approximations $x_0, x_1, x_2, \cdots$, approaches the real root of $f(x)$.

Note: If the iteration scheme converges to the fixed point of $g(x)$ (which is the intersection point of $y = x$ & $y = g(x)$), the x-coordinate of the fixed point gives the root of $f(x)$.

Convergence :-

For the given non-linear function $f(x)$, we can make iteration function/scheme $g(x)$ in different ways. If $x = g_1(x)$, $x = g_2(x)$, ... are the different iteration schemes from which the fixed point x can be obtained. The x-coordinate of the fixed point of $f(x)$ gives the root of $f(x)$.

Convergence of the method depends on selection of initial guess and the iteration function $g(x)$. Following are the conditions for convergence of the method.

(a) The initial guess x_0 lies in the interval $[a,b]$ in which the real root of $f(x)$ lies and $f(x)$ is continuous over $[a,b]$.

(b) The iteration function $g(x)$ is choosen such that $|g'(x_0)| < 1$.

If we have two iteration functions $g_1(x)$ & $g_2(x)$ satisfying the condition (b), we must select one for which $g'(x_0)$ is smaller for faster convergence.

Date
Page

Let $f(x)$ be a non-linear function and r be the real root of $f(x)$. Let $x = g(x)$ be the iteration function and x_0 be the initial guess for the root, where $x_0 \in [a,b]$ and $[a,b]$ is the interval containing the root r.

Here, iteration function is

$$x_{n+1} = g(x_n) \quad \text{--- (1)}$$

If r is the root of $f(x)$ (ie. fixed point of $g(x)$),

$$r = g(r) \quad \text{--- (ii)}$$

Subtracting,

$$x_{n+1} - r = g(x_n) - g(r)$$

or, $$x_{n+1} - r = \frac{g(x_n) - g(r)}{x_n - r}(x_n - r)$$

Using mean value theorem,

or, $$x_{n+1} - r = g'(c)(x_n - r), \quad c \in (x_n, r)$$

or, $$\boxed{e_{n+1} = g'(c)\, e_n}$$

Here, we can see that convergence can be guaranteed only if $e_{n+1} < e_n$. For this $|g'(c)|$ must be less than 1. If the ~~value~~ condition $|g'(c)| < 1$ is satisfied, the rate of convergence is linear.

Date
Page

Algorithm

1. Start
2. Define the function $f(x)$, error (e).
3. Define the function $g(x)$ in $x = g(x)$ as iteration function.
4. Define the function $g'(x)$.
5. Input an initial guess (x_0)
6. Find ~~first~~ ~~ap~~ $|g'(x_0)|$.
7. If $|g'(x_0)| \geq 1$, go to step-13.
8. Compute the first approximation as $x_1 = g(x_0)$.
9. Compute the next approximation as $x_2 = g(x_1)$.
10. Compute the error as:

$$error = |x_2 - x_1|$$

11. If error $> e$, set $x_1 = x_2$ and go to Step-9.
12. Display x_2 (or x_1) as root.
13. Stop.

Advantages:-

It uses only one initial guess.
It is easy to program.
It has high accuracy.

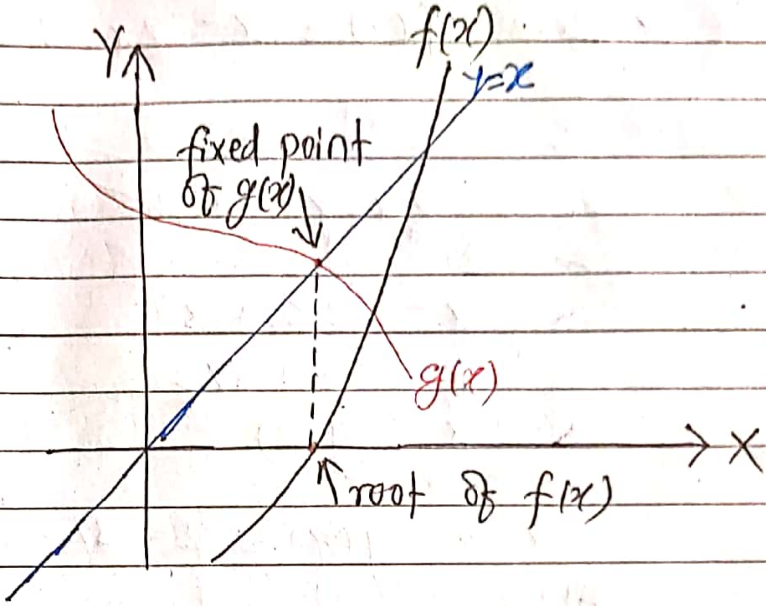
Disadvantages:-

The rate of convergence is slower & not guaranteed.
Selection of iteration function $g(x)$ is not simple.
Needs usually larger no. of iterations.
Convergence depends on choice of initial guess (x_0) & the ~~initial guess~~ iteration function $g(x)$.

Geometrical Meaning

Let $f(x)$ be a non-linear function.
$x = g(x)$ be the iteration function.

The root of $f(x)$ is the point on x-axis at which $f(x) = 0$.

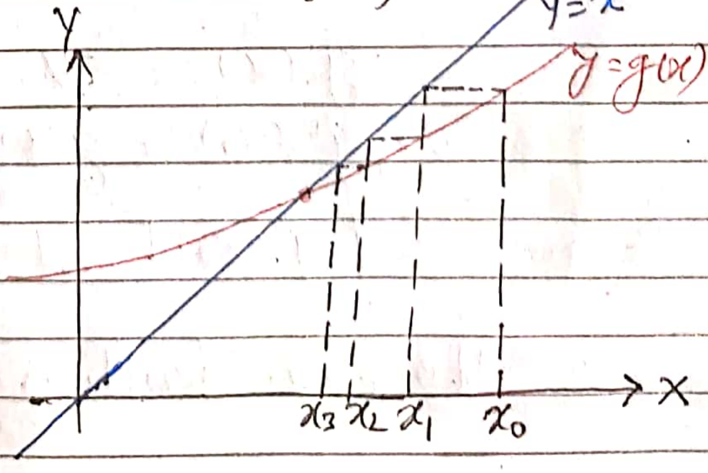


Fixed point of $g(x)$ is the point at which $x = g(x)$.

Fixed point of $g(x)$ is obtained by taking the intersection point between $y = x$ & $y = g(x)$.

The x-~~intercept~~ coordinate of the fixed point of $g(x)$ is same as the root of $f(x)$.

The fixed point iteration method can be shown geometrically as in the following diagram.



The method is all about finding the fixed point of $g(x)$.

CSIT-086

Q. ~~find~~ Given a function $f(x) = x^2 - 2x - 3$, rearrange the function in such a way that the iteration method converges to its real root. Also find the root of $f(x)$.

→ Here, $f(x) = x^2 - 2x - 3 = 0$

or, $x^2 = 2x + 3$

$x = 2 + 3/x$

$x = g(x)$

where $g(x) = 2 + 3/x$

Also,

$f(2) = -3$

$f(4) = 5$

root lies between 2 & 4.

Let initial guess be $x_0 = 2.5 \in (2,4)$.

$g'(x) = -3/x^2$

$g'(x_0) = g'(2.5) = -0.48$

$\therefore |g'(x_0)| < 1 \Rightarrow$ Convergence is guaranteed.

Iteration function is

$$\boxed{x_{n+1} = 2 + 3/x_n}$$

Tabulating the iterations,

n	x_n	x_{n+1}
0	$x_0 = 2.5$	$x_1 = 3.2$
1	$x_1 = 3.2$	$x_2 = 2.937$
2	$x_2 = 2.937$	$x_3 = 3.021$
3	$x_3 = 3.021$	$x_4 = 2.993$
4	$x_4 = 2.993$	$x_5 = 3.002$
5	$x_5 = 3.002$	$x_6 = 2.999$
6	$x_6 = 2.999$	$x_7 = 3.000$
7	$x_7 = 3.000$	$x_8 = 3.000$

$\therefore \boxed{x = 3}$ Ans