Unit-5

PAGE NO.:
DATE: / /

Solution of Ordinary Diff Equations

In many cases, the relationship between dependent & independent variables can be expressed in terms of normal algebraic equations. However, it is not always possible to model every problem using the usual algebraic expressions. Some complex scientific & engineering problems can be modeled by using the mathematical expressions involving the dependent & independent variables along with their derivatives of different order. Such equations are called ~~ord~~ differential equations. ⇒ what is a diff eqn.

An ordinary diff. eqn is a diff eqn containing one or more functions of an independent variable and the derivative of such functions.

The solution of an ordinary diff equation is a function or a number of functions that satisfy the given differential equation.

For eg: Consider a differential equation:

(a) $y' - x = 0$

→ The solution of this diff. eqn is a function $y=f(x)$ such that $\frac{dy}{dx} = x$

i.e. $y = \frac{1}{2}x^2 + C$ may be a possible solution.

(b) $y' - y = 0$

→ The solution is $y = ce^x$

as $\frac{d}{dx}(ce^x) - ce^x = 0$

∴ $0 = 0$ satisfied.

DATE: / /

Order & Degree of ODE:-

The highest order derivative present in an differential equation is called as the order of that diff. ODE.

Eg: $yy''' + 3(y'')^2 + x = 0 \Rightarrow$ order $= 3$

Similarly, the degree of an ODE is the power of the highest order derivative term.

Eg: Degree of above ODE is 1.

Other Examples:

ODE	Order	Degree
$y'' + (y')^3 + 5 = 0$	2	1
$x' + 3\sin t + 2t^2 = 0$	1	1

Note:-

- In any differential equation, the variable whose different order derivatives are present is called dependent variable. (जैसे y)

- The variable w.r.t. which the dependent variable is differentiated, is called independent variables. (जैसे x)

- A differential eqn containing only one independent variable with one or more dependent variables is called ODE.
- A diff. eqn containing more than one independent variable is called partial diff. eqns.

Initial Value Problem (IVP)

The solution of a diff. eqn gives a family of equations that satisfy the given diff equation. The solution involves an integration constant & hence doesn't give an exact solution.

Initial value problem is a problem of finding the exact solution of the given differential eqn based on the given initial condition.

Eg: Solve the given initial value problem.

$$\frac{dy}{dx} = x^2 + 2x \quad , \quad y(1) = 0$$

→ Here,

$$\frac{dy}{dx} = x^2 + 2x$$

or, $dy = x^2 dx + 2x\, dx$

Integrating both sides,

$$\int dy = \int x^2 dx + \int 2x\, dx$$

$$y = \frac{x^3}{3} + x^2 + c \quad \Big\} \text{ General sol}^n \text{ of ODE.}$$

from initial condition, $y(1) = 0$: $x = 1 \Rightarrow y = 0$

or, $0 = \frac{1}{3} + 1 + c$

or, $c = -4/3$

$\therefore$ Exact sol^n is $y = \frac{x^3}{3} + x^2 - \frac{4}{3}$ } particular sol^n of ODE.

DATE : / /

Q. Find the ~~initial~~ unique solution for the following initial value problem.

$$y' = 2y \quad , \quad y(0) = 3$$

→ Here $\frac{dy}{dx} = 2y$

or, $\frac{1}{2y}\frac{dy}{dx} = 1$

or, $\frac{1}{2y} dy = dx$

or, $\frac{1}{y} dy = 2\, dx$

Integrating both sides

$$\int \frac{1}{y} dy = \int 2\, dx$$

or, $\ln y = 2x + c$

or, $y = e^{2x+c}$

$= e^{2x} \cdot e^{c}$

$\therefore y = c\, e^{2x}$

From initial condition $y(0) = 3$ ie $(x = 0 \Rightarrow y = 3)$

or, $3 = c\, e^{0}$

$\Rightarrow c = 3$

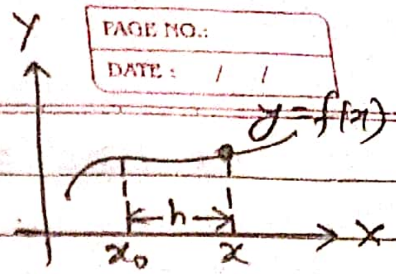
$\therefore y = 3e^{2x}$ is the exact solution.

Q. Solve the following initial value problem:

ukul $\frac{dy}{dx} = y^2(1+x^2)$, $y(0) = 1$.

$\therefore y = \frac{-1}{x + \frac{1}{3}x^3 - 1}$

$c = -1$.



Taylor's Series Method:-

Consider an initial value problem

$$y' = f(x, y) \quad , \quad y(x_0) = y_0 \quad \text{———} \quad (i)$$

Above initial value problem can be solved by using the Taylor's series expansion

$$y(x) = y_0 + (x-x_0)\, y_0' + \frac{(x-x_0)^2}{2!}\, y_0'' + \frac{(x-x_0)^3}{3!}\, y_0''' + \cdots \quad \text{———} \quad (ii)$$

If the values of $y_0, y_0', y_0'', \ldots$ are known, then the eq^n (ii) gives the solution of ~~bound~~ ininitial value problem (i); and we can easily calculate $y(x)$ at any given value of x.

Q. Using Taylor's series method, find the value of y at $x = 1.1$ (i.e. $y(1.1) = ?$). Given that

$$y' = xy - 1, \quad y(1) = 2$$

→ Given $y' = xy - 1$ for $x_0 = 1, y_0 = 2$

$$y_0' = x_0 y_0 - 1 = 1.2 - 1 = 1$$

$$y'' = xy' + y$$

$$y_0'' = x_0 y_0' + y_0 = 1 \times 1 + 2 = 3$$

$$y''' = xy'' + y' + y'$$

$$y_0''' = x_0 y_0'' + 2y_0' = 1.3 + 2.1 = 5$$

. . . .

DATE : / /

We have from Taylor's series:

$$y = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \dots$$

or, $y = 2 + (x-1).1 + \frac{(x-1)^2}{2}.3 + \frac{(x-1)^3}{6}.5 + \dots$

or, $y = 2 + x - 1 + \frac{3}{2}(x-1)^2 + \frac{5}{6}(x-1)^3$

or, $y = 2 + (x-1)\left(1 + \frac{3}{2}(x-1) + \frac{5}{6}(x-1)^2\right)$ ——— (i)

which is the numerical solution for the given ~~boundary~~ initial value problem.

Again using (i) for $x = 1.1$,

$$y = 2 + (1.1-1)\left(1 + \frac{3}{2}(1.1-1) + \frac{5}{6}(1.1-1)^2\right)$$

$= \cancel{2 + 0.1}$ (solution....)

$= 2.12$

SIT-066

Q. Find the value of $y(x)$ at point $x = 0.1$ & $x = 0.2$ for the O.D.E. $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$.

→ Here, $x_0 = 0$

$y_0 = 1$

$y' = x^2y - 1$

$y_0' = x_0^2y_0 - 1 = -1$

$y'' = x^2y' + 2xy$

$y_0'' = x_0^2y_0' + 2x_0y_0 = 0$

$y''' = x^2y'' + 2xy' + 2xy' + 2y$

$y_0''' = x_0^2y_0'' + 2x_0y_0' + 2x_0y_0' + 2y_0 = 2$

ukul

CamScanner

$$y^{iv} = x^2y''' + 2xy'' + 2xy'' + 2y' + 2xy'' + 2y' + 2y'$$
$$= x^2y''' + 6xy'' + 6y'$$
$$y_0^{iv} = x_0^2 y_0''' + 6x_0 y_0'' + 6y_0' = -6$$

From Taylor's Series

$$y = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0^{iv} + \cdots$$

or, $y = 1 + (x-0)(-1) + \frac{(x-0)^2}{2}\cdot 0 + \frac{(x-0)^3}{6}\cdot 2 + \frac{(x-0)^4}{24}(-6)$

or, $y = 1 - x + \frac{x^3}{3} - \frac{x^4}{4}$ ——— $(*)$

Here eqn $(*)$ is the numeric solution of the given O.D.E.

Again
at $x = 0.1$

$$y = 1 - 0.1 + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4}$$
$$= 0.90031$$

at $x = 0.2$

$$y = 1 - 0.2 + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4}$$
$$= 0.802$$

(See Taylor's Series method with improved accuracy).
This method uses interative method to find more accurate result.

CST-12 Q. Find the solution of following ODE using Taylor's series method $y' = (x^3 + xy^2)\, e^{-x}$ $y(0) = 1$ to find y at $x = 0.1, 0.2, 0.3$.

PAGE NO.:
DATE: / /

Picard Method:-

Consider an initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0$$

Here,

$$\frac{dy}{dx} = f(x,y)$$

or, $$dy = f(x,y)\,dx$$

Integrating

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

or, $$y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

or, $$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

We can rewrite above expression to determine y in successive iterations, as

$$\boxed{y_i = y_0 + \int_{x_0}^{x} f(x, y_{i-1})\,dx}$$

This is called Picard's successive approximation formula.

PAGE NO.:
DATE: / /

CST-067

Q. Obtain a solution upto the fifth approximation of the equation

$$\frac{dy}{dx} = y + x \quad , \quad y = 1 \text{ when } x = 0. \quad | \; y(0) = 1$$

using Picard's method. (If not mentioned, go upto 3rd approx). Also find the value (of y) at $x = 1$.

→ Here, $x_0 = 0$, $y_0 = 1$

$$f(x,y) = x + y$$

We have the picard's formula

$$y_i = y_0 + \int_{x_0}^{x} f(x, y_{i-1})\, dx$$

First approximation $(i=1)$

$$y_1 = y_0 + \int_{0}^{x} f(x, y_0)\, dx$$

$$= 1 + \int_{0}^{x} (x+1)\, dx$$

$$= 1 + \left[\frac{x^2}{2} + x\right]_0^x$$

$$= 1 + x + \frac{x^2}{2}$$

Second approximation $(i=2)$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$= 1 + \int_{0}^{x} \left(x + 1 + x + \frac{x^2}{2}\right) dx$$

$$= 1 + \int_{0}^{x} \left(1 + 2x + \frac{x^2}{2}\right) dx$$

Gurukul

DATE : / /

or $y_2 = 1 + \left[x + x^2 + \frac{x^3}{6}\right]_0^x$

$= 1 + x + x^2 + \frac{x^3}{6}$

Third approximation $(i=3)$

$y_3 = y_0 + \int_0^x f(x, y_2)\, dx$

$= 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{6}\right) dx$

$= 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{6}\right) dx$

$= 1 + \left[x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right]_0^x$

$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$

Fourth approximation $(i=4)$

$y_4 = y_0 + \int_0^x f(x, y_3)\, dx$

$= 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx$

$= 1 + \left[\frac{x^2}{2} + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right]_0^x$

$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$

Fifth approximation $(i=5)$

$y_5 = y_0 + \int_0^x f(x, y_4)\, dx$

ukul

or, $y_5 = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right) dx$

$= 1 + \left[\frac{x^2}{2} + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}\right]_0^x$

$\therefore y_5 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}$

Here y_5 is the required solution of given initial value problem (ODE) in 5th approximation.

at $x = 1$

$y = 1 + 1 + 1^2 + \frac{1^3}{3} + \frac{1^4}{12} + \frac{1^5}{60} + \frac{1^6}{720}$

$= 3 + \frac{1}{3} + \frac{1}{12} + \frac{1}{60} + \frac{1}{720}$

$= 3.435$

Euler's Method & It's Accuracy

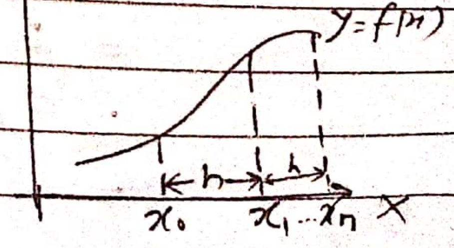
Consider the initial value problem

$\frac{dy}{dx} = f(x,y) \quad , y(x_0) = y_0$ ——— (i)

Let $y = f(x)$ be the solution of given ODE.

From Taylor's series,

$y = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$



Negleting higher power's of h $(h \to 0)$,

$y = y_0 + h y_0'$

or, $y = y_0 + h f(x_0, y_0)$

$\boxed{y_i = y_{i-1} + h f(x_{i-1}, y_{i-1})}$ This is Euler's formula to approximate $y(x)$.

Gurukul

Q. Solve the given IVP using Euler's method to find $y(1)$. Take $h = 0.2$.

$$y' = x + y \quad , \quad y(0) = 0.$$

→ Here,

$y' = x + y$

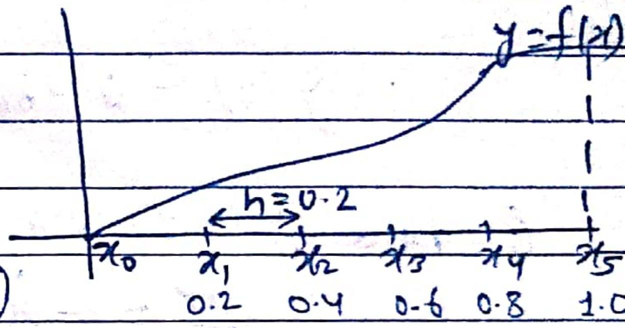
$f(x,y) = x + y$

$x_0 = 0 \quad , \quad y_0 = 0$

$x_n = 1 \quad\quad y_n = ? \quad$ (i.e. $y(1) = ?$)

$h = 0.2$

$$n = \frac{x_n - x_0}{h} = \frac{1-0}{0.2} = 5$$



from Euler's Formula,

$$y_i = y_{i-1} + h f(x_{i-1}, y_{i-1}) \quad \text{—— (i)}$$

i	x_0	y_0
0	0	0
1	0.2	0
2	0.4	0.04
3	0.6	0.128
4	0.8	
5	1	

For $i = 1$, $x = 0.2$

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= 0 + 0.2\,(x_0 + y_0)$$
$$= 0.2\,(0 + 0)$$
$$= 0$$

For $i = 2$, $x = 0.4$

$$y_2 = y_1 + h f(x_1, y_1)$$
$$= 0 + 0.2\,(x_1 + y_1)$$
$$= 0.2\,(0.2 + 0)$$
$$= 0.04$$

For $i = 3$, $x = 0.6$

$$y_3 = y_2 + h f(x_2, y_2)$$
$$= 0.04 + 0.2\,(0.4 + 0.04)$$
$$= 0.128$$

DATE: / /

For $i = 4$, $x = 0.8$

$$y_4 = y_3 + hf(x_3, y_3)$$
$$= 0.128 + 0.2(0.6 + 0.128)$$
$$= 0.2736$$

For $i = 5$, $x = 1$

$$y_5 = y_4 + hf(x_4, y_4)$$
$$= 0.2736 + 0.2(0.8 + 0.2736)$$
$$= 0.48832$$

$\therefore y(1) = 0.48832.$

using conventional method it is found
$y = e^x - x - 1$
$\therefore y(1) = 0.718$ Error $= |0.718 - 0.48|$

Q. Find $y(2.2)$ using Euler's method from the equation
$$\frac{dy}{dx} = -xy^2, \quad y(2) = 1.$$

Ans: $y(2.2) \simeq 0.69$

→ Solve it.

Note:-

- If h is not given in question, we can choose ourself. The result obtained will be more accurate if we choose h as small as possible.
- We can solve the given ODE using conventional (analytical) method & also check the error/accuracy.

Q. Solve the IVP $y' = x + y^2$, $y(0) = 0$ using Euler's method to find y when $x = 0.8$. Use step size 0.2.

Ans: 0.2432

Euler's Modified Method (Heun's Method)

Euler's method is found to be very slow and is less accurate if step size is large. A very small step size $(h \to 0)$ gives the accurate result. The solution would be correct only if the function is linear. To overcome such problems regarding the Euler's method, modified Euler's method is used.

It is not completely new concept, rather is a procedure that uses Euler's formula and modifies the result obtained by Euler's formula in each step.

Working procedure :-

1. Decide the interval & step size as in Eulers method.

$x_0 \quad x_1 \quad x_2 \cdots x_n$

2. Determine the value of y_i using Euler's formula

$$y_i = y_{i-1} + hf(x_{i-1}, y_{i-1}) \qquad (\text{put } i = 1)$$

3. Modify the value of y_i using Euler's Modified formula

$$y_i^m = y_{i-1} + \frac{h}{2}\left[f(x_{i-1}, y_{i-1}) + f(x_i, y_i^{m-1})\right]$$

4. Repeate step-3 for $m = 1, 2, 3, \ldots$ (practically don't repeat)

5. Repeat step-2, 3 & 4 for $i = 2, 3, \ldots, n$.

DATE : / /

Q. From the given ODE, determine y when $x = 0.1$.

$$y' = x^2 + y, \quad y(0) = 1$$

using modified Euler's method (Heun's method).

Take $h = 0.05$.

$\swarrow f(x, y)$

→ Here,

$$y' = x^2 + y$$

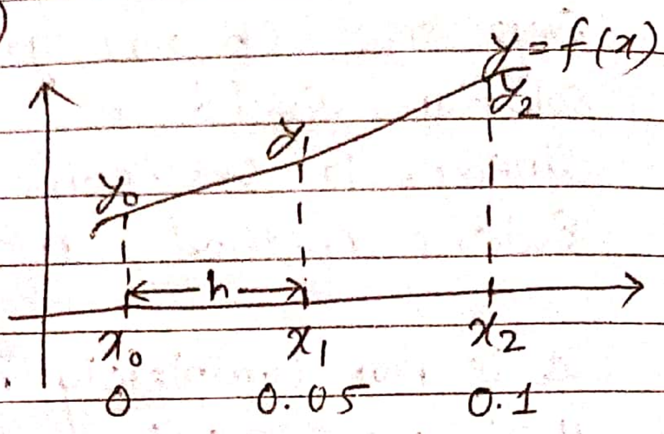
$$x_0 = 0$$

$$y_0 = 1$$

$$x_n = 0.1$$

$$y_n = ?$$

$$h = 0.05$$



$$n = \frac{x_n - x_0}{h} = \frac{0.1 - 0}{0.05} = 2$$

Using Euler's method

$$y_i = y_{i-1} + h f(x_{i-1}, y_{i-1}) \quad \text{——— (i)}$$

For $i = 1$,

$$y_1 = y_0 + h f(x_0, y_0)$$

$$= 1 + 0.05 (x_0^2 + y_0)$$

$$= 1 + 0.05 (0 + 1)$$

$$= 1.05 = y_1^0 \text{ (unmodified } y_1)$$

Now modifying y_1 using Euler's modified formula

$$y_i^m = y_{i-1} + \frac{h}{2}\left[f(x_{i-1}, y_{i-1}) + f(x_i, y_i^{m-1}) \right] \quad \text{——— (ii)}$$

put $i = 1$, $m = 1$

$$y_1^1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^0) \right]$$

$$= 1 + \frac{0.05}{2}\left[x_0^2 + y_0 + x_1^2 + y_1^0 \right]$$

DATE: / /

or, $y_1^1 = \cancel{y_0}\, 1 + \frac{0.05}{2}\left[0 + 1 + (0.05)^2 + 1.05\right]$

$= 1.0513125$

put $i=1$, $m=2$, (2^{nd} modification of y_1)

$y_1^2 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1')\right]$

$= 1 + \frac{0.05}{2}\left[x_0^2 + y_0 + x_1^2 + y_1'\right]$

$= 1 + 0.025\left(0 + 1 + (0.05)^2 + 1.0513125\right)$

$= 1.0513453$

$\therefore y_1 = 1.0513453$

For $i = 2$,

$y_2 = y_1 + h f(x_1, y_1)$

$= 1.0513453 + 0.05\left(x_1^2 + y_1\right)$

$= 1.0513453 + 0.05\left((0.05)^2 + 1.0513453\right)$

$= 1.104038 = y_2^0$ (unmodified y_2).

Modifying y_2 using Euler's modified formula
put $i=2$, $m=1$ in eq^n (ii)

$y_2^1 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^0)\right]$

ikul

or, $y_2^1 = 1.0513453 + 0.025\left[(0.05)^2 + 1.0513453 + (0.1)^2 + 1.104038\right]$

$= 1.10554$

put $i = 2$, $m = 2$ in eqn (ii), (second modification of y_2)

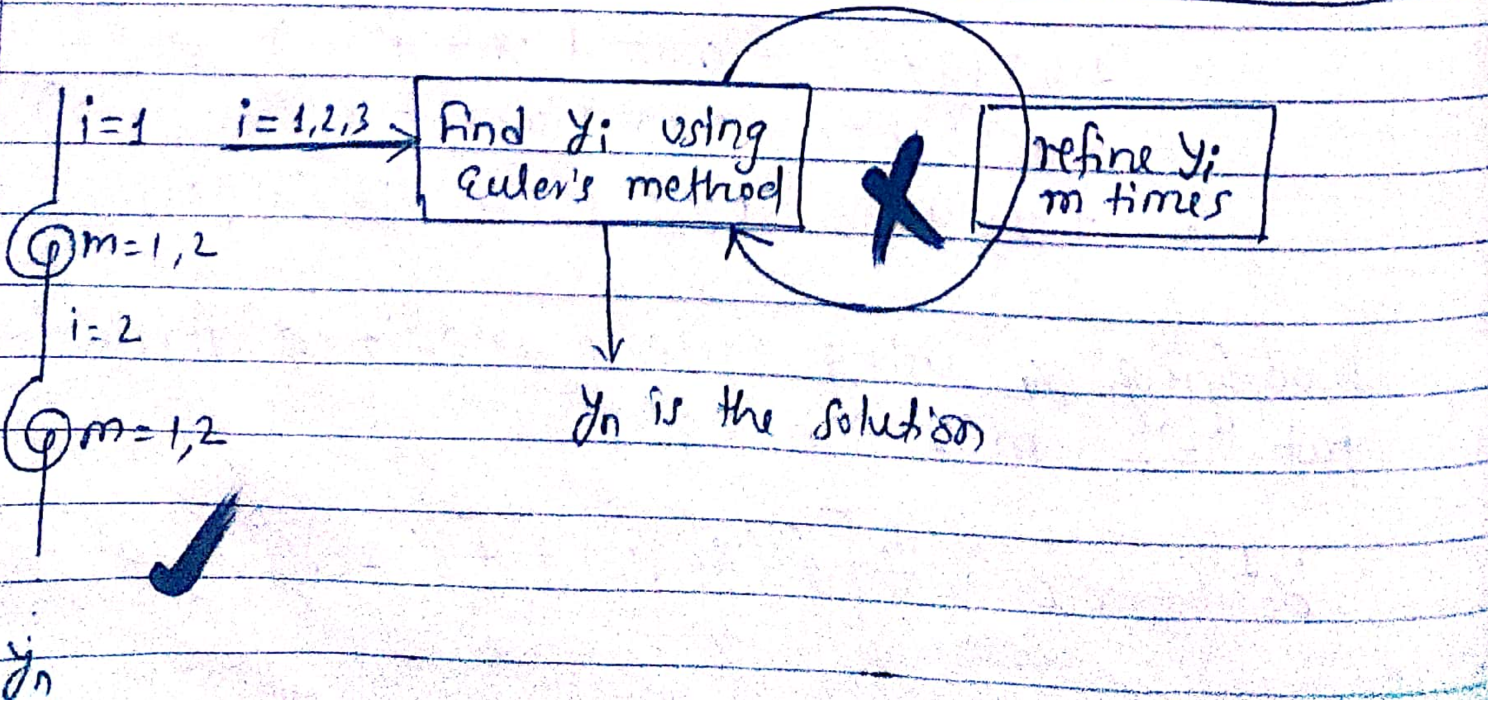
$$y_2^2 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^1)\right]$$

$$= 1.0513453 + 0.025\left[(0.05)^2 + 1.0513453 + (0.1)^2 + 1.10554\right]$$

$$= 1.10558$$

which is required solution. i.e. $y(0.1) = 1.10558$.

Note:- In exam we can neglet/remove/skip the second modification of y_1 & y_2.



DATE : / /

Runge-Kutta Method (Carl Runge, Martin Kutta)

It is a method of solving IVP in which was first proposed by C. Runge & later modified by M.W. Kutta.

1. Runge-Kutta Method of Order Two (RK2 - Method)

Consider an IVP (ODE) as

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$$

The given ODE is solved by using RK2 formula given by

$$\boxed{y_i = y_{i-1} + \frac{m_1 + m_2}{2}}$$

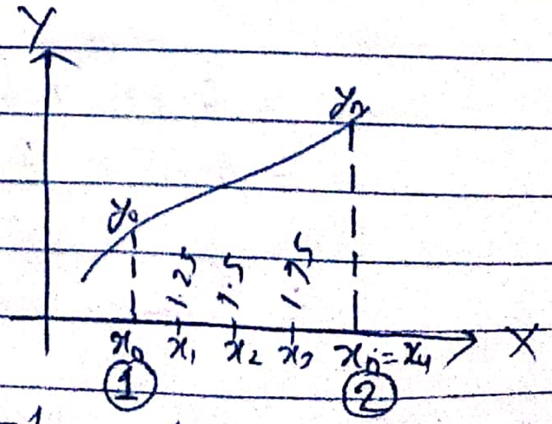
Where,

$$\cancel{m = \frac{1}{2}(m_1 + m_2)}$$

$$m_1 = h f(x_{i-1}, y_{i-1})$$

$$m_2 = h f(x_{i-1} + h, \; y_{i-1} + m_1) \quad , i = 1, 2, 3, \ldots, n$$

Q. Solve $y' = 2y/x$, $y(1) = 2$ for $x = 2$. Take $h = 0.25$.



→ Here, $y' = \frac{2y}{x}$

$f(x,y) = 2y/x$

$x_0 = 1$, $y_0 = 2$

$x_n = 2$, $y_n = ?$

$h = 0.25$, $n = \frac{x_n - x_0}{h} = \frac{2-1}{0.25} = 4$

PAGE NO.:
DATE: / /

Formula to be used in RK2 method is

$$y_i = y_{i-1} + \frac{m_1 + m_2}{2}$$

where, $m_1 = hf(x_{i-1}, y_{i-1})$

$m_2 = hf(x_{i-1} + h, y_{i-1} + m_1)$, $i = 1, 2, 3, 4$

For $i = 1$,

$$y_1 = y_0 + \frac{1}{2}(m_1 + m_2)$$

$$= \cancel{y_0 + \frac{1}{2}[hf(x_0, y_0) + hf(x_0+h, y_0+m_1)]}$$

$m_1 = hf(x_0 + y_0)$

$$= 0.25 \times \frac{2y_0}{x_0}$$

$$= 0.25 \times \frac{2 \times 2}{1}$$

$$= 1$$

$m_2 = hf(x_0 + h, y_0 + m_1)$

$$= 0.25\ f(1.25, 3)$$

$$= 0.25 \times \frac{2 \times 3}{1.25}$$

$$= 1.2$$

$\therefore y_1 = y_0 + \frac{1}{2}(m_1 + m_2)$

$$= 2 + \frac{1}{2} \times (1 + 1.2)$$

$$= 3.1$$

kul

For $i=2$

$$y_2 = y_1 + \frac{1}{2}(m_1 + m_2)$$

$$m_1 = hf(x_1, y_1)$$

$$= 0.25 \times \frac{2 \times y_1}{x_1} = 0.25 \times 2 \times \frac{3.1}{1.25} = 1.24$$

$$m_2 = hf(x_1 + h, y_1 + m_1)$$

$$= 0.25 \times f(1.5, 4.34)$$

$$= 0.25 \times 2 \times \frac{4.34}{1.5}$$

$$= 1.447$$

$$\therefore y_2 = y_1 + \frac{1}{2}(m_1 + m_2)$$

$$= 3.1 + \frac{1}{2}(1.24 + 1.447)$$

$$= 4.444$$

For $i=3$

$$y_3 = y_2 + \frac{1}{2}(m_1 + m_2)$$

$$m_1 = hf(x_2, y_2) = 0.25 \times 2 \times 4.444 / 1.5 = 1.481$$

$$m_2 = hf(x_2 + h, y_2 + m_1)$$

$$= 0.25 \times f(1.75, 5.925)$$

$$= 1.692$$

$$\therefore y_3 = y_2 + \frac{1}{2}(m_1 + m_2) = 4.444 + \frac{1}{2}(1.481 + 1.692) = 6.031$$

for $i = 4$

$$y_4 = y_3 + \frac{1}{2}(m_1 + m_2)$$

$$m_1 = hf(x_3, y_3) = 0.25 \times f(1.75, 6.031) = 0.25 \times \frac{2 \times 6.031}{1.75} = 1.72$$

$$m_2 = hf(x_3 + h, y_3 + m_1)$$
$$= 0.25 \times f(2, 7.754) = 0.25 \times \frac{2 \times 7.754}{2} = 1.939$$

$$\therefore y_4 = y_3 + \frac{1}{2}(m_1 + m_2)$$

$$= 6.031 + \frac{1}{2}(1.723 + 1.939)$$

$$= 7.862$$

$$\therefore y(2) = 7.862$$

Runge-Kutta Method of Order Four (RK-4 Method)

Consider a diff. eqn as

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

The given ODE problem can be solved by using RK-4 formula given by

$$y_i = y_{i-1} + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

Where,

$$K_1 = hf(x_{i-1}, y_{i-1})$$
$$K_2 = hf\left(x_{i-1} + \frac{h}{2}, y_{i-1} + \frac{K_1}{2}\right)$$

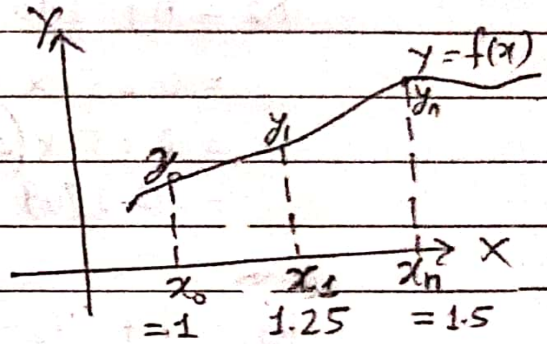
$$K_3 = hf\left(x_{i-1} + \frac{h}{2}, y_{i-1} + \frac{K_2}{2}\right)$$

$$K_4 = hf\left(x_{i-1} + h, y_{i-1} + K_3\right)$$

19/13

Q. Obtain $y(1.5)$ from given differential equation using Runge-kutta 4th order method.

$$\frac{dy}{dx} + 2x^2y = 1 \quad , \quad y(1) = 0 \, , \, h = 0.25$$



→ Here,

$$y' = 1 - 2x^2y$$
$$= f(x,y)$$

$x_0 = 1$, $y_0 = 0$, $x_n = 1.5$

$h = 0.25$, $n = \frac{x_n - x_0}{h} = \frac{1.5-1}{0.25} = 2$

We have from Runge-kutta method of order 4.

$$y_i = y_{i-1} + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

where,
$$K_1 = hf(x_{i-1}, y_{i-1})$$
$$K_2 = hf\left(x_{i-1} + \frac{h}{2}, y_{i-1} + \frac{K_1}{2}\right)$$
$$K_3 = hf\left(x_{i-1} + \frac{h}{2}, y_{i-1} + \frac{K_2}{2}\right)$$
$$K_4 = hf\left(x_{i-1} + h, y_{i-1} + K_3\right)$$

PAGE NO.:
DATE: / /

For i= 1

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$K_1 = hf(x_0, y_0)$$
$$= 0.25 \times (1 - 2 \times x_0^2 y_0)$$
$$= 0.25 \times (1 - 2 \times 1^2 \times 0)$$
$$= 0.25$$

$$K_2 = hf(x_0 + h/2, y_0 + \frac{K_1}{2})$$
$$= 0.25 \times f(1.125, 0.125)$$
$$= 0.25 \times (1 - 2 \times 1.125^2 \times 0.125)$$
$$= 0.1709$$

$$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$$
$$= 0.25 \times f(1.125, 0.0855)$$
$$= 0.25 \times (1 - 2 \times 1.125^2 \times 0.0855)$$
$$= 0.1959$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$
$$= 0.25 f(1.25, 0.1959)$$
$$= 0.25 \times (1 - 2 \times 1.25^2 \times 0.1959)$$
$$= 0.097$$

$$\therefore y_1 = 0 + \frac{1}{6}[0.25 + 2 \times 0.1709 + 2 \times 0.1959 + 0.097]$$
$$= 0.1801$$

For $i=2$

$$y_2 = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$K_1 = hf(x_1, y_1)$$
$$= 0.25 \times (1 - 2 \times 1.25^2 \times 0.1801)$$
$$= 0.3593$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right)$$

$$= 0.25 \times f(1.375, 0.3598)$$
$$= 0.25 \times (1 - 2 \times 1.375^2 \times 0.3598)$$
$$= 0.2247$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right)$$

$$= 0.25\, f(1.375, 0.2925)$$
$$= 0.25 \times (1 - 2 \times 1.375^2 \times 0.2925)$$
$$= 0.0265$$

$$K_4 = hf(x_1 + h, y_1 + K_3)$$
$$= 0.25 \times f(1.5, 0.2066)$$
$$= 0.25 \times (1 - 2 \times 1.5^2 \times 0.2066)$$
$$= 0.0176$$

$$y_2 = 0.1801 + \frac{1}{6}[0.3593 + 2 \times 0.2247 + 2 \times 0.0265 + 0.0176]$$

$$= 0.3267$$

$$\therefore y(1.5) = 0.3267$$

PAGE NO.:
DATE: / /

Solving System of Ordinary Differential Equations

Many complex scientific & engineering problems involve a number of ordinary differential equations. To solve such problems, we need to solve a no. of ordinary differential equations. If the system of ordinary differential equations is such that the individual equation can be solved separately, then such ~~equations~~ system of ODEs is called as uncoupled. If the system of ODEs such that the individual equation cannot be solved separately, then it is called coupled system of ODEs.

Eg. $y' = 2x + y - z$

$z' = y + z - 4x^2$ with $y(0) = 1$ & $z(0) = 0$

The above example shows a system of ODE which is coupled. Where; each ODE contains two dependent variables (y & z) & one independent variable (x). We cannot solve each equation separately.

Eg: $y' = 2x - x^2 + y$

$z' = z - 4x^2$ with $y(0) = 1$, $z(0) = 0$.

The above system of ODE is uncoupled, sinch each ODE can be solved separately.

PAGE NO.:
DATE: / /

The differential equations of the form

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots, y_m), \quad y_1(x_0) = y_{10}$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \dots, y_m), \quad y_2(x_0) = y_{20}$$

$$\dots\dots\dots\dots$$

$$\frac{dy_m}{dx} = f_m(x, y_1, y_2, \dots, y_m), \quad y_m(x_0) = y_{m0}$$

are called system of differential equations or simultaneous differential equations. These equations are solved for $y_1, y_2, \dots, y_m$ (all are functions of x) at point x

Solving the system of ODE's using Runge-Kutta Method of order-2.

Consider a system of ODEs as

$$y' = f(x, y, z), \quad y(x_0) = y_0$$

$$z' = g(x, y, z), \quad z(x_0) = z_0$$

We use the following RK-2 formula to solve the given system of ODE's.

$$\boxed{y_i = y_{i-1} + \frac{1}{2}(m_1 + m_2)} \qquad \boxed{z_i = z_{i-1} + \frac{1}{2}(K_1 + K_2)}$$

$$\boxed{m_1 = hf(x_{0_{i-1}}, y_{0_{i-1}}, z_{0_{i-1}})} \qquad \boxed{K_1 = hg(x_{0_{i-1}}, y_{0_{i-1}}, z_{0_{i-1}})}$$

$$\boxed{m_2 = hf(x_{0_{i-1}} + h, y_{0_{i-1}} + m_1, z_{0_{i-1}} + k_1)} \qquad \boxed{K_2 = hg(x_{0_{i-1}} + h, y_{0_{i-1}} + m_1, z_{0_{i-1}} + K_1)}$$

Q. Solve the following system of ODEs at $x = 0.2$ by using RK-2 method.

$$y' = x + y + z \qquad y(0) = 1$$
$$z' = 1 + y + z \qquad z(0) = -1$$

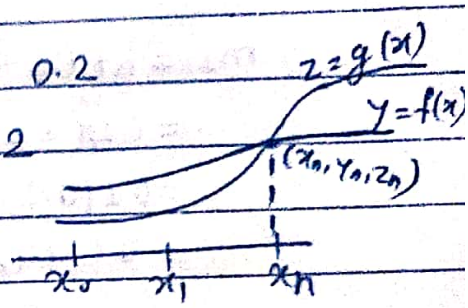
→ Let $y' = f(x, y, z) = x + y + z$

$z' = g(x, y, z) = 1 + y + z$

Here,

$x_0 = 0$, $y_0 = 1$, $z_0 = -1$, $x_n = 0.2$

Let $h = 0.1$ then $n = \dfrac{x_n - x_0}{h} = \dfrac{0.2 - 0}{0.1} = 2$



For $i = 1$

$y_1 = y_0 + \frac{1}{2}(m_1 + m_2)$

$z_1 = z_0 + \frac{1}{2}(k_1 + k_2)$

$m_1 = hf(x_0, y_0, z_0)$
$= 0.1\,(x_0 + y_0 + z_0)$
$= 0.1\,(0 + 1 - 1)$
$= 0$

$k_1 = hg(x_0, y_0, z_0)$
$= 0.1\,(1 + y_0 + z_0)$
$= 0.1$

$m_2 = hf(x_0 + h, y_0 + m_1, z_0 + k_1)$
$= 0.1\,(0.1 + 1 + (-0.9))$
$= 0.02$

$k_2 = hg(x_0 + h, y_0 + m_1 + z_0 + k_1)$
$= 0.1\,(1 + 1 - 0.9)$
$= 0.11$

$y_1 = 1 + \frac{1}{2}(0 + 0.02)$
$= 1.01$

$z_1 = -1 + \frac{1}{2}(0.1 + 0.11)$
$= -0.895$

For $i=2$

$$y_2 = y_1 + \frac{1}{2}(m_1 + m_2) \qquad z_2 = z_1 + \frac{1}{2}(k_1 + k_2)$$

$$m_1 = hf(x_1, y_1, z_1) \qquad k_1 = hg(x_1, y_1, z_1)$$
$$= 0.1(x_1 + y_1 + z_1) \qquad = 0.1(1 + y_1 + z_1)$$
$$= 0.1(0.1 + 1.01 + (-0.895)) \qquad = 0.1(1 + 1.01 - 0.895)$$
$$= 0.0215 \qquad = 0.1115$$

$$m_2 = hf(x_1 + h, y_1 + m_1, z_1 + k_1) \qquad k_2 = hg(x_1 + h, y_1 + m_1, z_1 + k_1)$$
$$= 0.1f(0.2, 1.0315, -0.7835) \qquad = hf(0.2, 1.0315, -0.7835)$$
$$= 0.1(0.2 + 1.0315 + (-0.7835)) \qquad = 0.1(1 + 1.0315 - 0.7835)$$
$$= 0.0448 \qquad = 0.1248$$

$$y_2 = 1.01 + \frac{1}{2}(0.0215 + 0.0448) \qquad z_2 = -0.895 + \frac{1}{2}(0.1115 + 0.1\ldots$$
$$= 1.04215 \qquad = -0.77685$$

$$y(0.2) = 1.04215$$
$$z(0.2) = -0.77685$$

Q. Solve the following simultaneous differential equations using RK-2 method at $x = 0.1$ and $x = 0.2$

$$y' = xz + 1, \quad y(0) = 0$$
$$z' = -xy, \quad z(0) = 1$$

Upto here, we studied different methods of solving first order IVP.

Ans: $y(0.1) = 0.105$
$y(0.2) = 0.2199$
$z(0.1) = 0.9995$
$z(0.2) = 0.9968$

Solution of Higher Order Differential Equations (IVP only)

A differential equation of order two or more is called higher order differential equation.

A second order differential equation (IVP) is of the form

$$y'' = f(x, y, y'), \quad y(x_0) = y_0 \quad \& \quad y'(x_0) = y_0'$$

To solve above equation, we introduce a new variable z as

$$z = y'$$

Then, the given differential equation of order 2 is reduced into a system of differential equations as

$$\cancel{z' = f(x, y, z)}$$

$$\cancel{z = y' = z}$$

$$y' = z$$
$$z' = f(x, y, z)$$

for example, Newton's 2nd law of motion is a second order differential equation.

$$f = m \cdot \frac{d^2x}{dt^2}$$

or, $$\frac{d^2x}{dt^2} = f/m$$

Let $y = \frac{dx}{dt}$, then, the above equation reduces to a system of ODE as

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = f/m$$

After reducing the given higher order differential equation into a system of equations of order 1, we can apply any known numerical methods (Eg: RK-2, ~~Euler's method~~ etc) to solve the system.

Working procedure:

1. Reduce the given higher order ODE into a system of differential equations of order 1.
2. Rearrange the initial conditions.
3. Apply RK-2 method to solve the system of ODEs. (Or you can apply normal Euler's method also).

Q. Solve the given differential equation at $x = 0.2$.

$$y'' = xy^2 - y^2, \quad y(0) = 1, \quad y'(0) = 0$$

→ Here, $y'' = xy^2 - y^2$ with $y(0) = 1$, $y'(0) = 0$

Let $y' = z \Rightarrow y'' = z'$

then the given eqn reduces into

$$\left.\begin{aligned} y' &= z, & y(0) &= 1 \\ z' &= xy^2 - y^2, & z(0) &= 0 \end{aligned}\right\} \text{System of ODE.}$$

Now, we can apply RK-2 method to solve the above system of ODE.

The RK-2 formula is,

$$\boxed{y_i = y_{i-1} + \frac{1}{2}(m_1 + m_2)} \qquad \boxed{z_i = z_{i-1} + \frac{1}{2}(k_1 + k_2)}$$

$m_1 = hf(x_{i-1}, y_{i-1}, z_{i-1})$ $\qquad$ $k_1 = hg(x_{i-1}, y_{i-1}, z_{i-1})$

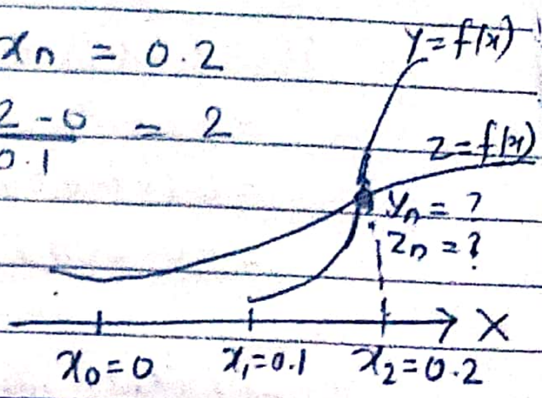
$m_2 = hf(x_{i-1} + h, y_{i-1} + m_1, z_{i-1} + k_1)$ $\qquad$ $k_2 = hg(x_{i-1} + h, y_{i-1} + m_1, z_{i-1} + k_1)$

Let, $y' = f(x,y,z) = z$. $y(0) = 1$.

$z' = g(x,y,z) = xy^2 - y^2$ $z(0) = 0$

Here, $x_0 = 0$, $y_0 = 1$, $z_0 = 0$, $x_n = 0.2$

Let $h = 0.1 \Rightarrow n = \frac{x_n - x_0}{h} = \frac{0.2-0}{0.1} = 2$



For i=1

$y_1 = y_0 + \frac{1}{2}(m_1 + m_2)$ $z_1 = z_0 + \frac{1}{2}(k_1 + k_2)$

$m_1 = hf(x_0, y_0, z_0)$
$= 0.1 f(0,1,0)$
$= 0.1 \times 0$
$= 0$

$k_1 = hg(x_0, y_0, z_0)$
$= 0.1\, g(0,1,0)$
$= 0.1 \times (0 \times 1^2 - 1^2)$
$= -0.1$

$m_2 = hf(x_0 + h, y_0 + m_1, z_0 + k_1)$
$= 0.1 f(0.1, 1, -0.1)$
$= 0.1 \times (-0.1)$
$= -0.01$

$k_2 = hg(x_0 + h, y_0 + m_1, z_0 + k_1)$
$= 0.1\, g(0.1, 1, -0.1)$
$= 0.1 \times (0.1 \times 1^2 - 1^2)$
$= -0.09$

$y_1 = 1 + \frac{1}{2}(0 - 0.01)$
$= 0.995$

$z_1 = 0 + \frac{1}{2}(-0.1 - 0.09)$
$= -0.095$

(Note: we can put $h = 0.2$ to get answer in $i=1$ only).
(It is recomended to do in one step in exam as in question $x_0 = 0$ & $x_n = 0.2$, we can put $h = 0.2$)

DATE : / /

For $i = 2$

$y_2 = y_1 + \frac{1}{2}(m_1 + m_2)$

$z_2 = z_1 + \frac{1}{2}(k_1 + k_2)$

$m_1 = hf(x_1, y_1, z_1)$
$= 0.1 \times z_1$
$= 0.1 \times (-0.095)$
$= -0.0095$

$k_1 = hg(x_1, y_1, z_1)$
$= 0.1 \times (x_1 y_1^2 - y_1^2)$
$= 0.1 \times (0.1 \times 0.995^2 - 0.995^2)$
$= -0.089$

$m_2 = hf(x_1 + h, y_1 + m_1, z_1 + k_1)$
$= 0.1 \times (z_1 + k_1)$
$= 0.1 \times (-0.184)$
$= -0.0184$

$k_2 = hg(x_1 + h, y_1 + m_1, z_1 + k_1)$
$= 0.1 \times g(0.1, 0.9855, -0.18$
$= 0.1 \times (0.1 \times 0.9855^2 - 0.9855^2)$
$= -0.0874$

$y_2 = 0.995 + \frac{1}{2}(-0.0095 - 0.0184)$
$= 0.995 - 0.01395$
$= 0.98105$

$z_2 = -0.095 + \frac{1}{2}(-0.089 - 0.0874)$
$= -0.1832$

$\therefore y(0.2) = 0.98105$ Ans

IT 069

Q. How can you solve higher order differential equation? Solve the following differential equation within $0 \le x \le 1$ using Heun's method (Or RK-2 method).

$$y'' + 3y' + 2xy = 1 \quad \text{with} \quad y(0) = 1 \quad \& \quad y'(0) = 1.$$

Take $h = 0.5$.

PAGE NO.:
DATE: / /

Boundary Value Problem (BVP)

The differential equation along with boundary conditions is called boundary value problem. The boundary conditions are the known values of dependent variables at different (usually two) ~~given~~ values of independent variable.

For example: Following is a boundary value problem

$$y'' = y' + 2xy \quad , y(0) = 0 \quad y'(1) = 12$$

A boundary value problem usually is of the following form

$$y'' = f(x, y, y')$$

with boundary conditions $y(a) = y_a$ & $y(b) = y_b$.
The problem is to find the value of y in the range $a \leq x \leq b$.

Two popular methods of solving boundary value problem are:

1. Shooting Method
2. Finite difference Method.

Note: BVP is the problem of solving second order differential equations with given boundary conditions.

Shooting Method:-

It is a popular numerical method of solving boundary value problem. In this method, the given BVP is transformed into an equivalent initial value problem and then we can apply any of the methods to solve the IVP (such as Euler's method, RK-2 method etc)

Working procedure:

Given BVP is of the form

$$y'' = f(x, y, y') \text{ with } y(x_0) = y_0, \; y(x_n) = y_n$$

1. Reduce the BVP into IVP system as
$$y' = f(x, y, z) \quad \text{with } y(x_0) = y_0$$
$$z' = g(x, y, z) \quad \text{with } z(x_0) = z_0$$
2. Guess the value of z_0 and solve the IVP system to check whether the given boundary condition is satisfied or not.
3. Repeat step-2 ~~until~~ with different guess untill the boundary condition is satisfied.
4. Present the values $y(x_0), y(x_1), \ldots, y(x_n)$ as solution of the given B.V.P.

IT 070 0.

Q. Solve the following boundary value problem using shooting method.

$$\frac{d^2y}{dx^2} - 2x^2 y = 1, \text{ with } y(0) = 1 \;\&\; y(1) = 1$$

Take $h = 0.5$.

kul

Here,

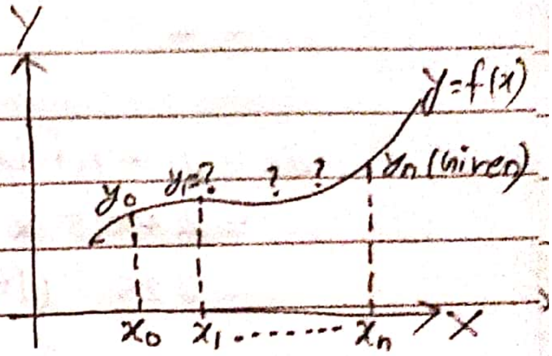
$$y'' = 2x^2y + 1 \quad \text{with} \quad y(0) = 1 \ \& \ y(1) = 1$$

$x_0 = 0$, $y_0 = 1$, (Initial Condition)

$x_n = 1$, $y_n = 1$ (Boundary condition)

$h = 0.5$

$$n = \frac{x_n - x_0}{h} = \frac{1-0}{0.5} = 2$$



let $y' = z$

or. $y'' = z'$

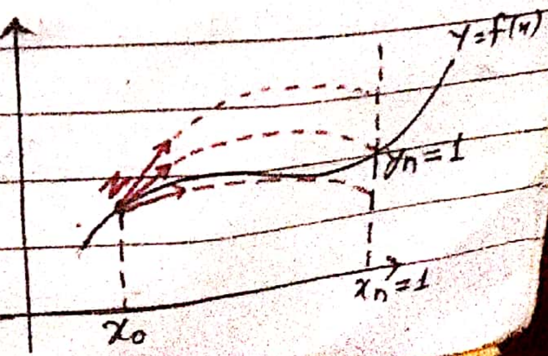
The given BVP can be transformed into the following IVP

$$\left.\begin{aligned} y' &= z \quad \text{with} \quad y(0) = 1 \\ &= f(x,y,z) \\ z' &= 2x^2y + 1 \quad \text{with} \quad z(0) = G \ (\text{Some guess value}) \\ &= g(x,y,z) \end{aligned}\right\} \text{-----} (*)$$

We have to decide the value of G by hit & trial such that the boundary condition $y(1) = 1$ is satisfied.

Now, make a guess for G and check whether boundary condition is satisfied or not (using Euler's method for simplicity).

(You may use RK-2 method as well).



let $G_1 = 2$ (initial guess for $z(0)$).

Now, solving eqn (*) by using Euler's method for $y(1)$

$$y_1 = y_0 + hf(x_0, y_0, z_0) \qquad z_1 = z_0 + hg(x_0, y_0, z_0)$$
$$= 1 + 0.5 \times 2 \qquad = 2 + 0.5 \times (2 \times 0^2 + 1)$$
$$= 2 \qquad = 2.5$$

$$y_2 = y_1 + hf(x_1, y_1, z_1)$$
$$= 2 + 0.5 \times 2.5$$
$$= 3.25$$ (larger than given boundary condition)

let $G_2 = -2$ (Another guess for $z(0)$)

Again using Euler's method,

$$y_1 = y_0 + hf(x_0, y_0, z_0) \qquad z_1 = z_0 + hg(x_0, y_0, z_0)$$
$$= 1 + 0.5 \times (-2) \qquad = -2 * (0.5 \times (2 \times 0^2 + 1))$$
$$= 0 \qquad = -1.5$$

$$y_2 = y_1 + hf(x_1, y_1, z_1)$$
$$= 0 + 0.5 \times (-1.5)$$
$$= -0.75$$ (Smaller than given boundary condition)

Let us estimate the third guess as

$$G_3 = G_2 - \frac{V_2 - V}{V_2 - V_1}(G_2 - G_1)$$

$$= -2 - \frac{-0.75 - V}{-0.75 - 3.25}(-2 - 2)$$

$$= -0.25$$

PAGE NO.:
DATE : / /

Again using Euler's method

$y_1 = y_0 + hf(x_0, y_0, z_0)$
$= 1 + 0.5 \times (-0.25)$
$= 0.875$

$z_1 = z_0 + hg(x_0, y_0, z_0)$
$= -0.25 + 0.5(2 \times 0^2 + 1)$
$= 0.25$

$y_2 = y_1 + hf(x_1, y_1, z_1)$
$= 0.875 + 0.5 \times 0.25$
$= 1$ (Satisfied the boundary condition).

Hence the solution ~~is~~ of given boundary value problem in the range $[0, 1]$ with $h = 0.5$ is

$y(0) = 1$
$y(0.5) = 0.875$
$y(1) = 1$

Q. Solve the given BVP using shooting method (take h yourself)

(i) $y'' = 6x + 4$ with $y(0) = 2$ & $y(1) = 5$

Ans:

(ii) $y'' = 12x^2$ with $y(1) = 2$ & $y(2) = 17$

Ans:

(iii) $y'' - 2y = 8x(9 - x)$ with $h = 3$, & $y(0) = 0$, $y(9) = 0$.

Ans: $y(0) = 0$, $y(3) = -61.7$, $y(6) = -123.42$, $y(9) = 0$

Gurukul