Chapter-6

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Partial Differential Equations (PDE)

A differential equation involving more than one independent variables is called partial differential equation. A partial differential equation includes the partial derivatives of various order.

Let u denote a dependent variable and x, y denote the independent variables then the general form of a second order P.D.E. is of the form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = F(x,y)$$

Where A, B & C are the functions of x & y & $D = f(x, y, u, \partial u/\partial x, \partial u/\partial y)$.

The PDE is ~~parabolic~~ elliptical, parabolic or hyperbolic based on the value of B^2-4AC.

Condition		Equation
$B^2-4AC<0$	$\Rightarrow$	Elliptic Equation (Eg Laplace Eqn)
$B^2-4AC=0$	$\Rightarrow$	Parabolic Equation (Eg. Heat eqn in 1D)
$B^2-4AC>0$	$\Rightarrow$	Hyperbolic Equation (Eg. Wave eqn)

Homogeneous & Non-homogeneous PDE:-

A partial diff. equation is said to be homogeneous if all of it's terms contains the dependent variable (u) or its partial derivatives (i.e. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ & $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ or $\frac{\partial^2 u}{\partial x \partial y}$ etc).

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For example:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} = 0 \Rightarrow \text{Homogeneous}$$

$$\frac{\partial^2 u}{\partial x^2} + 2xy + \frac{\partial^2 u}{\partial y^2} = x^2 y \Rightarrow \text{Non-Homogeneous}$$

Linear & Non-linear PDE

A PDE is linear if the ~~unknown function~~ dependent variable u & it's derivatives has power 1. The PDE is non-linear if the ~~unknown function~~ dependent variable & it's derivatives have quadratic or higher powers. The PDE is non-linear if it includes non-linear functions such as trigonometric ratios, exponential terms, logarithms etc.

Eg:

$$\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0 \Rightarrow \text{Linear}$$

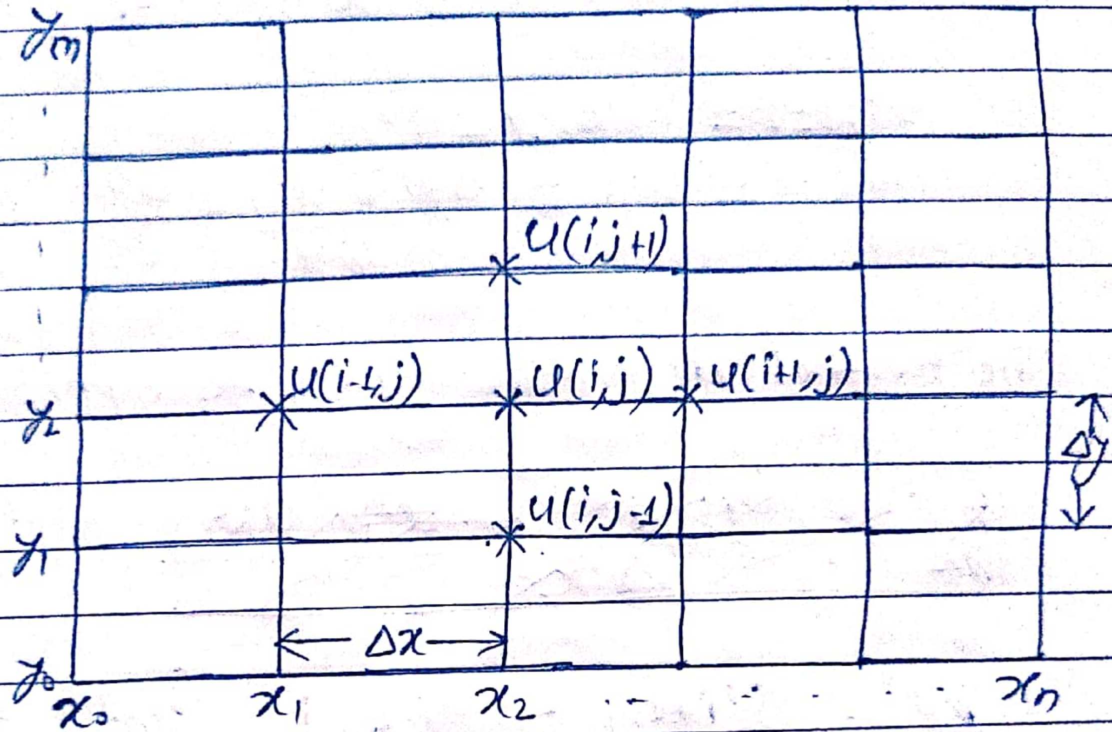
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \text{Linear}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} = 5u\frac{\partial u}{\partial y} \Rightarrow \text{Non linear}$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + 3x\frac{\partial u}{\partial y} + 4\frac{\partial u}{\partial y} = 0 \Rightarrow \text{Non linear.}$$

Difference Equations:- (Numerical Soln of PDE)

consider a rectangular domain D as follows



~~Here, $x_i = i \times \Delta x$ and $y_j = j * \Delta y$~~

The domain is splitted into a rectangular grid of width Δx & height Δy. The point of intersectio ~~pu~~ can be written as

~~$P_{i,j} = P(x_i, y_j)$~~ $u_{i,j} = u(x_i, y_j)$

where $x_i = x_0 + i \times \Delta x$

$y_j = y_0 + j \times \Delta y$

If $u(x,y)$ has continuous derivative in the domain

then,

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$$u_{i\pm1,j} = u(x \pm \Delta x, y)$$
$$= u_{i,j} \pm \Delta x \frac{\partial^2 u_{i,j}}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 u_{i,j}}{\partial x^2} \pm \cdots$$

$$u_{i,j\pm1} = u(x, y \pm \Delta y)$$
$$= u_{i,j} \pm \Delta y \frac{\partial^2 u_{i,j}}{\partial y} + \frac{\Delta y^2}{2!} \frac{\partial^2 u_{i,j}}{\partial y^2} \pm \cdots$$

the partial derivatives can be approximated as,

$$\cancel{\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{\Delta x}}$$

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} \quad \text{(central difference formula)}$$

$$= \frac{u_{i+1,j} - u_{i,j}}{\Delta x} \quad \text{(forward difference formula)}$$

$$= \frac{u_{i,j} - u_{i-1,j}}{\Delta x} \quad \text{(Backward difference formula)}$$

Assume $\Delta x = h$ & $\Delta y = k$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} \quad \text{(central diff.)}$$

$$= \frac{u_{i,j+1} - u_{i,j}}{\Delta y} \quad \text{(forward diff)}$$

$$= \frac{u_{i,j} - u_{i,j-1}}{\Delta y} \quad \text{(Backward diff)}$$

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Similarly

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \\[2ex] \frac{\partial^2 u}{\partial y^2} &= \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}\end{aligned}\right\} \text{used in laplace eqn!}$$

Making use of above approximations, the PDE are converted into difference equations and the resultant system of equations are solved using any method. Since, analytical solution of PDE depends on the type of PDE, the numerical solution also depend on the type of PDE.

~~Laplacian Equation~~

not needed

Geometric Meaning of partial differentiation:-

let $u = f(x,y)$. Then, the partial derivatives of u w.r.t. x & y are:

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}$$

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Laplacian Equation

A partial diff. eqn of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——— (i)}$$

is called as Laplaces equation. It is an standard equation that corresponds to many physical situations such as

- steady state temperature destribution
- steady state stress destribution
- steady state potential destribution
- steady state diffusion problem etc.

Solution of Laplace's Equation

From the difference equations, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}$$

Let $\Delta x = \Delta y = h$, then the laplace equation can be written as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

or, $\dfrac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = 0$

or, $\boxed{u_{i,j} = \dfrac{u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}}{4}}$

Which is a difference equation to represent the Laplace's equation. This can be used to solve the Laplace's equation numerically.

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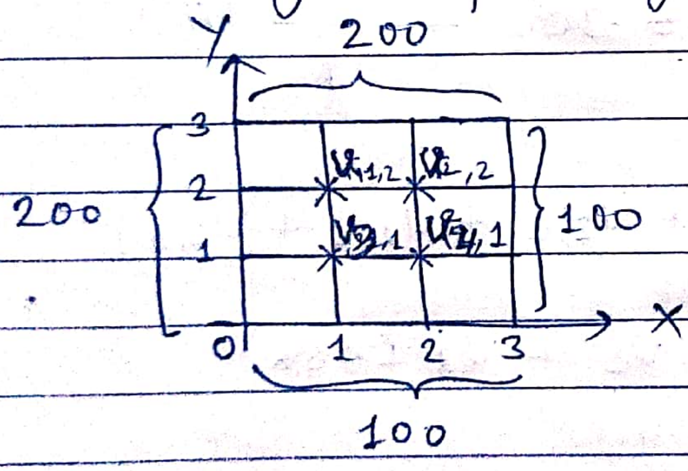
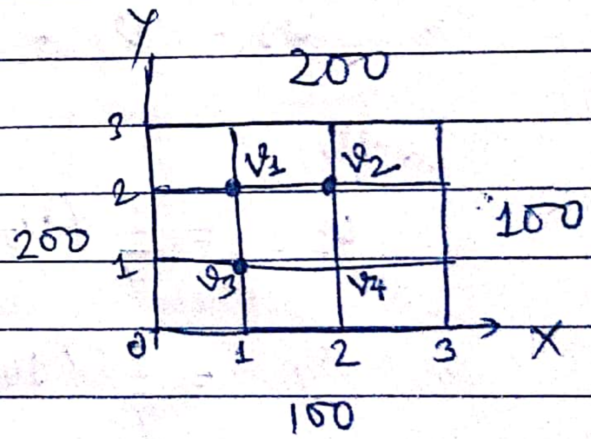
Q. Define a difference eqn to represent a laplace eqn. Solve the following Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ within $0 \le x \le 3$, $0 \le y \le 3$ for the rectangular plate given as





→ From finite difference equation,

$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}\right)$$

$$u_{1,2} = \frac{1}{4}\left(u_{2,2} + u_{1,3} + u_{0,2} + u_{1,1}\right)$$

or,

$$4v_1 - v_2 - 200 - 200 - v_3 = 0$$

or, $4v_1 - v_2 - v_3 = 400$ ———— (i)

Similarly,

$$u_{2,2} = \frac{1}{4}\left(u_{3,2} + u_{2,3} + u_{1,2} + u_{2,1}\right)$$

or, $$v_2 = \frac{1}{4}\left(100 + 200 + v_1 + v_4\right)$$

or, $$4v_2 - v_1 - v_4 = 300$$

or, $$v_1 - 4v_2 + v_4 = -300 \quad \text{--- (ii)}$$

Again,

$$u_{1,1} = \frac{1}{4}\left(u_{2,1} + u_{1,2} + u_{0,1} + u_{1,0}\right)$$

or, $$v_3 = \frac{1}{4}\left(v_4 + v_1 + 200 + 100\right)$$

or, $$4v_3 - v_4 - v_1 = 300$$

or, $$v_1 - 4v_3 + v_4 = -300 \quad \text{--- (iii)}$$

lastly,

$$u_{2,1} = \frac{1}{4}\left(u_{3,1} + u_{2,2} + u_{1,1} + u_{2,0}\right)$$

or, $$v_4 = \frac{1}{4}\left(100 + v_2 + v_3 + 100\right)$$

or, $$4v_4 - v_2 - v_3 = 200$$

or, $$v_2 + v_3 - 4v_4 = -200 \quad \text{--- (iv)}$$

Representing equations i, ii, iii & iv in the form of augmented matrix,

$$\left[\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 400 \\ 1 & -4 & 0 & 1 & -300 \\ 1 & 0 & -4 & 1 & -300 \\ 0 & -1 & -1 & 4 & -200 \end{array}\right]$$

Solving by using Gaussian ellimination method, we get

$\vartheta_1 = 158.33$

$\vartheta_2 = 116.67$

$\vartheta_3 = 116.67$

$\vartheta_4 = 8.33$

OST. 068

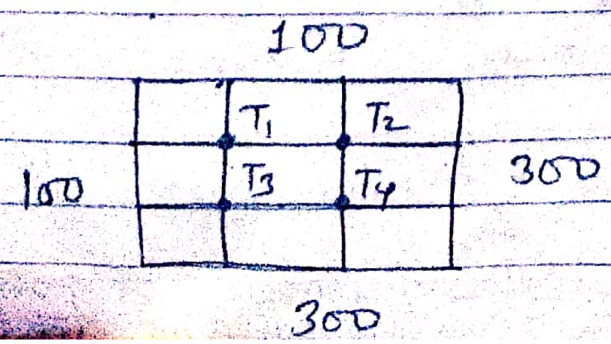
Q. How can you solve Laplace's equation? Explain. The steady state two dimensional heat flow in a metal plate is defined by $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$

A steel plate of size 30 x 30 cm is given. Two adjecent sides are at 100°c & other sides held at 0°c. Find the temperature at interior points, assuming the grid size of 10 x 10 cm.

OST. 069

Q. How can you obtain numerical solution of a PDE? Explain. The steady state two dimensional heat flow in a metal plate is defined by $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$

Given the boundary conditions as shown in the figure, find the temp at T_1, T_2, T_3 & T_4.



Poisson's Equation

A P.D.E. of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x,y) \quad \left[\text{or } \nabla^2 u = g(x,y) \text{ where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]$$

is known as poisson's Equation. It is more general form of Laplace eqn.

Solution of ~~Laplace~~ Poisson's Equation

The second order partial derivative of $u = f(x,y)$ can be expressed by difference formula as

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \quad \&$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}$$

let $\Delta x = \Delta y = h$. Then poisson's eqn becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x,y)$$

or, $$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = g(x,y)$$

or, $$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 g(x,y)$$

$$\therefore \boxed{u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 g(x,y)}{4}}$$

which is the ~~difference formula for Poisson's eqn~~ that can be used to solve the poisson's eqn numerically.

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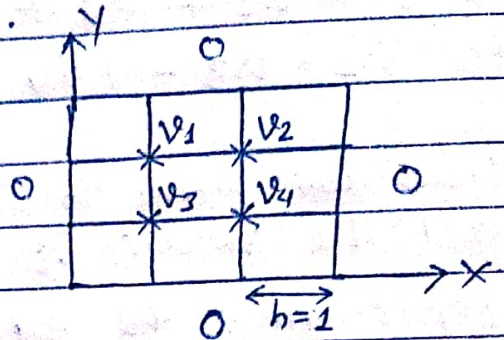
SCT Q. Derive a difference equation to represent a poisson's equation. Solve the poisson's equation $\nabla^2 u = 2x^2y^2$ over the domain $0 \le x \le 3$, $0 \le y \le 3$ with $u = 0$ at the boundary and $h = 1$.



→ The finite difference formula (or 5-points formula) for poisson's eqn is,

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = h^2 g(x,y) \quad \text{——— (i)}$$

For $v_1 = u_{1,2}$ (ie. $i=1, j=2$)

$$u_{2,2} + u_{1,3} + u_{0,2} + u_{1,1} - 4u_{1,2} = 1^2 . (2x^2y^2)$$

or, $v_2 + 0 + 0 + v_3 - 4v_1 = 1.(2\times1^2\times2^2)$

or, $-4v_1 + v_2 + v_3 = 8$ ——— (ii)

For $v_2 = u_{2,2}$ (ie. $i=2, j=2$)

$$u_{3,2} + u_{2,3} + u_{1,2} + u_{2,1} - 4u_{2,2} = 1^2.(2x^2y^2)$$

or, $0 + 0 + v_1 + v_4 - 4v_2 = 1.(2\times2^2\times2^2)$

or, $v_1 - 4v_2 + v_4 = 32$ ——— (iii)

For $v_3 = u_{11}$ (ie. $i=1, j=1$)

$$u_{2,1} + u_{1,2} + u_{0,1} + u_{1,0} - 4u_{1,1} = 1^2. (2x^2y^2)$$

or, $v_4 + v_1 + 0 + 0 - 4v_3 = 1.(2\times1^2\times1^2)$

or, $v_1 - 4v_3 + v_4 = 2$ ——— (iv)

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For $v_4 = u_{2,1}$ (ie. $i=2, j=1$)

$u_{3,1} + u_{2,2} + u_{1,1} + u_{2,0} - 4u_{2,1} = 1^2 (2x^2y^2)$

or, $0 + v_2 + v_3 + 0 - 4v_4 = 1.(2 \times 2^2 \times 1^2)$

or, $v_2 + v_3 - 4v_4 = 8$ ——— (v)

Now, we have to solve eqns (ii), (iii) (iv) & (v) to get the values of v_1, v_2, v_3 & v_4.

Representing the equations in augmented matrix,

$$\left[\begin{array}{cccc:c} -4 & 1 & 1 & 0 & 8 \\ 1 & -4 & 0 & 1 & 32 \\ 1 & 0 & -4 & 1 & 2 \\ 0 & 1 & 1 & -4 & 8 \end{array}\right]$$

Solving by Gaussian elimination method, we get

$v_1 = -5.5$

$v_2 = -10.75$

$v_3 = -3.25$

$v_4 = -5.5$

IT-073

Q. Write the finite difference formula for solving Poisson's equation. Hence Solve the poisson's equation $\nabla^2 u = 3x^2y$ over the domain $0 \le x \le 1.5$ & $0 \le y \le 3$ with $u=0$ on the boundary and $h = 0.5$.

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