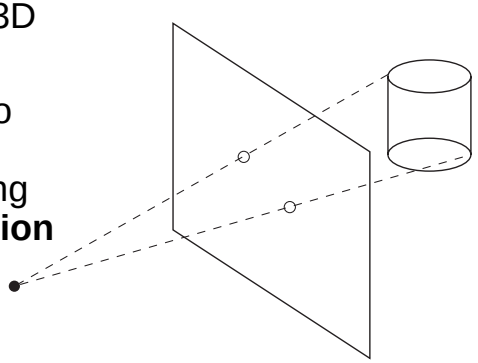
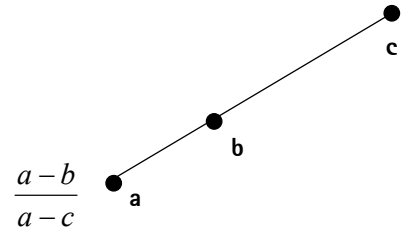
Projections

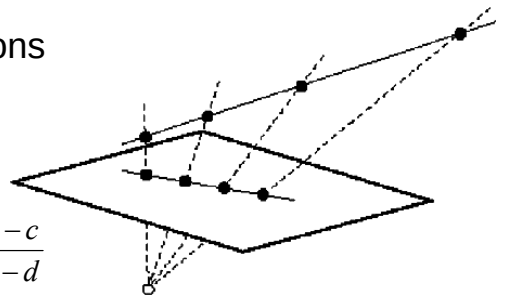
- **Projections** transform points in *n*-space to *m*-space, where $m < n$
- In graphics, we map from 3D to 2D
- Map points from 3-space to the **projection plane (PP)** along **projectors** emanating from the **center of projection (COP)**



Properties: projections vs. Euclidean

- Preserved with Euclidean transformations but *not* with projections
  - Distance
  - Angle
  - Parallelism
  - Ratios
- Preserved with projections
  - Lines map to lines
  - Intersections
  - Tangency
  - Cross-ratios

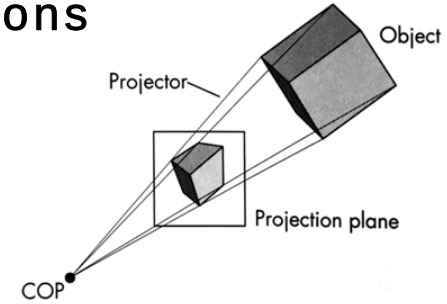


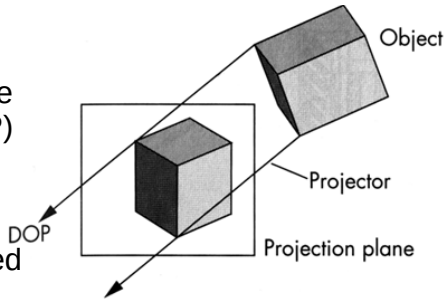


$$\{a,b;c,d\} = \frac{a-c}{a-d} : \frac{b-c}{b-d}$$

Perspective vs. parallel projections

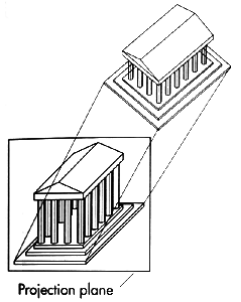
- Perspective projections
  - Distance from COP to PP finite
  - \+ Size varies inversely with distance - looks realistic
  - – Distance and angles are not (in general) preserved
  - – Parallel lines do not (in general) remain parallel
- Parallel projection
  - Distance from COP to PP infinite (use direction of projection DOP)
  - – Less realistic looking
  - \+ Good for exact measurements
  - \+ Parallel lines remain parallel
  - – Angles not (in general) preserved

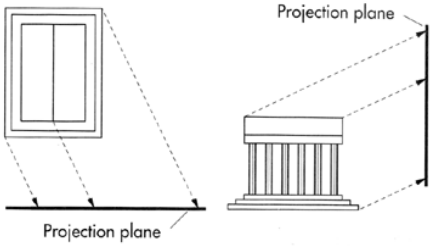




Parallel projections

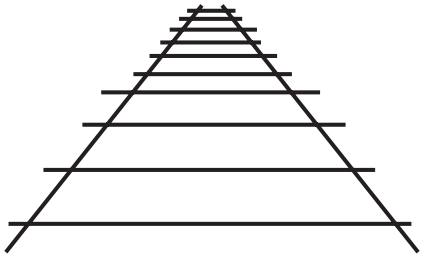
- Two types:
  - **Orthographic projection** — DOP perpendicular to PP
  - **Oblique projection** — DOP not perpendicular to PP
- Two especially useful obliques
- **Cavalier projection**
  - DOP makes 45° angle with PP
  - Does not foreshorten lines perpendicular to PP
- **Cabinet projection**
  - DOP makes 63.4° angle with PP
  - Foreshortens lines perpendicular to PP by one-half





Vanishing points

- Under perspective projections, any set of parallel lines that are not parallel to the PP will converge to a **vanishing point**



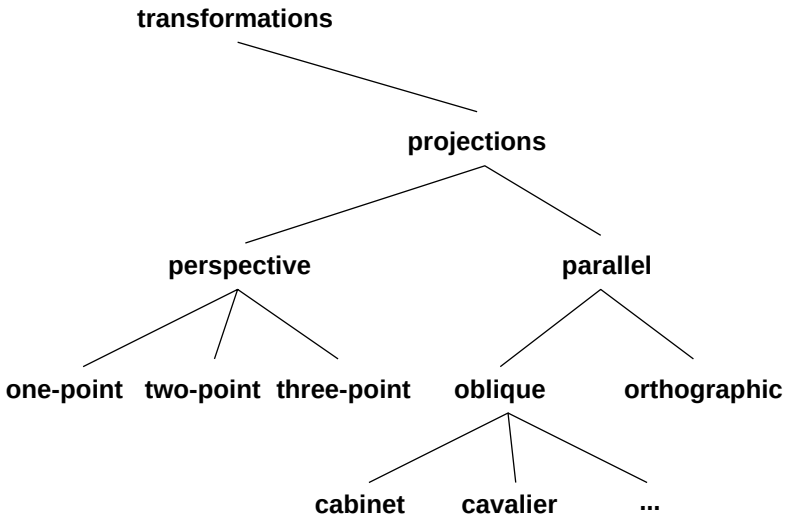
- Vanishing points of lines parallel to a principal axis *x, y,* or *z* are called **principal vanishing points**
- Q:
  - How many of these can there be? **3 (as many as there are principal axes)**
  - How many regular vanishing points? **infinite number, as many as pairs of parallel lines in 3D**

Types of perspective drawing

- Perspective drawings are often classified by the number of principal vanishing points
  - One-point perspective — simplest to draw
  - Two-point perspective — gives better impression of depth
  - Three-point perspective — most difficult to draw

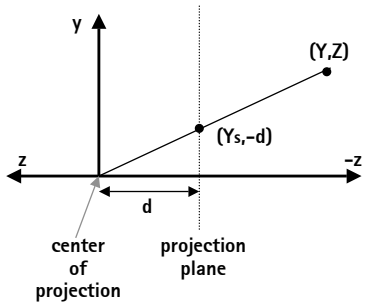
- All three types are equally simple with computer graphics

Projection taxology



Simple perspective

- Let's analyze central projection
  - center of projection (COP) at origin
  - projection plane perpendicular to z-axis, distance *d* away (on negative side)
- Project point (*Y*,*Z*)
  - draw a straight line through COP
  - intersect the line with projection plane
  - projects to (*Ys*,*-d*)
- Calculate *Ys* using similar triangles
  - $\frac{Y}{Z} = \frac{Y_s}{-d} \Leftrightarrow Y_s = \frac{-d}{Z} Y$
  - projection means multiplication with *-d* / *Z*
  - *–d* is constant, *Z* depends on the point
  - nonlinear because of the division with *Z*!

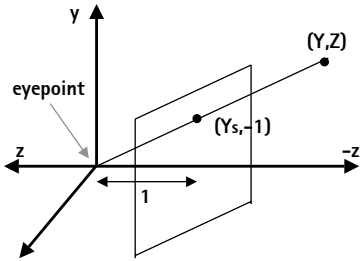


A typical eye space

- **Eye**
  - Acts as the COP
  - Placed at the origin
  - Looks down the negative *z*-axis
- **Perspective projection**
  - **[x y z]' -> [-1/z x, -1/z y, -1]'**
- Switch to homogen. coords
  - **[-1/z x, -1/z y, -1, 1]'**
  - **multiply with -z (same homog. point!)**
  - **[x, y, z, -z]'**

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- **Screen**
  - Lies in the PP
  - Perpendicular to *z*-axis
  - At distance *1* from the eye

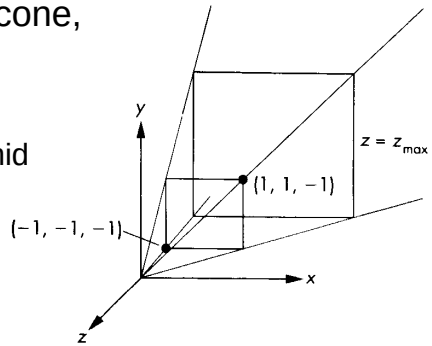


What about Z?

- Now all points project to Z=-1
  - a mapping from "eye space" to image plane
- What do we really want?
  - project X and Y coordinates as stated
  - keep Z-information around for figuring out relative depths
- Solution
  - choose which part of the world should be visible
  - map the z-values of the visible part between –1 (near) and 1 (far)
  - introduce a mapping from "eye space" to "projection space" or "normalized device coordinates"
- Invertible projection matrix
  - as a side product, the projection matrix becomes invertable
  - allows to project the points back to original "eye space"

Viewing frustum

- **View cone**
  - starts from the view point, goes through the view port
  - for now, let the sides of the cone be planes
    z = x, z = -x, z = y, z = -y
    so we get 90 degree vertical and horizontal opening angles
  - view through a window at –1 in *z*, from –1 to 1 in *x* and *y*
- **View frustum** is the view cone, cut by two planes
  - near (hither)
  - far (yonder)
  - frustum = a truncated pyramid
- Only things within view frustum are visible



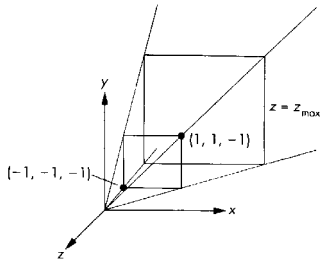
Perspective normalization

- Map the view frustum into the unit box [-1,1]x[-1,1]x[-1,1]
- With the previous page's view cone *x* and *y* are ok
- Need to map *z* of *near (-n)* to -1 and *z* of *far (-f)* to 1
  - don't let *x* or *y* influence *z*
  - calculate what the matrix must be!

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad z' = (\alpha z + \beta)/-z = -\alpha - \frac{\beta}{z} \Rightarrow$$

$$\begin{cases} -1 = -\alpha + \frac{\beta}{n} \\ 1 = -\alpha + \frac{\beta}{f} \end{cases} \Rightarrow \begin{cases} \alpha = -\frac{f+n}{f-n} \\ \beta = -\frac{2fn}{f-n} \end{cases}$$

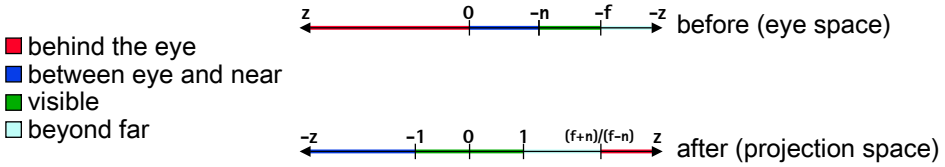
2 constraints: for the near and far planes



What does it mean?

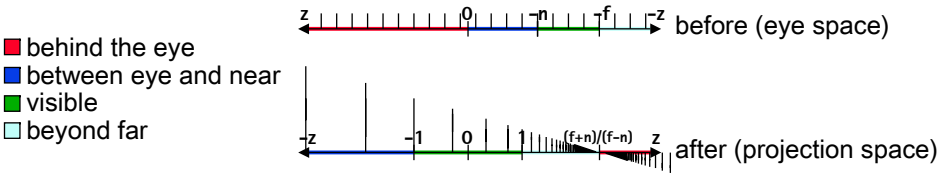
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Let's transform some points
  - **P** [x, y, -*n*, 1]' =[*x, y,* (*fn*+*nn*-2*fn*)/(*f*-*n*), *n*]'=[*x, y,-n, n*]' =[*x*/*n*, *y*/*n*,-1, 1]' point in near plane moves to -1
  - **P** [x, y, -*f*, 1]' = [*x, y,* (*ff*+*fn*-2*fn*)/(*f*-*n*), *f*]' =[*x, y, f, f*]' =[*x*/*f*, *y*/*f*, 1, 1]' point in far plane moves to 1
  - **P** [0, 0, 0, 1]' = [0, 0, -2*fn*/(*f*-*n*), 0]' eyepoint moves to infinity
  - **P** [0, 0, -1, 0]' = [0, 0, (*f*+*n*)/(*f*-*n*), 1]' point infinitely far becomes local
- Mapping of depths



More illustrations

- What about a line of telephone poles?
  - equidistant, equal heights in metric space
  - decreasing distances, shortening, and flipping in projective space



- Perspective foreshortening is visible
  - lines get shorter (the y component)
  - but also z gets "denser" with distance
  - what's the effect in z resolution?

Z resolution

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \alpha = -\frac{f+n}{f-n} \quad \beta = -\frac{2fn}{f-n}$$

- Assume all objects are between 4 and 5 units from camera
- z-buffer resolution to cover z in [-1,1]
  - 24 bits: resolution in z is 1.2 * 10^-7 (after projection)
  - 16 bits: resolution in z is 3.1 * 10^-5 (after projection)
- Set *near* = 0.001, *far* = 1000
  - α = -1, β = -0.002
  - 4 maps to 0.9995, 5 maps to 0.9996 `(-1*4-0.002*1) / -4 ~ 0.9995`
  - need over 14 bits [$\log_2$(2 / (0.9996 – 0.9995)) = 14.3] to locate the visible part in z, i.e., wasting over 14 bits, leaving < 10 useful bits with 24 bit, < 2 bits with 16 bit buffer!
- Set *near* = 0.001, *far* = 10
  - α = -1, β = -0.02
  - 4 maps to 0.9997, 5 maps to 0.9998
  - same waste of z resolution!

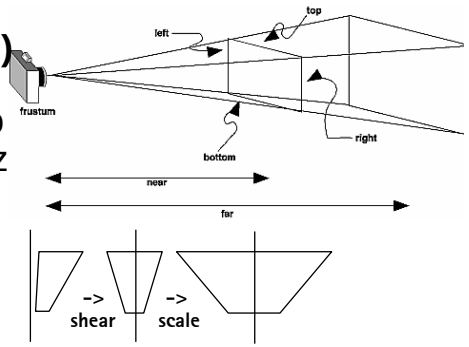
Z resolution, cont.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \alpha = -\frac{f+n}{f-n} \quad \beta = -\frac{2fn}{f-n}$$

- Set *near* = 1, *far* = 10
  - α = -1.2222, β = -2.2222
  - 4 maps to 0.66667, 5 maps to 0.77778
  - now waste only a bit more than 4 bits
- Set *near* = 1, *far* = 1000
  - α = -1.0020, β = -2.0020
  - 4 maps to 0.50150, 5 maps to 0.60160
  - almost the same situation!
- Lesson:
  - beginners set *near* close to camera and *far* really far
    => too little effective z resolution
    => objects' relative depth ordering becomes random
  - set *near* as far from camera as you can!
  - *far* doesn't matter much at all

General view frustum

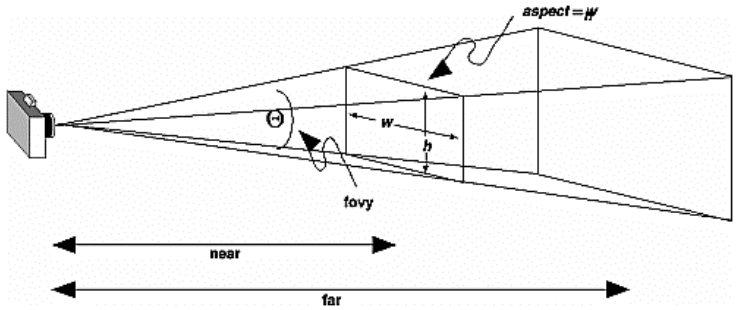
- **`glFrustum(l,r,b,t,n,f)`**
- Shear asymmetric frustum to become symmetric around -z
  - map [(r+l)/2, (t+b)/2, -n, 1]' to [0, 0, -n, 1]'
- Scale sides to $x = \pm z, y = \pm z$
  - map [(r-l)/2, (t-b)/2, -n, 1]' to [n, n, -n, 1]'



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{2n}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2n}{t-b} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{r+l}{2n} & 0 \\ 0 & 1 & \frac{t+b}{2n} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

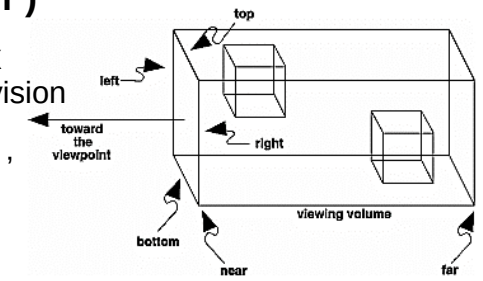
The easy way in OpenGL

- **`gluPerspective(fovy, aspect, n,f)`**



General orthographic matrix

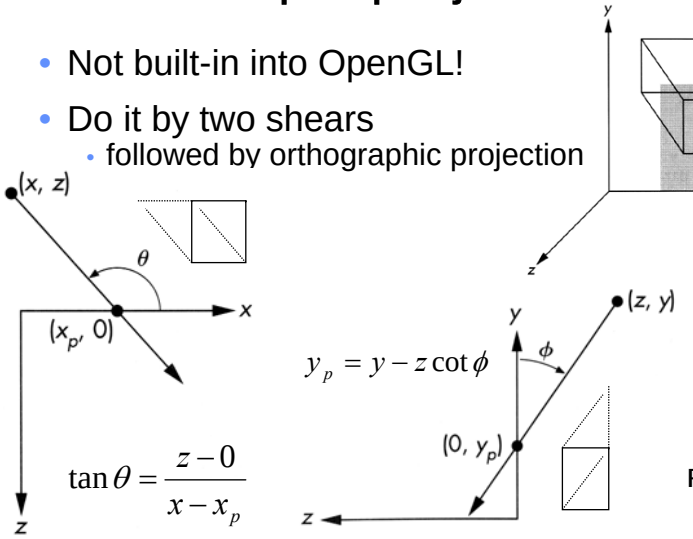
- **`glOrtho(l,r, b,t, n,f)`**
- View frustum already a box
  - no need for the projective division
- Translate center to origin
  - map [(r+l)/2, (t+b)/2, -(f+n)/2] ' to [0,0,0]'
- Scale to a [-1,1] box
  - map [(r-l)/2, (t-b)/2, -(f-n)/2] ' to [1,1,1]'



$$\begin{bmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{-2}{f-n} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -\frac{r+l}{2} \\ 0 & 1 & 0 & -\frac{t+b}{2} \\ 0 & 0 & 1 & \frac{f+n}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{-2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Oblique projection matrix

- Not built-in into OpenGL!
- Do it by two shears
  - followed by orthographic projection



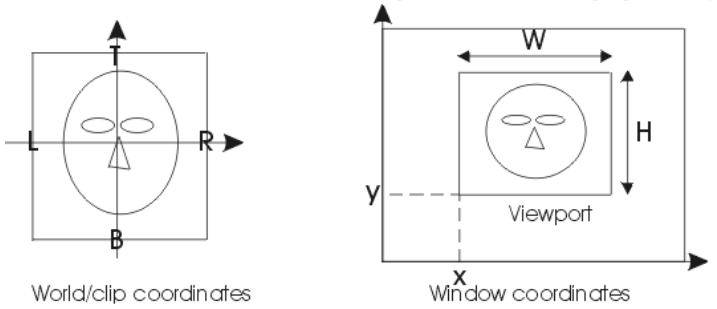
$$\mathbf{P}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x - z\cot\theta \\ y - z\cot\phi \\ z \\ 1 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & -\cot\theta & 0 \\ 0 & 1 & -\cot\phi & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Perspective transformations in OpenGL

- `glMatrixMode(GL_PROJECTION)`
- `glLoadIdentity()`
- `glOrtho(left,right, bottom,top, near,far)`
- `glFrustum(left,right, bottom,top, near,far)`
- get / set:
  - `glLoadMatrix(mat)`
  - `mat = glGetDoublev(GL_PROJECTION_MATRIX)`
  - mat is a float[16], in column-major order (like Fortran)!
- `glMultMatrixf(mat)`
- `glPushMatrix(), glPopMatrix()`
- `gluPerspective(fovy, aspect, near,far)`
- `gluOrtho2D(left,right, bottom,top)`

Window-to-viewport mapping



- Want to do a linear mapping

Match origins $\frac{x-\mathbf{L}}{\mathbf{R}-\mathbf{L}} = \frac{sx-\mathbf{x}}{\mathbf{W}}$

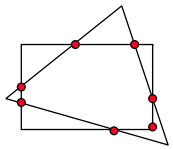
Match widths $\frac{y-\mathbf{B}}{\mathbf{T}-\mathbf{B}} = \frac{sy-\mathbf{y}}{\mathbf{H}}$

Check Chapter 3 of Red Book, section Viewport Transformation

Clipping

- Dropping out things that are not visible
  - points are stay or go
  - lines may get shorter
  - parts of polygons can be clipped off
  - if new vertices introduced (lines, polygons) properties (color, etc.) will have to be interpolated
- Order of processing in OpenGL
  - modeling transformations (=> camera coordinates)
  - user-defined clip planes
  - lighting and shading
  - apply projection matrix (=> clip coordinates)
  - clipping
  - projective divide (division by w) (=> normalized device coordinates)
  - rasterization
  - scissor test (if enabled, modifies only pixels within scissor box)

How many new vertices possible?



Each clip can grow number of vertices by 1

Cohen-Sutherland clipping

- How to make clipping cheap?
  - avoid it!
- Detect whether need any clipping
- Calculate **outcodes** for line end points
  - in 2D: 4 bits
  - how many in 3D? **3x2=6**
- Easy to do in hardware

1010	1000	1001
0010	0000	0001
0110	0100	0101

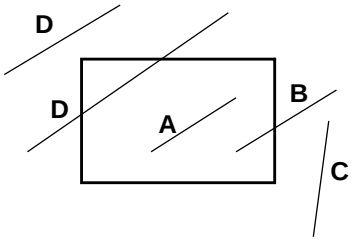
Meaning of bits?

1XXX	*above viewport*
X1XX	*below viewport*
XX1X	*left of viewport*
XXX1	*right of viewport*

Cohen-Sutherland cases

- Compare (bitwise) outcodes o1 and o2 of the end points of an edge
- Four cases
- **A:** o1 = o2 = 0
  - accept: the line is completely inside
- **B:** o1 != 0, o2 = 0 (or vice versa)
  - clip: one end inside, the other outside
- **C:** o1 & o2 != 0
  - reject: line segment fully outside one edge
- **D:** o1 & o2 = 0
  - can't tell: both ends outside, but not of the same edge!

1010	1000	1001
0010	0000	0001
0110	0100	0101



Parametric line representation

- Represent a line as an affine combination of two points

$$\mathbf{p}(t) = (1-t)\,\mathbf{p}_1 + t\,\mathbf{p}_2$$

- Coordinate version

$$x(t) = (1-t)\,x_1 + t\,x_2$$
$$y(t) = (1-t)\,y_1 + t\,y_2$$

- What if
  - $t = 0$? ***at p_1***
  - $t = 1$? ***at p_2***
  - t between 0, 1 ? ***between p_1 and p_2***
  - otherwise ? ***on the line of p_1 and p_2, but elsewhere***

Intersect line with window

- Example: clip with line y = 0
- Solve t from the equation for y:

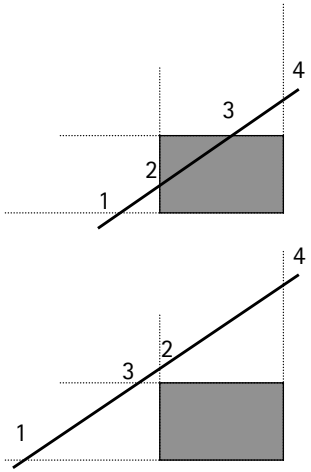
$$y(t) = (1-t)\,y_1 + t\,y_2$$
$$\Rightarrow \quad 0 = (1-t)\,y_1 + t\,y_2$$
$$\Rightarrow \quad 0 = y_1 + t\,(y_2 - y_1)$$
$$\Rightarrow \quad t = y_1 / (y_1 - y_2)$$

- get x(t)

$$x(t) = (1-t)\,x_1 + t\,x_2$$

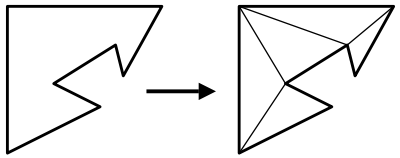
Liang-Barsky clipping

- Solve t_1,t_2,t_3,t_4 in order
- If $t < 0$ or $t > 1$, ignore
- In the first example the order is $0<t_1<t_2<t_3<t_4<1$
  - in this case we only need to solve the intersection points for y2 and x3
- In the second example the order is $0<t_1<t_3<t_2<t_4<1$
  - see from the order that the line does not hit the window and can be rejected

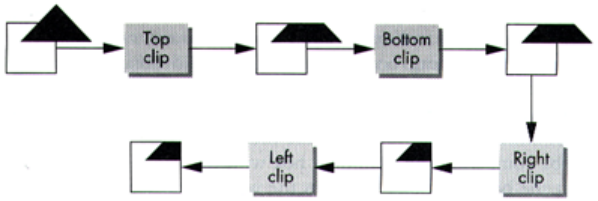


Clipping polygons

- Ignore concave polygons
  - split them into convex parts

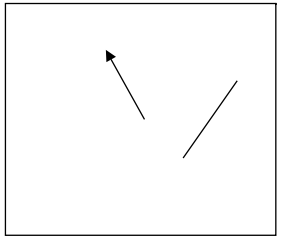


- Clip against one window edge at a time



Plane definition

- Plane
  - is the set of points whose difference (a vector)
  - is perpendicular to the normal vector
- Assume we have
  - normal vector **n** = (a,b,c)
  - point $\mathbf{p}_0$ on the surface
  - an arbitrary point **p** = (x,y,z)
- Let's write this out
  - vector on the surface: $\mathbf{p} - \mathbf{p}_0$
  - that's perpendicular to **n**



$$0 = \mathbf{n}\cdot(\mathbf{p}-\mathbf{p}_0) = \mathbf{n}\cdot\mathbf{p} - \mathbf{n}\cdot\mathbf{p}_0 = ax+by+cz+d,$$
$$\text{where } d = -\mathbf{n}\cdot\mathbf{p}_0$$

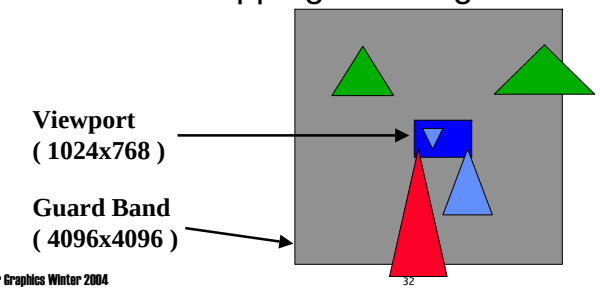
Extra clipping planes

- OpenGL allows to define extra clipping planes
- `glClipPlane(plane, equation)`
  - plane = `GL_CLIP_PLANEi`
  - equation = [a,b,c,d]
  - test for being on the positive side: $ax+by+cz+d \geq 0$
  - clipping done in camera coordinates
  - as usual, need to enable
- The way it's done: Clipping a line with a plane:
  - **line:** $\mathbf{p}(t) = (1-t)\,\mathbf{p}_1 + t\,\mathbf{p}_2$
  - **plane:** $\mathbf{n}\cdot(\mathbf{p}(t)-\mathbf{p}_0) = 0$
  - insert line equation and solve for parameter t:

$$\mathbf{n}\cdot(t(\mathbf{p}_2-\mathbf{p}_1)+\mathbf{p}_1-\mathbf{p}_0)=0$$
$$\Rightarrow t\,\mathbf{n}\cdot(\mathbf{p}_2-\mathbf{p}_1)=\mathbf{n}\cdot(\mathbf{p}_0-\mathbf{p}_1)$$
$$\Rightarrow t=\frac{\mathbf{n}\cdot(\mathbf{p}_0-\mathbf{p}_1)}{\mathbf{n}\cdot(\mathbf{p}_2-\mathbf{p}_1)}$$

Guard band clipping

- Clipping is slow
  - new vertices are introduced
  - must interpolate values (colors, alpha, texture coordinates, fog, …)
  - can destroy coherency of triangle strips, etc.
- Scissoring can be faster
  - just skip the pixels outside the viewport
- Near and far clipping unchanged



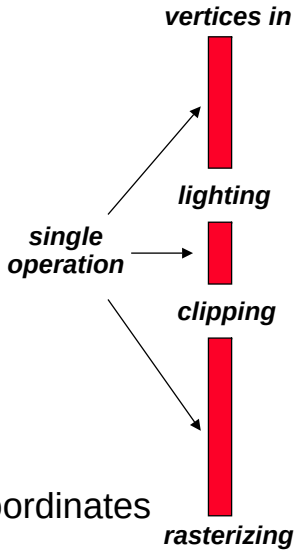
Green triangles can be trivially rejected

Blue triangles can be trivially accepted

Red triangle must be clipped

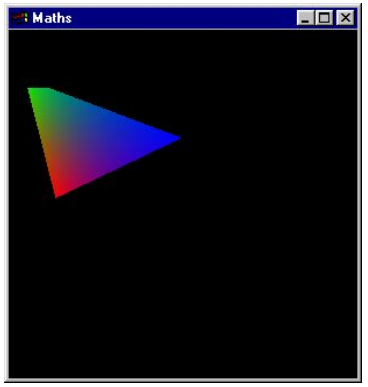
Transformation pipeline

- Object coordinates
  ⇩ (Modelview matrix)
- World coordinates
  ⇩ (Modelview matrix)
- Camera/Eye coordinates
  ⇩ (Projection matrix)
- Clipping coordinates
  ⇩ (Homogeneous division)
- Normalized device coordinates
  ⇩ (Viewport transformation)
- Window/Screen/Image/Viewport coordinates



Example: Analyze the maths

```
glViewport(10,50,150,200)
glClear(GL_COLOR_BUFFER_BIT)
glMatrixMode(GL_PROJECTION)
glLoadIdentity()
glFrustum(0,2,-1,.5,1,10)
glMatrixMode(GL_MODELVIEW)
glLoadIdentity()
gluLookAt(2,7,2, 6,6,6, 0,1,0)
glTranslatef(4,5,7)
glRotate(60, 0,0,1)
glBegin(GL_TRIANGLES)
glColor3f(1,0,0)
glVertex3f(0,0,0)
glColor3f(0,1,0)
glVertex3f(5,0,0)
glColor3f(0,0,1)
glVertex3f(0,4,0)
glEnd()
```



Window is 300x300

Projection matrix

`glFrustum(0,2,-1,.5,1,10)`

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

- Projection from eye to screen
- Map -1 to -1 and -10 to 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad z' = (\alpha z + \beta)/-z = -\alpha - \frac{\beta}{z} \Rightarrow$$

$$\begin{cases} -1 = -\alpha + \beta/1 \\ 1 = -\alpha + \beta/10 \end{cases} \Rightarrow \begin{cases} 2 = -\beta 9/10 \Rightarrow \beta = -20/9 \\ \alpha = 1 + \beta = -11/9 \end{cases} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-11}{9} & \frac{-20}{9} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

- Shear the center of near plane onto z axis ([1,-1/4,-1] -> [0,0,-1])
- Resize: (2-0, 0.5- -1)/2=(1,3/4) to (1,1)

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection...

- Multiply out
  - scale & shear

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 4/3 & -1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - and project

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-11}{9} & \frac{-20}{9} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 4/3 & -1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 4/3 & -1/3 & 0 \\ 0 & 0 & \frac{-11}{9} & \frac{-20}{9} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Camera transformation

- `gluLookAt(2,7,2, 6,6,6, 0,1,0)`
- Camera position is (2,7,2)
- Z is (2,7,2) – (6,6,6) = (-4,1,-4)
- X is VUP x Z = (0,1,0) x (-4,1,-4)= $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ -4 & 1 & -4 \end{vmatrix}$ = (-4,0,4)
- Y is Z x X = (-4,1,-4) x (-4,0,4) = $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & -4 \\ -4 & 0 & 4 \end{vmatrix}$ = (4,32,4)
- Check: dot(X,Y) = dot(X,Z) = dot(Y,Z) = 0
- Normalize and get cam->world
  - X = (-1,0,1) / sqrt(2)
  - Y = (1,8,1) / sqrt(66)
  - Z = (-4,1,-4) / sqrt(33)

$$\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{66} & -4/\sqrt{33} & 2 \\ 0 & 8/\sqrt{66} & 1/\sqrt{33} & 7 \\ 1/\sqrt{2} & 1/\sqrt{66} & -4/\sqrt{33} & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Camera transformation

- Invert to get world->cam:
  - transpose the rotation part
  - multiply that with neg. translation

$$\mathbf{M}\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}' & -\mathbf{R}'\mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{66} & -4/\sqrt{33} & 2 \\ 0 & 8/\sqrt{66} & 1/\sqrt{33} & 7 \\ 1/\sqrt{2} & 1/\sqrt{66} & -4/\sqrt{33} & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{66} & 8/\sqrt{66} & 1/\sqrt{66} & -60/\sqrt{66} \\ -4/\sqrt{33} & 1/\sqrt{33} & -4/\sqrt{33} & 9/\sqrt{33} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Modeling transformation

```
glTranslatef(4,5,7)
glRotate(60, 0,0,1)
```

- Translation

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rotation
  - cos 60 = .5, sin 60 = sqrt(1-.5*.5) = sqrt(3)/2
  - around z axis

$$\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Multiply

$$\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 & 4 \\ \sqrt{3}/2 & 1/2 & 0 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transform vertices

- First vertex: (0,0,0)
  - modeling

$$\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 & 4 \\ \sqrt{3}/2 & 1/2 & 0 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \\ 1 \end{bmatrix}$$

  - to camera

$$\begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{66} & 8/\sqrt{66} & 1/\sqrt{66} & -60/\sqrt{66} \\ -4/\sqrt{33} & 1/\sqrt{33} & -4/\sqrt{33} & 9/\sqrt{33} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{2} \\ -9/\sqrt{66} \\ -30/\sqrt{33} \\ 1 \end{bmatrix}$$

  - and projection

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 4/3 & -1/3 & 0 \\ 0 & 0 & \frac{-11}{9} & \frac{-20}{9} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3/\sqrt{2} \\ -9/\sqrt{66} \\ -30/\sqrt{33} \\ 1 \end{bmatrix} \approx \begin{bmatrix} -3.101 \\ 0.264 \\ 4.161 \\ 5.222 \end{bmatrix}$$

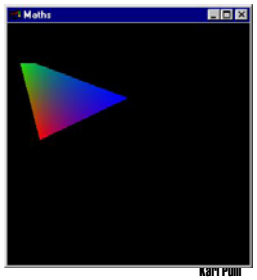
To viewport

`glViewport(10,50,150,200)`

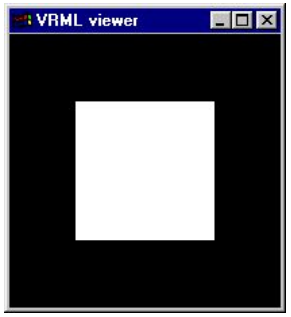
- Homogenous division to NDC:

$$\begin{bmatrix} -3.101 \\ 0.264 \\ 4.161 \\ 5.222 \end{bmatrix} \equiv \begin{bmatrix} -0.594 \\ 0.050 \\ 0.797 \\ 1 \end{bmatrix}$$

- x: 10 + 150 * (-0.594-(-1))/2 = 40
- y: 50 + 200 * (0.050-(-1))/2 = 155
- Second vertex in NDC: (-.943, 1.077, .864)
  - needs clipping!
    screen coordinates would have been (14, 258)
  - calculate t from y:
    (1.077 – 0.050) * t = (1.000 – 0.050) => 0.925
  - new screen coordinates
    x: 40 + (14-40)*t = 16
    y: 155 + (258-155)*t = 250
  - color: (1,0,0) + ((0,1,0) – (1,0,0))*t = (0.075, 0.925, 0)



Virtual trackball



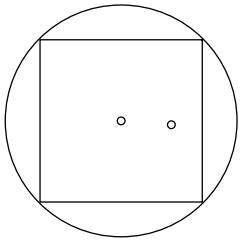
Inspect an object by

- rotating
- panning
- zooming

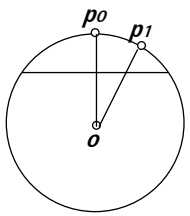
Rotating

- Rotate an object around its center
  - p' = R (p – c) + c = T_c R T_{-c} p
- What's R?
  - imagine a unit sphere over the viewport
  - when you click a point, project it onto the sphere
  - the rotation is the one that rotates the sphere from the first point to the second
  - get angle from dot product
    - $\arccos(p_0 \cdot p_1)$
  - get axis from cross product
    - $p_0 \times p_1$
  - use quaternion (or glRotate)

Top view



Side view



Panning

- Effect:
  - the point you click,
  - at the distance of rotation center,
  - should stay glued to the mouse pointer
- In other words:
  - how big a translation of object center corresponds to one pixel?
  - if that scale factor is *s*,
  - and you move *dx, dy* pixels,
  - translate by *s dx, s dy*
- What is s?
  - project object center to image
  - move by one pixel
  - project back to 3D
  - calculate distance

```
win = gluProject( obj_c[0], obj_c[1], obj_c[2] )
obj = gluUnProject( win[0], win[1] + 1, win[2] )
s   = dist(obj, obj_c)
```

Zooming

- For zooming
  - modify the distance from camera to object
  - so that the same mouse motion corresponds to the same relative zooming, must change distance by a multiplicative factor
  - e.g., multiply distance by 2^(dy / a)
  - if dy == a pixels, distance doubles
  - if dy == -a pixels, distance halves