THE SLAB WAVEGUIDE

6.1 GUIDED WAVES IN A METAL GUIDE

It is now time to consider more effective guiding structures for electromagnetic waves. To introduce the basic ideas, we shall start with a very simple geometry, the **metal-walled guide**. Though this is not actually used at optical frequencies, it contains nearly all the features of more useful guides and can be analysed with little mathematics. The metal-walled guide consists of two plane mirrors held parallel to one another in free space, as shown in Figure 6.1-1. From our knowledge of mirrors, we would expect plane waves to be reflected and re-reflected at each interface, and hence bounce up and down between the mirrors. This zig-zag pathway effectively results in travel only in the z-direction, so the field is guided down the z-axis.
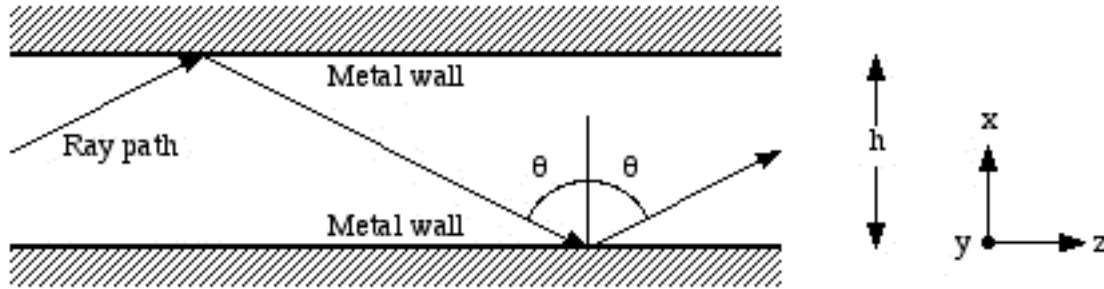

Figure 6.1-1   A metal-walled waveguide

We must begin by working out what kind of field can exist in the guide. First, we recall that the time- independent field due to a y-polarized plane wave travelling at an angle θ to the z-axis (either upwards or downwards) is:

$$E_y = E \exp\{-jk_0[z \sin(\theta) \pm x \cos(\theta)]\}$$

6.1-1

Since we can guess that the field inside the guide is a combination of an upward- and a downward- travelling wave, we will assume the solution:

$$E_y = E_+ \exp\{-jk_0[z \sin(\theta) + x \cos(\theta)]\} + E_- \exp\{-jk_0[z \sin(\theta) - x \cos(\theta)]\}$$

6.1-2

where E_+ and E_- are unknown constants. Now, we must satisfy the usual set of boundary conditions at any interfaces. If we adopt the approximations used in Chapters 3, 4 and 5 for a good conductor, these reduce to a requirement that the electric field vanishes at the metal walls (so that $E_y = 0$ at $x = 0$ and $x = h$). We can satisfy the first condition if $E_- = - E_+$, i.e. if the solution has the form:

$$E_y = E \sin[k_0x \cos(\theta)] \exp[-jk_0z \sin(\theta)]$$

6.1-3

The solution is therefore a wave with a sinusoidal amplitude envelope that travels in the z-direction. It is called a **guided mode**, because it has its energy confined within the guide walls. Note that the field can once again be written in the standard form:

$$E_y = E(x) \exp(-j\beta z)$$

6.1-4

where E(x) is the **transverse field distribution**, and β is the **propagation constant**. Clearly, in this case, β is given by:

$$\beta = k_0 \, \sin\theta$$

6.1-5

so the smaller the ray angle inside the guide, the smaller the propagation constant. The exact value of θ (and thus of β) is fixed by the second boundary condition, namely:

$$\sin[k_0 h \, \cos(\theta)] = 0$$

6.1-6

Now, Equation 6.1-6 is satisfied whenever:

$$2k_0 h \, \cos(\theta) = 2\nu\pi \quad \text{(where } \nu = 1, 2 \dots \text{ etc.)}$$

6.1-7

This is called the **eigenvalue equation** of the guide. It may be viewed as a condition for transverse resonance, since it implies that the round-trip phase accumulated by bouncing up and down between the walls must be a whole number of multiples of 2π.

Each solution corresponds to a particular guided mode, defined by the **mode index** ν. However, in general, only a fixed number of modes can be supported by the guide. For example, if $\nu\pi/k_0 h > 1$, there is no solution, so the condition for a guided mode of order ν to exist is that:

$$h > \nu\pi/k_0$$

6.1-8

Consequently, if the guide width h is too small, no modes are supported all. This occurs when:

$$h < \pi/k_0 \quad \text{or} \quad h < \lambda_0/2$$

6.1-9

This puts a lower limit on the useable value of h of half the optical wavelength. If h is slightly bigger, just one mode is supported, and the guide is then called **single-moded**. If h is bigger still, a finite number of modes can propagate, so the guide is **multi-moded**. Any mode that cannot propagate is described as being **cut off**. The higher the order of the mode, the smaller the value of θ, and, exactly at cut-off, the ray angle is zero, so at this point the rays just bounce up and down between the metal walls, making no progress down the guide. Each guided mode has its own particular transverse field distribution. The field patterns of the first two modes are shown in Figure 6.1-2. Note that the lowest-order mode has no sign reversals, while the second-order mode has one, and so on.
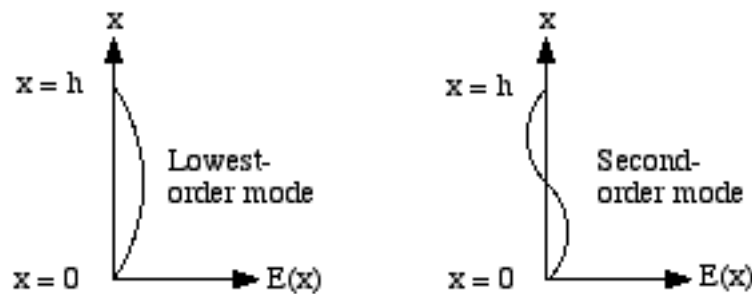

Figure 6.1-2   Transverse field distributions of the two lowest order modes in a metal-walled guide.

The metal-walled guide is very effective at microwave frequencies, when the conductivity of the walls is high, and the reflectivity is correspondingly good. However, at optical frequencies, σ is lower, causing unacceptable propagation loss. Fortunately, an alternative guiding mechanism is available, based on total internal reflection at a dielectric interface.

6.2 GUIDED WAVES IN A SLAB DIELECTRIC WAVEGUIDE

Figure 6.2-1 shows more useful structure, a **dielectric waveguide**. This consists of three layers of dielectric: layer 1 (which has thickness h, and refractive index n_1), and layers 2 and 3 (which are both semi-infinite, and which have indices n_2 and n_3 respectively). We shall assume that $n_1 > n_2$ and $n_1 > n_3$, so that total internal reflection can occur at each interface. Often, layer 1 is referred to as the **guiding layer**, while layers 2 and 3 are the **substrate** and the **cover layer**. A guide of this type can be formed simply by depositing a high-index guiding layer onto a polished substrate - the cover can then often be air. Because of this common geometry, it is usual to describe the structure as an **asymmetric slab guide**, and take $n_1 > n_2 > n_3$.
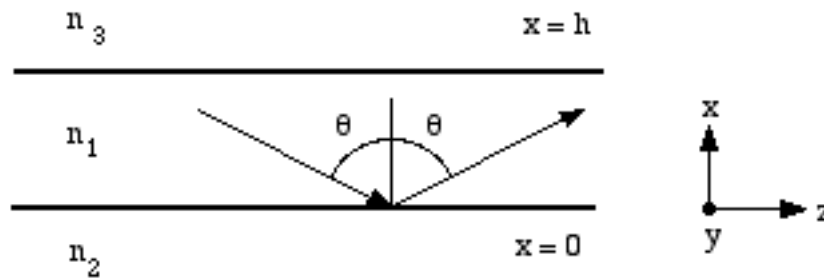

Figure 6.2-1   A slab dielectric waveguide.

We can represent the refractive index variation using a discontinuous function n(x), as shown in Figure 6.2-2a. Other, smoother profiles can result from using different fabrication methods; for example, Figure 6.2-2b shows the graded-index profile obtained when a guide is made by diffusion of metal atoms into a crystalline substrate. In this case, the index distribution below the crystal surface is a Gaussian function (as will be shown in Chapter 13), given by:

$$n(x) = n_2 + (n_1 - n_2) \exp[-(x - h)^2/L_D{}^2] \qquad \text{for } x < h$$

6.2-1

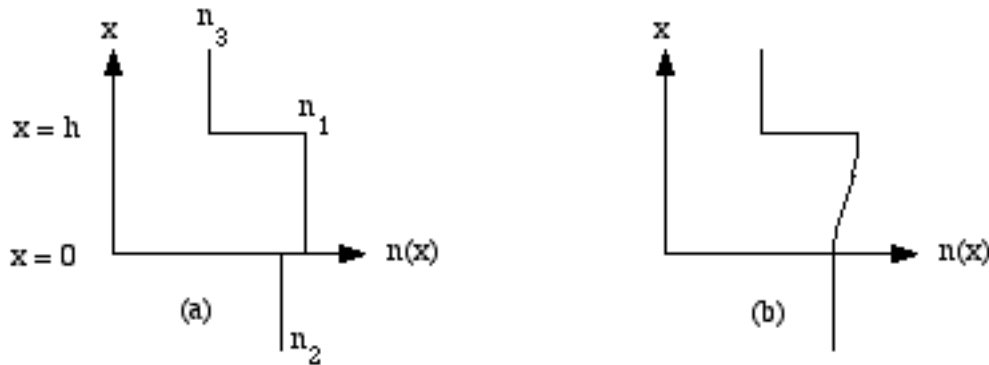where L_D is a constant related to the diffusion conditions, the **diffusion length**.

Figure 6.2-2   Refractive index profiles for asymmetric 1-D waveguides.

However, because many of these methods are also used to fabricate more complicated channel guide structures, we shall postpone their detailed discussion, and stay with the simple slab guide for the time being.

Through total internal reflection, waves may bounce to and fro between the guide walls much as before. However, the form of solution used above - combinations of plane waves - rapidly becomes too complicated when used with dielectric guides. Instead, we shall base the analysis on the modal solution introduced in Chapter 5, and merely look for solutions moving at constant speed in the z-direction. For simplicity, we assume y-polarization once again, so that the mode is **Transverse Electric**, or **TE**. In each layer, the scalar wave equation we must solve is therefore given by:

$$\nabla^2 E_{yi}(x, z) + n_i^2 k_0^2 E_{yi}(x, z) = 0 \quad (i = 1, 2, 3)$$

6.2-2

while the solutions can be written in the form:

$$E_{yi}(x, z) = E_i(x) \exp(-j\beta z) \quad (i = 1, 2, 3)$$

6.2-3

The waveguide equation (which links the transverse field E_i with the propagation constant β) is then given in each layer by:

$$d^2 E_i/dx^2 + [n_i^2 k_0^2 - \beta^2] E_i = 0$$

6.2-4

Since we are interested in fields that are confined within the guide (which will be standing waves inside the guiding layer, and evanescent fields outside) we will assume the following trial solutions:

In layer 1: $E_1 = E \cos(\kappa x - \phi)$
In layer 2: $E_2 = E' \exp(\gamma x)$
In layer 3: $E_3 = E'' \exp[-\delta(x - h)]$

6.2-5

Where the constants κ, γ and δ are as given in Chapter 5, namely:

$$\kappa = \sqrt{(n_1^2 k_0^2 - \beta^2)}$$
$$\gamma = \sqrt{(\beta^2 - n_2^2 k_0^2)}$$
$$\delta = \sqrt{(\beta^2 - n_3^2 k_0^2)}$$

6.2-6

Once more, we must satisfy the boundary conditions, which require continuity of $E_i(x)$ and its gradient dE_i/dx at each interface. Matching fields at $x = 0$ gives the answer we found in Chapter 5, namely:

$$E' = E \cos(\phi)$$

6.2-7

while matching the field gradients gives:

$$\gamma E' = -\kappa E \sin(-\phi)$$

6.2-8

Dividing Equations 6.2-7 and 6.2-8, we may eliminate the field amplitudes from the problem, and obtain a closed-form expression for ϕ:

$$\tan(\phi) = \gamma/\kappa \ , \ \text{or} \ \ \phi = \tan^{-1}(\gamma/\kappa)$$

6.2-9

Similarly, matching fields at x = h gives:

$$E'' = E \cos(\kappa h - \phi)$$

6.2-10

while matching field gradients gives:

$$-\delta E'' = -\kappa E \sin(\kappa h - \phi)$$

6.2-11

Dividing Equations 6.2-10 and 6.2-11, we therefore obtain:

$$\tan(\kappa h - \phi) = \delta/\kappa$$

6.2-12

Now, self-consistent solutions are only possible when Equations 6.2-7 - 6.2-12 are all satisfied simultaneously. The most important point is that the trigonometric relations in Equations 6.2-9 and 6.2-12 are both satisfied. Once this is done, the field amplitudes E' and E'' can be found in terms of E, simply by substituting values for ϕ and κ into Equations 6.2-7 and 6.2-10.

We can reduce the two main equations to a single one, as follows. Using the standard trigonometrical identity:

$$\tan(A - B) = (\tan A - \tan B) / (1 + \tan A \tan B)$$

6.2-13

we can convert the left-hand side of Equation 6.2-12 to:

$$\tan(\kappa h - \phi) = [\tan(\kappa h) - \tan\phi] / [1 + \tan(\kappa h) \tan\phi]$$

6.2-14

Substituting the value of tanϕ from Equation 6.2-9, we then obtain:

$$[\tan(\kappa h) - \gamma/\kappa] / [1 + \tan(\kappa h) \gamma/\kappa] = \delta/\kappa$$

6.2-15

Finally, after some rearrangement, we get:

$$\tan(\kappa h) = \kappa[\gamma + \delta] / [\kappa^2 - \gamma\delta]$$

6.2-16

This is the eigenvalue equation for the dielectric guide. Once again, it can be shown that only certain values of β can satisfy it, so this guide will also only support a discrete set of guided modes. Since the parameters κ, γ and δ are all functions of the propagation constant β, the eigenvalue equation is obviously a function of β as well. However, it is a transcendental equation (which means the solution cannot be written in closed form), so the β-values must be found numerically. This is not difficult, but before we examine the solutions themselves, we will examine the connection between the eigenvalue equation and the transverse resonance condition mentioned previously.

THE TRANSVERSE RESONANCE CONDITION

We first recall the definition of the parameter ϕ, from our discussion of total internal reflection in Chapter 5. ϕ first appeared when the reflection coefficient Γ_E was written in the form:

$$\Gamma_E = \exp(j2\phi)$$

6.2-17

Since this is a complex exponential, -2ϕ can be interpreted as a phase-shift between the incident and reflected waves (note the negative sign here). We can therefore define two new parameters $-2\phi_{12}$ and $-2\phi_{13}$ for our three-layer guide. These are the shifts that occur on reflection from the interfaces between layers 1 and 2, and layers 1 and 3, respectively. ϕ_{12} can be identified with our previous parameter ϕ, so we may write $\tan(\phi_{12}) = \gamma/\kappa$. By analogy, we can also put $\tan(\phi_{13}) = \delta/\kappa$. Equation 6.2-12 can therefore be written as:

$$\tan(\kappa h - \phi_{12}) = \tan(\phi_{13})$$

6.2-18

Taking the inverse tangent of both sides of Equation 6.2-18, allowing for the periodic nature of the tan function (which repeats every π radians), and multiplying both sides by a factor of two, we get:

$$2\kappa h - 2\phi_{12} - 2\phi_{13} = 2\nu\pi$$

6.2-19

Equation 6.2-19 is now a transverse resonance condition, much like Equation 6.1-7. However, there are some differences. Firstly, the term κh has replaced $k_0 h \sin(\theta)$. This is not a real change, since it is easy to show that $\kappa = k_0 n_1 \sin(\theta)$, the new term therefore merely reflects the fact that the guiding layer is a medium of refractive index n_1. Secondly, the phase terms $-2\phi_{12}$ and $-2\phi_{13}$ have been introduced. The overall interpretation of Equation 6.2-19 is therefore that the total phase accumulated in bouncing between the guide walls, including the phase changes experienced on reflection, must be a whole number of multiples of 2π. The astute reader will be able to explain why this extra feature (the phase changes due to reflection) did not apparently show up in our analysis of the metal-walled guide.

THE TRANSVERSE FIELD PATTERNS

Once the values of β are found, the transverse field patterns can be drawn. It is easiest to do this for a symmetric guide, which can only support modes with symmetric or antisymmetric field patterns. In this case, it can be shown that the eigenvalue equation 6.2-16 reduces to:

$$\tan(\kappa h/2) = \gamma/\kappa$$

6.2-20

for all modes with symmetric fields, and:

$$\tan(\kappa h/2) = -\kappa/\gamma$$

6.2-21

for all antisymmetric modes. If Equations 6.2-20 and 6.2-21 are solved numerically, the following results are obtained. At low optical frequencies (when λ_0 is large and k_0 is low), the guide is **single-moded** and only symmetric patterns of the type shown in Figure 6.2-3 can be supported.

Note that the field varies cosinusoidally inside the guide, and exponentially outside, as expected. At extremely low frequencies, a significant amount of power propagates in the evanescent field. A mode of this type is described as poorly confined. As the frequency rises, however, the field concentrates more towards the centre of the guide. We shall see later that this improved confinement of the mode is advantageous, since it reduces the propagation loss that occurs when the waveguide is formed into a bend. As the frequency rises further, the guide becomes **two-moded** and a second, antisymmetric field solution becomes possible. This is shown in Figure 6.2-4.
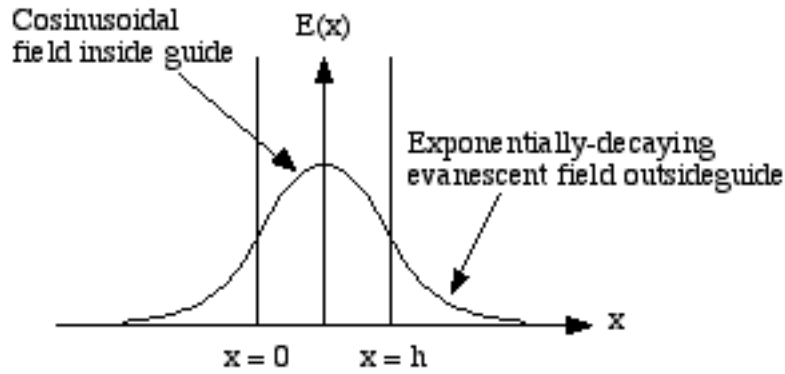
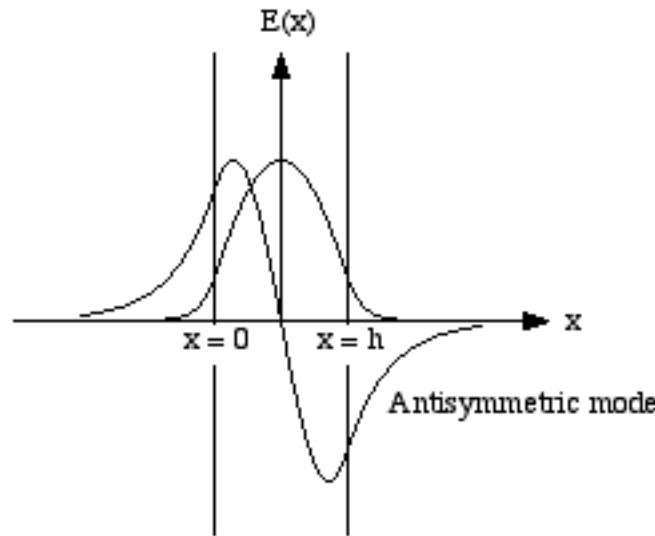Figure 6.2-3   The lowest order symmetric mode of a slab waveguide.

Figure 6.2-4   Mode patterns for a two-moded slab guide.

CUTOFF CONDITIONS

At higher frequencies still, even more patterns are possible. We can work out how many modes can propagate at any given frequency as follows. Particular modes cease to be guided when the ray angle inside the guide tends to the critical angle (note that this is different from the metal-walled guide, where $\theta_{cut-off} = 0°$). Since $\beta = n_1 k_0 \sin\theta$, and Snell's law requires $n_1 \sin\theta = n_2$ at $\theta = \theta_c$, this implies that $\beta \to n_2 k_0$ at cutoff. At the same time, we can show that $\gamma \to 0$, so the confinement of the field tends to zero at this point. For symmetric modes, the cutoff condition then reduces to:

$$\tan(\kappa h/2)_{c.o.} = 0$$

6.2-22

and hence:

$$(\kappa h/2)_{c.o.} = 0, \pi, 2\pi \ ....$$

6.2-23

Similarly, for antisymmetric modes, the cutoff condition is that

$$(\kappa h/2)_{c.o.} = \pi/2, 3\pi/2 \ ....$$

6.2-24

So in general we must have:

$$(\kappa h/2)_{c.o.} = \nu\pi/2$$

6.2-25

where ν is again the mode number. This is the cutoff condition for all the modes in a symmetric slab guide. However, it can be expressed more conveniently with a little further manipulation, as we now show. Since $\kappa = \sqrt{(n_1^2 k_0^2 - \beta^2)}$, then at cutoff we must have:

$$\kappa_{c.o.} = k_0 \sqrt{(n_1^2 - n_2^2)}$$

6.2-26

so that:

$$(\kappa h/2)_{c.o.} = (k_0 h/2) \sqrt{(n_1^2 - n_2^2)}$$

6.2-27

This parameter is often called the **'V' value** of the guide; it is a dimensionless number, which can be used to characterise a guide - generally, a guide will be heavily multimoded if $V \gg 1$. Combining Equations 6.2-25 and 6.2-27, we get:

$$(k_0 h/2) \sqrt{(n_1^2 - n_2^2)} = \nu\pi/2$$

6.2-28

This now represents the cutoff condition for all possible guided modes in a guide of width h, constructed from dielectric layers of refractive indices n_1 and n_2. We can use it as follows. For the lowest order mode (corresponding to $\nu = 0$), the cutoff condition is satisfied when:

$$(k_0 h/2) \sqrt{(n_1^2 - n_2^2)} = 0$$

6.2-29

Since h, n_1 and n_2 are all finite, it follows that k_0 must be zero to satisfy Equation 6.2-29. Effectively, therefore, there is no cutoff for the lowest order mode; this is a particular property of the symmetric slab guide. Similarly, for the second lowest order mode ($\nu = 1$) cutoff occurs when:

$$(k_0 h/2) \sqrt{(n_1^2 - n_2^2)} = \pi/2$$

6.2-30

and so on. We can also use the cutoff condition to work out the dimensions needed for a guide to be single-moded, given the guide indices and the optical wavelength. In this case, we require the second-order mode to be just cut off. Rearranging Equation 6.2-30 slightly, we then find that:

$$h < \lambda_0/2\sqrt{(n_1^2 - n_2^2)}$$

6.2-31

There is therefore a strict upper limit on the allowable width of the guide.

DESIGN EXAMPLE

We can now put in some example numbers to illustrate typical dimensions of single-mode guides. We choose $\lambda_0 = 0.633$ μm, corresponding to the He-Ne laser wavelength, and guide indices of $n_1 \approx n_2 \approx 1.5$; these are typical for different types of glass. With the relatively large index difference of $n_1 - n_2 = 0.01$, we then get $h < 1.8$ μm. For the same parameters, but with the smaller index difference of $n_1 - n_2 = 0.001$, we get $h < 5.8$ μm. Single-mode guides are clearly small in practical materials!

PHASE VELOCITY, GROUP VELOCITY AND WAVEGUIDE DISPERSION

Naturally enough, we will be concerned with the information-carrying capacity of waveguides. This will prove especially important for optical fibres, which we will discuss in Chapter 8. Now, our previous analysis of plane waves (in Chapter 2) showed that dispersion occurs whenever v_{ph} (or ω/k) varies with frequency. The analogous quantity can be found for the νth guided mode, simply by inserting the relevant value for the propagation constant. In this case, we get:

$$v_{ph\nu} = \omega/\beta_\nu$$

6.2-32

It is also often convenient to define an **effective index** $n_{eff\nu}$ for the mode, such that the phase velocity is given by:

$$v_{ph\nu} = c/n_{eff\nu}$$

6.2-33

A moment's thought then shows that the effective index and the propagation constant are related by:

$$\beta_\nu = 2\pi n_{eff\nu}/\lambda_0$$

6.2-34

For a given mode, the effective index may therefore be interpreted as the refractive index of an equivalent bulk medium, which would give identical values for the phase velocity and propagation constant to those obtained in the guide.

Once more, the dispersion characteristics of modes can be represented by a graph of ω against β. For the symmetric slab guide, it is as shown in Figure 6.2-5. Here the lowest order mode has been labelled the TE_0 mode, the second lowest the TE_1 mode, and so on.
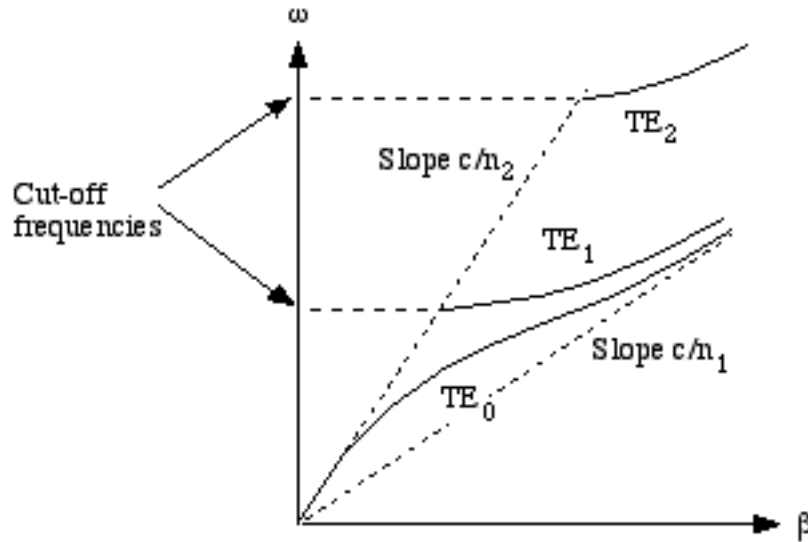

Figure 6.2-5   Dispersion diagram for the symmetric slab guide.

Note that each characteristic lies between two lines, which have slopes of c/n_1 and c/n_2, respectively, so that the phase velocity of any mode cannot be less than c/n_1 or greater than c/n_2. The former value is approached if the mode is well-confined, when most of the field is travelling inside the guide (layer 1, which has refractive index n_1). The latter is approached as the mode tends towards cut-off, when most of the field is travelling outside the guide (in

a medium of index n_2). Consequently, the effective index of all guided modes must lie between n_1 and n_2.

Since the phase velocity clearly depends on frequency (and also on the mode number), the guide must be dispersive. For each mode, we may define a group velocity v_{gv} as:

$$v_{gv} = d\omega/d\beta_v$$

6.2-35

In general, the group velocity will also depend on frequency and mode number. Note that this effect (known as **waveguide dispersion**) is distinct from the material dispersion discussed in Chapters 2 and 3, which for simplicity was ignored in Figure 6.2-5; it forms an extra contribution to the overall dispersion, and one which may be extremely large. Often, the two components of waveguide dispersion are referred to separately, as **inter-** and **intra-modal** dispersion, respectively. The former refers to the variation in v_g between the different modes, the latter to changes in a particular mode's group velocity with frequency. We shall return to both aspects in Chapter 8.

6.3 OTHER TYPES OF MODE

The modal solutions found so far are, of course, descriptive of light confined inside the guide. However, further solutions must exist which account for light propagating <u>outside</u> the guide. These are known as **radiation modes**, and we now describe the way they are calculated. Remember that we solved Equation 6.2-4 to find the TE modes of a slab guide. This is, of course, a general second-order differential equation, which can be written in the form:

$$d^2E_i/dx^2 + C_i{}^2 E_i = 0$$

6.3-1

In our previous analysis, the form of the solution was exponential or sinusoidal, depending on the sign of the term $C_i{}^2 = (n_i{}^2k_0{}^2 - \beta^2)$. If we consider all possible values of β, it turns out that a wider range of solutions can be found. If we again take $n_1 > n_2 > n_3$, which is often the case in real guides, the complete set can be represented as a diagram in β-space as shown in Figure 6.3-1. The essential features of the diagram are that:

1) For $\beta > k_0n_1$, the solutions are exponential in all three layers. Since this implies infinite field amplitudes at large distances from the guide (which are physically unrealistic) we will ignore these solutions for the time being.

2) For $k_0n_1 > \beta > k_0n_2$, there are a discrete number of bound or guided modes, which are the solutions already found. These vary cosinusoidally inside the guide core, and decay exponentially outside the guide.

3) For $k_0n_2 > \beta > k_0n_3$, the solutions vary exponentially in the cover (layer 3), and cosinusoidally in both the guide (layer 1) and substrate (layer 2). Since these fully penetrate the substrate region, they are called **substrate modes**. Any value of β is allowed, between the two limits given above, so the set forms a continuum.

4) For $k_0n_3 > \beta$, solutions vary cosinusoidally in all three layers. These particular field patterns are known as **radiation modes**. Once again, any value of β is allowed in the range above, so the set forms another continuum.
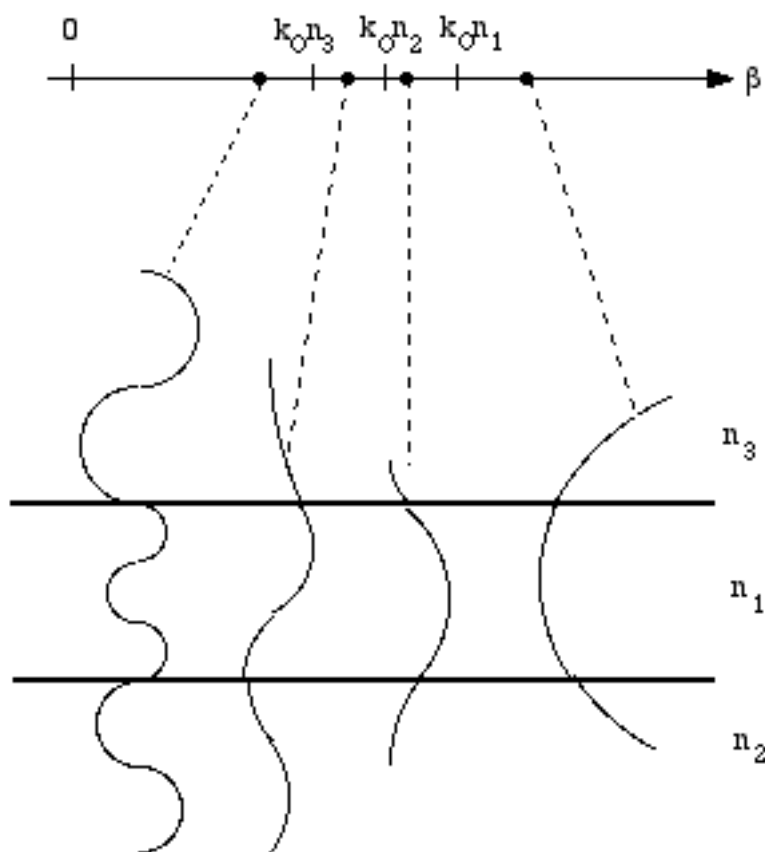
Figure 6.3-1   Modal solutions represented in β-space.

The introduction of radiation modes suggests that we should alter our previous dispersion diagram to account for them. Equally, we should allow for the possibility of backward-travelling modes (which have negative values of β). Both modifications are simple to carry out, and the complete set of modes for a symmetric slab guide is then as shown in the modified ω - β diagram of Figure 6.3-2. We will find this representation useful in the solution of coupled mode problems in Chapter 10.
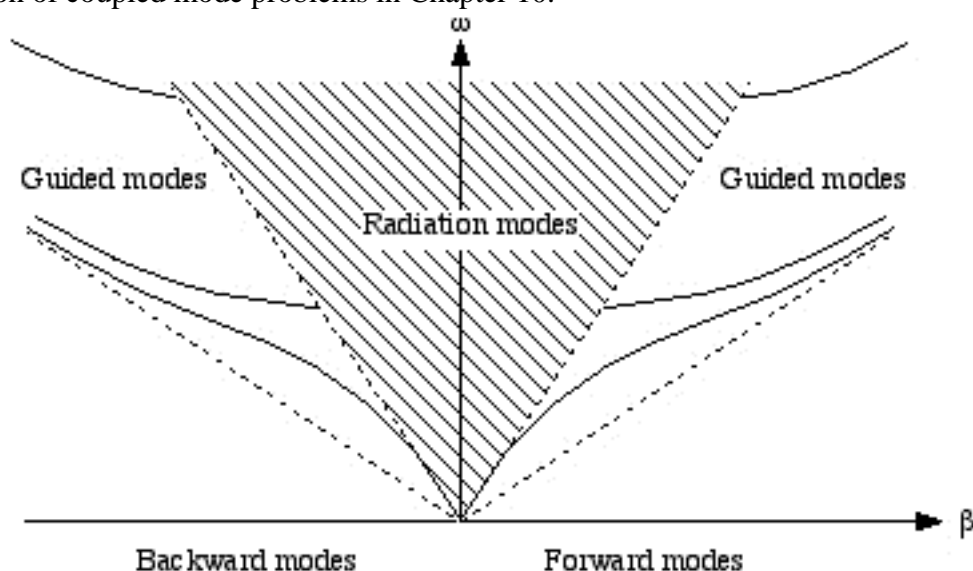

Figure 6.3-2   Full dispersion diagram for the symmetric slab guide.

THE MAGNETIC FIELD PATTERNS

Having found the complete set of all possible solutions for the electric field, it is simple to obtain the corresponding magnetic field patterns. This gives a self-consistent, vectorial solution for the entire guided electromagnetic field. First, recall that the TE solution for $\underline{E}$ is a y-polarized guided mode, written as $\underline{E} = j\ E(x)\ \exp(-j\beta z)$. Now, from the curl relation between $\underline{E}$ and $\underline{H}$, we can show that the associated magnetic field has two components, given by:

$$H_x = (1/j\omega\mu_0)\ \partial E_y/\partial z = (-\beta/\omega\mu_0)\ E(x)\ \exp(-j\beta z)$$

6.3-2

and:

$$H_z = (-1/j\omega\mu_0)\ \partial E_y/\partial x = (-1/j\omega\mu_0)\ dE/dx\ \exp(-j\beta z)$$

6.3-3

Unlike the electric field, the magnetic field therefore has a longitudinal component (i.e. acomponent in the direction of propagation) as well as a transverse one.

THE VECTORIAL REPRESENTATION OF MODES

It is often convenient to represent both the fields in terms of vectorial transverse fields. Thus, for the νth mode, we would write:

$$\underline{E} = \underline{E}_\nu\ \exp(-j\beta_\nu z) \quad \text{and} \quad \underline{H} = \underline{H}_\nu\ \exp(-j\beta_\nu z).$$

6.3-4

where $\underline{E}_\nu$ and $\underline{H}_\nu$ are vectorial descriptions of the transverse fields. This notation will prove useful in some of the later proofs in this section. We may find expressions for $\underline{E}_\nu$ and $\underline{H}_\nu$ for TE modes by comparing Equation 6.3-4 with the results above. If this is done, we get:

$$\underline{E}_\nu = j\ E_\nu \quad \text{and} \quad \underline{H}_\nu = \underline{i}\ (-\beta/\omega\mu_0)\ E_\nu + \underline{k}\ (-1/j\omega\mu_0)\ dE_\nu/dx$$

6.3-5

We can see from Equation 6.3-5 that (unlike the transverse electric field, which is real), the transverse magnetic field is complex, since $H_{\nu z}$ is imaginary.

TM MODES

A similar analysis can be performed for the case when the magnetic field lies in the y-direction. The field patterns are then known as **Transverse Magnetic**, or **TM** modes, and are complementary to the TE solutions already found. The results are qualitatively similar, but this time the eigenvalue equation for guided modes takes the form:

$$\tan(\kappa h) = \kappa(\gamma' + \delta') / (\kappa^2 - \gamma'\delta')$$

6.3-6

where:

$$\gamma' = \gamma\ (n_1^2/n_2^2) \quad \text{and} \quad \delta' = \delta\ (n_1^2/n_3^2)$$

6.3-7

Broadly speaking, the behaviour of TM modes is similar to that of TE modes. However, detailed analysis shows that the field profiles no longer have the smooth, continuous form found earlier. Similarly, the detailed shape of the dispersion diagram is slightly different, because of the refractive index ratios that appear in Equation 6.3-7. However, if the index differences are small (so that the ratios approach unity) the two are practically identical.

This implies that a useful simplification can be made in this case, known as the **weak-guidance approximation**.

6.4 THE WEAK-GUIDANCE APPROXIMATION

The vectorial analysis used so far is very complicated, and a simpler approach is often desirable. This is especially true for the two-dimensional refractive index distributions that occur in channel guides or optical fibres. Luckily, it turns out that if the index difference Δn forming the guide is small, a scalar approximation can be used instead. Since this is often the case in practical guides, the approximation is a very useful one. It is based on the observation that, for a small value of Δn, the critical angle θ_c at a single interface is very large. Hence, for a guided mode, the ray directions inside the guide are almost parallel to the axis of propagation, as shown in Figure 6.4-1.

Since the total field inside the guide may be considered as a summation of similar bouncing waves, the combined E- and H-vectors must be almost exactly orthogonal to each other, and to the axis of propagation. This will be the case anyway for our TE mode, assuming that dE/dx is small in Equation 6.3-3 (the physical justification being that the field inside a weak guide must be a fringe pattern of large periodicity). The resulting modes are known as **TEM modes** (implying that both the electric and the magnetic field components are approximately transverse).
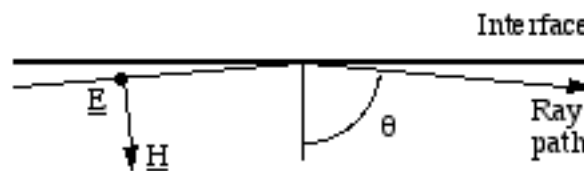

Figure 6.4-1   Field directions for total internal reflection at a low-Δn interface.

As it turns out, the following analysis can also be used to treat two-dimensional guides of arbitrary cross-section, provided the index changes involved are small. They may also be gradual changes, rather than the discrete steps we have encountered so far. To illustrate this, let us consider a guide formed by a 2-D index distribution n(x, y), lying in the z-direction as shown in Figure 6.4-2.
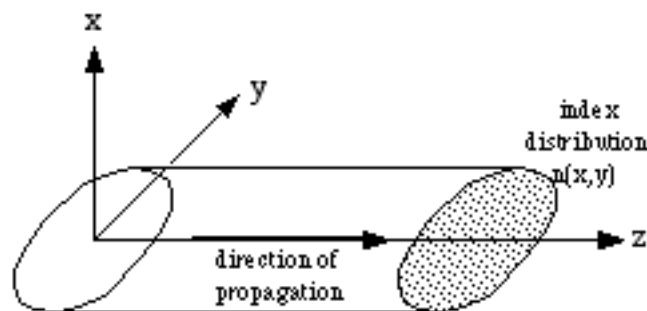

Figure 6.4-2   A two-dimensional waveguide.

If we pick one field component to work with (say $\underline{E}$), then this can point in either the x- or the y-direction. For example, let us suppose that it points in the y-direction. The scalar amplitude $E_y(x, y, z)$ then satisfies (at least, approximately) the three-dimensional scalar wave equation:

$$\nabla^2 E_y \,(x, y, z) + n^2(x, y)k_0^{\,2}\, E_y \,(x, y, z) = 0$$

6.4-1

We now assume a modal solution in the usual form, namely as a product of a transverse electric field distribution and an exponential propagation term. This time, the transverse field must also be two-dimensional, so we take:

$$E_y \,(x, y, z) = E(x, y) \exp(-j\beta z)$$

6.4-2

where E(x, y) is the transverse electric field. Substituting into Equation 6.4-1, we obtain the following waveguide equation linking E with β:

$$\nabla^2_{x,y} E(x, y) + [n^2(x, y)k_0^{\,2} - \beta^2]\, E(x, y) = 0$$

6.4-3

Here the subscripts on the Laplacian indicate that differentiation is to be performed with respect to x and y only. Though Equation 6.4-3 is very similar to our previous waveguide equation, it is also valid for 2-D guides. Although several standard techniques exist for solving this type of equation, we will postpone any attempt at a solution until Chapter 8. For now, we simply note that a complete set of two-dimensional modes - both guided and radiation modes - can exist. However, we can assume directly that the transverse <u>magnetic</u> field points in the x-direction, and that its amplitude can be found from that of the transverse electric field as:

$$H(x, y) = -(\beta/\omega\mu_0)\, E(x, y)$$

6.4-4

The two transverse fields are therefore related by a simple scaling factor.

6.5 THE ORTHOGONALITY OF MODES

Since the modes above form a complete set of solutions to Maxwell's equations for a waveguide geometry, we must be able to describe an arbitrary field by a weighted summation of all types of mode. Before we detail how this is done, we must examine some further general properties of the solutions. The most useful one is that the modes are **orthogonal** to each other. This means that an integral over the whole guide cross-section, of a product of the transverse field of two modes, is zero. This can be proved using the power conservation theorem, as we now show.

The power propagating in a guide can be found by integrating the normal component of the time- averaged Poynting vector over the guide cross-section. For a guide lying in z-direction, we obtain:

$$P_z = 1/2 \text{ Re } [\,\iint_A (\underline{E} \times \underline{H}^*) \cdot \underline{k}\, dx\, dy]$$

6.5-1

Since power is conserved in a lossless system, this implies that $dP_z/dz = 0$, or:

$$d/dz \,[1/4 \iint_A (\underline{E} \times \underline{H}^* + \underline{E}^* \times \underline{H}) \cdot \underline{k}\; dx\, dy] = 0$$

6.5-2

where we have used the identity Re(z) = 1/2 (z + z*). Let us now assume that any field can be written as a sum of modes. For simplicity, we restrict the expansion to forward-travelling guided modes (although generally we should include backward-travelling and radiation modes as well). Suppose that the μth mode has transverse fields $\underline{E}_\mu$ and $\underline{H}_\mu$, and propagation constant β_μ. We can then write:

$$\underline{E} = \sum_{\mu} a_{\mu} \underline{E}_{\mu} \exp(-j\beta_{\mu}z) \quad \text{and} \quad \underline{H} = \sum_{\mu} a_{\mu} \underline{H}_{\mu} \exp(-j\beta_{\mu}z)$$

6.5-3

Here the terms a_{μ} are **expansion coefficients**, which describe the amplitude of each mode contributing to the total field. Substituting these expressions into Equation 6.5-2, we get:

$$d/dz \; [1/4 \iint_A \sum_{\mu,\nu} \{ a_{\mu}a_{\nu}{}^* \exp[j(\beta_{\nu} - \beta_{\mu})z] \; \underline{E}_{\mu} \; x \; \underline{H}_{\nu}{}^*$$
$$+ \; a_{\nu}{}^*a_{\mu} \exp[j(\beta_{\nu} - \beta_{\mu})z] \; \underline{E}_{\nu}{}^* \; x \; \underline{H}_{\mu} \}. \; \underline{k} \; dx \; dy] = 0$$

6.5-4

Or:

$$\sum_{\mu,\nu} \; a_{\mu}a_{\nu}{}^* \; j(\beta_{\nu} - \beta_{\mu}) \exp[j(\beta_{\nu} - \beta_{\mu})z] \iint_A [\underline{E}_{\mu} \; x \; \underline{H}_{\nu}{}^* + \underline{E}_{\nu}{}^* \; x \; \underline{H}_{\mu}] \; . \; \underline{k} \; dx \; dy = 0$$

6.5-5

Now, the coefficients a_{μ} and $a_{\nu}{}^*$ are arbitrary. Equally, the propagation constants of different modes are unequal, so that $\beta_{\mu} \neq \beta_{\nu}$. The only way Equation 6.5-5 can be satisfied for all z is therefore if:

$$\iint_A (\underline{E}_{\mu} \; x \; \underline{H}_{\nu}{}^* + \underline{E}_{\nu}{}^* \; x \; \underline{H}_{\mu}) \; . \; \underline{k} \; dx \; dy = 0$$

6.5-6

This called the **orthogonality condition**. It is also possible to show (by including backward-travelling modes as well) that each of the two parts of the integral are themselves zero, so that:

$$\iint_A (\underline{E}_{\mu} \; x \; \underline{H}_{\nu}{}^*) \; . \; \underline{k} \; dx \; dy = 0$$

6.5-7

We will now show how this condition is used in practice, using a number of examples.

DESIGN EXAMPLE # 1

We begin with the case of TE modes in a slab guide, for which there is clearly no need to integrate in the y-direction. The orthogonality condition therefore reduces to:

$$\int_x (\underline{E}_{\mu} \; x \; \underline{H}_{\nu}{}^*) \; . \; \underline{k} \; dx = 0$$

6.5-8

Or, in terms of the relevant vectorial components:

$$\int_x -E_{\mu y} \; H_{\nu x}{}^* \; dx = 0$$

6.5-9

Substituting for $H_{\nu x}$, and assuming that $E_{\nu y}$ is real, we then obtain:

$$\int_x (E_{\mu y} \; E_{\nu y}) \; dx = 0$$

6.5-10

In other words, the integral of the product of two different modes, when taken over the cross-section of the guide, is zero. This is confirmed by Figure 6.2-4, which shows the field distributions for the lowest-order symmetric and antisymmetric modes in a symmetric guide. Clearly, the integral of the product of these two functions must be zero.

DESIGN EXAMPLE # 2

We can obtain a similar result for TEM modes in a weak two-dimensional graded-index guide, by using the relation between the transverse electric and magnetic fields given in Equation 6.4-4. Substituting into Equation 6.5-7, and making no assumption that the fields are real, we get:

$$\iint_A (E_{\mu y} E_{\nu y}^*) \, dx \, dy = 0$$

6.5-11

In both cases described above, the orthogonality condition is often written in shorthand as:

$$< E_{\mu y}, E_{\nu y} > = 0$$

6.5-12

The bracket notation employed here is often used in quantum mechanics. The operation of multiplying two functions together and integrating over a range is called an **inner product**.

6.6 THE POWER CARRIED BY A MODE

We can use the orthogonality relation to define the power carried down the guide by each mode. From the previous section, we know that the total power flow P_z in the z-direction is given by:

$$P_z = 1/4 \iint_A \sum_{\mu,\nu} \{ a_\mu a_\nu^* \exp[j(\beta_\nu - \beta_\mu)z] \, \underline{E}_\mu \times \underline{H}_\nu^* \\ + a_\nu^* a_\mu \exp[j(\beta_\nu - \beta_\mu)z] \, \underline{E}_\nu^* \times \underline{H}_\mu \} \cdot \underline{k} \, dx \, dy$$

6.6-1

However, from the orthogonality relation, we also know that many of the terms in Equation 6.6-1 must be zero. Removing these gives:

$$P_z = \sum_\mu a_\mu a_\mu^* \, [1/4 \iint_A \{ \underline{E}_\mu \times \underline{H}_\mu^* + \underline{E}_\mu^* \times \underline{H}_\mu \} \cdot \underline{k} \, dx \, dy]$$

6.6-2

We can now simplify this expression by defining a **modal power coefficient** P_μ, such that:

$$P_\mu = 1/4 \iint_A [\underline{E}_\mu \times \underline{H}_\mu^* + \underline{E}_\mu^* \times \underline{H}_\mu] \cdot \underline{k} \, dx \, dy$$

6.6-3

In which case, we obtain:

$$P_z = \sum_\mu a_\mu a_\mu^* \, P_\mu$$

6.6-4

This is a very simple result. It implies that the power carried by forward-travelling modes is found as the sum of the modulus-squares of the mode amplitudes, weighted by the coefficients P_μ.

DESIGN EXAMPLE

Calculation of the modal power coefficients is quite simple. For TE modes in a slab guide, for example, we can define P_μ per unit width in the y-direction as:

$$P_\mu = 1/4 \int_x (\underline{E}_\mu \times \underline{H}_\mu^* + \underline{E}_\mu^* \times \underline{H}_\mu) \cdot \underline{k} \, dx$$

6.6-5

Inserting the relevant vectorial components, and substituting for $H_{\mu x}$ using Equation 6.3-5, we get:

$$P_\mu = (\beta/4\omega\mu_0) \int_x (E_{\mu y} E_{\mu y}{}^* + E_{\mu y}{}^* E_{\mu y}) \, dx$$

6.6-6

It is obvious that each component in the integral above contributes an equal amount to the total. Furthermore, the transverse fields $E_{\mu y}$ are real in this case. We therefore obtain:

$$P_\mu = (\beta/2\omega\mu_0) \int_x E_{\mu y}{}^2 \, dx$$

6.6-7

Note that this can of course be written in inner product notation, as $P_\mu = (\beta/2\omega\mu_0) < E_{\mu y} , E_{\mu y} >$.

6.7 THE EXPANSION OF ARBITRARY FIELDS IN TERMS OF MODES

The orthogonality condition can be also used to show how arbitrary transverse fields can be written in terms of the modal solutions, by finding the expansion coefficients themselves. Suppose we wish to expand a forward-travelling field $\underline{E}(x, y, 0)$ at $z = 0$ in this way. We first put:

$$\underline{E}(x, y, 0) = \sum_\mu a_\mu \underline{E}_\mu (x, y)$$

6.7-1

If we now take vector products of both sides of Equation 6.7-1 with the complex conjugate of the transverse magnetic field of one mode (say, the νth), and integrate over the cross-section, we get:

$$\iint_A [\underline{E} \times \underline{H}_\nu{}^*] \cdot \underline{k} \, dx \, dy = \iint_A \sum_\mu a_\mu [\underline{E}_\mu \times \underline{H}_\nu{}^*] \cdot \underline{k} \, dx \, dy$$

6.7-2

Now, since the field is forward travelling, we could also have put:

$$\underline{H} = \sum_\mu a_\mu \underline{H}_\mu$$

6.7-3

Performing a similar operation, but this time using the complex conjugate of the transverse <u>electric</u> field of the νth mode, we get:

$$\iint_A [\underline{E}_\nu{}^* \times \underline{H}] \cdot \underline{k} \, dx \, dy = \iint_A \sum_\mu a_\mu [\underline{E}_\nu{}^* \times \underline{H}_\mu] \cdot \underline{k} \, dx \, dy$$

6.7-4

Adding Equations 6.7-2 and 6.7-4 together then gives:

$$\iint_A [\underline{E} \times \underline{H}_\nu{}^* + \underline{E}_\nu{}^* \times \underline{H}] \cdot \underline{k} \, dx \, dy = \iint_A \sum_\mu a_\mu [\underline{E}_\mu \times \underline{H}_\nu{}^* + \underline{E}_\nu{}^* \times \underline{H}_\mu] \cdot \underline{k} \, dx \, dy$$

6.7-5

Now, from the orthogonality condition, we know that the right-hand side of Equation 6.7-5 must be zero, unless $\mu = \nu$. Hence, we can rearrange it to extract the modal coefficient aν as:

$$a_\nu = \iint_A [\underline{E} \times \underline{H}_\nu{}^* + \underline{E}_\nu{}^* \times \underline{H}] \cdot \underline{k} \, dx \, dy \ / \iint_A [\underline{E}_\nu \times \underline{H}_\nu{}^* + \underline{E}_\nu{}^* \times \underline{H}_\nu] \cdot \underline{k} \, dx \, dy$$

6.7-6

The coefficients can therefore be found by evaluating Equation 6.7-6 for each mode in turn. Once more, we shall illustrate this with an example.

DESIGN EXAMPLE

To expand a TE field $E_y(x)$ in terms of the TE modes of a slab guide, we follow the usual procedure of dropping the integration with respect to the y-variable. Substituting the relevant vectorial components into Equation 6.7-6, we get:

$$a_\nu = \int_x E_y \, E_{\nu y} \, dx \, / \int_x \, E_{\nu y}{}^2 \, dx$$

6.7-7

Note that the integral in the denominator of Equation 6.7-7 really just represents a normalisation factor. The integral in the numerator is more important; it is often called an **overlap integral**, because it represents the 'overlap' of the two fields E_y and $E_{\nu y}$. This is really a measure of how similar the total field is to the νth mode, rather like a correlation. Clearly, if the two are actually identical, we obtain $a_\nu = 1$ (and $a_\mu = 0$ for $\nu \neq \mu$). This is common sense. Finally, we note that Equation 6.7-7 can be written in inner product notation, as $a_\nu = <E_y , E_{\nu y}> / <E_{\nu y} , E_{\nu y}>$.

6.8 APPLICATION OF THE OVERLAP PRINCIPLE

Overlap integrals are particularly useful in calculating the coupling efficiency in **end-fire couplers**, when an external field is used to excite a guided mode. They can also be used to explain what happens when a mode passes through a waveguide discontinuity or a taper. We shall postpone a description of end-fire coupling until Chapter 8, concentrating for now on discontinuities and tapers.

DISCONTINUITIES

Figure 6.8-1 shows a number of different waveguide discontinuities. In each case, the substrate is continuous, but there is some difference in the guiding layer on either side of the junction.
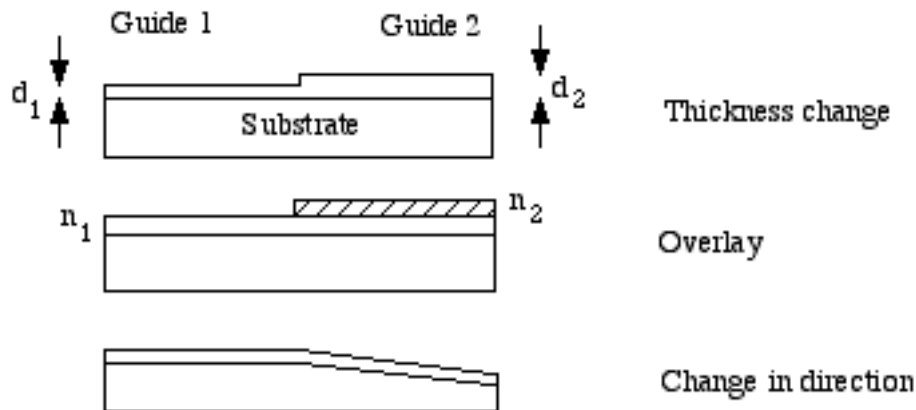

Figure 6.8-1   Examples of waveguide discontinuities.

This could involve a change in layer thickness, or the addition of an overlay of a different material. Both can be achieved easily using conventional planar processing techniques. Alternatively, the orientation of the guide can change. This requires machining of the substrate surface to the desired topology before deposition of the guiding layer. A junction can also be formed between two entirely different guides, with no common substrate. In this case, the process is called **butt coupling**.

In both cases, Equation 6.7-6 can be used to calculate the effect of the discontinuity. For example, let us assume in a TEM model that the set of modes on the input side of the discontinuity have transverse field distributions E_{vy}. On the other side, the corresponding fields are E_{vy}'. We will now show how to find the efficiency with which an input mode (for example, the lowest-order mode E_{0y}) is coupled across the junction. We may take the input field at the junction as $a_0 E_{0y}$, where a_0 is the input mode amplitude. The power P_{in} carried by this field is then:

$$P_{in} = (\beta/2\omega\mu_0) \, |a_0|^2 < E_{0y}, E_{0y} >$$

6.8-1

After the junction, the portion of the total field carried by the lowest-order mode is now $a_0' E_{0y}'$, where a_0' is the new mode amplitude. This can be found as:

$$a_0' = <a_0 E_{0y}, E_{0y}'> / < E_{0y}', E_{0y}' >$$

6.8-2

The power P_{out} carried by this field is (by analogy with Equation 6.8-1):

$$P_{out} = (\beta/2\omega\mu_0) \, |a_0'|^2 < E_{0y}', E_{0y}' >$$

6.8-3

Substituting for a_0', this reduces to:

$$P_{out} = (\beta/2\omega\mu_0) \, |a_0|^2 \, |< E_{0y}, E_{0y}' >|^2 / < E_{0y}', E_{0y}' >$$

6.8-4

If we now define the **coupling efficiency** η as the ratio P_{out}/P_{in}, we obtain:

$$\eta = |< E_{0y}, E_{0y}' >|^2 / \{< E_{0y}, E_{0y} >< E_{0y}', E_{0y}' >\}$$

6.8-5

Clearly, η is a dimensionless number, which tends to unity as E_{0y} tends to E_{0y}'. Although this is what one would intuitively expect, the implication is that the input mode will cross a discontinuity without much in the way of conversion to other modes, merely provided it is small enough. We can also deduce that the most effective waveguide joint will be one between two guides with matched fields. Though we have not included radiation modes in the calculation, this would be required in a rigorous model; it would then be found that some power is lost to radiation at any discontinuity.

TAPERS

We can extend this principle to discover what happens in a tapered waveguide. The upper diagram in Figure 6.8-2 shows a typical slow taper, which involves a gradual change in the guide cross-section. The lower one shows a discrete approximation to this shape, consisting of a series of steps. Once again, we might wish to discover how a mode of a particular order travels through the taper. If we can avoid any mode conversion, we might then be able to design the structure to enlarge the cross-section of a particular mode in a slow and controlled manner. The taper would then act as a beam expander, for use as a matching section between two different guides.

The theory we need to analyse the problem is called the method of **local normal modes**. Using a TEM model, the strategy is to proceed as follows. We start by arranging that the steps in our discrete approximation are at regular intervals Δz apart. We then consider the field at three different points. The first is point 1, just to the left of the discontinuity at $z = 0$ in Figure 6.8-2. Here it is assumed that the local normal modes have transverse field functions $E_{\mu y}$ and propagation constants β_μ.
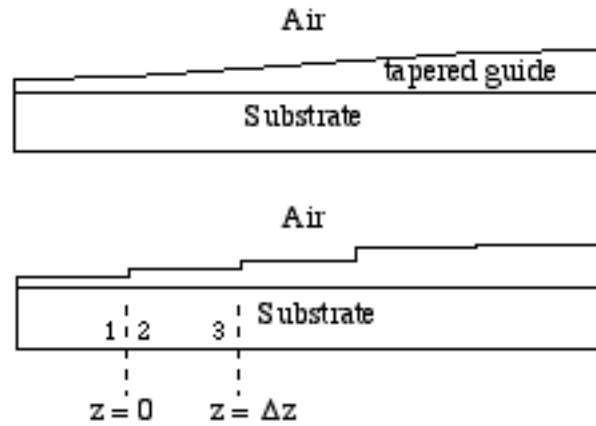
Figure 6.8-2   Continuous and discrete models of a waveguide taper.

If we expand the field E_y in terms of these modes, we can write:

$$E_y = \Sigma_\mu \; a_\mu E_{\mu y}$$

6.8-6

The second place of interest is point 2, just to the right of the discontinuity at z = 0. In this region, the local transverse fields and propagation constants have the slightly different values of $E_{\mu y}'$ and β_μ'. If we expand the same field in terms of these modes instead, we can put:

$$E_y = \Sigma_\mu \; b_\mu E_{\mu y}'$$

6.8-7

Here the terms b_μ are an entirely different set of mode amplitudes, but we can relate them to our original a-values using the overlap principle. If this is done, we get:

$$b_\mu = \Sigma_\nu \; a_\nu < E_{\nu y} \, , E_{\mu y}' > / < E_{\mu y}' \, , E_{\mu y}' >$$

6.8-8

This relation between the b-values and the a-values effectively tells us how the mode amplitudes change simply in crossing the step at z = 0. Now, the next place of interest is point 3. This is just to the left of the discontinuity at z = Δz, and we call field here E_y'. This can be found by propagating the field at point 2 over the distance Δz between points 2 and 3. All that happens is that the phase of each mode changes slightly en route, so we must have:

$$E_y' = \Sigma_\mu \; b_\mu \; E_{\mu y}' \exp(-j\beta_\mu'\Delta z)$$

6.8-9

Now, our aim is to see how the mode amplitudes change with distance. We should, therefore, have expressed the field at point 3 in a form comparable with that used at point 1, i.e. as:

$$E_y' = \Sigma_\mu \; a_\mu' \; E_{\mu y}'$$

6.8-10

Here the terms a_μ' represent the amplitudes of the local normal modes at the start of the next discontinuity at z = Δz. These new amplitudes now can be extracted as follows. First, we equate 6.8-10 and 6.8-9, which gives:

$$a_\mu' = b_\mu \exp(-j\beta_\mu'\Delta z)$$

6.8-11

Then we substitute for b_μ using Equation 6.8-8, to get:

$$a_\mu' = \{ \Sigma_\nu \, a_\nu < E_{\nu y} \, , E_{\mu y}' > / < E_{\mu y}' \, , E_{\mu y}' > \} \exp(-j\beta_\mu'\Delta z)$$

6.8-12

At this point, we note that the quantity $a_\mu' - a_\mu$ is actually the <u>change</u> in a_μ that has occurred over the distance Δz. We might call this Δa_μ. Using Equation 6.8-12, we may write it as:

$$\Delta a_\mu = \{ \, a_\mu < E_{\mu y} \, , E_{\mu y}' > / < E_{\mu y}' \, , E_{\mu y}' > \} \, \{\exp(-j\beta_\mu'\Delta z) - 1\} +$$
$$\{ \, \Sigma_{\nu, \, \nu \neq \mu} \, a_\nu < E_{\nu y} \, , E_{\mu y}' > / < E_{\mu y}' \, , E_{\mu y}' > \} \exp(-j\beta_\mu'\Delta z)$$

6.8-13

We now make a number of approximations, which will be valid provided the distance Δz is short enough and the discontinuity is small. We start by assuming that the transverse modal field changes only a small amount in the distance Δz, so we can put $E_{\mu y}' = E_{\mu y} + \Delta E_{\mu y}$. To reasonable accuracy, we can then say that:

$$< E_{\mu y} \, , E_{\mu y}' > / < E_{\mu y}' \, , E_{\mu y}' > \approx 1$$

6.8-14

Using the orthogonality principle, we may also show that:

$$< E_{\nu y} \, , E_{\mu y}' > / < E_{\mu y}' \, , E_{\mu y}' > \approx < E_{\nu y} \, , \Delta E_{\mu y} > / < E_{\mu y} \, , E_{\mu y} >$$

6.8-15

We now assume that the propagation constant also changes only a small amount in the distance Δz, so we can put $\beta_\mu' = \beta_\mu + \Delta\beta_\mu$. To first-order approximation, we can then write:

$$\exp(-j\beta_\mu'\Delta z) \approx (1 - j\beta_\mu \, \Delta z \,) \approx 1$$

6.8-16

However, we note that a slightly different approximation must be used for the other exponential term in Equation 6.8-13, because it is the <u>difference</u> between two similar quantities. We therefore write:

$$\exp(-j\beta_\mu'\Delta z) - 1 \approx -j\beta_\mu \, \Delta z$$

6.8-17

When all these values are substituted into Equation 6.8-13, it reduces to:

$$\Delta a_\mu = -j\beta_\mu\Delta z \, a_\mu + \Sigma_{\nu, \, \nu \neq \mu} \, a_\nu < E_{\nu y} \, , \Delta E_{\mu y} > / < E_{\mu y} \, , E_{\mu y} >$$

6.8-18

Dividing both sides by Δz, and letting Δz tend to zero, we then get:

$$da_\mu/dz = -j\beta_\mu a_\mu + \Sigma_{\nu, \, \nu \neq \mu} \, a_\nu < E_{\nu y} \, , \partial E_{\mu y}/\partial z > / < E_{\mu y} \, , E_{\mu y} >$$

6.8-19

Where a_μ and β_μ are functions of z, and $E_{\mu y}$ is a function of x, y and z. Equation 6.8-19 is now a differential equation, which describes the way the amplitudes of the local normal modes change with distance. In fact, there is one such equation for each value of μ (i.e., for each mode of interest) and all the equations must be solved simultaneously for a given set of boundary conditions. Because each mode amplitude appears in every equation, they are known as **coupled mode equations**.

In principle, the equations can be solved as follows. The local normal modes must first be found at all points along the taper, so that the functions $E_{\mu y}(x, y, z)$ and $\beta_\mu(z)$ are known. The equations are then integrated numerically. However, this process is exceptionally tedious, so we will not perform it here; instead, we will simply draw some broad

conclusions. Firstly, we note that the solution for a particular mode must be $a_\mu = A_\mu \exp(-j\beta_\mu z)$ (where A_μ is a constant) if the summation in Equation 6.8-19 is zero. Under these conditions, the mode amplitude does not change, only its phase. To design a conversion-free taper, we must simply arrange for it to be slow enough, so that:

$$< E_{\nu y} , \partial E_{\mu y}/\partial z > \rightarrow 0$$

6.8-20

Even if the taper does not satisfy these conditions, we can extract a significant result from the mathematics. We have already seen that the transverse fields of a symmetric slab guide are either symmetric or antisymmetric patterns. If we taper such a guide symmetrically, the same must be true for the local normal modes. More importantly, we can also say that $\partial E_{\mu y}/\partial z$ must be symmetric for a symmetric mode, and antisymmetric otherwise. It therefore follows that $< E_{\nu y} , \partial E_{\mu y}/\partial z > = 0$ if one of the modes is symmetric and the other antisymmetric. Consequently, such a taper cannot convert a symmetric mode to an antisymmetric one, or vice versa. We will find this rule useful in Chapter 9.

PROBLEMS

6.1. Sketch the dispersion diagram for guided modes in a metal-walled waveguide.

6.2. Find the cutoff frequency of a planar waveguide, formed from a slab of dielectric with metallized walls. The slab has thickness 0.5 μm and refractive index 1.5.
[2×10^{14} Hz]

6.3. Show that the eigenvalue equation for symmetric modes in a symmetric slab dielectric waveguide is given by: $\tan(\kappa h/2) = \gamma/\kappa$, where γ, κ and h have their usual meanings.

6.4. A symmetric slab guide is to be used at 1.5 μm wavelength. What is the V-value of the guide, if its thickness is 6 μm, and the refractive indices of the layers are defined by $n_1 \approx n_2 \approx 1.5$, and (a) $n_1 - n_2 = 0.05$, and (b) $n_1 - n_2 = 0.005$? How many modes are supported in each case?
[(a) 4.866; (b) 1.539; 4 modes; 1 mode]

6.5. Show that the eigenvalue equation for TM modes in an asymmetric slab dielectric waveguide is given by: $\tan(\kappa h) = \kappa(\gamma' + \delta') / (\kappa^2 - \gamma'\delta')$, where γ, κ and h have their usual meanings, and γ' and δ' are given by: $\gamma' = \gamma (n_1^2 / n_2^2)$ and $\delta' = \delta (n_1^2 / n_3^2)$.

6.6. Sketch the variation of the transverse magnetic field for the lowest-order TM mode of a symmetric slab dielectric waveguide (a) far from cutoff, and (b) near to cutoff.

6.7. A symmetric slab guide has refractive indices $n_1 = 1.505$, $n_2 = 1.5$. If the guide thickness is 1 μm, estimate the effective index of the lowest-order TE mode of the guide at $\lambda_0 = 0.633$ μm.
[1.50125]

6.8. Show by direct integration that two symmetric TE modes of a symmetric slab dielectric waveguide are orthogonal to one another, if the modes are of different order.

6.9. Two identical planar waveguides are butt-coupled together, as shown below. Assuming that the transverse variation of the electric field in the left-hand guide is $E_y(x) = E_0 \exp(-\alpha x^2)$, calculate and sketch the dependence of the coupling efficiency on any accidental lateral misalignment δ.
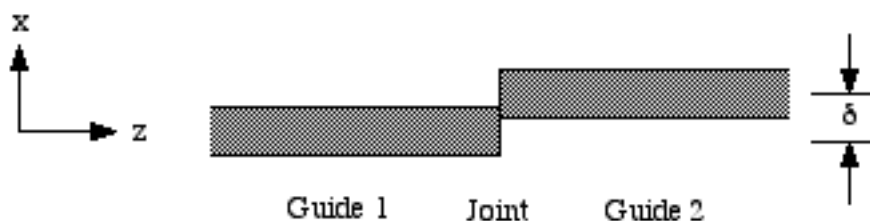
6.10. (i) Figure I) below shows a discontinuity between two different, symmetric slab dielectric waveguides. Describe qualitatively what you expect to occur when the lowest-order guided mode is incident from the left, assuming that the discontinuity is (a) small, and (b) large.

(ii) The two guides are now joined by a taper section, as shown in Figure ii). Describe what will occur under similar circumstances, if the taper is (a) fast and (b) slow.

Guide 1 | Guide 2

(i)

Guide 1 | taper | Guide 2

(ii)
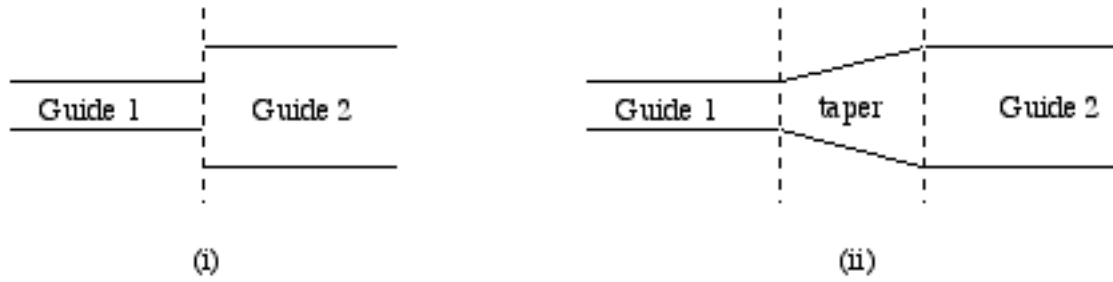
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