# MTH607 Lecture Notes

## TODO

- (01-12-2026) Create a system that makes connecting theorems and definitions back to their original declaration easier.

## 01-12-2026

### Pre-Lecture

### Lecture

**Definitions:** Graph — Adjacency — Incidence — Order — Size — $(p, q)$-Graph — Null Graph — Finite Graph — Infinite Graph — Self Loop — Multiple Edges — Parallel Edges — Simple Graph — Degree — Isolated Vertex — Pendant Vertex — Internal Vertex — Minimum Degree — Maximum Degree — Degree Sequence — Graphical Sequence — Subgraph — Spanning Subgraph — Complete Graph — Bipartite Graph — Complete Bipartite Graph — Regular Graph — Induced Subgraph — Edge-Induced Subgraph — Isomorphic Graph

**Definition: Graph**

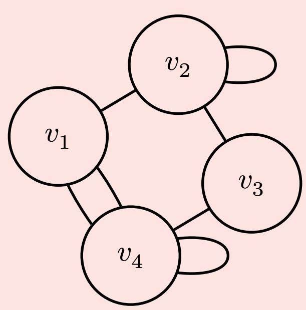
A graph $G$ is an ordered Triple (V, E, $\psi$) such that:

$v = \{v_1, v_2, v_3, \ldots\}$ = set of all vertices/nodes.

$e = \{e_1, e_2, e_3, \ldots\}$ = set of all edges/line/arc.

$\psi$ is the adjacency relation $\psi : E \to V \times V$ defining the association between each edge with each vertex pair of $G$.

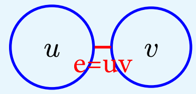
**Example: Graph**




**Definition: Adjacency, Incidence**

If $e = uv$ or $vu$, then:

1. $u$ and $v$ are **adjacent** vertices.
2. $e$ is **incident** with vertices $u$ and $v$




**Definition: Order**

The order of a graph $G$, denoted by $V(G)$ is the number of the vertices in $G$.

In other words, the cardinality of V, $|V|$.

**Definition: Size**

The size of G, $\varepsilon(G)$ is the number of edges in $G$.

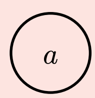
In other words, the cardinality of E, $|E|$.

**Definition: (p, q)-Graph**

A graph with $p$ vertices and $q$ edges is called the (p, q)-graph.

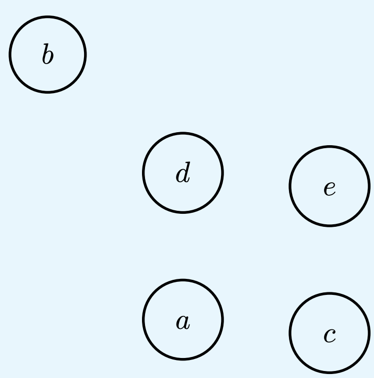
**Example: Trivial Graph**

The trivial graph, also known as the (1, 0)-graph, is a graph with only 1 vertex.




**Definition: Null Graph**

A graph without edges is called the null graph.

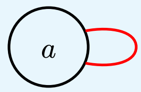



**Definition: Finite, Infinite Graphs**

A graph with a finite number of vertices and edges is called a finite graph. Otherwise, it is called an infinite graph.

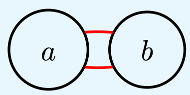
**Definition: Self Loop**

An edge of a graph that joins a vertex to itself.



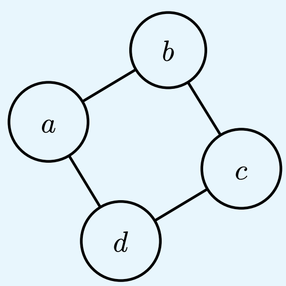

**Definition: Multiple/Parallel Edges**

The edges connecting the same pair of vertices are called multiple edges or parallel edges.



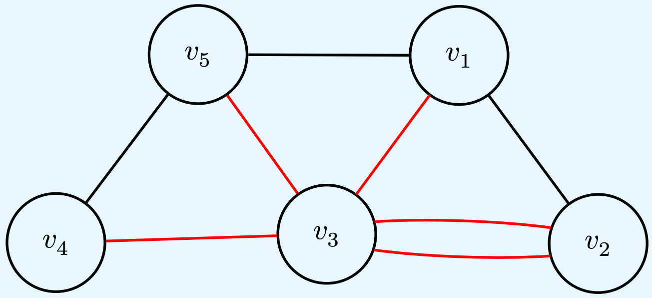

**Definition: Simple Graph**

A graph with no loop or parallel edges. Otherwise it is called a **multi-graph**.




**Definition: Degree**

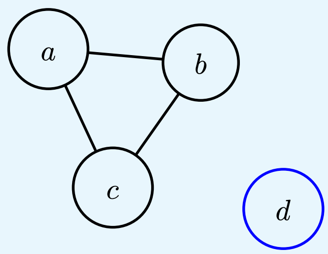
The degree of a vertex, denoted by $d(v_i)$, are the number of edges incident to that vertex. In the example below, $d(v_3) = 5$.




**Note:** Self-loops are counted twice.

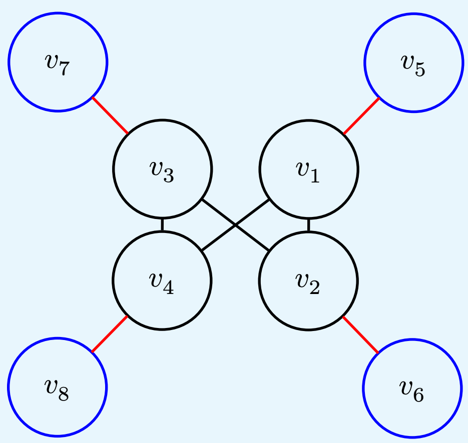
**Definition: Isolated Vertex**

A vertex with no incident edges. If a vertex $v$ is isolated, $d(v) = 0$.



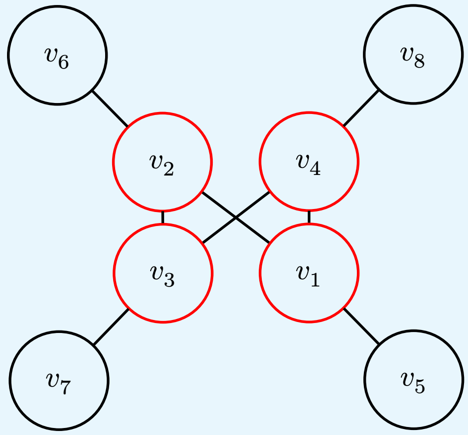

**Definition: Pendant Vertex**

A vertex with 1 incident edge. If a vertex $v$ is pendant, $d(v) = 1$.




**Definition: Internal Vertex**

A vertex that is neither isolated nor pendant.




**Definition: Minimum Degree of $G$**

$\delta(G) = \min\{d(v) \mid v \in V\}$

**Definition: Maximum Degree of $G$**

$\Delta(G) = \max\{d(v) \mid v \in V\}$

Knowing the above two, $\forall d(v)$, it follows that $d(v) \in [\delta(G), \Delta(G)]$.

**Theorem: Sum of all Degrees**

In a graph $G$, the sum of degrees of all the vertices is equal to twice the number of edges.

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} d(v_i) = 2|E|$$

**Proof:** Every edge contributes 2 vertices to the graph.

**Theorem:**

For any graph $G$:

$$\delta(G) \leq \frac{2|E|}{|V|} \leq \Delta(G)$$

**Proof:**

$$\delta(G) \leq d(v_1) \leq \Delta(G)$$
$$\delta(G) \leq d(v_2) \leq \Delta(G)$$
$$\vdots$$
$$\delta(G) \leq d(v_n) \leq \Delta(G)$$

Take the sum, you get:

$$n\delta(G) \leq \sum_{i=1}^{n} d(v_i) \leq n\Delta(G)$$
$$n\delta(G) \leq n\frac{2|E|}{|V|} \leq n\Delta(G)$$
$$\delta(G) \leq \frac{2|E|}{|V|} \leq \Delta(G)$$

**Theorem:**

For any graph $G$, the number of vertices with odd degrees is always even.

**Proof:**

$$\sum_{i=1}^{n} d(v_i) = d(v_1) + d(v_2) + \ldots + d(v_n) = 2|E|$$

Note that $2|E|$ is even.

Divide the vertices into two sets, even vertices and odd vertices.

$$2|E| = \sum_{i=1}^{m} d(v_i) + \sum_{j=1}^{k} d(v_j)$$

Note that $k + m = n$.

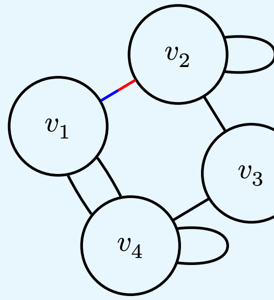
$d(v_i)$ is even. $d(v_j)$ is odd.

We need to have an even number of odd vertices for the second term to be even. In order words, $k$ has to be even.

**Definition: Degree Sequence**

A graph of order $n$ is the n-term sequence of the vertex degrees.

TOOD: Color this graph to show the degree counts. Hint: [color="blue;0.5:red"] splits the color in half. Use whatever colors you want, except red and blue (as those are your default edge and node colors respectively).




Let's get the degrees of each vertex:

$d(v_1) = 3$

$d(v_2) = 4$

$d(v_3) = 2$

$d(v_4) = 5$ (recall that self-loops are counted twice)

Let's calculate $\Delta$ and $\delta$:

$\Delta(G) = 5$

$\delta(G) = 2$

$(v_4, v_2, v_1, v_2) = (5, 4, 3, 2)$

**Note:** Some books write degree sequences in descending order, some in ascending order.

**Definition: Graphical Sequence**

An integer sequence is said to be graphical if it is the degree sequence of some graph.

**Example: Graphical Sequence**

Let $S$ be a graphical sequence:

$$S = (5, 4, 3, 3, 2, 2, 2, 1, 1, 1, 1)$$

Is $S$ a graphical sequence? No, because the number of the odd vertices is odd.

**Example: Graphical Sequence**

Let $S$ be a graphical sequence:

$$S = (9, 9, 8, 7, 7, 6, 6, 5, 5)$$

Is $S$ a graphical sequence? No, there are 9 vertices and the degree of at least one of the vertices is $\geq$ 9. It is not possible because the maximum possible degree for a vertex $v$ in a simple graph is $(n-1)$ where $n$ is the number of the vertices.

**Definition: Subgraph**

A graph $H(v_1, E_1)$ is said to be a subgraph of $G(V, E)$ if $v_1 \subseteq V$ and $E_1 \subseteq E$.

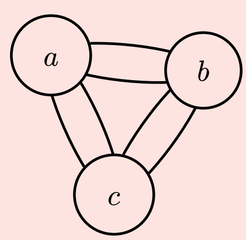
**Definition: Spanning Subgraph**

A graph $H(v, E_1)$ is said to be a spanning subgraph of $G(V, E)$ if $E_1 \subseteq E$, but not $v_1 \subseteq V$

**Definition: Complete Graph**

A simple graph in which every pair of distinct vertices are connected by a unique edge. A complete graph of order $n$ is denoted as $k_n$.

**Example: $k_3$**




**Theorem:**

A $k_n$ graph has $\frac{n(n-1)}{2}$ edges.

**Proof:** TLDR; For every vertex $v$, $d(v) - (n-1)$,

**Definition: Bipartite Graph**

A graph $G$ is bipartite if its vertex set $V$ can be partitioned into two sets $v_1$ and $v_2$ such that no two same vertices in the same partition are adjacent.

$$v_1 \cup v_2 = v$$
$$v_1 \cap v_2 = \emptyset$$

**Definition: Complete Bipartite Graph**

A bipartite graph in which every vertex of one partition is adjacent to every vertex of the other.

**Example: Complete Bipartite Graph**

TODO: Add the example in your notebook.

A complete bipartite graph with $a$ (white vertices) and $b$ (black vertices) denoted by $k_{\text{a, b}}$ has $ab$ edges.

**Definition: Regular Graph**

A graph is said to be $r$-regular if the degree of all its vertices is equal to

**Example: 2-Regular Graph**

**Example: Petersen Graph**

A special type of 3-regular graph.

**Theorem:**

A $k_n$ graph is an $(n-1)$-regular graph.

**Proof:**

**Definition: Induced Subgraph**

Suppose $v'$ is a subset of $V$ of graph $G(V, E)$. Then the subgraph of $G$ whose vertex set is $v'$ and whose edge set contains the edges of $G$ with both end vertices in $v'$ is called an induced subgraph.

**Definition: Edge-Induced Subgraph**

Suppose $E'$ is a subset of $E$ of a graph $G(V, E)$. Then the subgraph of $G$ whose edge set is $E'$ and whose vertex set is the set of the end vertices of the edges in $E'$ are called an edge-induced subgraph of $G$.

**Definition: Isomorphic Graph**

An isomorphism of two graphs $G$ and $H$ is a bijection $f : V(G) \to V(H)$ such that any two vertices $u$ and $v$ of $G$ are adjacent in $G$ if and only if $f(u)$ and $f(v)$ are adjacent in $H$.

In other words, $G$ and $H$ are isomorphic if:

$$|V(G)| = |V(H)|$$
$$|E(G)| = |E(H)|$$
$$v_i v_j \in E(G) \leftrightarrow f(v_i) f(i_j) \in E(H)$$

Note: If two graphs are isomorphic, they have the same degree sequence. However, it doesn't guarantee it.

**Example: Graphs That Are Isomorphic**

$G_1$ and $G_2$

$f : V(G_1) \to V(G_2)$

$$v_1 \longrightarrow f \longrightarrow u_1 : f(v_1) = u_1$$
$$v_2 \longrightarrow f \longrightarrow u_2 : f(v_2) = u_2$$
$$v_3 \longrightarrow f \longrightarrow u_3 : f(v_3) = u_3$$
$$v_4 \longrightarrow f \longrightarrow u_4 : f(v_4) = u_4$$
$$v_5 \longrightarrow f \longrightarrow u_5 : f(v_5) = u_5$$

Prof checks all edges to make sure that the bijection is there.

**Exercise: Graphs That Aren't Isomorphic**

Explain why the following two graphs are not isomorphic:

$G$, $H$

**Solution:**

Consider $G$:

abs(V(G)) = 8 abs(E(G)) = 10

$$d(v_1) = 3$$
$$d(v_2) = 2$$
$$d(v_3) = 3$$
$$d(v_4) = 2$$
$$d(v_5) = 2$$
$$d(v_6) = 3$$
$$d(v_7) = 2$$
$$d(v_8) = 3$$

$S_G = (3, 3, 3, 3, 2, 2, 2, 2)$

Consider $H$:

abs(V(G)) = 8 abs(E(G)) = 10

$$d(u_1) = 3$$
$$d(u_2) = 3$$
$$d(u_3) = 2$$
$$d(u_4) = 2$$
$$d(u_5) = 3$$
$$d(u_6) = 3$$
$$d(u_7) = 2$$
$$d(u_8) = 2$$

$S_H = (3, 3, 3, 3, 2, 2, 2, 2)$

Note: In graph $H$, $u_7$ and $u_8$ are adjacent and are both of degree 2. However, in graph $G$, no two vertices of degree 2 are adjacent.

The problem is $v_7$ in $G$. It isn't adjacent to any vertices of degree 2. However, $v_7$ in $G$ IS adjacent to a vertex of degree 2.

Thus, $G$ and $H$ cannot be isomorphic.

### Post-Lecture

## 01-19-2026

**Definitions:** Union of 2 Graphs — Intersection of 2 Graphs — Ringsum of 2 Graphs — Edge-Disjoint — Graph Complement — Self-Complementary Graph — Edge Deletion — Vertex Deletion — Walk — Trail — Path — Cycle — Length — Vertex Distance — Connected Graph — Component

### Pre-Lecture

Cycles, paths, maybe connectivity? Maybe trees? Maybe forests?

### Lecture

#### Operations on Graphs

**Definition: Union of 2 Graphs**

$G = G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$

**Definition: Intersection of 2 Graphs**

$G = G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$

**Definition: Ringsum of 2 Graphs**

$G = G_1 \oplus G_2 = (V_1 \cup V_2, (E_1 \cup E_2) - (E_1 \cap E_2))$.

Note: This is the equivalent of applying XOR between the two edge sets $E_1$ and $E_2$.

**Definition: Edge-Disjoint**

$G_1, G_2$ are called edge-disjoint if $G_1 \cap G_2$ is a null graph and $G_1 \oplus G_2$. It's called edge-disjoint because it must pass that $E(G_1) \cap E(G_2) = \emptyset$

**Example: Diestal Figure 1.1.2**

TODO

**Definition: Graph Complement**

The complement of graph $G$, denoted by $\overline{G}$ in a graph with $V\left(\overline{G}\right) = V(G)$ such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in $G$.

**Definition: Self-Complementary Graph**

A self-complementary graph is a graph that is isomorphic to its complement.

Note: $G \cap \overline{G} = k_n$.

**Example: Diestal Figure 1.1.4**

TODO

Recall that a homomorphism is a map $\varphi : V \to V'$ from $G = (V, E)$ to $G' = (V', E')$ such that if $\{x, y\} \in E$, then $\{\varphi(x), \varphi(y)\} \in E'$. For $\varphi$ to be isomorphic means that it's a bijection.

**Example:**

Prove that for any self-complementary graph, $G$ of order $n$, $n \equiv 0$ or $n \equiv 1$.

In other words, the remainder of the divison of $n$ by 4 is either 0 or 1.

Solution: Recall:

$$|E(G)| + \left|E\left(\overline{G}\right)\right| = \frac{n(n-1)}{2}$$

Since $G$ is self-complementary, $|E(G)| = \left|E\left(\overline{G}\right)\right|$. It follows that $|E(G)| = \frac{n(n-1)}{4}$.

**Definition: Edge Deletion**

If $e \in E(G)$, then $G - \{e\}$ (otherwise marked $G - e$) is a graph obtained by removing edge $e$.

**Definition: Vertex Deletion**

If $v \in V(G)$, then $G - \{v\}$ (otherwise marked $G - v$) is a graph obtained by removing vertex $v$ and all the edges incident on $v$.

**Walks, Paths, Cycles**

**Definition: Walk**

A walk in a graph $G$ is an alternating sequence of vertices and connecting edges in $G$.

**Definition: Walk (Diestal)**

A walk (of length $k$) in a graph $G$ is a non-empty alternating sequence $v_0 e_0 v_1 e_1 \ldots e_{k-1} v_k$ of vertices and edges in graph $G$ such that $e_i = \{v_i, v_{i+1}\}, \forall i < k$.

**Definition: Trail**

A trail is a walk that doesn't pass over the same edge twice.

**Definition: Path**

A path is a walk that doesn't include the same vertex twice.

**Definition: Cycle**

A cycle is the same as a path except that the initial and final vertices are the same.

**Definition: Length**

The length of a path or cycle is the number of edges in it.

**Definition: Vertex Distance**

The distance between two vertices $u, v$ denoted by $d(u, v)$ is the length of the shortest path connecting $u$ to $v$.

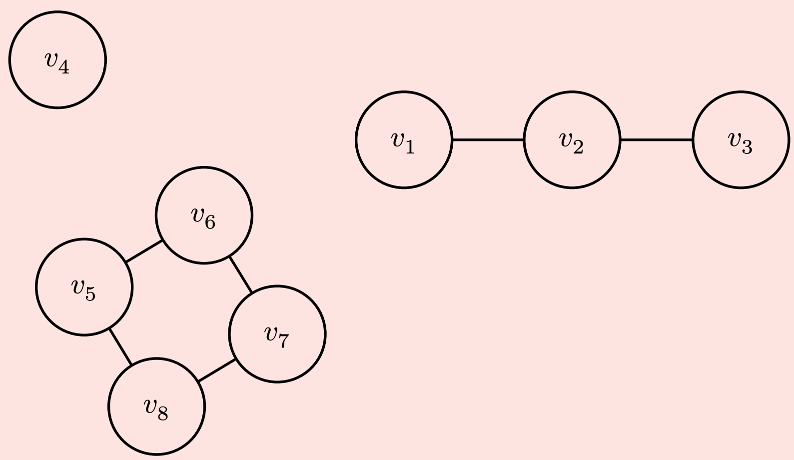
**Definition: Connected Graph**

A connected graph is a graph in which there is at least one path between any two vertices.

**Definition: Component**

A component of a graph is a maximal connected subgraph of $G$. In other words, it is a connected subgraph that is not part of any larger connected subgraph.

**Example:**




$G = G_1 \cup G_2 \cup G_3$. $G$ has 3 components.

**Theorem:**

A connected graph $G$ is bipartite if and only if $G$ doesn't have an odd-length cycle.

Note: In Diestal, this is Proposition 1.6.1

*Proof:* Both sides must be proven.

(A connected graph $G$ is bipartite $\rightarrow$ $G$ doesn't have an odd-length cycle)

Let $G = ((V_1, V_2), E)$, $V_1 \cap V_2 = \emptyset$. For every edge $xy \in E$, $x \in V_1, y \in V_2$. Let $C^n$ be an odd cycle. It's implied that $n$ (the cycle length) is odd. For every $v_i \in C$, consider two conditions:

$$v_i \in V_1, \text{if } i \text{ is odd}$$
$$v_i \in V_2, \text{if } i \text{ is even}$$

Since $n$ is odd, $v_n \in V_1$. There also exists edge $v_n v_1$. Yet, by the condition above, $v_1 \in V_1$. The graph is supposed to be bipartite. Yet, two vertices in the same set are adjacent. Contradiction.

($G$ doesn't have an odd-length cycle $\rightarrow$ A connected graph $G$ is bipartite) (Proof by Counter-Example)

Let $v \in V$. Partition $V(G)$ into $V_1, V_2$:

$$V_1 = \{x \in V(G) \mid d(v, x) \text{ is even}\}$$
$$V_2 = \{y \in V(G) \mid d(v, y) \text{ is odd}\}$$

Recall that $d(v_1, v_2)$ is the distance between both vertices. We can take advantage of this by denoting this as the shortest path between $v_1$ and $v_2$.

Now, we need to prove that if $xy$ is in $E$, $x \in V_1$ and $y \in V_2$.

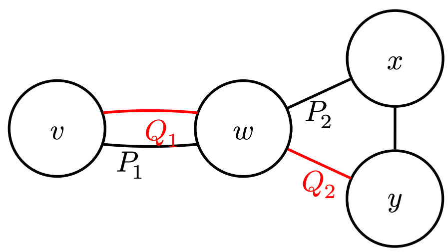
Consider two vertices $x, y \in V_1$. We show that this results in a contradiction.

$P(v \to x)$: The shortest path from $v$ to $x$.

$Q(v \to y)$: The shortest path from $v$ to $y$.

The length $P$ denoted by $|P|$ is even. Moreover, the length of $Q$ denoted by $|Q|$ is even.

Note: Paths $P$ and $Q$ share at least one vertex, $v$. We assume $w$ be that last vertex that $P$ and $Q$ share.




$|P_1| = |Q_1|$. Why? Because if weren't (eg, $|P_1| < |Q_1|$), we could take $P_1 + Q_2$ as the shortest path from $v$ to $y$ whereas we assumed that the shortest path from $v$ to $y$ is $Q_1 + Q_2$.

$$|P| = |P_1| + |P_2|$$
$$|Q| = |Q_1| + |Q_2|$$

$|P_2|$ and $|Q_2|$ are either both even or odd.

Thus, the cycle $w \to y \to x \to w$ is an odd cycle.

We assumed that $G$ has no odd cycle. This is a contradiction.

**Theorem:**

If $G$ has exactly 2 odd vertices, there exists a path joining them.

*Proof:* Consider two cases,

Case: $G$ is connected

There exists at least one path connecting any 2 vertices including the two odd vertices.

Case: $G$ is disconnected

Then both of these odd vertices should be in the same component. Since each component is a connected graph, we return to the previous case.

Note: These two vertices cannot belong to different components because the number of odd vertices in each component must be even.

**Lemma:**

$$\sum_{i=1}^{k} (n_i)^2 \leq n^2 - (k-1)(2n-k)$$

Proof: Assume $n_i$ is the number of the vertices in the $k^{\text{th}}$ component.

$$\begin{aligned} \sum_{i=1}^{k} (n_i - 1)^2 &= (n-k)^2 \\ &= n^2 + k^2 - 2nk \end{aligned}$$

$$\sum_{i=1}^{k} (n_i - 1)^2 + k + \text{non-negative terms} = n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 + \text{non-negative terms}$$

$$\begin{aligned} n^2 + k^2 - 2nk + 2n - k &= n^2 - k(k-2n) + (2n-k) \\ &= n^2 - (k-1)(2n-k) \end{aligned}$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k)$$

$$\left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = \left[(n_1-1)^2 + (n_2-1)^2 + \ldots + (n_k-1)^2\right]^2$$

TODO: Complete the expansion above.

**Theorem:**

Assume $G$ is a graph with $n$ vertices and k-components. Then $|E(G)| \leq \frac{1}{2}(n-k)(n-k+1)$.

*Proof:* We show that the max number of edges in component $i$ is $\frac{n_i(n_i-1)}{2}$. Plugging it into the sum proved above:

$$\begin{aligned}\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} &= \frac{1}{2}\left[\sum_{i=1}^{k}(n_i^2 - n_i)\right] \\ &= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i \\ &= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}n\end{aligned}$$

$$\frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}n \leq \frac{1}{2}(n^2 - (k-1)(2n-k)) - \frac{n}{2} = \frac{1}{2}(n-k)(n-k+1)$$

### Post-Lecture

Some concepts to really drill in:

- Spanning Subgraph
- Bipartite Graph
- Isomorphisms

## 01-26-2026

### Pre-Lecture

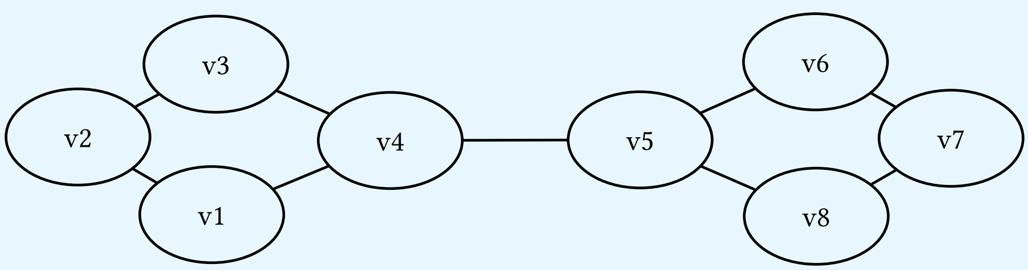
### Lecture

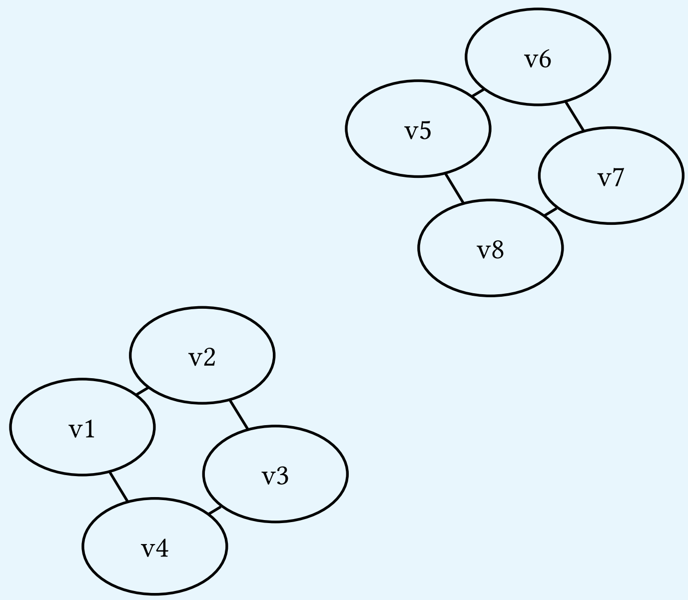
**Definitions:** Cut Edge/Bridge — Cut Vertex — Eulerian Graph — Eulerian Cycle — Fleury Algorithm — Edge-Disjoint Cycles — Semi-Eulerian Graph

**Theorems:** Euler's Theorem — Cycle Decomposition

**Definition: Cut Edge/Bridge**

An edge $e \in E$ of a graph $G \in (E, V)$ is said to be a cut edge or a bridge if $G - e$ is a disconnected graph.

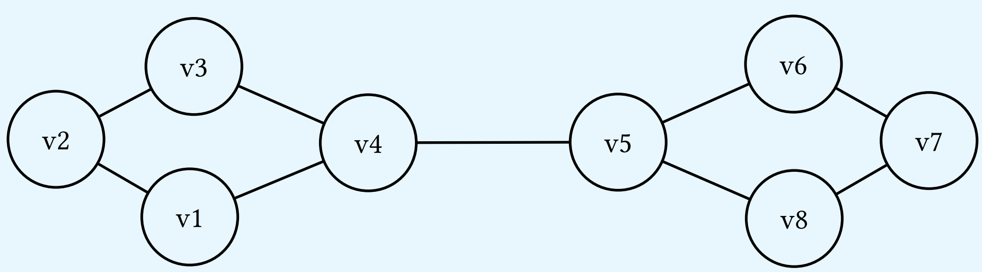




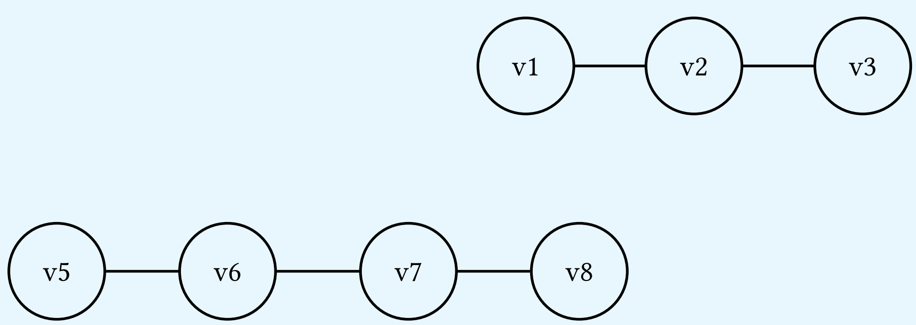

**Definition: Cut Vertex**

An vertex $v \in V$ of a graph $G \in (E, V)$ is said to be a cut vertex if $G - v$ is a disconnected graph.

$G$




$G - v_4$




**Definition: Eulerian Graph**

A connected graph $G = (V, E)$ is called Eulerian if there exists a closed trail that includes every edge of $G$.

**Definition: Eulerian Cycle/Circuit**

The closed trail of an Eulerian Graph is called its Eulerian cycle/circuit.

**Theorem: Euler's Theorem**

A connected graph $G$ is Eulerian if and only if every vertex in $G$ has an even degree.

**Proof:**

*Lemma:* If for a graph $G = (V, E)$, $\deg(v) \geq 2, \forall v \in V$, then $G$ contains a cycle.

Proof: Pick an arbitrary vertex $v \in V$. Choose a vertex $v_1 \in V$ such that $vv_1$ exists. Now, for each $i > 1$, choose $v_{i+1}$ to be any vertex adjacent to $v_i$ except $v_{i-1}$. Note the existence of $v_{i+1}$ is guranteed because of the assumption that the degree of every vertex is $\geq 2$. Since $G$ has a finite number of vertices, we will eventually encounter a vertex that we have met before. In other words, we have traversed a cycle.

($G$ is Eulerian $\Rightarrow$ It contains an Eulerian cycle)

*Claim:* If $G$ is Eulerian, then it contains an Eulerian cycle.

Proof: Assume $G$ is Eulerian. This implies the existence of a cycle, $C \subseteq G$. $\forall v \in C$, everytime $v$ is entered through an edge, it needs to be left through another edge. Thus, $\deg(v), \forall v \in C$ is even. It should be noted that an Eulerian cycler contains all the vertices at least once. Thus, $\deg(v)$ is even for all vertices $v \in G$.

($G$ contains an Eulerian cycle $\Rightarrow$ $G$ is Eulerian)

*Claim:* If $G$ contains an Eulerian cycle, then $G$ is Eulerian

Proof: Proof by strong induction on the number of edges of $G = (V, E)$, $m = |E|$.

*Claim (Base Step):* The null graph, $m = 0$

Proof: The degree of every vertex is 0, an even number. Thus, every vertex forms an Eulerian cycle. It follows that the null graph is Eulerian.

*Assume (Induction Step):* Assume that for any graph with $k < m$ edges if the degree of every vertex is even, the graph is Eulerian. Following, assume that we have a graph $G$ with $m$ edges where the degree of every vertex is even. By the lemma, this implies that $G$ contains a cycle, $D$. We create a new graph $G' = G - E(D)$. In other words, we take the cycle $D$ out of the graph. Note that every vertex of $G'$ is also of even degree. $G'$ is a graph $G$ with fewer than $m$ edges where the degree of each vertex is even.

By the induction step, $G'$ is Eulerian. In other words, it contains an Eulerian cycle.

Now, we pick a vertex of the cycle $D$ and start traversing the cycle $D$ and start traversing the cycle each time we encounter a component of $G'$, we detour the Eulerian cycle of that component. Once we get back to where we started, we will have traversed an Eulerian cycle of the whole graph $G$.

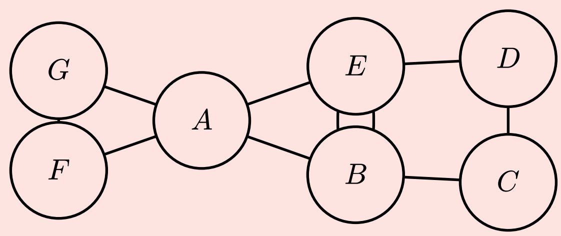
Thus, $G$ is Eulerian.

**Definition: Fleury Algorithm**

To find an Eulerian cycle in an Eulerian graph $G$. Start at any vertex $u$ and traverse the edges in any arbitrary manner subject only to the following rules:

1. Erase the edges as they traversed and erase any isolated vertices.
2. sAt each stage, use a cut vertex, if there is no other alternative.

**Example:**




Start from vertex $A$.

- Delete $AB \Rightarrow AB$
- Delete $BE \Rightarrow ABE$

Note: When we reach $E$, we can't delete $EA$ because it's a cut vertex. So we delete $ED$.

- Delete $ED \Rightarrow ABED$
- Delete $DC \Rightarrow ABEDC$
- Delete $CB \Rightarrow ABEDCB$
- Delete $BE \Rightarrow ABEDCBE$
- Delete $EA \Rightarrow ABEDCBEA$
- Delete $AG \Rightarrow ABEDCBEAG$
- Delete $GF \Rightarrow ABEDCBEAGF$
- Delete $FA \Rightarrow ABEDCBEAGFA$

$ABEDCBEAGFA$ is a Eulerian cycle.

**Definition: Edge-Disjoint Cycles**

Cycles in a common graph $G$ with no common edges.

**Theorem: Cycle Decomposition**

$G$ is Eulerian if and only if $G$ can be decomposed into edge-disjoint cycles.

**Proof:**

($G$ can be decomposed into edge-disjoint cycles $\Rightarrow G$ is Eulerian)

*Claim:* If $G$ can be decomposed into edge-disjoint cycles, then it is Eulerian.

Proof: Assume that $G$ can be decomposed into edge-disjoint cycles. Therefore, any vertex $v \in G$ falls into one of two cases,

1. $v$ is isolated, $\deg(v) = 0$
2. $v$ belongs to one or more cycles

For every cycle that $v$ belongs to, $v$ has two degrees, recalling that every vertex of a cycle is of degree 2. Thus, $\deg(v)$ is even. Thus, $G$ is Eulerian.

($G$ is Eulerian $\Rightarrow G$ can be decomposed into edge-disjoint cycles)

*Claim:* If $G = (V, E)$ is Eulerian, then $G$ can be decomposed into edge-disjoint cycles

Proof: $G$ is Eulerian. Use strong induction on the number of edges $m = |E|$.

*Claim (Base Step):* $m = 0$, no edges.

Proof: You can define a cycle on each vertex.

*Claim (Induction Step):* Assume that for all Eulerian Graphs with $h \in [0, m)$, there is a cycle decomposition.

Proof: Consider an Eulerian graph $G$ with $m$ edges. We take all the isolated vertices of $G$ and consier each to be a cycle. The resulting graph $F$ has vertices that are all even ($\geq 2$). Thus, there is a cycle $C$ in $F$. Remove the edges of $C$ from $F$ and we have a new graph $G' = F - E(C)$, we know that every vertex of $G'$ is even and the number of the edges of $G'$ is less then $m$. By the induction step, $G'$ has a cycle decomposition. Now, if we add the cycle $C$ back in $G'$, we will have a cycle decomposition on $F$. Therefore, there will be a cycle decomposition on $G$.

**Definition: Semi-Eulerian Graph**

A graph in which exists a non-closed trail that includes every edge of the graph.

**Theorem:**

A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.

Proof:

($G$ is semi-Eulerian $\Rightarrow$ There is a non-closed trail that contains all the edges of $G$)

*Claim:* If $G$ is semi-Eulerian, then there is a non-closed trail that contains all the edges of $G$

Proof: If I connect the first and the last vertices together, I will have a graph $G'$ which contains an Eulerian cycle, that is consequently Eulerian. Each vertex of $G'$ is even. Since the addition of an edge means adding two degrees to the graph. Thus, there must have been two odd degrees in $G$.

(There is a non-closed trail that contains all the edges of $G \Rightarrow G$ is semi-Eulerian)

*Claim:* If there exists a non-closed trail that contains all the edges of $G$, then $G$ is semi-Eulerian

Proof: Given as lab exercise.