Static Nuel Games with Terminal Payoff

Symeon Mastrakoulis and Athanasios Kehagias

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Abstract

In this paper we study a variant of the *Nuel* game (a generalization of the *duel*) which is played in turns by N players. In each turn a single player *must* fire at one of the other players and has a certain probability of hitting and killing his target. The players shoot in a fixed sequence and when a player is eliminated, the "move" passes to the next surviving player. The winner is the last surviving player. We prove that, for every $N \geq 2$, the Nuel has a stationary Nash equilibrium and provide algorithms for its computation.

1 Introduction

In this paper we study a variant of the *nuel* game (a generalization of the *duel*) which is played in turns by N players. In each turn a single player *must* fire at one of the other players (in other words, *abstention* is not allowed) and has a certain probability of hitting and killing his target. The players shoot in a fixed sequence and when a player is eliminated, the "move" passes to the next surviving player. The winner is the last surviving player.

In what follows, we will use the term "N-uel" to describe the N-player game; hence the 2-uel or *duel* involves two players, the 3-uel or *truel* involves three players etc.

Early works on the static 3-uel are [11, 16, 18, 19, 20] in which the postulated game rules guarantee the existence of *exactly one* survivor ("winner"). A more general analysis appears in [17] which considers the possibility of "cooperation" between the players. This idea is further studied in [13, 14, 15, 25]. Recent papers on the truel include [1, 2, 3, 5, 6, 7, 8, 9, 22, 23, 24]. Only a few of these papers [1, 2, 3, 25] make short remarks on the general N-uel (i.e., for $N \geq 3$) and of these only [25] but the matter is not pursued in depth. [1]

By *solving* the N-uel, we mean establishing the existence of one or more *Nash equilbria* (NE) and computing these equilbria. As will be explained in a later section, solving the N-uel involves solving a system of nonlinear equations or, equivalently, minimizing the total *equilibrium error*. Hence the problem is essentially one of global optimization.

The paper is organized as follows. In Section 2 we present the rules of the game. In Section 3 we set up the N-uel *system of payoff equations* and prove that, for every N and under appropriate conditions, this system always has a unique solution, i.e., every N-uel has uniquely

[1] Let us also note the existence of an extensive literature on a quite different type of duel games, which essentially are *games of timing* [4, 10, 12]. However, this literature is not relevant to the game studied in this paper.

defined payoffs for all players. In Section 4 we prove that for every N, under appropriate conditions, the N-uel has a *stationary equilbrium strategy profile* and we provide algorithms to compute this profile (equivalently, to solve the payoff equations system). In Section 5 we present illustrative experiments. In Section 6 we conclude and present some future research directions.

2 The N-uel Game

We will now define the N-uel game rigorously for every $N \in \{2, 3, ...\}$. The game involves N players, who will be denoted by $P_1, ..., P_N$ or by $1, ..., N$, and evolves in discrete times (turns) $t \in \{0, 1, 2, ...\}$. For each $n \in \{1, ..., N\}$, P_n has a *marksmanship* p_n, which is the probability that he hits (and kills) the opponent whom he shoots. In what follows we assume that, for all $n \in \{1, ..., N\}$, p_n is *strictly* between 0 and 1. In the next subsections we define the components of the game.

2.1 States and Actions

Each *game state* has the form $\mathbf{s} = s_0 s_1 ... s_N$, where $s_0 \in \{0, 1, ..., N\}$ and, for $n \in \{1, ..., N\}$, $s_n = 0$ (resp. 1) means that P_n is dead (resp. alive). Now suppose that at time $t \in \{0, 1, 2, ...\}$ the game state is $\mathbf{s}(t) = s_0(t) ... s_N(t)$. We have the following possibilities.

1. If $s_0(t) = n \in \{1, ..., N\}$, then the game is in progress and P_n "has the move", i.e., P_n is the single player who will shoot in the current turn. In this case we will also have $s_1(t) ... s_N(t) \in \{0, 1\}^N$; $s_i(t) = 1$ means that P_i is alive and $s_i(t) = 0$ means that he is dead.

2. If $s_0(t) = 0$, then the game has terminated (i.e., there is a single alive player) and we must actually have $s_1(t) ... s_N(t) = 0...010...0$; i.e., there exists a single n such that $s_n(t) = 1$ (P_n is the single alive player) and, for $m \neq n$, $s_m(t) = 0$ (P_m is dead).

We exclude from consideration *inadmissible* states, i.e., states which will never be visited during a play of the game[2]. It will be useful to define the following sets of *admissible* states:

$$\forall k \in \{2, ..., N\} : S_k = \left\{ \mathbf{s} = s_0 s_1 ... s_N : s_0 \neq 0 \text{ and } s_{s_0} = 1 \text{ and } \sum_{n=1}^{N} s_n = k \right\},$$
$$\forall k \in \{1, ..., N\} : \widetilde{S}_k = \{\mathbf{s} = 0 s_1 ... s_N : \text{and } s_k = 1 \text{ and } s_n = 0 \text{ for } n \neq k\}.$$

S_k is the set of states in which k players are alive and P_{s_0} is the (alive) player who has the move; $\widetilde{S}_k$ is the set of states in which the sole surviving player is P_k and no player has the move. Letting $S_1 = \bigcup_{k=1}^{N} \widetilde{S}_k$, the set of all admissible states is $S = \bigcup_{k=1}^{N} S_k$.

Game actions have the form $a{=}m \in \{\lambda, 1, ..., N\}$. Suppose that at time $t \in \{1, 2, ...\}$ the game action is $a(t) = m$; if $m > 0$ then the player who has the move (as determined by the

[2]For example, with $N = 3$ players, the state $s_0 s_1 s_2 s_3 = 1011$ will never occur because it corresponds to the dead player P_1 having the move.

corresponding game state $\mathbf{s}$) will fire at P_m. Depending on the current state, an admissible action must satisfy the following: a player cannot fire either at himself or at an already dead player. We write $a = \lambda$ to denote the "*null action*", i.e., no shooting takes place.

We assume that the game has *perfect information*, i.e., at every t all players know all previous states and actions.

2.2 State Transitions

The game evolution is described in terms of state transitions.

1. The game starts at some initial state $\mathbf{s}(0) = s_0(0)\, s_1(0)\, s_2(0) \ldots s_N(0)$.
2. Assume at $t \in \{1, 2, \ldots\}$ we have $\mathbf{s}(t-1) = s_0(t-1)\, s_1(t-1) \ldots, s_N(t-1)$ with $n = s_0(t-1) \neq 0$ and P_n performs action $a(t) = m \in \{1, \ldots, N\}$ where $s_m(t-1) = 1$. Then the next state is $\mathbf{s}(t) = s_0(t)\, s_1(t) \ldots s_N(t)$ and is obtained by the following rules.
   (a) For the $s_1(t) \ldots s_N(t)$ part of the state we have:
   $$\begin{aligned} &\Pr(s_m(t) = 0) = p_n \text{ (i.e., } P_n \text{ hit and killed } P_m\text{)}, \\ &\Pr(s_m(t) = 1) = 1 - p_n \text{ (i.e., } P_n \text{ missed } P_m\text{)}, \\ &\forall i \in \{1, \ldots, m-1, m+1, \ldots, N\} : \Pr(s_i(t) = s_i(t-1)) = 1, \end{aligned}$$
   where we assume that the marksmanships $p_n \in (0, 1)$ for all $n \in \{1, \ldots, N\}$.
   (b) The $s_0(t)$ part of the state specifies the next player who has the move. This will be the "next alive player" and is best understood by some examples for the three player case (analogous things hold for $N > 3$ players).
      i. Suppose $\mathbf{s}(t-1) = 1111$, $a(t) = 2$ and $s_1(t)\, s_2(t)\, s_3(t) = 111$. This means that P_1 had the move, fired at P_2 and missed him; hence the next player to have the move is P_2, i.e., $s_0(t) = 2$.
      ii. Suppose $\mathbf{s}(t-1) = 1111$, $a(t) = 2$ and $s_1(t)\, s_2(t)\, s_3(t) = 101$. This means that P_1 had the move, fired at P_2 and killed him; hence the next player to have the move is P_3, i.e., $s_0(t) = 3$.
      iii. Suppose $\mathbf{s}(t-1) = 3111$, $a(t) = 1$ and $s_1(t)\, s_2(t)\, s_3(t) = 111$. This means that P_3 had the move, fired at P_1 and missed him; hence the next player to have the move is P_1, i.e., $s_0(t) = 1$.
      iv. Suppposse $\mathbf{s}(t-1) = 1101$, $a(t) = 3$ and $s_1(t)s_2(t)s_3(t) = 100$. This means that P_1 had the move, fired at P_3 and killed him; hence there is no next player to have the move i.e., the game has terminated and $s_0(t) = 0$.
3. Finally, if $\mathbf{s}(t-1) = 0 \ldots 010 \ldots 0$ then the game has terminated and there is not either a next action $a(t)$ or a next state $\mathbf{s}(t)$.

Using the above rules, for every $N \in \{2, 3, ..\}$ we can construct a *controlled Markov chain* with state space S and transition probability matrix $\Pi(a)$, with

$$\begin{aligned} &\forall \mathbf{s}' \in S \backslash S_1 : \Pi_{\mathbf{s}', \mathbf{s}''}(a) = \Pr\left(\mathbf{s}(t) = \mathbf{s}'' | \mathbf{s}(t-1) = \mathbf{s}', a(t) = a\right), \\ &\forall \mathbf{s}' \in S_1 : \Pi_{\mathbf{s}', \mathbf{s}'}(\lambda) = 1. \end{aligned}$$

The state sequence obtained from the above controlled Markov chain corresponds exactly to the state sequence of a N-uel, except for the fact that, in the Markov chain, every terminal state $\mathbf{s}' \in S_1$ loops back to itself with probability one, producing an infinite state sequence, while in the N-uel as soon as a terminal state is entered the game is over, resulting in a finite state sequence. Note that when $\mathbf{s}' \in S_1$ the only admissible action is $a = \lambda$ (no-shoot action).

Because in each turn a player must shoot (unless he is the sole survivor) and $p_n > 0$ for all $n \in \{1, ..., N\}$, it is easy to prove the following.

Proposition 2.1 For all N, the probability that the N-uel terminates in finite time equals one.

The above complete the description of the N-uel, which we will also denote by $\Gamma_N(\mathbf{p})$, where $\mathbf{p} = (p_1, ..., p_N)$.

2.3 Histories, Payoffs and Strategies

We now proceed to define and discuss *histories*, *payoffs* and *strategies* used in $\Gamma_N(\mathbf{p})$. In view of Proposition 2.1, we only need to consider finite length histories. The game history at time t is $\mathbf{h}(t) = \mathbf{s}(0)\, a(1)\, \mathbf{s}(1) ... \mathbf{s}(t)$ and its *length* is t. The set of all *admissible terminal histories* is

$$H_0 = \{\mathbf{h}(t) = \mathbf{s}(0)\, a(1)\, \mathbf{s}(1) ... \mathbf{s}(t) : \ \mathbf{h}(t) \text{ can occur in the game and } \mathbf{s}(t) \in S_1\}.$$

The set of all *admissible nonterminal histories* is

$$H_1 = \{\mathbf{h}(t) = \mathbf{s}(0)\, a(1)\, \mathbf{s}(1) ... \mathbf{s}(t) : \mathbf{h}(t) \text{ can occur in the game } \mathbf{s}(t) \notin S_1\}.$$

The set of all *admissible histories* is $H = H_0 \cup H_1$.

For $n \in \{1, ..., N\}$, P_n's *payoff* function is $\mathsf{Q}_n : H \to \mathbb{R}$ and is defined as follows:

$$\forall n \in \{1, ..., N\}, \forall \mathbf{h} \in H : \mathsf{Q}_n(\mathbf{h}) = \begin{cases} 1 & \text{iff } \mathbf{h} = \mathbf{s}(0)\, a(1) ... \mathbf{s}(t) \in H_0 \text{ and } s_n(t) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

I.e., upon game termination, the single surviving player receives one payoff unit.

Let Δ_N denote the set of N-long probability vectors $\mathbf{x} = (x_1, ..., x_N)$. A *strategy* is a function $\sigma : H_1 \to \Delta_N \cup \{(0, ..., 0)\}$. Suppose $\mathbf{h} = \mathbf{s}(0)a(1)...a(t-1)ns1...sN$ and P_n uses strategy $\mathbf{x}_n = (x_{n1}, ..., x_{nN}) = \sigma_n(\mathbf{h})$, then

$$\forall m \in \{1, ..., N\} : x_{nm} = \Pr(\text{“}P_n \text{ shoots } P_m\text{”}).$$

Note that a strategy needs to be defined only for nonterminal histories, since at the end of a terminal history no shooting takes place. We will only consider *admissible strategies*, i.e., those which assign positive probability only to admissible actions[3]. In addition, when P_n uses an admissible σ_n and $m \neq n$, we must have

$$\forall \mathbf{h} = \mathbf{s}(0)\, a(1) ... a(t-1)\, m s_1 ... s_N : \sigma_n(\mathbf{h}) = (0, ..., 0)$$

(since P_n does not have the move he cannot fire at anybody).

[3]For example a strategy which assigns positive probability to shooting a dead player is inadmissible.

The probabilities $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_N) = \sigma_n(\mathbf{h})$ depend, in general, on the entire game history (up to the current time). A *stationary strategy* is one that only depends on the current state and then we write $\sigma_n(\mathbf{s}) = \mathbf{x}_n = (x_{n1}, ..., x_{nN})$. A *pure strategy* is a function that maps histories to probability vectors which concentrate all probability to a single component, equal to one; with a slight abuse of notation we then also write $\sigma_n(\mathbf{h}) = m$, meaning that P_n shoots P_m with probability one ($\sigma_n(\mathbf{h}) = (0, ..., 0)$ can be abbreviated as $\sigma_n(\mathbf{h}) = \lambda$, meaning that P_n does not shoot). A *pure stationary strategy* is a pure strategy which only depends on the current state.

A *strategy profile* is a vector $\sigma = (\sigma_1, ..., \sigma_N)$, where σ_n is the strategy used by P_n (for $n \in \{1, ..., N\}$). An initial state $\mathbf{s}(0)$ and a strategy profile σ, define a probability measure on all histories, hence the *expected payoff* to P_n is well defined by

$$Q_n(\mathbf{s}(0), \sigma) = \mathbb{E}_{\mathbf{s}(0),\sigma}(\mathsf{Q}_n(\mathbf{h})).$$

Note that $Q_n(\mathbf{s}(0), \sigma)$ equals the probability that P_n is the sole remaining survivor or, equivalently, the winner of $\Gamma_N(\mathbf{p})$.

3 N-uel Payoff System

For each $N \in \{2, 3, ...\}$ and each fixed strategy profile σ, the payoffs $(Q_i(\mathbf{s}, \sigma))_{i\in\{1,...,N\},\mathbf{s}\in S}$ satisfy a system of *payoff equations*. In this section we will formulate the *payoff system*, study its form and properties, and will introduce appropriate algorithms for its solution. For clarity of presentation, we will first study the 2-uel and 3-uel as special cases and then deal with the general case of the N-uel.

3.1 The 2-uel

With $N = 2$ players we have

$$S_1 = \{010, 001\} \text{ and } S_2 = \{111, 211\},$$
$$S = S_1 \cup S_2 = \{010, 001, 111, 211\}.$$

For each player, the only admissible strategy is to keep shooting at his opponent, until one player is eliminated. The only nonzero state transition probabilities are

$$\begin{aligned}
\Pr(\mathbf{s}(t) = 010|\mathbf{s}(t-1) = 111, a(t) = 2) &= p_1 \\
\Pr(\mathbf{s}(t) = 211|\mathbf{s}(t-1) = 111, a(t) = 2) &= 1 - p_1 \\
\Pr(\mathbf{s}(t) = 001|\mathbf{s}(t-1) = 211, a(t) = 1) &= p_2 \\
\Pr(\mathbf{s}(t) = 111|\mathbf{s}(t-1) = 211, a(t) = 1) &= 1 - p_2
\end{aligned}$$

We can represent the above information compactly by the following state transition graph, where actions and corresponding transition probabilities are written next to the respective edges.
The partition of states into the sets S_1 and S_2 is reflected in the structure of the state transition graph: each $\mathbf{s} \in S_2$ has two successors: a state $\mathbf{s}' \in S_2$ and a state $\mathbf{s}'' \in S_1$; each $\mathbf{s} \in S_1$ has no outgoing transitions.

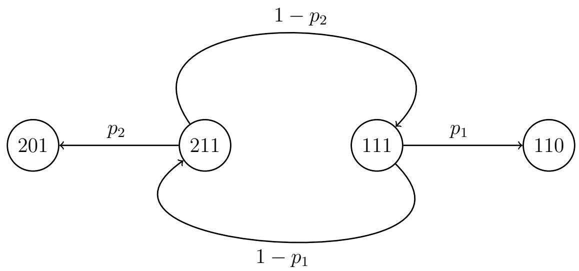


Figure 1: The 2-uel state transition graph.

Since the players have no choice of strategies, we do not have have here a strategic game, but a *game of chance*. In fact, the 2-uel is essentially a Markov chain[4]. For every initial state, a player's payoff equals his winning probability, which we will now compute. For notational brevity, we set

$$\forall i, n \in \{1, 2\}, \text{ for all admissible } ns_1, s_2 : V_{i,ns_1s_2} = Q_i \left(ns_1s_2, \left(\sigma_1', \sigma_2' \right) \right),$$

i.e., the payoff to P_i when the game starts in state $\mathbf{s}(0) = ns_1s_2$, and both players use the same strategy $\sigma_1' = \sigma_2'$ of always shooting at the opponent. It is easily seen that the V_{1,ns_1s_2}'s satisfy the following system of equations:

$$\begin{aligned} V_{1,010} &= 1 \\ V_{1,001} &= 0 \\ V_{1,111} &= (1-p_1) V_{1,211} + p_1 V_{1,010} \\ V_{1,211} &= (1-p_2) V_{1,111} + p_2 V_{1,200} \end{aligned} \tag{1}$$

This system has the unique solution

$$V_{1,010} = 1, \quad V_{1,001} = 0, \quad V_{1,111} = \frac{p_1}{p_1 + p_2 - p_1 p_2}, \quad V_{1,211} = \frac{p_1 (1-p_2)}{p_1 + p_2 - p_1 p_2}. \tag{2}$$

A similar system can be set up for the V_{2,ns_1s_2} variables and has the unique solution

$$V_{2,010} = 0, \quad V_{2,001} = 1, \quad V_{2,111} = \frac{p_2 (1-p_1)}{p_1 + p_2 - p_1 p_2}, \quad V_{2,211} = \frac{p_2}{p_1 + p_2 - p_1 p_2}. \tag{3}$$

The formulas (2)-(3) provide the solution to the $\Gamma_2(\mathbf{p})$, i.e., the winning probability for each player and for each starting state.

3.2 The 3-uel

We will limit our analysis to the case where all players use stationary strategies. Suppose $\sigma_n(\mathbf{s})$ is a stationary strategy used by P_n. This can be characterized as follows.

1. For all $\mathbf{s} \in S_1$ (one player alive) there is no need to define $\sigma_n(\mathbf{s})$.

[4]Except for the fact that the terminal states have no associated state transitions.

2. For all $\mathbf{s} \in S_2$ (two players alive) the only admissible strategy $\sigma_n(\mathbf{s})$ is to shoot at the sole alive opponent.

3. Finally, consider $\sigma_n(\mathbf{s}) = \mathbf{x}_n = (x_{n1}, x_{n2}, x_{n3})$ when $\mathbf{s} = s_0 s_1 s_2 s_3 \in S_3$.

   (a) When $s_0 \neq n$, we will necessarily have $\mathbf{x}_n = (0, 0, 0)$.

   (b) When $s_0 = n$, we will necessarily have $x_{nn} = 0$.

Hence, for every $n \in \{1, 2, 3\}$, an admissible stationary strategy σ_n for P_n is fully determined by the two positive numbers

$$\forall m \neq n : x_{nm} = \Pr\left(\text{“}P_n \text{ fires at } P_m\text{”} \,|\, \text{“the game state is } n111\text{”}\right)$$

which must satisfy $\sum_{m \neq n} x_{nm} = 1$. Using the x_{mn}'s we can draw the state transition graph shown in Figure 2.

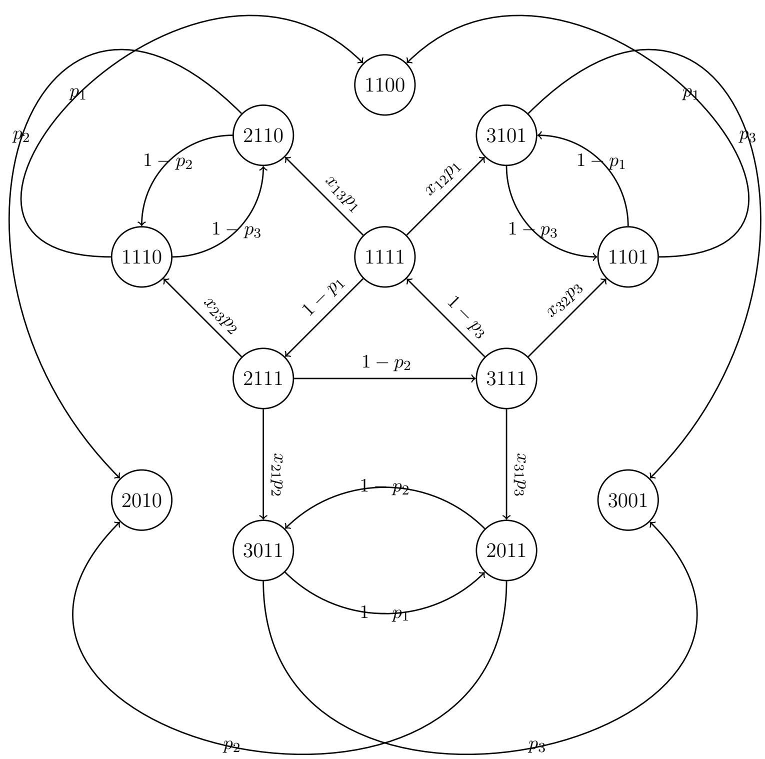


Figure 2: The 3-uel state transition graph.

We will now show the following.

Proposition 3.1 For any p_1, p_2, p_3 such that $m \neq n \Rightarrow p_m \neq p_n$, and every admissible strategy profile σ, the 3-uel payoff system has a unique solution.

Proof. Our goal is to compute each player's expected payoff (equivalently, his winning probability). To this end, similarly to the two-player game, we define

$$\forall i, n \in \{1,2,3\}\,, \text{for all admissible } ns_1s_2s_3 : V_{i,ns_1s_2s_3} = Q_i\left(ns_1s_2s_3, \sigma\right),$$

i.e., the payoff to P_i when the game starts at state $ns_1s_2s_3$ and the strategy profile $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is used[5].

Let us momentarily focus on P_1's payoffs. It is easily seen that the variables $\left(V_{1,ns_1s_2s_3}\right)_{ns_1s_2s_3 \in S}$ must satisfy the following equations:

$$\begin{aligned}
V_{1,0100} &= 1 \\
V_{1,0010} &= 0 \\
V_{1,0001} &= 0 \\
V_{1,1101} &= (1-p_1)V_{1,3101} + p_1 V_{1,0100} \\
V_{1,3101} &= (1-p_3)V_{1,1101} + p_3 V_{1,0001} \\
V_{1,1110} &= (1-p_1)V_{1,2110} + p_1 V_{1,0100} \\
V_{1,2110} &= (1-p_2)V_{1,1110} + p_2 V_{1,0010} \\
V_{1,2011} &= (1-p_2)V_{1,3011} + p_2 V_{1,0010} \\
V_{1,3011} &= (1-p_3)V_{1,2011} + p_3 V_{1,0001} \\
V_{1,1111} &= (1-p_1)V_{1,2111} + x_{12}p_1 V_{1,3101} + x_{13}p_1 V_{1,2110} \\
V_{1,2111} &= (1-p_2)V_{1,3111} + x_{23}p_2 V_{1,1110} + x_{21}p_2 V_{1,3011} \\
V_{1,3111} &= (1-p_3)V_{1,1111} + x_{31}p_3 V_{1,2011} + x_{32}p_3 V_{1,1101}
\end{aligned} \tag{4}$$

The above system can be solved in a stepwise fashion. The first three equations immediately yield the values of $V_{1,0100}$, $V_{1,0100}$, $V_{1,0100}$. The fourth and fifth equations can be solved to obtain the values of $V_{1,1101}$ and $V_{1,3101}$:

$$V_{1,1101} = \frac{p_1}{p_1 + p_3 - p_1 p_3}, \quad V_{1,3101} = \frac{p_1\left(1-p_3\right)}{p_1 + p_3 - p_1 p_3}; \tag{5}$$

naturally, these are exactly the payoffs for a duel between P_1 and P_3. Similarly, the sixth and seventh equations yield $V_{1,1110}$ and $V_{1,2110}$ (expressions for $V_{1,s_0s_1s_2s_3}$ similar to those of (5)) and the eight and ninth equations yield $V_{1,2011} = V_{1,3011} = 0$.

The final three equations involve the unknowns $V_{1,1111}$, $V_{1,2111}$, $V_{1,3111}$ and the previously computed $V_{1,ns_1s_2s_3}$'s. The system can be rewritten as

$$\begin{aligned}
V_{1,1111} - (1-p_1)V_{1,2111} &= x_{12}p_1 V_{1,3101} + x_{13}p_1 V_{1,2110}, \\
V_{1,2111} - (1-p_2)V_{1,3111} &= x_{23}p_2 V_{1,1110} + x_{21}p_2 V_{1,3011}, \\
V_{1,3111} - (1-p_3)V_{1,1111} &= x_{31}p_3 V_{1,2011} + x_{32}p_3 V_{1,1101}.
\end{aligned}$$

[5] The dependence on σ is omitted from the notation, for the sake of brevity.

Letting

$$\begin{aligned} A_1 &= x_{12}p_1V_{1,3101} + x_{13}p_1V_{1,2110}, \\ A_2 &= x_{23}p_2V_{1,1110} + x_{21}p_2V_{1,3011}, \\ A_3 &= x_{31}p_3V_{1,2011} + x_{32}p_3V_{1,1101}, \end{aligned} \tag{6}$$

the system becomes

$$\begin{bmatrix} 1 & -(1-p_1) & 0 \\ 0 & 1 & -(1-p_2) \\ -(1-p_3) & 0 & 1 \end{bmatrix} \begin{bmatrix} V_{1,1111} \\ V_{2,1111} \\ V_{3,1111} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

The determinant of the system is $D = 1 - (1-p_1)(1-p_2)(1-p_3) > 0$. Hence the system has a unique solution which is

$$V_{1,1111} = \frac{A_1 + A_2 + A_3 - p_1A_2 - p_2A_3 - p_1A_3 + p_1p_2A_3}{1 - (1-p_1)(1-p_2)(1-p_3)} \tag{7}$$

$$V_{1,2111} = \frac{A_2 + A_3 + A_1 - p_2A_3 - p_3A_1 - p_2A_1 + p_2p_3A_1}{1 - (1-p_1)(1-p_2)(1-p_3)} \tag{8}$$

$$V_{1,3111} = \frac{A_3 + A_1 + A_2 - p_3A_1 - p_1A_2 - p_3A_2 + p_3p_1A_2}{1 - (1-p_1)(1-p_2)(1-p_3)} \tag{9}$$

In the same manner we can prove that the payoff systems for P_2 and P_3 have unique solutions and this completes the proof. Note that $(V_{i,n111})_{i,n\in\{1,2,3\}}$ are actually functions of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$; for brevity, the dependence has been suppressed from the notation. ■

The structure and stepwise solution of the payoff equations correspond to the structure of the state transition diagram. Namely, the vertices of the state transition graph are the game states and the possible transitions are as follows: S_1 states are terminal, each S_2 state can transit either to a single other S_2 state or to a single S_1 state, and each S_3 state can transit either to a single other S_3 state or one of two S_2 states. Furthermore, the S_3 states form a cycle, i.e.,

$$... \to (1111) \to (2111) \to (3111) \to (1111) \to ...$$

These facts, clearly, correspond to the stepwise procedure of solving the payoff system. For each $n \in \{1,2,3\}$ separately, we first obtain the payoffs $(V_{n,\mathbf{s}})_{\mathbf{s}\in S_1}$, then the $(V_{n,\mathbf{s}})_{\mathbf{s}\in S_2}$ and finally the $(V_{n,\mathbf{s}})_{\mathbf{s}\in S_3}$.

3.3 The N-uel

We will now write and solve the payoff system which, analogously to (1) and (4), governs the N-uel expected total payoffs. We use, for all $n \in \{1,...,N\}$ and all $ms_1...s_N \in S$, the notation

$$V_{n,ms_1...s_N} = Q_n(ms_1...s_N, \sigma).$$

In addition we introduce the following notations. For all $\mathbf{s} = ms_1...s_N \in S$, we define

$$\text{the set of alive players} : L(\mathbf{s}) = \{n : s_n = 1\},$$
$$\text{the set of alive players other than } m : L_m(\mathbf{s}) = \{n : s_n = 1 \text{ and } n \neq m\}.$$

For all $\mathbf{s} = ms_1...s_N \in S$, we will use $\mathbf{N}(\mathbf{s})$ to denote the state following $\mathbf{s}$ when no player is killed. For example, when $N = 4$ we have

$$\mathbf{N}(11111) = 21111, \quad \mathbf{N}(11011) = 31011, \quad \mathbf{N}(41111) = 11111, \quad ... \quad .$$

Furthermore, for all $\mathbf{s} = ms_1...s_N \in S$, we will use $\mathbf{N}_i(\mathbf{s})$ to denote the the state following $\mathbf{s}$ when P_i is killed. For example, when $N = 4$ we have

$$\mathbf{N}_2(11111) = 31011, \quad \mathbf{N}_4(31111) = 11110, \quad ... \quad .$$

Finally, for all $n \in \{1, ..., N\}$ and all $\mathbf{s} = ms_1...s_N \in S$, the probability that P_m shoots P_n, when the state is $\mathbf{s}$ and P_m uses σ_m, is:

$$x_{ms_1...s_N,n} = \Pr(a = n | P_m \text{ uses strategy } \sigma_m, \text{ current state is } \mathbf{s} = ms_1...s_N).$$

Using the above notations and assuming a given strategy profile σ (which determines all the shooting probabilities $x_{ms_1...s_N,n}$) the expected total payoffs for P_1 satisfy the following equations.

1. At terminal states we have:

$$V_{1,0s_1...s_N} = 1 \text{ when } s_1 = 1 \text{ and } V_{1,0s_1...s_N} = 0 \text{ when } s_1 = 0. \tag{10}$$

2. At all admissible states with two alive players P_1 and P_m we have:

$$V_{1,1s_1...s_N} = \frac{p_1}{p_1 + p_m - p_1 p_m} \text{ and } V_{1,ms_1...s_N} = \frac{p_1(1-p_m)}{p_1 + p_m - p_1 p_m}, \tag{11}$$

   which are P_1's winning probabilities in a duel against P_m.

3. At all admissible states $is_1...s_N$ with two alive players, both different from P_1, we have:

$$V_{1,is_1...s_N} = 0 \tag{12}$$

4. At all admissible states with more than two alive players: for all $k \in \{3, ..., N\}$, $\mathbf{s} = ms_1...s_N \in S_k$, we have:

$$\text{when } 1 \in L(\mathbf{s}) : V_{1,ms_1...s_N} = (1-p_m) V_{1,\mathbf{N}(\mathbf{s})} + \sum_{n \in L_m(\mathbf{s})} x_{ms_1...s_N,n} p_m V_{1,\mathbf{N}_n(\mathbf{s})}, \tag{13}$$

$$\text{when } 1 \notin L(\mathbf{s}) : V_{1,ms_1...s_N} = 0. \tag{14}$$

The payoff system which must be satisfied by the $V_{1,ms_1...s_N}$'s consists of the equations (10)-(14). Similar systems are satisfied by the $V_{n,ms_1...s_N}$'s, for $n \in \{2, ..., N\}$.

Proposition 3.2 For every $N \in \{2, 3, ...\}$, for any p_1, ..., p_N such that $m \neq n \Rightarrow p_m \neq p_n$, and every admissible strategy profile σ, the N-uel payoff system has a unique solution.

Proof. We only consider the payoff systems regarding $(V_{1,\mathbf{s}})_{s\in S}$ (the cases $(V_{n,\mathbf{s}})_{s\in S}$ with $n\geq 2$ are treated similarly). The proof is by induction. Clearly, for $N=2$, the payoff system has a unique solution, given by (2). Now suppose the $(N-1)$-uel payoff system has a unique solution and consider the N-uel payoff system (10)-(14).

Take any $K\in\{2,...,N-1\}$; for any state $\mathbf{s}=s_0s_1...s_N\in S_K$, we want to determine (for all $n\in\{1,...,N\}$) the corresponding $V_{1,s_0s_1...s_N}=Q_1(s_0s_1...s_N,\sigma)$. This is P_1's payoff in a N-uel involving himself, $K-1$ other alive and $N-K$ dead players, which is the same as P_1's payoff in a K-uel involving himself and $K-1$ other alive players. Hence $V_{n,s_0s_1...s_N}$ can be computed by solving the respective K-uel with K alive players and relabeling the K-uel players and their payoffs so as to correspond with the K alive players of the N-uel. By the inductive assumption, the K-uel has a unique solution, hence the value $V_{1,s_0s_1...s_N}$ is also uniquely determined.

Consequently the $V_{1,s_0s_1...s_N}$'s are uniquely determined for all $\mathbf{s}=s_0s_1...s_N\in\cup_{K=2}^{N-1}S_K$. It remains to show that the $V_{1,s_0s_1...s_N}$'s with $\mathbf{s}=s_0s_1...s_N\in S_N$ are also uniquely determined. Now, $\mathbf{s}\in S_N$ means there exists N alive players; hence $s_1=...=s_N=1$ and there exist exactly N such states:

$$S_N=\{11...1,...,N1...1\}.$$

Also, since in such states all players are alive, we have

$$21...1=\mathbf{N}(11...1),\quad 31...1=\mathbf{N}(21...1),\quad ...,\quad 11...1=\mathbf{N}(N1...1).$$

Each of the above states appears once on the left side of an equation (13) and once on the right side of another equation (13). Let us rename the corresponding $V_{1,s_0s_1...s_N}$ variables as follows.

$$\forall m\in\{1,...,N\}:Z_m=V_{1,ms_1...s_N}.$$

Let us also define

$$\forall m\in\{1,...,N\}:A_m=\sum_{i\in L_m(\mathbf{s})}x_{ms_1...s_N,i}p_mV_{1,\mathbf{N}_i(ms_1...s_N)}.$$

It follows that, for all $\mathbf{s}\in S_N$, the equations (13) can be rewritten in the form

$$\begin{bmatrix} 1 & -(1-p_1) & 0 & ... & 0\\ 0 & 1 & -(1-p_2) & ... & 0\\ 0 & 0 & 1 & ... & 0\\ ... & ... & ... & ... & ...\\ -(1-p_N) & 0 & 0 & ... & 1 \end{bmatrix}\begin{bmatrix} Z_1\\ Z_2\\ Z_3\\ ...\\ Z_N\end{bmatrix}=\begin{bmatrix} A_1\\ A_2\\ A_3\\ ...\\ A_N\end{bmatrix}\tag{15}$$

Furthermore, since the states $\mathbf{N}_i(ms_1...s_N)\in S_{N-1}$, the $V_{1,\mathbf{N}_i(ms_1...s_N)}$'s are uniquely determined as solutions of an $(N-1)$-uel.

A necessary and sufficient condition for the system (15) to have a unique solution is that the determinant

$$D_N=\begin{vmatrix} 1 & -(1-p_1) & 0 & ... & 0\\ 0 & 1 & -(1-p_2) & ... & 0\\ 0 & 0 & 1 & ... & 0\\ ... & ... & ... & ... & ...\\ -(1-p_N) & 0 & 0 & ... & 1 \end{vmatrix}$$

is different from zero. It is easily proved that

$$D_N = 1 - (1 - p_1)(1 - p_2) \dots (1 - p_N).$$

Since, by assumption, for all n we have $p_n \in (0,1)$, it follows that $D_N > 0$ and the inductive step is completed. ■

Now our problem is to solve the system of payoff equations and the above proof suggests a solution method. In what follows we define, for every $i \in \{1, \dots, N\}$ and every $S' \subseteq S$, the *payoffs vector* $\mathbf{V}_{i,S'} = (V_{i,\mathbf{s}})_{\mathbf{s} \in S'}$, i.e., the vector of all payoffs indexed by players and states. For example, let $N = 3$; then the sets of all states with one and two surviving players are, respectively,

$$S_1 = \{0100, 0010, 0001\}, \quad S_2 = \{1110, 2110, 1101, 3101, 2011, 3011\}$$

and the corresponding payoff vectors are

$$\mathbf{V}_{1,S_1} = (V_{1,0100}, V_{2,0100}, V_{3,0100}, V_{1,0010}, V_{2,0010}, V_{3,0010}, V_{1,0001}, V_{2,0001}, V_{3,0001})$$
$$\mathbf{V}_{1,S_2} = (V_{1,1110}, V_{2,1110}, V_{3,1110}, V_{1,2110}, \dots, V_{2,3011}, V_{3,3011}).$$

Initialization of the $\mathbf{V}_{i,S_1}$'s, i.e., the set of payoffs (to all players) for states with a single surviving player, is immediate; for example when $N = 3$ we have

$$\begin{array}{lll} V_{1,0100} = 1, & V_{2,0100} = 0, & V_{3,0100} = 0, \\ V_{1,0010} = 0, & V_{2,0010} = 1, & V_{3,0010} = 0, \\ V_{1,0001} = 0, & V_{2,0001} = 0, & V_{3,0001} = 1, \end{array}$$

and similar values are obtained for $\mathbf{V}_{2,S_1}$ and $\mathbf{V}_{3,S_1}$.

The function SolveNuel, presented below in pseudocode, computes the $\mathbf{V}_{i,S}$ vectors (for $i \in \{1, \dots, N\}$) as follows.

1. The function inputs are:
   (a) the number of players N,
   (b) the player of interest i,
   (c) the vector of marksmanships $\mathbf{p} = (p_1, \dots, p_N)$ and
   (d) the strategy profile σ.
2. At inialization, the one-player payoff vector $\mathbf{V}_{i,S_1}$ is computed.
3. In the outer loop of the function, K is the number of living players, from $K = 2$ to $K = N$; for each K we create the set $\mathcal{C}$ of $\binom{N}{K}$ combinations of living players.
4. In the inner loop, we solve a K-uel for each player set $C = \{n_1, n_2, \dots, n_K\} \in \mathcal{C}$. This involves solving a system of the K unknown $\mathbf{V}_{S_C}$'s; when obtained these are used to gradually populate the elements of the "full" $\mathbf{V}_{i,S}$ vector.

5. On completion of both loops, all components of the $\mathbf{V}_{i,S}$ vector have been computed and the function returns $\mathbf{V}_{i,S}$.

Algorithm 1 Function for recursive N-uel solution

```
function SolveNuel(N, i, p, σ)
    Construct the state set S corresponding to the N-uel.
    Compute V_{i,S_1}
    for K = 2..N do
        Let 𝒞 be the set of all combinations of K players out of {1, ..., N}
        for C ∈ 𝒞 do
            Let S_C be the set of states corresponding to player subset C
            Compute V_{i,S_C} by solving a K-uel
        end for
    end for
    Return V_{i,S}
end function
```

The $\mathbf{V}_{i,S_k}$ values, for $k \in \{3, ..., N\}$, are obtained by solving $\binom{N}{K}$ systems, each involving K unknowns. Exact values can be obtained by Cramer's rule or matrix inversion. However, we have found that implementation is simpler when the following iterative algorithm is used. We first present the algorithm and then prove its correctness.

1. The function inputs are:
   (a) the number of alive players K,
   (b) the player of interest i,
   (c) the vector of marksmanships $\mathbf{p} = (p_1, ..., p_N)$,
   (d) the strategy profile σ,
   (e) the payoffs vector for $K-1$ players $\mathbf{V}_{i,S_{K-1}}$ and
   (f) the termination parameter ε.
2. We initialize, for all states $ms_1...s_N \in S_K$, at arbitrary values $V^{(0)}_{i,ms_1...s_N}$.
3. Then we iterate, for $t \in \{0, 1, 2, ...\}$ and for each state $ms_1...s_N \in S_K$, to obtain new $V^{(t+1)}_{i,ms_1...s_N}$ values by (16).
4. If at some iteration t we have $\max_{i\in\{1,...,K\},\mathbf{s}\in S_K} \left| V^{(t+1)}_{i,\mathbf{s}} - V^{(t)}_{i,\mathbf{s}} \right| < \varepsilon$, the algorithm terminates and returns $\mathbf{V}_{i,S} = \mathbf{V}^{(t+1)}_{i,S}$.

Algorithm 2 Iterative Solution of Payoff System

function ITERSOLVE$(K, i, \mathbf{p}, \sigma, \mathbf{V}_{S_{K-1}}, \varepsilon)$
  for $\mathbf{s} = ms_1...s_N \in S_K$ **do**
    $V^{(0)}_{i,ms_1...s_N}$ arbitrary
  end for
  for $t \in \{0, 1, 2, ...\}$ **do**
    for $\mathbf{s} = ms_1...s_N \in S_K$ **do**

$$V^{(t+1)}_{i,\mathbf{s}} = (1 - p_m) V^{(t)}_{i,\mathbf{N}(\mathbf{s})} + \sum_{n \in L_m(\mathbf{s})} x_{\mathbf{s},n} p_m V_{i,\mathbf{N}_n(\mathbf{s})} \tag{16}$$

    end for
    if $\max_{\mathbf{s} \in S_K} \left| V^{(t+1)}_{i,\mathbf{s}} - V^{(t)}_{i,\mathbf{s}} \right| < \varepsilon$ **then**
      Break
    end if
  end for
  $\mathbf{V}_{i,S_K} = \mathbf{V}^{(t+1)}_{i,S_K}$
  Return $\mathbf{V}_{i,S_K}$
end function

The following proposition shows that, for given strategy profile σ, the IPC algorithm yields in the limit the payoffs of the N-uel.

Proposition 3.3 For every $K \in \{2, 3, ...\}$, $i \in \{1, 2, ..., K\}$, for any p_1, ..., p_K such that $m \neq n \Rightarrow p_m \neq p_n$, and for every admissible strategy profile σ, the iterative solution of the payoff system always converges and we have

$$\forall \mathbf{s} \in \mathbf{S}_K : \lim_{t \to \infty} V^{(t+1)}_{i,\mathbf{s}} = V_{i,\mathbf{s}}.$$

Proof. Consider the same iteration starting from two different initial conditions $V^{(0)}_{i,S_K}$ and $U^{(0)}_{i,S_K}$. Then we have

$$\forall \mathbf{s} = ms_1...s_N \in S_K : V^{(t+1)}_{i,\mathbf{s}} = (1 - p_m) V^{(t)}_{i,\mathbf{N}(\mathbf{s})} + \sum_{n \in L_m(\mathbf{s})} x_{\mathbf{s},n} p_m V_{i,\mathbf{N}_n(\mathbf{s})}$$
$$\forall \mathbf{s} = ms_1...s_N \in S_K : U^{(t+1)}_{i,\mathbf{s}} = (1 - p_m) U^{(t)}_{i,\mathbf{N}(\mathbf{s})} + \sum_{n \in L_m(\mathbf{s})} x_{\mathbf{s},n} p_m V_{i,\mathbf{N}_n(\mathbf{s})}$$

We then have

$$\forall \mathbf{s} = ms_1...s_N \in S_K : \left| V^{(t+1)}_{i,\mathbf{s}} - U^{(t+1)}_{i,\mathbf{s}} \right| = (1 - p_m) \left| V^{(t)}_{i,\mathbf{N}(\mathbf{s})} - U^{(t)}_{i,\mathbf{N}(\mathbf{s})} \right| \Rightarrow$$
$$\sum_{\mathbf{s} \in S_K} \left| V^{(t+1)}_{i,\mathbf{s}} - U^{(t+1)}_{i,\mathbf{s}} \right| \leq \sum_{\mathbf{s} \in S_K} \left(1 - \min_m p_m\right) \left| V^{(t)}_{i,\mathbf{N}(\mathbf{s})} - U^{(t)}_{i,\mathbf{N}(\mathbf{s})} \right| \Rightarrow$$
$$\sum_{m \in L(\mathbf{s})} \left| V^{(t+1)}_{i,\mathbf{s}} - U^{(t+1)}_{i,\mathbf{s}} \right| \leq \left(1 - \min_m p_m\right) \sum_{\mathbf{s} \in S_K} \left| V^{(t)}_{i,\mathbf{N}(\mathbf{s})} - U^{(t)}_{i,\mathbf{N}(\mathbf{s})} \right| \Rightarrow$$
$$\sum_{m \in L(\mathbf{s})} \left| V^{(t+1)}_{i,\mathbf{s}} - U^{(t+1)}_{i,\mathbf{s}} \right| \leq \left(1 - \min_m p_m\right)^{t+1} \sum_{\mathbf{s} \in S_K} \left| V^{(0)}_{i,\mathbf{N}(\mathbf{s})} - U^{(0)}_{i,\mathbf{N}(\mathbf{s})} \right|.$$

Since $|1 - \min_m p_m| \in (0,1)$, we have

$$\lim_{t\to\infty} \sum_{m\in L(\mathbf{s})} \left| V_{i,\mathbf{s}}^{(t+1)} - U_{i,\mathbf{s}}^{(t+1)} \right| = 0$$

This means that, for every i and $\mathbf{s}$, the iteration tends to a unique limit

$$\forall i, \mathbf{s} : \overline{V}_{i,\mathbf{s}} = \lim_{t\to\infty} \sum_{m\in L(\mathbf{s})} V_{i,\mathbf{s}}^{(t)}$$

and we have

$$\forall i, \mathbf{s} : \overline{V}_{i,\mathbf{s}} = (1-p_m)\, \overline{V}_{i,\mathbf{N}(\mathbf{s})} + \sum_{n\in L_m(\mathbf{s})} x_{\mathbf{s},n} p_m V_{i,\mathbf{N}_n(\mathbf{s})}.$$

In other words $\overline{V}_{i,S_K}$ satisfies the payoff equations, which means that the iteration yields the unique solution of the payoff system. ■

It should be pointed out that Algorithms 1 and 2 are *guaranteed* to work (i.e., solve the N-uel) when the p_n marksmanships belong to $(0,1)$; but the algorithms *may* also work even when some of the p_n's are equal to zero or to one.

4 N-uel Stationary Equilibria

We are now ready to study the existence of N-uel equilibria. For every N we have to solve a separate system of *nonlinear* equations. For clarity of presentation, we will first deal with the 3-uel and then for the general N-uel.

Proposition 4.1 For any p_1, p_2, p_3 such that $m \neq n \Rightarrow p_m \neq p_n$, the 3-uel has a unique stationary deterministic Nash equilibrium $\widehat{\sigma}$, which can be described as follows

$$\widehat{\sigma}(1111) = (0, \widehat{x}_{12}, 1-\widehat{x}_{12}), \quad \widehat{\sigma}(2111) = (1-\widehat{x}_{23}, 0, \widehat{x}_{23}), \quad \widehat{\sigma}(3111) = (\widehat{x}_{31}, 1-\widehat{x}_{31}, 0),$$

where

$$\widehat{x}_{12} = \begin{cases} 1 & \text{iff } p_2 > p_3 \\ 0 & \text{else} \end{cases}, \quad \widehat{x}_{23} = \begin{cases} 1 & \text{iff } p_3 > p_1 \\ 0 & \text{else} \end{cases}, \quad \widehat{x}_{31} = \begin{cases} 1 & \text{iff } p_1 > p_2 \\ 0 & \text{else} \end{cases}.$$

In other words, when in equilibrium, at every turn each player shoots at his "strongest" opponent with probability one.

Proof. It suffices to consider P_1's point of view. His only strategy choice is when the game is in state $\mathbf{s} = 1111$ (in all other states, there exists a unique admissible strategy), i.e., P_1 must choose x_{12} and x_{13} (subject to $x_{12} \geq 0$, $x_{13} \geq 0$, $x_{12} + x_{13} = 1$) so as to maximize $V_{1,1111}$, $V_{1,2111}$ and $V_{1,3111}$. Recall that $V_{1,1111}$ is given from (7):

$$V_{1,1111} = \frac{A_1 + A_2 + A_3 - p_1 A_2 - p_2 A_3 - p_1 A_3 + p_1 p_2 A_3}{1 - (1-p_1)(1-p_2)(1-p_3)}$$

where A_1, A_2, A_3 are given from (6). Note that x_{12} and x_{13} only appear in A_1. Hence P_1 wants to maximize

$$\begin{aligned}
F_1(x_{12}, x_{13}) &= \frac{A_1}{1-(1-p_1)(1-p_2)(1-p_3)} \\
&= \frac{x_{12}p_1V_{1,3101} + x_{13}p_1V_{1,2110}}{1-(1-p_1)(1-p_2)(1-p_3)} \\
&= \frac{x_{12}p_1\frac{p_1(1-p_3)}{p_1+p_3-p_1p_3} + x_{13}p_1\frac{p_1(1-p_2)}{p_1+p_2-p_1p_2}}{1-(1-p_1)(1-p_2)(1-p_3)} \\
&= p_1^2\frac{x_{12}(1-p_3)(p_1+p_2-p_1p_2) + x_{13}(1-p_2)(p_1+p_3-p_1p_3)}{(1-(1-p_1)(1-p_2)(1-p_3))(p_1+p_2-p_1p_2)(p_1+p_3-p_1p_3)}
\end{aligned}$$

subject to the constraints $x_{12} + x_{13} = 1, x_{12} \geq 0, x_{13} \geq 0$. Since the denominator is positive, it suffices to choose x_{12} (and consequently $x_{13} = 1 - x_{12}$) so as to maximize

$$x_{12}(1-p_3)(p_1+p_2-p_1p_2) + x_{13}(1-p_2)(p_1+p_3-p_1p_3).$$

Finally, since

$$(1-p_3)(p_1+p_2-p_1p_2) - (1-p_2)(p_1+p_3-p_1p_3) = p_2 - p_3$$

we have

$$\begin{aligned}
p_2 > p_3 &\Leftrightarrow (1-p_3)(p_1+p_2-p_1p_2) > (1-p_2)(p_1+p_3-p_1p_3) \\
p_2 < p_3 &\Leftrightarrow (1-p_3)(p_1+p_2-p_1p_2) < (1-p_2)(p_1+p_3-p_1p_3)
\end{aligned}$$

Hence the optimization rule for $V_{1,1111}$ is simple:

$$\begin{aligned}
&\text{if } p_2 > p_3 \text{ then } \widehat{x}_{12} = 1,\ \widehat{x}_{13} = 0 \\
&\text{if } p_2 < p_3 \text{ then } \widehat{x}_{12} = 0,\ \widehat{x}_{13} = 1
\end{aligned} \tag{17}$$

This rule also maximizes

$$\begin{aligned}
V_{1,2111} &= \frac{A_2 + A_3 + A_1 - p_2A_3 - p_3A_1 - p_2A_1 + p_2p_3A_1}{1-(1-p_1)(1-p_2)(1-p_3)} \\
&= \frac{A_1 - p_3A_1 - p_2A_1 + p_2p_3A_1 + A_2 + A_3 - p_2A_3}{1-(1-p_1)(1-p_2)(1-p_3)} \\
&= \frac{(1-p_2)(1-p_3)A_1 + A_2 + A_3 - p_2A_3}{1-(1-p_1)(1-p_2)(1-p_3)}
\end{aligned}$$

and

$$\begin{aligned}
V_{1,3111} &= \frac{A_3 + A_1 + A_2 - p_3A_1 - p_1A_2 - p_3A_2 + p_3p_1A_2}{1-(1-p_1)(1-p_2)(1-p_3)} \\
&= \frac{A_1 - p_3A_1 + A_3 + A_2 - p_1A_2 - p_3A_2 + p_3p_1A_2}{1-(1-p_1)(1-p_2)(1-p_3)} \\
&= \frac{(1-p_3)A_1 + A_3 + A_2 - p_1A_2 - p_3A_2 + p_3p_1A_2}{1-(1-p_1)(1-p_2)(1-p_3)}
\end{aligned}$$

Hence the rule (17) simulataneously maximizes $V_{1,1111}$, $V_{1,2111}$ and $V_{1,3111}$. This is the "strongest opponent" strategy and is P_1's best response to any P_2 and P_3 strategy. By a similar analysis we can prove that the "strongest opponent" strategy is also P_2's and P_3's best response. This completes the proof. ■

Proposition 4.2 For every $N \in \{3, 4, ..., \}$, the N-uel has a stationary deterministic Nash equilibrium.

Proof. It suffices to consider P_1's problem of determining his equilibrium strategy $\widehat{\sigma}_1$. The proof will be inductive.

For all states $\mathbf{s} \in S_3$, P_1 can determine the value of his optimal strategy $\widehat{\sigma}_1(\mathbf{s})$ as follows. For every such state in which he is not alive he has no strategy choice. For every state in which he is alive, he must solve a 3-uel against the other two alive players; he can do this without considering the value of $\widehat{\sigma}_1(\mathbf{s})$ for states $\mathbf{s} \notin S_1 \cup S_2 \cup S_3$.

Suppose that P_1 has determined $\widehat{\sigma}_1(\mathbf{s})$ for all $\mathbf{s} \in \cup_{k=3}^{K-1} S_k$; now he wants to determine $\widehat{\sigma}_1(\mathbf{s})$ for all states $\mathbf{s} \in S_K$. He has nothing to determine for states $\mathbf{s} \in S_K$ in which he is not alive. There exist $\binom{N-1}{K-1}$ sets of states with P_1 and $K-1$ other players are alive. Let S' be any such set and let the alive players be $n_1, ..., n_K$; in particular, let $n_1 = 1$. With an appropriate state reordering, we can write S' as

$$S' = \{\mathbf{s}_1, ..., \mathbf{s}_K\}$$

where P_{n_k} is the player having the move in $\mathbf{s}_k$; in particular, $\mathbf{s}_1$ is the state in which P_1 has the move. Now, letting

$$\forall k \in \{1, ..., K\} : \begin{cases} Z_k = V_{1,\mathbf{s}_k} \\ A_k = \sum_{i \in L_k(\mathbf{s}_k)} x_{\mathbf{s}_k,i} p_{n_k} V_{1,\mathbf{N}_i(\mathbf{s}_k)} \end{cases},$$

the following payoff system must be satisfied

$$\begin{bmatrix} 1 & -(1-p_{n_1}) & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ -(1-p_{n_K}) & 0 & ... & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ ... \\ Z_K \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ ... \\ A_K \end{bmatrix}. \tag{18}$$

The Z_k's are the unknowns and, since each state $\mathbf{N}_i(\mathbf{s}_k)$ belongs to S_{K-1}, the A_k's are known (but depending on the $x_{\mathbf{s}_k,i}$'s). P_1 wants to maximize $Z_k = V_{1,\mathbf{s}_k}$ (for all $k \in \{1, ..., K\}$). We can solve (18) using Cramer's rule. We have

$$\begin{vmatrix} 1 & 1-p_{n_1} & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 1-p_{n_K} & 0 & ... & 1 \end{vmatrix} = 1-(1-p_{n_1})(1-p_{n_2})...(1-p_{n_K}) > 0,$$

and, expanding with respect to the first column, we have

$$V_{1,\mathbf{s}_1} = Z_1 = \frac{\begin{vmatrix} A_1 & 1-p_{n_1} & 0 & ... & 0 \\ A_2 & 1 & 1-p_{n_2} & ... & 0 \\ A_3 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ A_K & 0 & 0 & ... & 1 \end{vmatrix}}{1-(1-p_{n_1})(1-p_{n_2})...(1-p_{n_K})} \tag{19}$$

or

$$V_{1,\mathbf{s}_1} = Z_1 = \frac{A_1 \begin{vmatrix} 1 & 1-p_{n_2} & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{vmatrix} - A_2 D_2 + A_3 D_3 - ...}{1-(1-p_{n_1})(1-p_{n_2})...(1-p_{n_K})}$$
$$= \frac{\sum_{i\in L_1(\mathbf{s}_1)} x_{\mathbf{s}_1,i} p_{n_1} V_{1,\mathbf{N}_i(\mathbf{s}_1)} + B}{1-(1-p_{n_1})(1-p_{n_2})...(1-p_{n_K})}.$$

In the above, D_k is the determinant of the submatrix obtained by removing the first column and the k-th row ($k \in \{2, ..., K\}$) in (21) and $B = \sum_{k=2}^{K} (-1)^{k-1} A_k D_k$; neither the A_k's nor B contain P_1's shooting probabilities $x_{\mathbf{s}_1,i}$. Hence P_1 needs only to maximize $\sum_{i\in L_1(\mathbf{s}_1)} x_{\mathbf{s}_1,i} p_{n_1} V_{1,\mathbf{N}_i(\mathbf{s}_1)}$. The rule to achieve this is simple:

$$\widehat{i} = \arg \max_{i\in L_1(\mathbf{s}_1)} V_{1,\mathbf{N}_i(\mathbf{s}_1)}, \tag{20}$$

$$\forall i \neq \widehat{i} : x_{\mathbf{s}_1,i} = 0 \text{ and } x_{\mathbf{s}_1,\widehat{i}} = 1. \tag{21}$$

In (20), $\arg\max_i$ is understood as the smallest i which achieves the maximum (there may exist more than one such and this may result in more than one Nash equilibria). After some additional algebra it can be verified that this rule also maximizes $V_{1,\mathbf{s}_k}$ for all remaining $k \in \{2, ..., K\}$.

This completes the inductive proof for P_1's equilibrium strategy σ_1. The proof for all other players works the same way. Let us define a *family* of rules $\mathbf{R}_k$ (for $K \in \{2, ..., N\}$):

$\mathbf{R}_K$: When K players are alive, P_n shoots at some P_i whose elimination results in a $(K-1)$-uel with highest payoff to P_n.

What we have proved is that, for all $N \in \{2, 3, ...\}$ and all $n \in \{1, ..., N\}$, the family $(\mathbf{R}_K)_{K=1}^{N}$ yields a deterministic NE for the N-uel. ■

The above proof also furnishes an algorithm for computing the equilibrium $\widehat{\sigma} = (\widehat{\sigma}_1, ..., \widehat{\sigma}_N)$

Algorithm 3 Computation of Nash Equilbrium

```
function NashCompute(N, p = (p_1, ..., p_N))
    Initialize σ̂_1(s), ..., σ̂_N(s) for all s ∈ S_1 ∪ S_2
    for K ∈ {3, ..., N} do
        Compute σ̂_1(s), ..., σ̂_N(s) for all s ∈ S_K
    end for K.
end function
```

If the original assumptions are violated there is no guarantee that Algorithm 3 will yield the equilibrium of the N-uel. However, it is worth noting that if Algorithm 3 terminates, it will always yield an equilibrium; i.e., the algorithm *may work* for combinations of p_1, ..., p_N values which violate some of our original assumptions (e.g., with some marksmanships equal to zero or to one).

As will be seen in the next section, the strongest opponent rule can result in rather interesting behaviors for certain combinations of p_1, ..., p_N.

5 Experiments

In this section we use computer simulation to present some interesting cases of N-uels. In all the following tables $\mathbf{s}_n$ denotes the state $(n, 1, 1, 1, \cdots, 1)$.

5.1 3-uels

We start with 3-uels in which, as mentioned, every player's optimal strategy is to shoot at his strongest opponent (and this holds for all states belonging to S_3).

5.1.1 Strongest Player Has Highest / Lowest Winning Probability

In Table 1 we see a case where the strongest player $P_{\max} = P_1$, i.e., the one with highest marksmanship, has the greatest expected payoff regardless of who has the first move (note that, for each player, the optimal strategy is the same for every state: shoot at your strongest opponent). While this may seem natural, it is actually the exception and not the rule, as one might expect.

Table 1: Strongest player has the greatest expected payoff.

n	1	2	3
p_n	0.90	0.10	0.20
$\widehat{\sigma}_n(\mathbf{s}_n)$	3	1	1
$V_{n,1111}$	0.86	0.12	0.02
$V_{n,2111}$	0.62	0.18	0.20
$V_{n,3111}$	0.69	0.16	0.15

In Table 2 we see that the player $P_{\max} = P_3$ does *not* have the highest expected payoff. In fact, if he does not have the first move, he has the *lowest* expected payoff. This is because each player $P_i \neq P_{\max}$ will shoot at $P_{\max}$ (by the optimal strategy of shooting at the strongest opponent) and he has a high probability of dying before he has a chance to shoot back. Hence, the "team" consisting of the two players with lowest marksmanship has a better survival probability than the $P_{\max}$ playing alone.

Table 2: Strongest player does not have the greatest expected payoff.

n	1	2	3
p_n	0.50	0.70	0.95
$\widehat{\sigma}_n(\mathbf{s}_n)$	3	3	2
$V_{n,1111}$	0.37	0.56	0.07
$V_{n,2111}$	0.56	0.30	0.14
$V_{n,3111}$	0.50	0.03	0.47

This is one of many cases in which the strongest player P_3 has the lowest expected payoff when he does not have the first move. And even when he does have the first move, P_1 has greatest expected payoff. On the other hand, P_1 has higher expected payoff when P_2 has the move, and P_2 has higher payoff when P_1 has the move.

5.1.2 Zugzwang

Consider the case $p_1 = p_2 = p_3 = 1$, i.e., every player has perfect markmanship. Without loss of generality, we assume that $\hat{\sigma}_n = \mathbf{N}(n111)$ meaning that each P_n shoots at the next player (actually, it makes no difference to P_n which player he will shoot at). Let P_n be the player who has the first move. P_n will always lose, no mater what strategy he uses, since, after killing his first target he will fight a 2-uel in which his opponent will have the first move *and* perfect markmanship and so will certainly kill P_n. These facts are illustrated in Table 3.

Table 3: Player who has the first move loses.

n	1	2	3
p_n	1.00	1.00	1.00
$\widehat{\sigma}_n(n111)$	2	3	1
$V_{n,1111}$	0.00	0.00	1.00
$V_{n,2111}$	1.00	0.00	0.00
$V_{n,3111}$	0.00	1.00	0.00

This resembles a *zugswang* position in chess, i.e., a position in which a player will necessarily lose if he moves *any* of his pieces, whereas he would not necessarily lose if he could pass (not move any piece).

5.1.3 Being Weaker May Increase Payoff

This example is a continuation of the previous one. In Figure 3 we see that P_1's probability of winning is a *decreasing* function of marksmanship p_1 in the interval $[0.5, 1.0]$, when $p_2 = p_3 = 1$. In other words, having a lower marksmanship can increase one's probability of winning (and this is true regardless of which player has the first move).

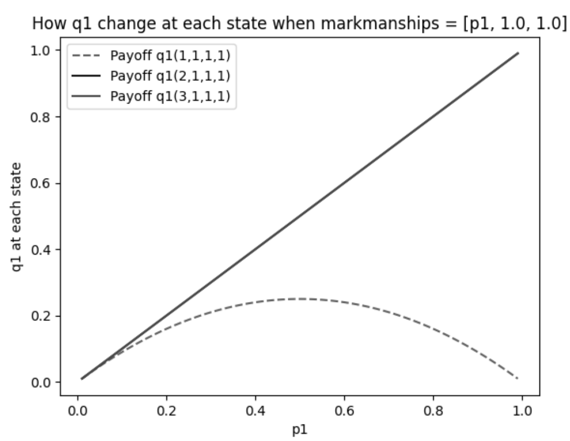


Figure 3: P_1's payoff is a decreasing function ofhis marksmanship.

5.2 4-uels

5.2.1 Shooting Weakest Opponent May Yield Maximum Payoff

Adding one more player with $p_4 < p_1$ at the experiment of Section 5.1.3, we end up with a paradoxical 4-uel. Consider the example of the following Table 4.

Table 4: Optimal strategy is to shoot at weakest player.

n	1	2	3	4
p_n	0.70	1.00	1.00	0.50
$\widehat{\sigma}_n(\mathbf{s}_n)$	4	3	4	2
$V_{n,11111}$	0.66	0.23	0.00	0.11

P_1's optimal strategy at $\mathbf{s}$ =11111 is to shoot at the weakest player. Taking the possible 3-uels in which P_1 can end up if he shoots successfully we have the following cases.

1. With $\sigma_1(11111) = 2$, if P_1 kills P_2 he ends up in a truel , similar to that of Section 5.1.1, where P_3 with $p_3 = 1.00$ has the move and P_1 is the second best player. Hence, the optimal strategy for P_3 is to shoot at P_1 and P_1 loses in the overall 4-uel, i.e., $V_{3,1011} = 0$.

2. With $\sigma_1(11111) = 3$, P_1 ends up in a similar 3-uel where P_2 always shoots and kills P_1.

3. With $\sigma_1(11111) = 4$, P_1 ends up in a truel similar to the one of Section 5.1.3, where he does not have the first move and achieves $V_{1,21110} = 0.66$. Hence this strategy yields the best possible payoff to P_1.

Note that in this example P_3's best strategy is also to shoot at the weakest player, because the probability of P_1 shooting successfully at P_2 after that is high.

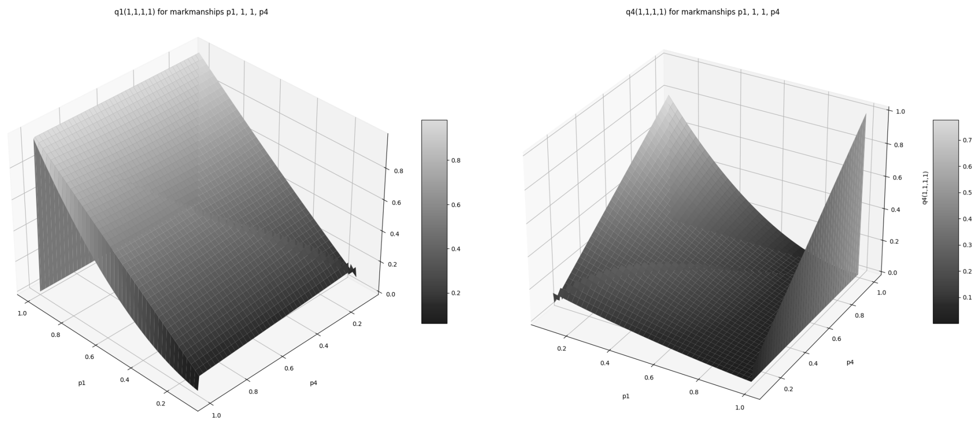
5.2.2 Payoffs as Functions of Marksmanships p_1 and p_4

In this example we generalize the results of Section 5.2.1. In particular, we assume that $p_2 = p_3 = 1$ and we study the dependence of the payoffs to the "nonperfect" players P_1 and P_4 on their marksmanships p_1 and p_4.

The surface in Figure 4(a) (resp. in Figure 4(b)) is $V_{1,11111}$ (resp. $V_{4,11111}$) as a function of p_1 and p_4, when $p_2 = p_3 = 1$. In both figures we have a discontinuity at $p_1 = p_4$. This is due to the fact that players change strategies. Taking $p_4 > p_1$ in the 3-uel starting at, for example, $\mathbf{s} = 21101$, P_4 is P_2's new target as he is the next strongest player after P_2.

5.2.3 Circular Shooting Sequences and Formation of Teams

We now present two examples in which we focus on the players' strategies rather than their payoffs. Taking $\mathbf{p} = (0.80, 0.40, 0.85, 0.50)$, we get the following optimal strategies.



(a) (b)

Figure 4: Payoffs $V_{1,11111}$ and $V_{4,11111}$ as functions of p_1 and p_4.

Table 5: An example of circular shooting and team formation.

n	1	2	3	4
p_n	0.80	0.40	0.85	0.50
$\widehat{\sigma}_n(\mathbf{s}_n)$	4	3	1	2

Taking $\mathbf{p} = (0.75, 0.25, 1.00, 0.50)$, we get the following optimal strategies.

Table 6: Another example of circular shooting and team formation.

n	1	2	3	4
p_n	0.75	0.25	1.00	0.50
$\widehat{\sigma}_n(\mathbf{s}_n)$	4	3	1	2

In both cases the optimal strategy profile follows a *circular shooting order:*

$$P_1 \underset{\text{shoots at}}{\rightarrow} P_4 \underset{\text{shoots at}}{\rightarrow} P_2 \underset{\text{shoots at}}{\rightarrow} P_3 \underset{\text{shoots at}}{\rightarrow} P_1$$

More generally, in all 4-uel experiments where we have a circular strategy, we can consider the players to be forming two teams. In this experiment the teams are $\{P_1, P_2\}$ and $\{P_3, P_4\}$; the optimal strategy for each member of each team is to shoot at his teammate's shooter.

5.2.4 Solidarity of the Weakest

In our final 4-uel example, taking $\mathbf{p} = (0.05, 0.10, 0.15, 0.70)$ we get a situation which resembles an economic model with three small businesses P_1, P_2 and P_3 and a large one P_4. The optimal strategy for the large business P_4 is to eliminate his strongest opponent, hence he shoots at P_3. The other three players are so weak that they do not want to lead the game to a truel similar to the experiment of Section 5.1.1, where the strongest player has the greatest payoff (for example, P_2 will not shoot P_1). Consequently, the three weak players (P_1, P_2 and P_3) cooperate to eliminate the strong player. The optimal strategies and respective payoffs are shown in Table 7. Note that P_4 has the greater expected payoff in each case and, in fact, his payoff is either very close or higher than 0.5.

Table 7: Solidarity of the weakest players.

n	1	2	3	4
p_n	0.05	0.10	0.15	0.70
$\widehat{\sigma}_n(\mathbf{s}_n)$	4	4	4	3
$V_{n,11111}$	0.18	0.20	0.13	0.49
$V_{n,21111}$	0.18	0.19	0.12	0.51
$V_{n,31111}$	0.16	0.18	0.09	0.57
$V_{n,41111}$	0.14	0.15	0.04	0.67

In N-uels of this type, we see a *solidarity* effect between the weakest players. However, it must be noted that such N-uels are a small, not representative, subset of the possible cases, as we have seen from our previous examples.

5.3 N-uels, $N \geq 5$

We conclude with two examples where, similar to the example of Section 5.2.3, we obtain circular shooting sequences. In the 5-uel with $\mathbf{p} = (0.25, 0.20, 0.10, 0.06, 0.04)$, computation shows that each player's optimal strategy is to shoot at the next player and the last shoot at the first.

Table 8: An example of circular shooting.

n	1	2	3	4	5
p_n	0.25	0.20	0.10	0.06	0.04
$\widehat{\sigma}_n(\mathbf{s}_n)$	2	3	4	5	1

Here is a 7-uel which has a similar effect.

Table 9: An example of circular shooting.

n	1	2	3	4	5	6	7
p_n	0.40	0.26	0.18	0.12	0.08	0.06	0.04
$\widehat{\sigma}_n(\mathbf{s}_n)$	4	1	5	6	2	7	3

While we have discovered many similar examples by brute force computation, we have not been able to obtain a condition on the $\mathbf{p}$ values which guarantees the emergence of circular shooting orders. Also, it is not obvious what shooting order will emerge as soon as one player is eliminated; it is not usually the case that the resultant shooting order will again be circular. In the future we intend to further study these questions.

6 Conclusion

We have studied a N-uel game (a generalization of the *duel*) in which finding a Nash equilibrium reduces to solving the system of nonlinear payoff equations. We have proved that this system has a solution (hence the N-uel has a stationary Nash equilibrium) and we have provided algorithms for its computational solution. In the future we want to study

1. the existence and computation of *nonstationary* Nash equilibria.
2. the properties of the N-uel variant in which each player can *abstain*, i.e., not shoot at any of his opponents.

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