MENG SUN

BOGOLONS

Contents

1
Basic properties

In what follows, we have $\hbar = k_B = 1$.

1.1 Bogoliubov coefficient

Bogoliubov coefficient is defined in [1]

[1]

$$u_{\mathbf{p}}^2 = 1 + v_{\mathbf{p}}^2 = \frac{1}{2}\left(1 + \left[1 + \left(\frac{Ms^2}{\omega_{\mathbf{p}}}\right)^2\right]^{1/2}\right), \tag{1.1}$$

$$u_{\mathbf{p}} v_{\mathbf{p}} = -\frac{Ms^2}{2\omega_{\mathbf{p}}}, \tag{1.2}$$

where the M is the effective mass of the condensed particle; s is the sound velocity of bogolons and $s = \sqrt{\frac{\kappa n_c}{M}}$; the n_c is the density of condensed particle; κ is the interaction strength; and the dispersion of the bogolons are[2]

$$\omega_{\mathbf{p}} = s\mathbf{p}\sqrt{1 + \mathbf{p}^2 \xi_h^2}. \tag{1.3}$$

The healing length is defined as $\xi_h = \frac{1}{2Ms}$.

[2] Usually, if we consider the exciton condensation, κ is exciton-exciton interaction, for indirect exciton, the result is $\kappa = \frac{e_0^2 d}{\epsilon}$.

1.2 Green's function

Bogolons are bosons. The Green's function is defined like phonons, First, introducing the following operators

$$A_{\mathbf{q}} = u_{\mathbf{q}} b_{\mathbf{p}} + v_{\mathbf{q}} b_{-\mathbf{q}}^{\dagger} \tag{1.4}$$

unlike the phonons where $u_{\mathbf{q}} = v_{\mathbf{q}} = 1$, we do **not** have $A_{\mathbf{q}}^{\dagger} = A_{-\mathbf{q}}$. These corresponding operators are

$$A_{-\mathbf{q}} = u_{\mathbf{q}} b_{-\mathbf{q}} + v_{\mathbf{q}} b_{\mathbf{q}}^{\dagger} \tag{1.5}$$

$$A_{\mathbf{q}}^{\dagger} = u_{\mathbf{q}} b_{\mathbf{q}}^{\dagger} + v_{\mathbf{q}} b_{-\mathbf{q}} \tag{1.6}$$

where we see from (**??**), $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$ are only magnitude depended.

From here, we have two definitions of the Green's function

$$\mathcal{F}(\mathbf{q},\tau) = -\langle T A_{\mathbf{q}}(\tau) A_{-\mathbf{q}}\rangle \tag{1.7}$$

The corresponding Green's function in Matsubara frequency is

$$\mathcal{D}(\mathbf{q}, i\omega_n) = u_{\mathbf{q}} v_{\mathbf{q}} \left[\frac{1}{i\omega_n - \omega_{\mathbf{q}}} - \frac{1}{i\omega_n + \omega_{\mathbf{q}}}\right] = \frac{2u_{\mathbf{q}} v_{\mathbf{q}} \omega_{\mathbf{q}}}{(i\omega_n)^2 - \omega_{\mathbf{q}}^2} \tag{1.8}$$

However, if we consider this

$$\mathcal{G}(\mathbf{q},\tau) = -\langle T A_{\mathbf{q}}(\tau) A^{\dagger}_{\mathbf{q}}\rangle \tag{1.9}$$

The corresponding result is

$$\mathcal{G}(\mathbf{q}, i\omega_n) = \frac{u_{\mathbf{q}}^2}{i\omega_n - \omega_{\mathbf{q}}} - \frac{v_{\mathbf{q}}^2}{i\omega_n + \omega_{\mathbf{q}}} \tag{1.10}$$

With another notation, we found

$$\mathcal{G}'(\mathbf{q},\tau) = -\langle T A^{\dagger}_{\mathbf{q}}(\tau) A_{\mathbf{q}}\rangle \tag{1.11}$$

the result is

$$\mathcal{G}'(\mathbf{p}, i\omega_n) = \frac{v_{\mathbf{q}}^2}{i\omega_n - \omega_{\mathbf{q}}} - \frac{u_{\mathbf{q}}^2}{i\omega_n + \omega_{\mathbf{q}}} \tag{1.12}$$

In principle we can write the following matrix form for the Green's function by notating

$$\Psi = (A_p, A^{\dagger}_{-p})^T \tag{1.13}$$

and

$$\hat{G} = -\langle T\Psi\Psi^{\dagger}\rangle = -\begin{bmatrix} \langle A_q A_q^{\dagger}\rangle & \langle A_q A_{-q}\rangle \\ \langle A^{\dagger}_{-q} A_q^{\dagger}\langle & \langle A^{\dagger}_{-q} A_q\rangle \end{bmatrix} \tag{1.14}$$

We can make a table for the result of the Green's function in Matsubara frequency

$\int d\tau e^{i\omega_n\tau}$	$-\langle TA_{\mathbf{q}}(\tau)\cdot$	$-\langle TA^{\dagger}_{\mathbf{q}}(\tau)\cdot$	$-\langle TA_{-\mathbf{q}}(\tau)\cdot$	$-\langle TA^{\dagger}_{-\mathbf{q}}(\tau)\cdot$	$\times$
	0	$\frac{v_{\mathbf{q}}^2}{i\omega_n-\omega_{\mathbf{q}}} - \frac{u_{\mathbf{q}}^2}{i\omega_n+\omega_{\mathbf{q}}}$	$\frac{u_{\mathbf{q}}v_{\mathbf{q}}}{i\omega_n-\omega_{\mathbf{q}}} - \frac{u_{\mathbf{q}}v_{\mathbf{q}}}{i\omega_n+\omega_{\mathbf{q}}}$	$\frac{v_{\mathbf{q}}^2}{i\omega_n-\omega_{\mathbf{q}}} - \frac{u_{\mathbf{q}}^2}{i\omega_n+\omega_{\mathbf{q}}}$	$A_{\mathbf{q}}\rangle$
	$\frac{u_{\mathbf{q}}^2}{i\omega_n-\omega_{\mathbf{q}}} - \frac{v_{\mathbf{q}}^2}{i\omega_n+\omega_{\mathbf{q}}}$	0	$\frac{u_{\mathbf{q}}^2}{i\omega_n-\omega_{\mathbf{q}}} - \frac{v_{\mathbf{q}}^2}{i\omega_n+\omega_{\mathbf{q}}}$	$\frac{u_{\mathbf{q}}v_{\mathbf{q}}}{i\omega_n-\omega_{\mathbf{q}}} - \frac{u_{\mathbf{q}}v_{\mathbf{q}}}{i\omega_n+\omega_{\mathbf{q}}}$	$A^{\dagger}_{\mathbf{q}}\rangle$
	$\frac{u_{\mathbf{q}}v_{\mathbf{q}}}{i\omega_n-\omega_{\mathbf{q}}} - \frac{u_{\mathbf{q}}v_{\mathbf{q}}}{i\omega_n+\omega_{\mathbf{q}}}$	$\frac{v_{\mathbf{q}}^2}{i\omega_n-\omega_{\mathbf{q}}} - \frac{u_{\mathbf{q}}^2}{i\omega_n+\omega_{\mathbf{q}}}$	0	$\frac{v_{\mathbf{q}}^2}{i\omega_n-\omega_{\mathbf{q}}} - \frac{u_{\mathbf{q}}^2}{i\omega_n+\omega_{\mathbf{q}}}$	$A_{-\mathbf{q}}\rangle$
	$\frac{u_{\mathbf{q}}^2}{i\omega_n-\omega_{\mathbf{q}}} - \frac{v_{\mathbf{q}}^2}{i\omega_n+\omega_{\mathbf{q}}}$	$\frac{u_{\mathbf{q}}v_{\mathbf{q}}}{i\omega_n-\omega_{\mathbf{q}}} - \frac{u_{\mathbf{q}}v_{\mathbf{q}}}{i\omega_n+\omega_{\mathbf{q}}}$	$\frac{u_{\mathbf{q}}^2}{i\omega_n-\omega_{\mathbf{q}}} - \frac{v_{\mathbf{q}}^2}{i\omega_n+\omega_{\mathbf{q}}}$	0	$A^{\dagger}_{-\mathbf{q}}\rangle$

By looking the (**??**) and (**??**), we realized that this two Green's function are identical

$$\mathcal{G}(\mathbf{q}, i\omega_n) = \mathcal{G}'(\mathbf{q}, -i\omega_n). \tag{1.15}$$

[3]

Using the (**??**), with some calculation we can write down the Green's function in a matrix form [3]

$$\hat{\mathcal{G}} = \begin{pmatrix} \mathcal{G} & \mathcal{F} \\ \mathcal{F} & \mathcal{G}' \end{pmatrix} \tag{1.16}$$

Another method to calculate the Green's function is based on the the discussion in [4] The result is given as the retarded Green's function

[4] ; and

$$\hat{\mathcal{G}}_{ret} = \begin{pmatrix} \frac{E+(q^2/2M)+\kappa n_c}{E^2-\omega_{\mathbf{q}}^2+i\delta} & \frac{-\kappa n_c}{E^2-\omega_{\mathbf{q}}^2+i\delta} \\ \frac{-\kappa n_c}{E^2-\omega_{\mathbf{q}}^2+i\delta} & \frac{-E+(q^2/2M)+\kappa n_c}{E^2-\omega_{\mathbf{q}}^2+i\delta} \end{pmatrix} \tag{1.17}$$

This result is given by the following approximation:

$$\begin{aligned} & \frac{u_{\mathbf{q}}^2}{i\omega_n + \omega_{\mathbf{q}}} - \frac{v_{\mathbf{q}}^2}{i\omega_n + \omega} \\ = & \frac{(u_{\mathbf{q}}^2 - v_{\mathbf{q}}^2) i\omega_n + (v_{\mathbf{q}}^2 + u_{\mathbf{p}}^2)\omega_{\mathbf{q}}}{(i\omega_n)^2 - (\omega_{\mathbf{q}})^2} \\ = & \frac{E + i\delta + (v_{\mathbf{q}}^2 + u_{\mathbf{q}}^2)\omega_{\mathbf{q}}}{E^2 + 2i\delta E - \delta^2 - \omega_{\mathbf{q}}^2} \\ \approx & \frac{E + (u_{\mathbf{q}}^2 + v_{\mathbf{q}}^2)\omega_{\mathbf{q}}}{E^2 - \omega^2 + i\delta \mathrm{sign}(E)} \end{aligned} \tag{1.18}$$

where we first apply the analytic continuous: $i\omega_n \rightarrow E + i\delta$ and assme $\delta E \rightarrow 0$. Further, for the rest part we consider

$$\begin{aligned} u_{\mathbf{q}}^2 &= \frac{1}{2}\left(1 + \frac{Ms^2}{\omega_{\mathbf{q}}}\sqrt{1 + \frac{\omega_{\mathbf{p}}^2}{M^2 s^4}}\right) \\ &\approx \frac{1}{2}\left(1 + \frac{Ms^2}{\omega_{\mathbf{q}}}\left(1 + \frac{\omega_{\mathbf{q}}^2}{2M^2 s^4}\right)\right) \\ &= \frac{1}{2}\left(1 + \frac{Ms^2}{\omega_{\mathbf{q}}} + \frac{\omega_{\mathbf{q}}}{2Ms^2}\right) \end{aligned} \tag{1.19}$$

$$v_{\mathbf{q}}^2 = \frac{1}{2}\left(\frac{Ms^2}{\omega_{\mathbf{q}}} + \frac{\omega_{\mathbf{q}}}{2Ms^2} - 1\right) \tag{1.20}$$

we can get the final result of the first element in (**??**).

2
Appendix

2.1 Fourier transform pairs

For a 3D functions, the Fourier transformation define

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \iiint E(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k} \tag{2.1}$$

and the inverse functions define

$$E(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \iiint f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r} \tag{2.2}$$

in the function are **spherically symmetric**, we have

$$f(r) = \sqrt{\frac{2}{\pi}} \frac{1}{r} \int_0^\infty E(k) \sin(kr) k dk \tag{2.3}$$

and

$$E(k) = \sqrt{\frac{2}{\pi}} \frac{1}{k} \int_0^\infty f(r) \sin(kr) r dr \tag{2.4}$$

For different integral function we have the following identity

$$e^{-\alpha r} \Leftrightarrow \sqrt{\frac{2}{\pi}} \frac{2\alpha}{(\alpha^2 + k^2)^2} \tag{2.5}$$

$$\frac{e^{-\alpha r}}{r} \Leftrightarrow \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + k^2} \tag{2.6}$$

$$1 \Leftrightarrow (2\pi)^{3/2} \delta(\mathbf{k}) \tag{2.7}$$

For 2D function, the Fourier transformation

$$f(\mathbf{r}) = \frac{1}{2\pi} \iint E(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k} \tag{2.8}$$

and the inverse Fourier transformation

$$E(\mathbf{k}) = \frac{1}{2\pi} \iint f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r} \tag{2.9}$$

An important pair is

$$\frac{1}{r} \Leftrightarrow \frac{1}{k} \tag{2.10}$$

If we consider the form, $\frac{1}{\mathbf{k}^2}$

$$
\begin{aligned}
& \frac{1}{2\pi} \iint \frac{1}{\mathbf{k}^2} e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k} \\
= & \int_0^{2\pi} \int_0^{\infty} \frac{e^{ikr\cos\theta}}{k} dk d\theta \\
= & \int_0^{\infty} \frac{J_0(kr)}{k} dk \qquad (2.11)
\end{aligned}
$$

Which does not converge.