MENG SUN

NOTES ON: CONDENSED MATTER FIELD THEORY –BY ALEXANDER ALTLAND

Contents

1

Second quatization

Second quntization provides a compact way of representing the many-body space of excitations; secondly, the properties of the ladder operatore, $\hat{a}_k$ were encoded in a simple set of commutation relations rather than in some explicit Hilbert space representation

1.1 Introduction to second quantization

1.1.1 Motivation

Consider the normalized set of wavefunctions $|\lambda\rangle$ of some single-particle Hamiltonian $\hat{H} : \hat{H}|\lambda\rangle = \epsilon_\lambda |\lambda\rangle$. With this definition, two-particle wavefunction in two leverls is

$$\psi_{F,B}(x_1, x_2) = \frac{1}{\sqrt{2}} \left(\langle x_1|\lambda_1\rangle \langle x_2|\lambda_2\rangle \mp \langle x_1|\lambda_2\rangle \langle x_2|\lambda_1\rangle \right) \tag{1.1}$$

More generally, an appropriately symmetrized N-particle wavefunction can be expressed in **Slater determinants**

$$|\lambda_1, \lambda_2, \ldots, \lambda_N\rangle \equiv \frac{1}{\sqrt{N! \prod_{\lambda=0}^{\infty} (n_\lambda!)}} \sum_{\mathcal{P}} \zeta^{(1-\mathrm{sgn}\mathcal{P})/2} \left|\lambda_{\mathcal{P}_1}\right\rangle \otimes \left|\lambda_{\mathcal{P}_2}\right\rangle \otimes \cdots \otimes \left|\lambda_{\mathcal{P}_N}\right\rangle \tag{1.2}$$

where η_λ represents the total number of particles in state λ. The summation runs over $N!$ permutations of the set of quantume numbers $\lambda_1, \ldots \lambda_N$ and sgn$\mathcal{P}$ denotes the sign of the permutation $\mathcal{P}$ [1].

While representations eq. (1.2) can be used to represent the full Hilbert space of many-body quanatume mechanics, it is not always convenient:

- Eq. (1.2) is cumbersome.
- For the problem with fixed number of particle
- A represention where the quantum numbers of individual quasi-particles rather than the entangled ste of quantum number of all consitituents are fundamental.

[1] Permutation form $(\mathcal{P}_1, \ldots \mathcal{P}_N) \to (1, \ldots, N)$.

1.1.2 The apparatus of second qunatization

OCCUPATION NUMBER REPRESENTATION, in this representation, the basis states of F^N are specified by $|n_1, n_2, \ldots\rangle$. Any state $|\Psi\rangle$ in F^N can be obtain by a linear superposition.

Define the **Fock space** as

$$\mathcal{F} \equiv \oplus_{N=0}^{\infty} \mathcal{F}^N \tag{1.3}$$

The complicated permutation "entanglement" implied in the definition (1.2) of the Fock state can be generated by application of a set of linear operators to a single reference state.

$$|n_1, n_2, \ldots\rangle = \prod_i \frac{1}{\sqrt{n_i!}} \left(a_i^\dagger\right)^{n_i} |0\rangle \tag{1.4}$$

For practical aspects we need to find out, for Fock space, how changes from one single-particle basis to another affect the operator algebra, and in what way generic operators acting in many-particle Hilbert spaces can be represented in terms of creation and annihilation operators.

Suppose that $\mathcal{A}$ is irreducibly represented in some vector space V, i.e. that there is a mapping assigning to each $a_i \in \mathcal{A}$ a linear mapping $a_i : V \to V$, such that every vecotr $|v\rangle \in V$ can be reached from and other $|w\rangle \in V$ by application fo r operatore a_i and $a_i^\dagger$. According to the **Stone-von Neumann theorem** (a) such a representation is unique upt to unitary equivalence; (b) there is a unique state $|0\rangle \in V$ that is annihilated by every a_i.

Change basis

$$a_{\lambda'}^\dagger = \sum_\lambda \langle \lambda | \lambda' \rangle a_\lambda^\dagger, \quad a_{\lambda'} = \sum_\lambda \langle \lambda' | \lambda \rangle a_\lambda \tag{1.5}$$

Representation of operators: Single-particle operators acting in the N-partilce Hilbert space generally take the form $\hat{\mathcal{O}}_1 = \sum_{n=1}^N \hat{o}_n$, where $\hat{o}_n$ is an ordinary single-particle operator acting on the nth particle. In general, from a representation to a general basis,

$$\hat{\mathcal{O}}_1 = \sum_{\mu\nu} \langle \mu | \hat{o} | \nu \rangle a_\mu^\dagger a_\nu \tag{1.6}$$

for two-bdy operator in general we have

$$\hat{\mathcal{O}}_2 = \sum_{\lambda\lambda'\mu\mu'} \langle \mu, \mu' | \hat{\mathcal{O}}_2 | \lambda, \lambda' \rangle a_\mu^\dagger a_{\mu'}^\dagger a_\lambda a_{\lambda'} \tag{1.7}$$

1.2 Application of second qunatization