APPENDIX

A. Intermediate matrices for the state-space model

$$\boldsymbol{A}_{11} = \begin{bmatrix} -\boldsymbol{\Gamma}^{-1} & -\boldsymbol{\Gamma}^{-1}\boldsymbol{K}_{\mathrm{I}} & -\boldsymbol{\Gamma}^{-1}\boldsymbol{K}_{\mathrm{P}}\hat{\boldsymbol{T}}_{\mathrm{FL,B}}\underline{\boldsymbol{E}}_{\mathcal{G}}\underline{\boldsymbol{E}}_{\mathcal{G}}^{T} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \hat{\boldsymbol{T}}_{\mathrm{FL,B}}\underline{\boldsymbol{E}}_{\mathcal{G}}\underline{\boldsymbol{E}}_{\mathcal{G}}^{T} & \boldsymbol{O} \\ \hat{\boldsymbol{T}}_{\mathrm{B,FL}} & \boldsymbol{O} & -\boldsymbol{T}_{\mathrm{B,L}}\boldsymbol{D}_{\mathrm{L}}^{-1}\boldsymbol{T}_{\mathrm{L,A}}\underline{\boldsymbol{L}}(\mathcal{G},\boldsymbol{W}_{\mathrm{p}})\boldsymbol{T}_{\mathrm{A,B}} & \boldsymbol{T}_{\mathrm{B,SF}} \\ \boldsymbol{O} & \boldsymbol{O} & -\underline{\boldsymbol{M}}_{\mathrm{SF}}^{-1}\boldsymbol{T}_{\mathrm{SF,A}}\underline{\boldsymbol{L}}(\mathcal{G},\boldsymbol{W}_{\mathrm{p}})\boldsymbol{T}_{\mathrm{A,B}} & -\underline{\boldsymbol{M}}_{\mathrm{SF}}^{-1}\boldsymbol{D}_{\mathrm{SF}} \end{bmatrix} \tag{42}$$

$$\boldsymbol{A}_{12} = \mathrm{col}(\boldsymbol{O}, \boldsymbol{O}, -\boldsymbol{T}_{\mathrm{B,L}}\boldsymbol{D}_{\mathrm{L}}^{-1}\boldsymbol{T}_{\mathrm{L,A}}\underline{\boldsymbol{L}}(\mathcal{G}, \boldsymbol{W}_{\mathrm{p}})\boldsymbol{T}_{\mathrm{A,FL}}, \boldsymbol{O}) \tag{43}$$

$$\boldsymbol{A}_{21} = \mathrm{row}(\boldsymbol{D}_{\mathrm{FL}} - \boldsymbol{\Gamma}^{-1}\boldsymbol{M}_{\mathrm{FL}}, -\boldsymbol{\Gamma}^{-1}\boldsymbol{M}_{\mathrm{FL}}\boldsymbol{K}_{\mathrm{I}}, -\boldsymbol{\Gamma}^{-1}\boldsymbol{M}_{\mathrm{FL}}\boldsymbol{K}_{\mathrm{P}}\hat{\boldsymbol{T}}_{\mathrm{FL,B}}\underline{\boldsymbol{E}}_{\mathcal{G}}\underline{\boldsymbol{E}}_{\mathcal{G}}^{T} + \boldsymbol{T}_{\mathrm{FL,A}}\underline{\boldsymbol{L}}(\mathcal{G}, \boldsymbol{W}_{\mathrm{p}})\boldsymbol{T}_{\mathrm{A,B}}, \boldsymbol{O}) \tag{44}$$

$$\boldsymbol{A}_{22} = \boldsymbol{T}_{\mathrm{FL,A}}\underline{\boldsymbol{L}}(\mathcal{G}, \boldsymbol{W}_{\mathrm{p}})\boldsymbol{T}_{\mathrm{A,FL}} \tag{45}$$

with $\boldsymbol{W}_{\mathrm{p}} = \boldsymbol{B}_{\mathrm{V}} \frac{\partial \sin(\underline{\boldsymbol{E}}_{\mathcal{G}}^{T}\boldsymbol{T}_{\mathrm{A}}^{T}\boldsymbol{\alpha}^{0})}{\partial(\underline{\boldsymbol{E}}_{\mathcal{G}}^{T}\boldsymbol{T}_{\mathrm{A}}^{T}\boldsymbol{\alpha}^{0})}$.

B. Notation of some matrices

TABLE I
NOTATION OF SOME MATRICES.

Matrix	Expression
$\boldsymbol{T}$	$\mathrm{row}(-\boldsymbol{1}_{\lvert\mathcal{V}_{\mathrm{FL}}\rvert+\lvert\underline{\mathcal{V}}\rvert}, \boldsymbol{I}_{\lvert\mathcal{V}_{\mathrm{FL}}\rvert+\lvert\underline{\mathcal{V}}\rvert}) \in \mathbb{R}^{(\lvert\mathcal{V}_{\mathrm{FL}}\rvert+\lvert\underline{\mathcal{V}}\rvert)\times(\lvert\mathcal{V}_{\mathrm{FL}}\rvert+\lvert\mathcal{V}\rvert)}$
$\boldsymbol{T}_{\mathrm{SF}}$	Submatrix of $\boldsymbol{T}$ consisting of columns corresponding to $\underline{\mathcal{V}}_{\mathrm{SF}}$
$\boldsymbol{T}_{\mathrm{L}}$	Submatrix of $\boldsymbol{T}$ consisting of columns corresponding to $\mathcal{V}_{\mathrm{L}}$
$\hat{\boldsymbol{T}}_{\mathrm{FL}}$	Submatrix of $\boldsymbol{T}$ consisting of columns corresponding to $\mathcal{V}_{\mathrm{FL}}$ (multiplying $\hat{\boldsymbol{\theta}}_{\mathrm{FL}}$)
$\boldsymbol{T}_{\mathrm{FL}}$	Submatrix of $\boldsymbol{T}$ consisting of columns corresponding to $\mathcal{V}_{\mathrm{FL}}$ (multiplying $\boldsymbol{\theta}_{\mathrm{FL}}$)
$\boldsymbol{T}_{\mathrm{A}}$	$\mathrm{diag}(\mathrm{col}(\boldsymbol{I}_{\lvert\underline{\mathcal{V}}_{\mathrm{SF}}\rvert+\lvert\mathcal{V}_{\mathrm{L}}\rvert}, \boldsymbol{O}_{\lvert\mathcal{V}_{\mathrm{FL}}\rvert\times(\lvert\underline{\mathcal{V}}_{\mathrm{SF}}\rvert+\lvert\mathcal{V}_{\mathrm{L}}\rvert)}), \boldsymbol{I}_{\lvert\mathcal{V}_{\mathrm{FL}}\rvert})$
$\boldsymbol{T}_{\mathrm{B}}$	$\mathrm{row}(\boldsymbol{T}_{\mathrm{SF}}, \boldsymbol{T}_{\mathrm{L}}, \hat{\boldsymbol{T}}_{\mathrm{FL}})$
$\boldsymbol{T}_{\mathrm{C}}$	$\mathrm{row}(\boldsymbol{T}_{\mathrm{SF}}, \boldsymbol{T}_{\mathrm{L}})$
$\boldsymbol{T}_{\mathrm{DS}}$	$\mathrm{row}(\boldsymbol{I}_{\lvert\mathcal{V}_{\mathrm{SF}}\rvert}, \boldsymbol{O}_{\lvert\mathcal{V}_{\mathrm{SF}}\rvert\times\lvert\mathcal{V}_{\mathrm{FL}}\rvert})$
$\boldsymbol{T}_{\mathrm{DL}}$	$\mathrm{row}(\boldsymbol{O}_{\lvert\mathcal{V}_{\mathrm{L}}\rvert\times\lvert\mathcal{V}_{\mathrm{SF}}\rvert}, \boldsymbol{I}_{\lvert\mathcal{V}_{\mathrm{L}}\rvert}, \boldsymbol{O}_{\lvert\mathcal{V}_{\mathrm{L}}\rvert\times\lvert\mathcal{V}_{\mathrm{FL}}\rvert})$
$\boldsymbol{T}_{\mathrm{D}}$	$\mathrm{col}(\mathrm{row}(\boldsymbol{I}_{\lvert\mathcal{V}_{\mathrm{SF}}\rvert}, \boldsymbol{O}_{\lvert\mathcal{V}_{\mathrm{SF}}\rvert\times\lvert\mathcal{V}_{\mathrm{L}}\cup\mathcal{V}_{\mathrm{FL}}\rvert}), \mathrm{row}(\boldsymbol{O}_{\lvert\mathcal{V}_{\mathrm{FL}}\rvert\times\lvert\mathcal{V}_{\mathrm{SF}}\cup\mathcal{V}_{\mathrm{L}}\rvert}, \boldsymbol{I}_{\lvert\mathcal{V}_{\mathrm{FL}}\rvert}))$
$\widetilde{\boldsymbol{T}}_{\mathrm{D}}$	$\mathrm{diag}(\mathrm{row}(\boldsymbol{I}_{\lvert\mathcal{V}_{\mathrm{SF}}\rvert}, \boldsymbol{I}_{\lvert\mathcal{V}_{\mathrm{SF}}\rvert}), \boldsymbol{I}_{\lvert\mathcal{V}_{\mathrm{L}}\rvert-\lvert\mathcal{V}_{\mathrm{SL}}\rvert}, \mathrm{row}(\boldsymbol{I}_{\lvert\mathcal{V}_{\mathrm{FL}}\rvert}, \boldsymbol{I}_{\lvert\mathcal{V}_{\mathrm{FL}}\rvert}))$
$\boldsymbol{T}_{\mathrm{SL}}$	$\mathrm{col}(\mathrm{row}(\boldsymbol{I}_{\lvert\mathcal{V}_{\mathrm{SF}}\rvert}, \boldsymbol{O}_{\lvert\mathcal{V}_{\mathrm{SF}}\rvert\times\lvert\mathcal{V}_{\mathrm{FL}}\rvert}), \boldsymbol{O}_{(\lvert\mathcal{V}_{\mathrm{L}}\rvert-\lvert\mathcal{V}_{\mathrm{SL}}\rvert)\times\lvert\mathcal{V}_{\mathrm{SL}}\rvert}, \mathrm{row}(\boldsymbol{O}_{\lvert\mathcal{V}_{\mathrm{FL}}\rvert\times\lvert\mathcal{V}_{\mathrm{SF}}\rvert}, \boldsymbol{I}_{\lvert\mathcal{V}_{\mathrm{FL}}\rvert}))$
$\boldsymbol{T}_{1}$	$\mathrm{col}(\boldsymbol{I}_{\lvert\mathcal{E}_{\mathrm{SF}}\rvert}, \boldsymbol{O}_{\lvert\mathcal{E}_{\mathrm{L}}\rvert\times\lvert\mathcal{E}_{\mathrm{SF}}\rvert})$
$\boldsymbol{T}_{2}$	$\mathrm{col}(\boldsymbol{O}_{\lvert\mathcal{E}_{\mathrm{SF}}\rvert\times\lvert\mathcal{E}_{\mathrm{L}}\rvert}, \boldsymbol{I}_{\lvert\mathcal{E}_{\mathrm{L}}\rvert}) \in \mathbb{R}^{(\lvert\mathcal{E}_{\mathrm{SF}}\rvert+\lvert\mathcal{E}_{\mathrm{L}}\rvert)\times\lvert\mathcal{E}_{\mathrm{L}}\rvert}$
$\boldsymbol{T}_{3}$	$\mathrm{col}(\boldsymbol{O}_{2\lvert\mathcal{V}_{\mathrm{FL}}\rvert\times(\lvert\mathcal{V}\rvert-1)}, \boldsymbol{I}_{\lvert\mathcal{V}\rvert-1}, \boldsymbol{O}_{(\lvert\mathcal{V}_{\mathrm{SF}}\rvert-1)\times(\lvert\mathcal{V}\rvert-1)})$
$\boldsymbol{T}_{4}$	$\mathrm{col}(\boldsymbol{O}_{2\lvert\mathcal{V}_{\mathrm{FL}}\rvert\times(\lvert\mathcal{V}\rvert+\lvert\mathcal{V}_{\mathrm{SF}}\rvert-2)}, \boldsymbol{I}_{\lvert\mathcal{V}\rvert+\lvert\mathcal{V}_{\mathrm{SF}}\rvert-2})$

C. Linear matrices in the reformulation of BMI

$$\widetilde{F}_1(\widetilde{\boldsymbol{Y}}) = -\widetilde{\boldsymbol{A}}_{\mathrm{b}}^{T}\boldsymbol{E}_{\mathrm{G}}\widetilde{\boldsymbol{B}} \; (i,j)\in\mathcal{E}_{\mathrm{L}} \left\{ \overbrace{\begin{bmatrix} \frac{1}{d_i}\widetilde{\boldsymbol{Y}}_{23}^{ij,k,0} - \frac{1}{d_j}\widetilde{\boldsymbol{Y}}_{23}^{ij,k,1} & \cdots \\ \vdots & \ddots \end{bmatrix}}^{k\in\{1,\cdots,n_{\mathrm{x}}\}} \right. \tag{46}$$

$$F_1(\boldsymbol{Y}) = -\boldsymbol{A}_{\mathrm{b}}^{T}\underline{\boldsymbol{E}}_{\mathcal{G},4}\boldsymbol{B}_{\mathrm{SF}} \; (i,j)\in\mathcal{E}_{\mathrm{SF}} \left\{ \overbrace{\begin{bmatrix} -\frac{1}{d_j}\boldsymbol{Y}_{23}^{ij,k,1} + \frac{1}{m_i'}\boldsymbol{Y}_{23}^{ij,k,0} & \cdots \\ \vdots & \ddots \end{bmatrix}}^{k\in\{1,\cdots,n_{\mathrm{x}}\}} \right. \tag{47}$$

Note that in $\boldsymbol{E}_{\mathrm{G}}$, entries for branch $(i,j)\in\mathcal{E}_{\mathrm{L}}$ are assigned 1 and -1 for buses i and j, respectively. $\frac{1}{m_i'} = 0$ if $i = i_0$ and $m_i' = m_i$ otherwise.

D. The dual problem in DSOS

To write the dual problem of (38), we first reformulate it into a more canonical form. By rearranging optimization variables in $\mathcal{X}$ as a block diagonal matrices:

$$\tilde{\boldsymbol{X}} = \mathrm{diag}\begin{pmatrix} \mathrm{diag}(\boldsymbol{z}), \mathrm{diag}(\boldsymbol{T}_{\mathrm{E}}\boldsymbol{p}_{\mathrm{in}}^{0}), \mathrm{diag}(\Re(\hat{\boldsymbol{u}})), \mathrm{diag}(\Im(\hat{\boldsymbol{u}})), \\ \mathrm{diag}(\check{\boldsymbol{u}}), \boldsymbol{X}, \boldsymbol{Z}_{\mathrm{PD}}, \boldsymbol{U}_{\mathrm{PSD1},ij}, \boldsymbol{U}_{\mathrm{PSD2},ij}, \\ \widetilde{\boldsymbol{Y}}^{ij,k,0}, \widetilde{\boldsymbol{Y}}^{ij,k,1}, \boldsymbol{Y}^{ij,k,0}, \boldsymbol{Y}^{ij,k,1}, \boldsymbol{Z}^{ij} \end{pmatrix} \tag{48}$$

where $\mathrm{diag}(\boldsymbol{V})$ is not needed for $\tilde{\boldsymbol{X}}$ since they are contained in $\boldsymbol{U}_{\mathrm{PSD1},ij}$. Then (38) can be rewritten as

$$\min_{\tilde{\boldsymbol{X}}} \; \mathrm{Tr}(\boldsymbol{B}\boldsymbol{B}^{T}\boldsymbol{X}) \tag{49a}$$

$$\text{s.t.} \; \mathrm{Tr}(\tilde{\boldsymbol{A}}_i\tilde{\boldsymbol{X}}) \trianglerighteq_i \tilde{b}_i \quad \forall i \in \mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2 \tag{49b}$$

$$\mathrm{Tr}(\boldsymbol{N}_{\mathrm{x}}^{(k)T}\boldsymbol{v}_{\mathrm{x},i}\boldsymbol{v}_{\mathrm{x},i}^{T}\boldsymbol{N}_{\mathrm{x}}^{(k)}\boldsymbol{X}) > 0 \quad \forall i \in \{1,\ldots,\dim^2(\boldsymbol{A})\} \tag{49c}$$

$$\mathrm{Tr}(\boldsymbol{N}_{\mathrm{z}}^{(k)T}\boldsymbol{v}_{\mathrm{z},i}\boldsymbol{v}_{\mathrm{z},i}^{T}\boldsymbol{N}_{\mathrm{z}}\boldsymbol{Z}_{\mathrm{PD}}) > 0 \quad \forall i \in \{1,\ldots,\lvert\mathcal{E}_{\mathrm{L},2}\rvert^2\} \tag{49d}$$

$$[\tilde{\boldsymbol{X}}]_j \succcurlyeq_j 0 \quad \forall j \in \mathcal{J}_1 \cup \mathcal{J}_3 \backslash \{j_1, j_2\} \tag{49e}$$

$$[\tilde{\boldsymbol{X}}]_j \in \mathbb{R} \quad \forall j \in \mathcal{J}_2 \tag{49f}$$

$$\boldsymbol{X} \in \mathbb{S}^{\dim(\boldsymbol{A})}, \boldsymbol{Z}_{\mathrm{PD}} \in \mathbb{S}^{\lvert\mathcal{E}_{\mathrm{L},2}\rvert} \tag{49g}$$

where "$\trianglerighteq_i$" equals "$\geq$" for $i \in \mathcal{I}_1$ and "$\trianglerighteq_i$" equals "$=$" for $i \in \mathcal{I}_2$; "$\succcurlyeq_j$" equals "$\geq$" for $j \in \mathcal{J}_1$, "$\geq$ or $\leq$" for $j \in \mathcal{J}_2$, "$\succeq$" for $j \in \mathcal{J}_3$, and "$\succ$" for $j \in \{j_1, j_2\}$; $[\tilde{\boldsymbol{X}}]_j$ denotes scalar entries of vector variables in $\tilde{\boldsymbol{X}}$, indexed by $j \in \mathcal{J}_1 \cup \mathcal{J}_2$, and $[\tilde{\boldsymbol{X}}]_j$ denotes matrix variables in $\tilde{\boldsymbol{X}}$, indexed by $j \in \mathcal{J}_3 \cup \{j_1, j_2\}$; and $[\tilde{\boldsymbol{X}}]_{j_1} = \boldsymbol{X}$ and $[\tilde{\boldsymbol{X}}]_{j_2} = \boldsymbol{Z}_{\mathrm{PD}}$. Then the dual of (49) is

$$\max_{\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{\alpha}}_{\mathrm{x}}, \tilde{\boldsymbol{\alpha}}_{\mathrm{z}}} \; \sum_{i\in\mathcal{I}} \tilde{y}_i \tilde{b}_i \tag{50a}$$

$$\text{s.t.} \; \tilde{y}_i \geq 0 \quad \forall i \in \mathcal{I}_1 \tag{50b}$$

$$\tilde{y}_i \in \mathbb{R} \quad \forall i \in \mathcal{I}_2 \tag{50c}$$

$$\tilde{\alpha}_{\mathrm{x},i} > 0 \quad \forall i \in \{1, \ldots, \dim^2(\boldsymbol{A})\} \tag{50d}$$

$$\tilde{\alpha}_{\mathrm{z},i} > 0 \quad \forall i \in \{1, \ldots, \lvert\mathcal{E}_{\mathrm{L},2}\rvert^2\} \tag{50e}$$

$$\sum_{i\in\mathcal{I}} y_i [\tilde{\boldsymbol{A}}_i]_j \leq 0 \quad \forall j \in \mathcal{J}_1 \tag{50f}$$

$$\sum_{i\in\mathcal{I}} y_i [\tilde{\boldsymbol{A}}_i]_j = 0 \quad \forall j \in \mathcal{J}_2 \tag{50g}$$

$$\sum_{i\in\mathcal{I}} y_i [\tilde{\boldsymbol{A}}_i]_j \preceq 0 \quad \forall j \in \mathcal{J}_3 \tag{50h}$$

$$\boldsymbol{B}\boldsymbol{B}^{T} - \sum_{i\in\mathcal{I}} y_i [\tilde{\boldsymbol{A}}_i]_{j_1} = \sum_{i=1}^{\dim^2(\boldsymbol{A})} \tilde{\alpha}_{\mathrm{x},i} (\boldsymbol{N}_{\mathrm{x}}^{(k)T}\boldsymbol{v}_{\mathrm{x},i})\boldsymbol{v}_{\mathrm{x},i}^{T}\boldsymbol{N}_{\mathrm{x}}^{(k)} \tag{50i}$$

$$-\sum_{i\in\mathcal{I}} y_i [\tilde{\boldsymbol{A}}_i]_{j_2} = \sum_{i=1}^{\lvert\mathcal{E}_{\mathrm{L},2}\rvert^2} \tilde{\alpha}_{\mathrm{z},i} (\boldsymbol{N}_{\mathrm{z}}^{(k)T}\boldsymbol{v}_{\mathrm{z},i})(\boldsymbol{N}_{\mathrm{z}}^{(k)T}\boldsymbol{v}_{\mathrm{z},i})^{T} \tag{50j}$$