Chapter 5.1 Systems of linear inequalities in two variables.

In this section, we will learn how to graph linear inequalities in two variables and then apply this procedure to practical application problems.

Graphing a linear inequality

- Our first example is to graph the linear equality

$$y < \frac{3}{4}x - 1$$

The following is the procedure to graph a linear inequality in two variables:

1. Replace the inequality symbol with an equal sign

$$y = \frac{3}{4}x - 1$$

2. Construct the graph of the line. If the original inequality is a > or < sign, the graph of the line should be dotted. Otherwise, the graph of the line is solid.

Continuation of Procedure

- Since the original problem contained the inequality symbol (<) the line that is graphed should be dotted.

  For our problem, the equation of our line $y = \frac{3}{4}x - 1$

  is already in slope-intercept form,(y=mx+b) so we easily sketch the line by first starting at the y-intercept of -1, then moving vertically 3 units and over to the right of 4 units corresponding to our slope of ¾. After locating the second point, we sketch the dotted line passing through these two points.

Continuation of Procedure

- 3. Now, we have to decide which half plane to shade. The solution set will either be (a) the half-plane above the line or (b) the half-plane below the graph of the line. To determine which half-plane to shade, we choose a **test point** that is not on the line. Usually, a good test point to pick is the origin (0,0), unless the origin happens to lie on the line. In this case, we choose the origin as a test point to see if this point satisfies the original inequality.

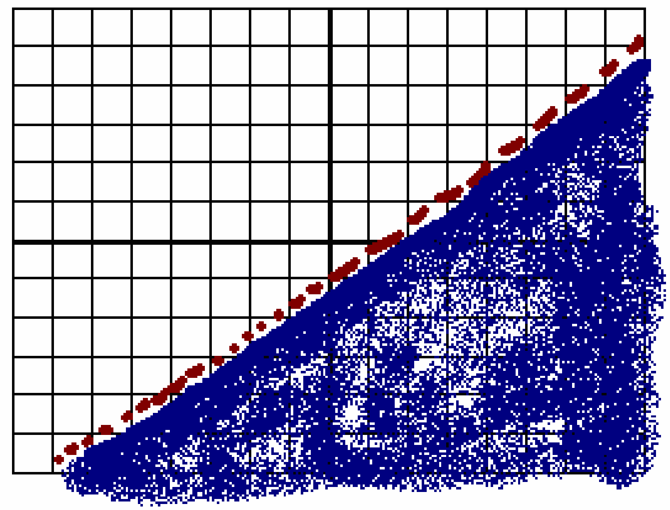
- Substituting the origin in the inequality

$$y < \frac{3}{4}x - 1$$

- produces the statement
- **0 < 0 – 1 or 0 < -1. Since this is a false statement, we shade the region on the side of the line NOT containing the origin. Had the origin satisfied the inequality, we would have shaded the region on the side of the line CONTAINING THE ORIGIN.**

Graph of Example 1

- Here is the complete graph of the first inequality:



Example 2

- For our second example, we will graph the inequality $3x - 5y \geq 15$

1. Step 1. Replace inequality symbol with equals sign: **3x – 5y = 15**
2. Step 2. Graph the line

3x – 5y = 15 Since 3 and -5 are divisors of 15, we will graph the line using the x and y intercepts: When x = 0 , y = -3 and y = 0 , x = 5. Plot these points and draw a **solid line** since the original inequality symbol is **less than or equal to which means that the graph of the line itself is included.**

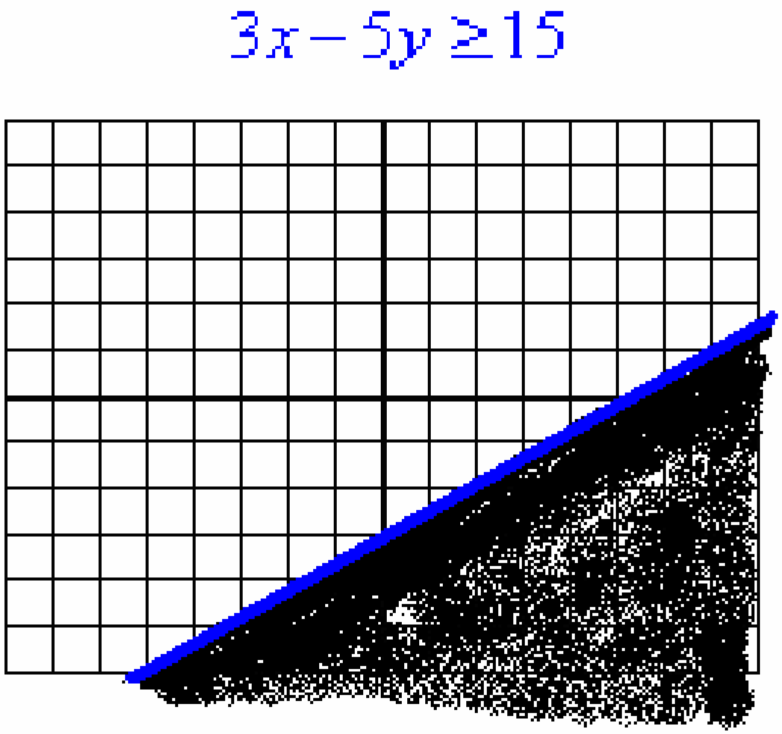
Example 2 continued:

- **Step 3.** Choose a point not on the line. Again, the origin is a good test point since it is not part of the line itself. We have the following statement which is clearly false.

  $$3(0)-5(0)\geq 15$$

- Therefore, we shade the region on the side of the line that **does not include the origin.**

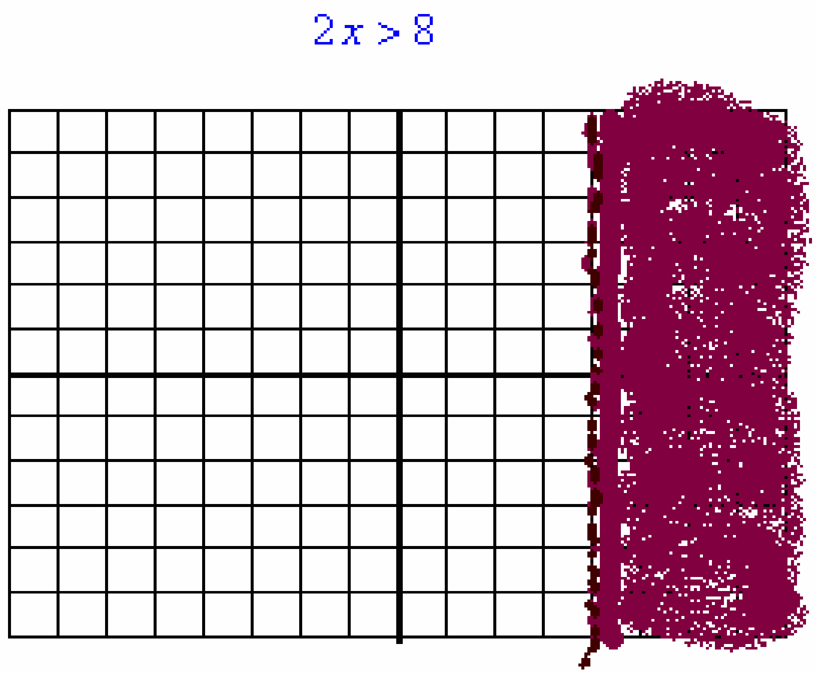
Graph of Example 2



Example 3 : $2x > 8$

- Our third example is unusual in that there is no y-variable present. The inequality $2x>8$ is equivalent to the inequality $x > 4$. How shall we proceed to graph this inequality? The answer is the same way we graphed previous inequalities:
- Step 1: Replace the inequality symbol with an equals sign.
- $x = 4$.
- Step 2: Graph the line $x = 4$. Is the line solid or dotted? The original inequality is > (strictly greater than- not equal to). Therefore, the line is **dotted.**
- Step 3. Choose the origin as a test point. Is $2(0)>8$? Clearly not.
- Shade the side of the line that does not include the origin. The graph is displayed on the next slide.

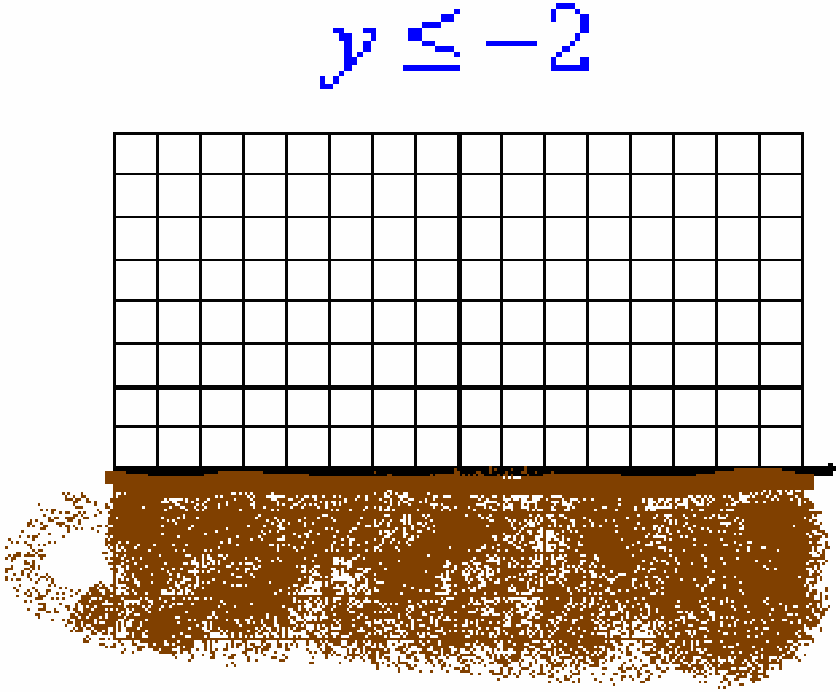
Graph of 2x>8



Example 4: $y \leq -2$

- This example illustrates the type of problem in which the x-variable is missing. We will proceed the same way.
- Step 1. Replace the inequality symbol with an equal sign
- $y = 2$
- Step 2. Graph the equation $y = 2$. The line is solid since the original inequality symbol is less than **or equal to**.
- Step 3. Shade the appropriate region. Choosing again the origin as the test point, we find that $0 \leq -2$ is a false statement so we shade the side of the line that **does not** include the origin.
- Graph is shown in next slide.

Graph of Example 4.



Graphing a system of linear inequalities- Example 5

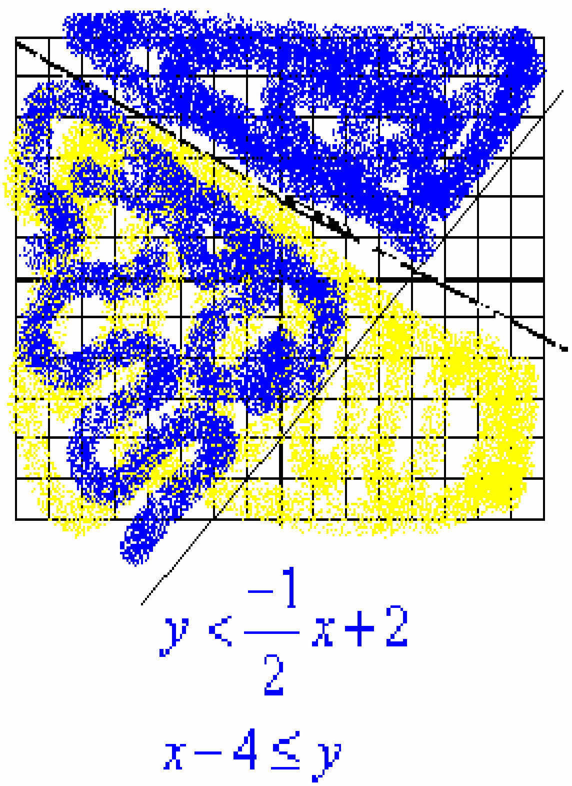
- To graph a system of linear inequalities such as

$$y < \frac{-1}{2}x + 2$$

$$x - 4 \leq y$$

- we proceed as follows:
- Step 1. Graph each inequality on the same axes. The solution is the set of points whose coordinates satisfy all the inequalities of the system. In other words, the solution is the **intersection** of the regions determined by each separate inequality.

Graph of example 5



- The graph is the region which is colored **both blue and yellow**. The graph of the first inequality consists of the region shaded yellow and lies below the dotted line determined by the inequality $y < \frac{-1}{2}x + 2$
- The blue shaded region is determined by the graph of the inequality $x - 4 \le y$
- and is the region above the line **x – 4 = y**

Graph of more than two linear inequalities

- To graph more than two linear inequalities, the same procedure is used. Graph each inequality separately. The graph of a system of linear inequalities is the area that is **common to all graphs, or the intersection of the graphs of the individual inequalities.**

$$x \geq 0$$

$$y \geq 0$$

$$x \leq 15$$

$$8x + 8y \leq 160$$

$$4x + 12y \leq 180$$

Application

- Before we graph this system of linear inequalities, we will present an application problem. Suppose a manufacturer makes two types of skis: a trick ski and a slalom ski. Suppose each trick ski requires 8 hours of design work and 4 hours of finishing. Each slalom ski 8 hours of design and 12 hours of finishing. Furthermore, the total number of hours allocated for design work is 160 and the total available hours for finishing work is 180 hours. Finally, the number of trick skis produced must be less than or equal to 15. How many trick skis and how many slalom skis can be made under these conditions? How many possible answers? Construct a set of linear inequalities that can be used for this problem.

Application

- Let x represent the number of trick skis and y represent the number o slalom skis. Then the following system of linear inequalities describes our problem mathematically. The graph of this region gives the set of ordered pairs corresponding to the number of each type of ski that can be manufactured. Actually, only whole numbers for x and y should be used, but we will assume, for the moment that x and y can be any positive real number.

- Remarks:

$x \geq 0$

$y \geq 0$

x and y must both be positive

$x \leq 15$

Number of trick skis has to be less than or equal to 15

$8x + 8y \leq 160$

Constraint on the total number of design hours

$4x + 12y \leq 180$

Constraint on the number of finishing hours

See next slide for graph of solution set.

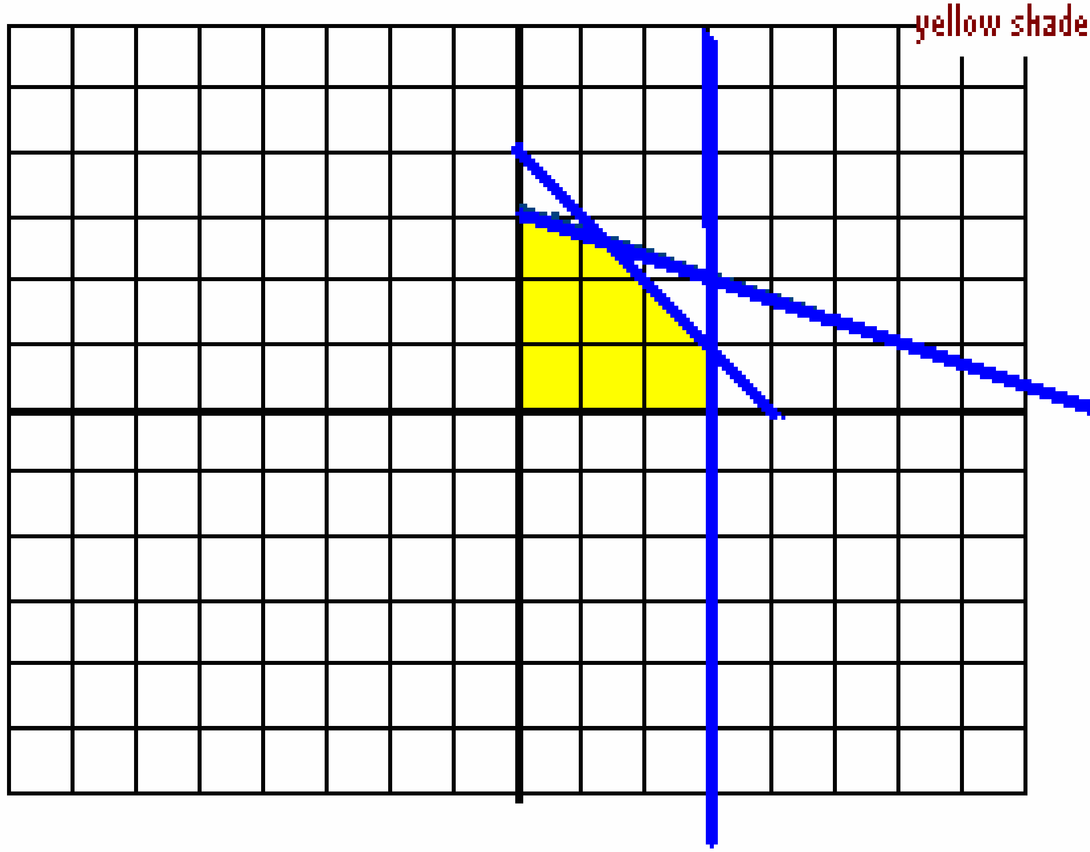
Scale: One square = 5 units.

$$x \geq 0$$
$$y \geq 0$$

These first two inequalities include only points in the first quadrant, including the x and y axes

$$x \leq 15$$
$$8x + 8y \leq 160$$
$$4x + 12y \leq 180$$

The origin satisfies all the inequalites. We shade the region to the left of the vertical line x = 15 and below the other lines. The intersection of all graphs is the yellow shaded region.



Some possible solutions include the points (1,1), (2,1) (1,2) . (1,1) means produce 1 o each type of ski. (2,1) means 2 trick skis and 1 slalom ski (1,2) means one trick ski and 2 slalom skis produced.

5.2 Linear Programming in two dimensions: a geometric approach

In this section, we will explore applications which utilize the graph of a system of linear inequalities.

A familiar example

- We have seen this problem before. An extra condition will be added to make the example more interesting. Suppose a manufacturer makes two types of skis: a trick ski and a slalom ski. Suppose each trick ski requires 8 hours of design work and 4 hours of finishing. Each slalom ski 8 hours of design and 12 hours of finishing. Furthermore, the total number of hours allocated for design work is 160 and the total available hours for finishing work is 180 hours. Finally, the number of trick skis produced must be less than or equal to 15. How many trick skis and how many slalom skis can be made under these conditions? Now, here is the twist: Suppose the profit on each trick ski is $5 and the profit for each slalom ski is $10. How many each of each type of ski should the manufacturer produce to earn the greatest profit?

Linear Programming problem

- This is an example of a **linear programming problem.** Every linear programming problem has two components:
- **1. A linear objective function is to be maximized or minimized. In our case the objective function is Profit = 5x + 10y (**5 dollars profit for each trick ski manufactured and $10 for every slalom ski produced).
- **2. A collection of linear inequalities that must be satisfied simultaneously.** These are called the **constraints** of the problem because these inequalities give limitations on the values of x and y. In our case, the linear inequalities
- are the **constraints.**

$$x \geq 0$$

x and y have to be positive

$$y \geq 0$$

$$x \leq 15$$

The number of trick skis must be less than or equal to 15

Profit = 5x + 10y

$$8x + 8y \leq 160$$

Design constraint: 8 hours to design each trick ski and 8 hours to design each slalom ski. Total design hours must be less than or equal to 160

$$4x + 12y \leq 180$$

Finishing constraint: Four hours for each trick ski and 12 hours for each slalom ski.

Linear programming

- 3. The **feasible set** is the set of all points that are possible for the solution. In this case, we want to determine the value(s) of x, the number of trick skis and y, the number of slalom skis that will yield the maximum profit. Only certain points are eligible. Those are the points within the common region of intersection of the graphs of the constraining inequalities. Let’s return to the graph of the system of linear inequalities. Notice that the feasible set is the yellow shaded region.
- Our task is to maximize the profit function
- $P = 5x + 10y$ by producing x trick skis and y slalom skis, but use only values of x and y that are within the yellow region graphed in the next slide.

Scale: One square = 5 units.

$x \geq 0$

$y \geq 0$

These first two inequalities include only points in the first quadrant, including the x and y axes

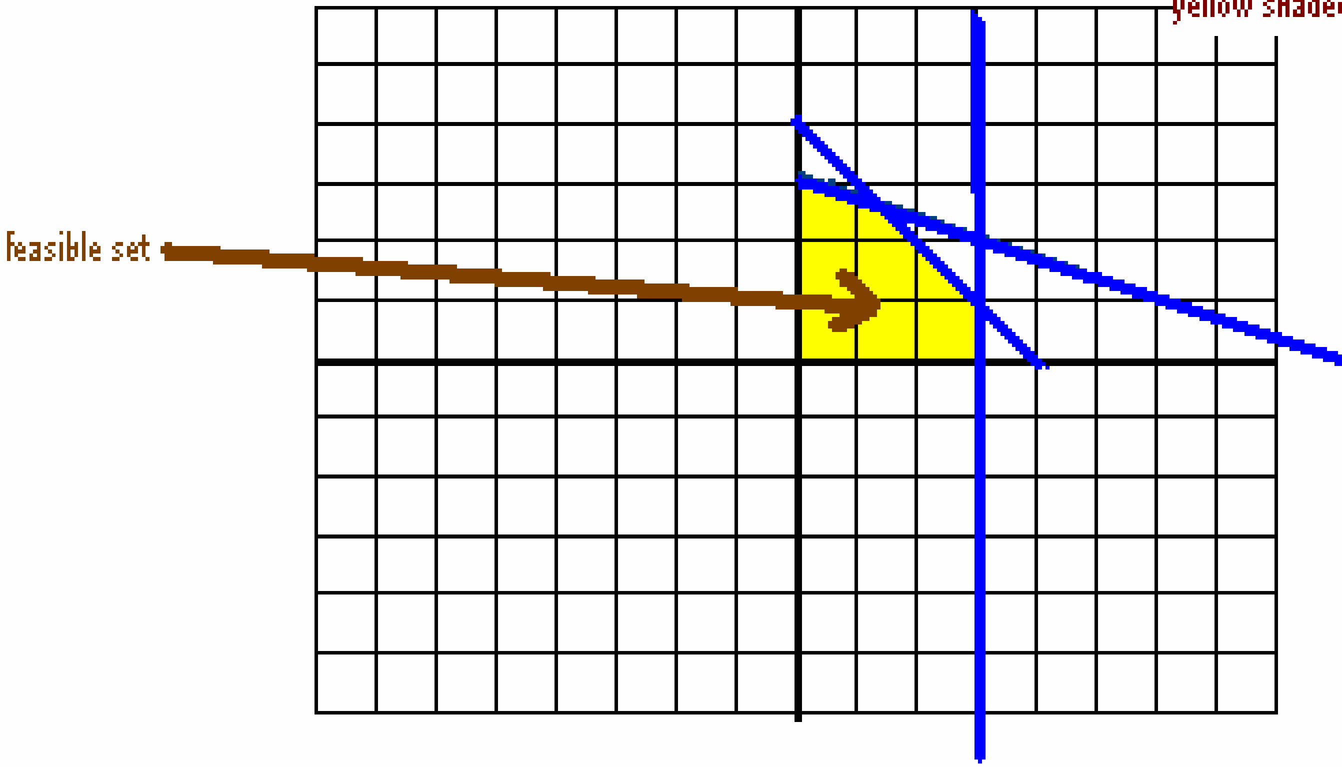
$x \leq 15$

$8x + 8y \leq 160$

$4x + 12y \leq 180$

$$Profit = 5x + 10y$$

The origin satisfies all the inequalites. We shade the region to the left of the vertical line x = 15 and below the other lines. The intersection of all graphs is the yellow shaded region.



Some possible solutions include the points (1,1), (2,1) (1,2) . (1,1) means produce 1 of each type of ski. (2,1) means 2 trick skis and 1 slalom ski (1,2) means one trick ski and 2 slalom skis produced.

Maximizing the profit

- Profit = 5x + 10y Suppose profit equals a constant value, say k . Then the equation
- k = 5x + 10y represents a family of parallel lines each with slope of one-half. For each value of k (a given profit) , there is a unique line. What we are attempting to do is to find the largest value of k possible. The graph on the next slide shows a few iso-profit lines. Every point on this profit line represents a production schedule of x and y that gives a constant profit of k dollars. As the profit k increases, the line shifts upward by the amount of increase while remaining parallel. The maximum value of profit occurs at what is called a corner point- a point of intersection of two lines. The exact point of intersection of the two lines is (7.5,12.5). Since x and y must be whole numbers, we round the answer down to (7,12). *See the graph in the next slide.*

Scale: One square = 5 units.

$$x \geq 0$$
$$y \geq 0$$

These first two inequalities include only points in the first quadrant, including the x and y axes

$$x \leq 15$$

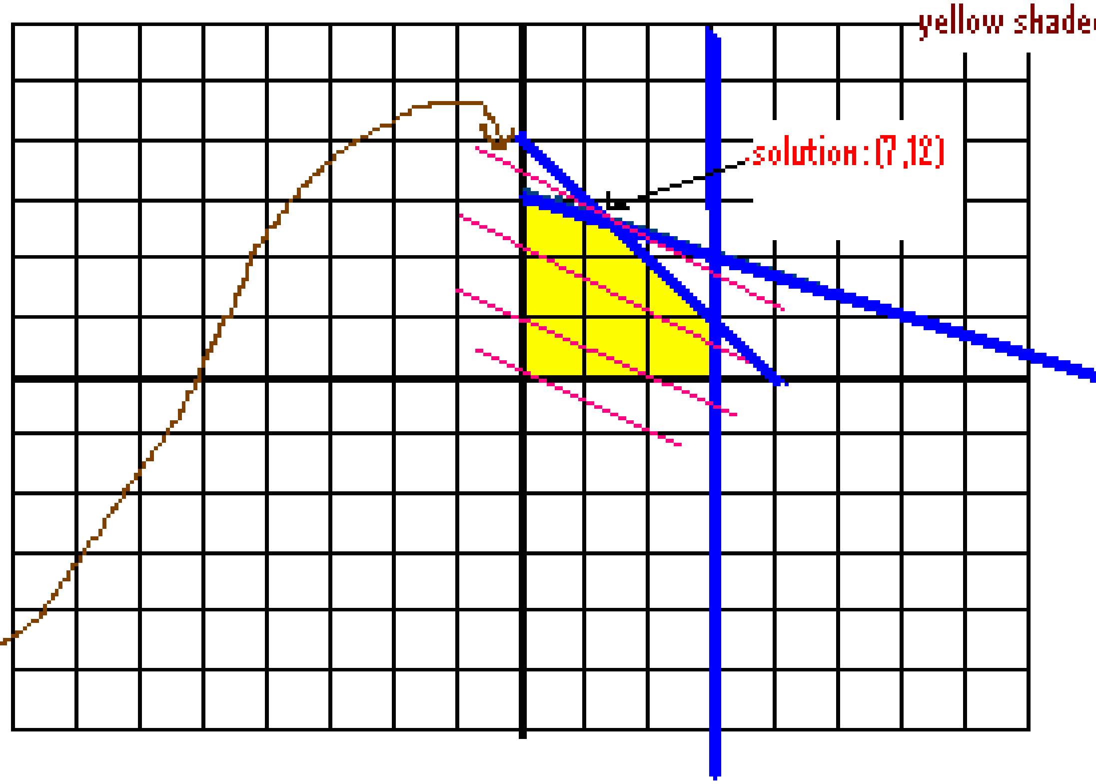
$$Profit = 5x + 10y$$

$$8x + 8y \leq 160$$

$$4x + 12y \leq 180$$

The origin satisfies all the inequalites. We shade the region to the left of the vertical line x = 15 and below the other lines. The intersection of all graphs is the yellow shaded region.

Various profit levels are indicated by the pink lines. Each line has a slope of -1/2 the slope of the line 5x + 10y = P, where P is a constant. The profit level that is greatest and yet still includes one or more points of the feasible set is indicated.



Some possible solutions include the points (1,1), (2,1) (1,2) . (1,1) means produce 1 o each type of ski. (2,1) means 2 trick skis and 1 slalom ski (1,2) means one trick ski and 2 slalom skis produced.

Maximizing the Profit

- Thus, the manufacturer should produce 7 trick skis and 12 slalom skis to achieve maximum profit. What is the maximum profit?

- $P = 5x + 10y \longrightarrow P=5(7)+10(12)=35 + 120 = 155$

General Result

- If a linear programming problem has a solution, it is located at a vertex of the set of feasible solutions. If a linear programming problem has more than one solution, at **least** one of them is located at a vertex of the set of feasible solutions.
- If the set of feasible solutions is bounded, as in our example, then it can be enclosed within a circle of a given radius. In these cases, the solutions of the linear programming problems will be unique.
- If the set of feasible solutions is not bounded, then the solution may or may not exist. Use the graph to determine whether a solution exists or not.

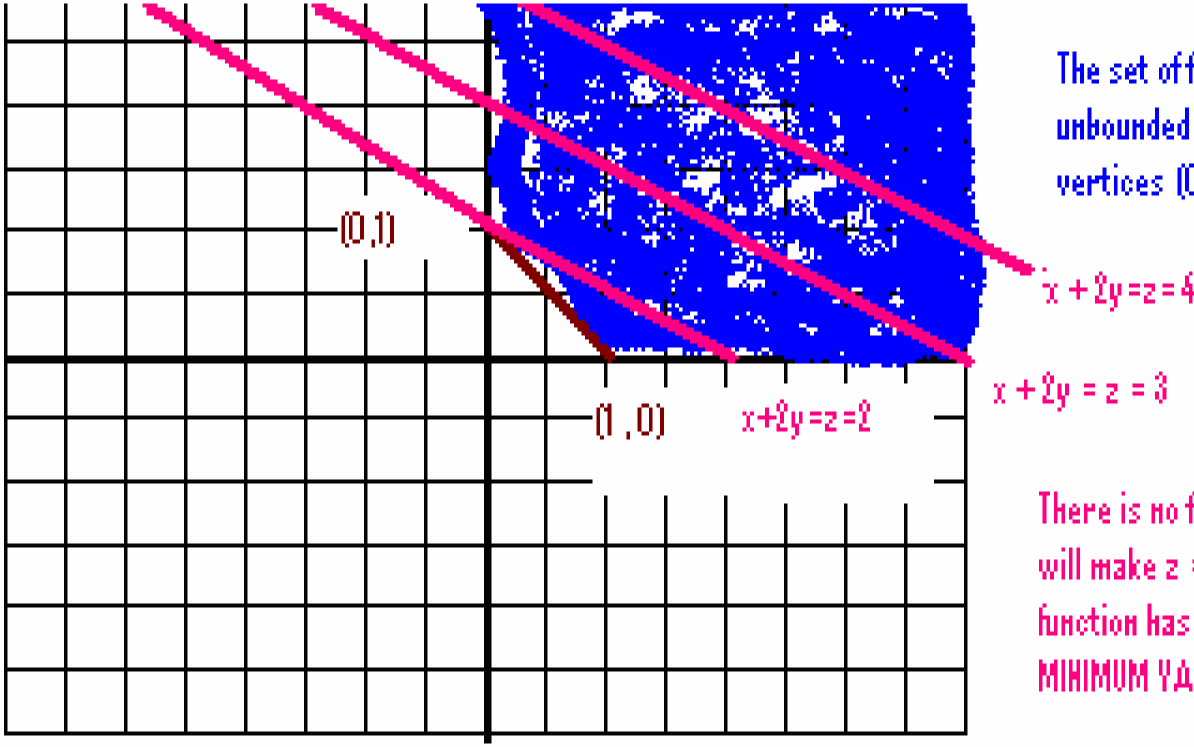
General Procedure for Solving Linear Programming Problems

- 1. Write an expression for the quantity that is to be maximized or minimized. This quantity is called the **objective function** and will be of the form $z = Ax + By$. In our case $z = 5x + 10y$.

- 2. Determine all the constraints and graph them
- 3. Determine the feasible set of solutions- the set of points which satisfy all the constraints simultaneously.
- 4. Determine the vertices of the feasible set. Each vertex will correspond to the point of intersection of two linear equations. So, to determine all the vertices, find these points of intersection.
- 5. Determine the value of the objective function at each vertex.

Linear programming problem with no solution

- Maximize the quantity **$z = x + 2y$** subject to the constraints
- $x + y \geq 1$, $x \geq 0$, $y \geq 0$
- 1. The objective function is **$z = x + 2y$** is to be maximized.
- 2. Graph the constraints: (see next slide)
- 3. Determine the feasible set (see next slide)
- 4. Determine the vertices of the feasible set. There are two vertices from our graph. (1,0) and (0,1)
- 5. Determine the value of the objective function at each vertex.
- 6. at (1,0): **$z = (1) + 2(0) = 1$**
- at (0, 1) : **$z = 0 + 2(1) = 2$.**
- We can see from the graph there is no feasible point that makes z largest. We conclude that the linear programming problem has no solution.



The set of feasible solutions is unbounded. The region has two vertices (0 ,1) and (1 ,0) .

There is no feasible point within the region that will make z = x + 2y the largest. The objective function has no maximum value BUT DOES HAVE A MINIMUM VALUE

Scale: two squares equals one unit

5.3 Geometric Introduction to the Simplex Method

The geometric method of the previous section is limited in that it is only useful for problems involving two decision variables and cannot be used for applications involving three or more decision variables. It is for this reason, that a more sophisticated method be developed for such situations. A man by the name George B. Dantzig developed such a method in 1947 while being assigned to the U.S. military.

An interview with George Dantzig, inventor of the simplex method

http://www.e-optimization.com/directory/trailblazers/dantzig/interview_opt.cfm

- How do you explain optimization to people who haven't heard of it?
- **GEORGE**
  I would illustrate the concept using simple examples such as the diet problem or the blending of crude oils to make high-octane gasoline.
- **IRV**
  What do you think has held optimization back from becoming more popular?
- **GEORGE**
  It is a technical idea that needs to be demonstrated over and over again. We need to show that firms that use it make more money than those who don't.
- **IRV**
  Can you recall when optimization started to become used as a word in the field?
- **GEORGE**
  From the very beginning of linear programming in 1947, terms like maximizing, minimizing, extremizing, optimizing a linear form and optimizing a linear program were used.

An interview with George Dantzig, inventor of the simplex method

- The whole idea of objective function, which of course optimization applies, was not known prior to linear programming. In other words, the idea of optimizing something was something that nobody could do, because nobody tried to optimize. So while you are very happy with it and say it's a very familiar term, optimization just meant doing it better than somebody else. And the whole concept of getting the optimum solution just didn't exist. So my introducing the whole idea of optimization in the early days was novel.
- **IRV**
  I understand that while programming the war effort in World War II was done on a vast scale, the term optimization as a word was never used. What was used instead?
- **GEORGE**
  A program can be thought of as a set of blocks, or activities, of different shapes that can to be fitted together according to certain rules, or mass balance constraints. Usually these can be combined in many different ways, some more, some less desirable than other combinations. Before linear programming and the simplex method were invented, it was not possible to computationally determine the best combination such as finding the program that maximizes the number of sorties flown. Instead, all kinds of ground rules were invented deemed by those in charge to be desirable characteristics for a program to have. A typical example of a ground rule that might have been used was: "Go ask General Arnold which alternative he prefers." A big program might contain hundreds of such highly subjective rules.

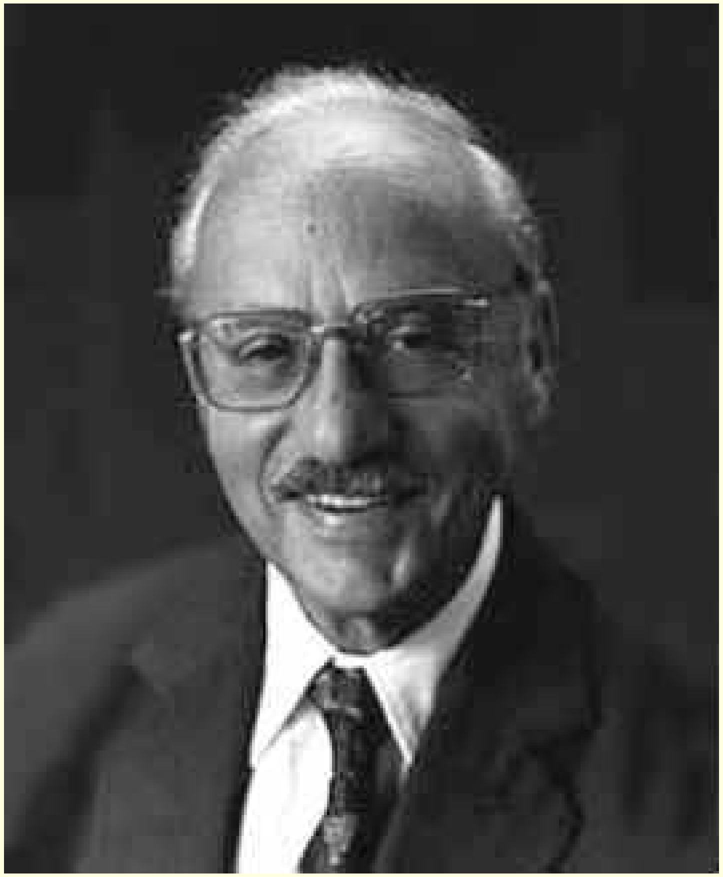
An interview with George Dantzig, inventor of the simplex method

- And I said to myself: "Well, we can't work with all these rules." Because what it meant was that you set up a plan. Then you have so many rules that you have to get some resolution of these rules and statements of what they were. To do this, you had to be running to the general, and to his assistants and asking them all kinds of questions.
- **IRV**
  Name some of your most important early contributions.
- **GEORGE**
  The first was the recognition that most practical planning problems could be reformulated mathematically as finding a solution to a system of linear inequalities. My second contribution was recognizing that the plethora of ground rules could be eliminated and replaced by a general objective function to be optimized. My third contribution was the invention of the simplex method of solution.

An interview with George Dantzig, inventor of the simplex method

- **IRV**
  And these were great ideas that worked and still do.
- **GEORGE**
  Yes, I was very lucky.
- **IRV**
  What would you say is the most invalid criticism of optimization?
- **GEORGE**
  Saying: "It's a waste of time to optimize because one does not really know what are the exact values of the input data for the program."
- **IRV**
  Ok, let's turn this around. What would you say is the greatest potential of optimization?
- **GEORGE**
  It has the potential to change the world.

George Dantzig



An example

- To see how this method works, we will use a modified form of a previous example. Consider the linear programming problem of maximizing z under the constraints. The problem constraints involve $\leq$ inequalities with positive constants to the right of the inequality symbol. Optimization problems that satisfy this condition are called standard maximization problems.

$$z = 5x + 10y$$

$$8x + 8y \leq 160$$

$$4x + 12y \leq 180$$

$$x \geq 0;\ y \geq 0$$

Slack Variable

- To use the simplex method of the next section, the constraint inequalities must be converted to a system of linear equations by using what are called **slack variables**. In particular, consider the two constraint inequalities $\begin{array}{l} 8x + 8y \le 160 \\ 4x + 12y \le 180 \end{array}$ To make this into a
- a system of two equations, two unknowns, we use the slack variables,
- s_1 s_2 as follows. They are called slack variables because they take up the slack between the left and right hand sides of the inequalities.

$$8x + 8y + s_1 = 160$$

$$4x + 12y + s_2 = 180$$

Slack variables

- We now have two equations, but four unknowns, x , y , s_1 s_2
- The system has an infinite number of solutions since there are more unknowns than equations. We can make the system be consistent by assigning two of the variables a value of zero and then solving for the remaining two variables. This is accomplished by dividing the four variables into two groups:
- 1. Basic variables
- 2. Non-basic variables.
- We are free to select any two of the four variables as basic variables while the remaining two variables automatically become non-basic variables. The non-basic variables are always assigned a value of zero. Then, solve the equations for the two basic solutions.

Basic Solutions and the Feasible Region

- There is a association between the basic solutions and the intersection points of the boundary lines of the feasible region. This is best illustrated by an example: Consider the feasible set determined by the constraints The graph of the feasible set is shown on the next slide along with the feasible region.

$$z = 5x + 10y$$

$$8x + 8y \leq 160$$

$$4x + 12y \leq 180$$

$$x \geq 0;\ y \geq 0$$

Scale: One square = 5 units.

P = 5x + 10y

$$x \geq 0$$

$$y \geq 0$$

$$8x + 8y \leq 160$$

$$4x + 12y \leq 180$$

Points of intersection of lines and feasible points

(0 , 0)

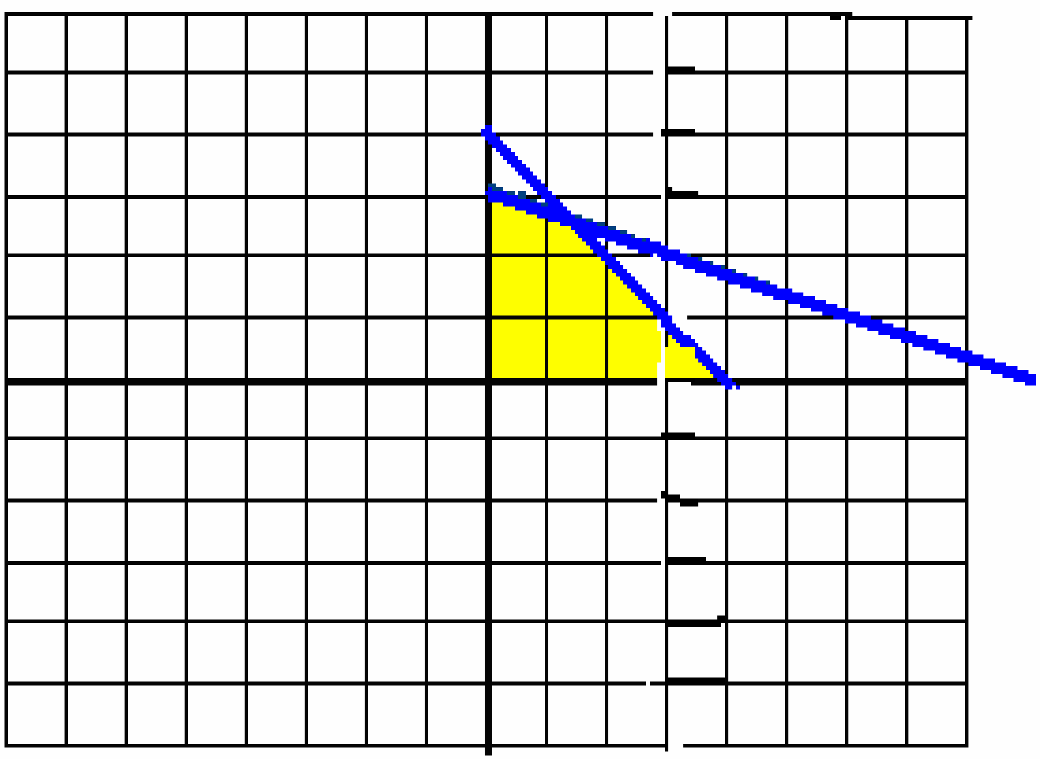
(0 , 15)

(0 , 20) not part of feasible set

(7.5 , 12.5) Optimal point

(20 , 0)

(45 , 0)



Basic Solutions and the Feasible Region

- We will start with our linear equations containing the two slack variables. We will systematically assign two of these variables a zero (non-basic variables) value and then solve for the remaining two variables (basic variables). The result is displayed in a table. To illustrate how the table is constructed, study the example: **Assign x = 0 and y = 0 as our non-basic variables**. Then **8(0) + 8(0) + s1 = 160** so s1 = 160. Substitute x = 0 and y = 0 in the second equation to obtain
- **4(0) + 12(0) + s2 = 180** . This implies that s2 = 180

x	y	s1	s2	point	feasible?
0	**0**	**160**	**180**	**(0,0)**	**yes**

- **There are six different rows of the table corresponding to six different ways you can 2 variables a value of zero out of four possible variables. The rest of the table is displayed in the next slide.**

$$8x + 8y + s_1 = 160$$

$$4x + 12y + s_2 = 180$$

Table: Basic Solutions

x	y	s_1	s_2	Point	Feasible ?
0	0	160	180	(0,0)	yes
0	20	0	-60	(0,20)	No
0	15	40	0	(0,15)	Yes
20	0	0	100	(20,0)	Yes
45	0	-200	0	(45,0)	no
7.5	12.5	0	0	**(7.5,12.5)**	**Yes- solution**

Discovery!

- In the previous table, you may observe that a basic solution that is not feasible includes **at least one negative value and that a basic feasible solution does not include any negative values.**

- **That is, we can determine the feasibility of a basic solution simply by examining the signs of all the variables in the solution.**

- **Observe that basic feasible solutions correspond to corner points (vertices) of the feasible region, which also include the optimum solution of the linear programming problem, which in this case is (7.5, 12.5)**

Generalization:

- Given a system of linear equations associated with a linear programming problem in which there will always be more variables than equations, the variables are separated into two mutually exclusive groups called basic variables and non-basic variables.

- Basic variables are selected arbitrarily with the restriction that there are as many basic variables as there are equations. The remaining variables are called non-basic variables.

- We obtain a solution by assigning the non-basic variables a value of zero and solving for the basic variables. If a **basic solution** has no negative values, it is a **basic feasible solution.**

- Theorem: If the optimal value of the objective function in a linear programming problem exists, then that value **must occur at one**
- **(or more) of the basic feasible solutions.**

Conclusion:

- This is simply the first step in developing a procedure for solving a more complicated linear programming problem. But it is an important step in that we have been able to identify all the corner points (vertices) of the feasible set without drawing its graph since the graphical method will not work for systems having 3 or more variables.

5.4 Simplex method: maximization with problem constraints of the form $\leq$

The procedures for the simplex method will be illustrated through an example. Be sure to read the textbook to fully understand all the concepts involved.

Example:

- We will solve the same problem that was presented earlier, but this time we will use the simplex method. We wish to maximize the Profit function subject to the constraints below. The method introduced here can be used to solve larger systems that are more complicated.

$$P = 5x + 10y$$

$$8x + 8y \le 160$$

$$4x + 12y \le 180$$

$$x \ge 0; y \ge 0$$

Introduce slack variables; rewrite objective function

- First step is to rewrite the system without the inequality symbols and introduce the slack variables, s_1 and s_2

$$8x+8y+s_1=160$$

$$4x+12y+s_2=180$$

$$-5x-10y+P=0$$

$$x,y,s_1,s_2\geq 0$$

Represent linear system using matrix

- The variable y was replaced the variable x_2

$$
\begin{array}{c}
\\
s_1 \\
s_2 \\
P
\end{array}
\begin{array}{c}
\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\
\left(\begin{array}{ccccc|c}
8 & 8 & 1 & 0 & 0 & 160 \\
4 & 12 & 0 & 1 & 0 & 180 \\
-5 & -10 & 0 & 0 & 1 & 0
\end{array}\right)
\end{array}
$$

Determine the pivot element

$$\begin{array}{c} \\ s_1 \\ s_2 \\ P \end{array}\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \left(\begin{array}{ccccc|c} 8 & 8 & 1 & 0 & 0 & 160 \\ 4 & 12 & 0 & 1 & 0 & 180 \\ -5 & -10 & 0 & 0 & 1 & 0 \end{array}\right) \end{array}$$

- In the last row, we see the most negative element is -10. Therefore, the column containing -10 is the pivot column.
- To determine the pivot row, we divide the coefficients above the -10 into the numbers in the rightmost column and determine the smallest quotient. Since 160 divided by 8 is 20 and 180 divided by 12 is 15, the constant 12 becomes the pivot element.

Using the pivot element and elementary row operations

- 1.Divide row 2 by 12 to get a 1 in the position of the pivot element.
- 2. Obtain zeros in the other two positions of the pivot column.
- 3. x_2 becomes the entering variable and s_2 exits.

$$\begin{array}{c} \\ s_1 \\ s_2 \\ P \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & P \end{array} \\ \left(\begin{array}{ccccc|c} 8 & 8 & 1 & 0 & 0 & 160 \\ 4 & 12 & 0 & 1 & 0 & 180 \\ -5 & -10 & 0 & 0 & 1 & 0 \end{array}\right) \end{array}$$

$$\downarrow$$

$$\begin{array}{c} \\ s_1 \\ x_2 \\ P \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & P \end{array} \\ \left(\begin{array}{ccccc|c} 8 & 8 & 1 & 0 & 0 & 160 \\ \frac{1}{3} & 1 & 0 & \frac{1}{12} & 0 & 15 \\ -5 & -10 & 0 & 0 & 1 & 0 \end{array}\right) \end{array}$$

$$\nearrow$$

$$\begin{array}{c} \\ s_1 \\ x_2 \\ P \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & P \end{array} \\ \left(\begin{array}{ccccc|c} \frac{16}{3} & 0 & 1 & \frac{-2}{3} & 0 & 40 \\ \frac{1}{3} & 1 & 0 & \frac{1}{12} & 0 & 15 \\ \frac{-5}{3} & 0 & 0 & \frac{5}{6} & 1 & 150 \end{array}\right) \end{array}$$

Find the next pivot element and repeat the process

- The process must continue since the last row of the matrix contains a negative value (-5/3) We use the same procedure to find the next pivot element. Divide 15 by 1/3 to obtain 45. Divide 7.5 by 1 to obtain 7.5. Since 7.5 is less than 45, our next pivot element is 1. The entering variable of x_1 replaces the exiting variable s_1.

x_1 enters, and s_1 exits

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ P \end{array} &
\left(\begin{array}{ccccc|c}
1 & 0 & \frac{3}{16} & \frac{-1}{8} & 0 & 7.5 \\
\frac{1}{3} & 1 & 0 & \frac{1}{12} & 0 & 15 \\
\frac{-5}{3} & 0 & 0 & \frac{5}{6} & 1 & 150
\end{array}\right)
\end{array}
$$

Obtain zeros in the remaining two entries of the pivot column:

- 1. Multiply row 1 by 5/3 and add the result to row 3. Replace row 3 by that sum.
- 2. Multiply row 1 by -1/3 and add result to row 2. Replace row 2.

s

$$\begin{array}{c} \\ x_1 \\ x_2 \\ P \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 0 & \frac{3}{16} & \frac{-1}{8} & 0 & 7.5 \\ 0 & 1 & \frac{-1}{16} & \frac{1}{8} & 0 & 12.5 \\ 0 & 0 & \frac{5}{16} & \frac{5}{8} & 1 & \frac{275}{2} \end{array}\right) \end{array}$$

Find the solution

- Since the last row of the matrix contains no negative numbers, we can stop the procedure and find the solution. IF the slack variables are assigned a value of zero, then we have x= 7.5 and y = x2=12.5. This is the same solution we obtained geometrically.

$$
\begin{array}{c} \\ x_1 \\ \\ x_2 \\ \\ P \end{array}
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\
\left(\begin{array}{ccccc|c}
1 & 0 & \frac{3}{16} & \frac{-1}{8} & 0 & 7.5 \\
0 & 1 & \frac{-1}{16} & \frac{1}{8} & 0 & 12.5 \\
0 & 0 & \frac{5}{16} & \frac{5}{8} & 1 & \frac{275}{2}
\end{array}\right)
\end{array}
$$

5.5 Dual problem: minimization with problem constraints of the form $\geq$

- Associated with each minimization problem with $\geq$ constraints is a maximization problem called the **dual problem**. The dual problem will be illustrated through an example. Read the textbook carefully to learn the details of this method. We wish to minimize the objective function subject to certain constraints:

$$C = 16x_1 + 9x_2 + 21x_3$$

$$x_1 + x_2 + 3x_3 \geq 12$$

$$2x_1 + x_2 + x_3 \geq 16$$

$$x_1, x_2, x_3 \geq 0$$

Initial matrix

- We start with an initial matrix , A, corresponds to the problem constraints:

$$A = \begin{bmatrix} 1 & 1 & 3 & 12 \\ 2 & 1 & 1 & 16 \\ 16 & 9 & 21 & 1 \\ & & & \end{bmatrix}$$

Transpose of matrix A

- To find the transpose of matrix A, interchange the rows and columns so that the first row of A is now the first column of A transpose.

$$A^T = \begin{bmatrix} 1 & 2 & 16 \\ 1 & 1 & 9 \\ 3 & 1 & 21 \\ 12 & 16 & 1 \end{bmatrix}$$

Dual of the minimization problem is the following maximization problem:

- Maximize P under the following constraints.

$$P = 12y_1 + 16y_2$$

$$y_1 + 2y_2 \leq 16$$

$$y_1 + y_2 \leq 9$$

$$3y_1 + y_2 \leq 21$$

$$y_1, y_2 \geq 0$$

Theorem 1: Fundamental principle of Duality

- A minimization problem has a solution if and only if its dual problem has a solution. If a solution exists, then the optimal value of the minimization problem is the same as the optimum value of the dual problem.

Forming the Dual problem with slack variables x_1, x_2, x_3

- result:

$$\begin{aligned}
y_1 + 2y_2 + x_1 \qquad\qquad &= 16 \\
y_1 + y_2 + \qquad x_2 \qquad &= 9 \\
3y_1 + y_2 + \qquad\qquad x_3 &= 21 \\
-12y_1 - 16y_2 + p \qquad &= 0
\end{aligned}$$

Form the simplex tableau for the dual problem and determine the pivot element

- The first pivot element is 2 (in red) because it is located in the column with the smallest negative number at the bottom(-16) **and** when divided into the rightmost constants, yields the smallest quotient (16 divided by 2 is 8)

$$
\begin{array}{c}
 \\ x_1 \\ x_2 \\ x_3 \\ P
\end{array}
\begin{array}{c}
\begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & P \end{array} \\
\begin{pmatrix}
1 & 2 & 1 & 0 & 0 & 16 \\
1 & 1 & 0 & 1 & 0 & 9 \\
3 & 1 & 0 & 0 & 1 & 21 \\
-12 & -16 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Divide row 1 by the pivot element (2) and change the exiting variable to y_2 (in red)

- Result:

$$
\begin{array}{c}
 \\
y_2 \\
x_2 \\
x_3 \\
P
\end{array}
\begin{array}{c}
\begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & P \end{array} \\
\begin{pmatrix}
.5 & 1 & .5 & 0 & 0 & 8 \\
1 & 1 & 0 & 1 & 0 & 9 \\
3 & 1 & 0 & 0 & 1 & 21 \\
-12 & -16 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Perform row operations to get zeros in the column containing the pivot element. Identify the next pivot element (0.5) (in red)

-1*row 1 + R2=R2

-1*row1+r3 =r3

16*r1+r4 = r4

$$\begin{array}{c|cccccc} & y_1 & y_2 & x_1 & x_2 & x_3 & P \\ \hline y_2 & .5 & 1 & .5 & 0 & 0 & 8 \\ x_2 & \color{red}{.5} & 0 & -.5 & 1 & 0 & 1 \\ x_3 & 2.5 & 0 & -.5 & 0 & 1 & 13 \\ P & \color{red}{-4} & 0 & 8 & 0 & 0 & 128 \end{array}$$

New pivot element

Pivot element located in this column

Variable y_1 becomes new entering variable

- Divide row 2 by 0.5 to obtain a 1 in the pivot position.

$$
\begin{array}{c}
\\
y_2 \\
{\color{red}y_1} \\
x_3 \\
P
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
y_1 & y_2 & x_1 & x_2 & x_3 & P
\end{array} \\
\left(\begin{array}{cccccc}
.5 & 1 & .5 & 0 & 0 & 8 \\
1 & 0 & -1 & 2 & 0 & 2 \\
2.5 & 0 & -.5 & 0 & 1 & 13 \\
-4 & 0 & 8 & 0 & 0 & 128
\end{array}\right)
\end{array}
$$

More row operations

- -0.5*row2 + row1=Row1
- -2.5*row 2 + row3=row3
- 4*row2+row4=row4

$$
\begin{array}{c|cccccc}
 & y_1 & y_2 & x_1 & x_2 & x_3 & P \\
\hline
y_2 & 0 & 1 & 1 & -1 & 0 & 8 \\
y_1 & 1 & 0 & -1 & 2 & 0 & 2 \\
x_3 & 0 & 0 & 2 & -5 & 1 & 8 \\
P & 0 & 0 & 4 & 8 & 0 & 136
\end{array}
$$

Solution: An optimal solution to a minimization problem can always be obtained from the bottom row of the final simplex tableau for the dual

problem.

- Minimum of P is 136. It occurs at $x_1 = 4$, $x_2 = 8$, $x_3 = 0$

$$\begin{array}{c} \\ y_2 \\ y_1 \\ x_3 \\ P \end{array} \begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & P \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & -1 & 0 & 8 \\ 1 & 0 & -1 & 2 & 0 & 2 \\ 0 & 0 & 2 & -5 & 1 & 8 \\ 0 & 0 & 4 & 8 & 0 & 136 \end{pmatrix} \end{array}$$

5.6 Maximization and Minimization with Mixed Problem Constraints

Introduction to the Big M Method

- In this section, a generalized version of the simplex method that will solve both maximization and minimization problems with any combination of
- $\leq$ $\geq$ $=$ constraints will be presented.

Definition: Initial Simplex Tableau

- For a system tableau to be considered an **initial simplex tableau**, it must satisfy the following two requirements:
  - 1. A variable can be selected as a basic variable only if it corresponds to a column in the tableau that has exactly one nonzero element and the nonzero element in the column is not in the same row as the nonzero element in the column of another basic variable.
  - 2. The remaining variables are then selected as non-basic variables to be set equal to zero in determining a basic solution.
  - 3. The basic solution found by setting the non-basic variables equal to zero is feasible.

Key Steps of the big M method

- Big M Method: Introducing slack, surplus, and artificial variables to form the modified problem
  - 1. If any problem constraints have negative constants on the right side, multiply both sides by -1 to obtain a constraint with a nonnegative constant. (remember to reverse the direction of the inequality if the constraint is an inequality).
  - 2. Introduce a **slack variable** for each constraint of the form $\leq$
  - 3. Introduce a **surplus variable** and an **artificial variable** in each $\geq$ constraint.
  - 4. Introduce an **artificial variable** in each = constraint.
  - 5. For each artificial variable **a,** add **–Ma** to the objective function. Use the same constant M for all artificial variables.

An example:

- Maximize $P = 3x_1 - 2x_2 + x_3$
- subject to :

$$x_1 - 2x_2 + x_3 \geq 5$$

$$-x_1 - 3x_2 + 4x_3 \leq -10$$

$$2x_1 + 4x_2 + 5x_3 \leq 20$$

$$3x_1 - x_2 - x_3 = -15$$

$$x_{1,} x_2 , x_3 \geq 0$$

Solution:

1) Notice that the second constraint has a negative number on the right hand side. To make that number positive, multiply both sides by -1 and reverse the direction of the inequality:

$$x_1 + 3x_2 - 4x_3 \geq 10$$

2. The fourth constraint has a negative number on the right hand side so multiply both sides of this equation by -1 to change the sign of -5 to + 15:

$$-3x_1 + x_2 + x_3 = 15$$

Solution continued:

3) Introduce a surplus variable and an artificial variable for the constraint:

$$x_1 - 2x_2 + x_3 \geq 5 \rightarrow$$

$$x_1 - 2x_2 + x_3 - s_1 + a_1 = 5$$

Solution continued:

- 4) Do the same procedure for the other
- constraint: $\geq$

$$x_1 + 3x_2 - 4x_3 - s_2 + a_2 = 10$$

5) Introduce surplus variable for less than or equal to constraint:

$$2x_1 + 4x_2 + 5x_3 + s_3 = 20$$

Solution continued:

- 6) Introduce the third artificial variable for the equation constraint:

$$-3x_1 + x_2 + x_3 + a_3 = 15$$

7) For each of the three artificial variables, we will add –Ma to the objective function:

$$P = 3x_1 - 2x_2 + x_3 - Ma_1 - Ma_2 - Ma_3$$

Final result

- The modified problem is:

Maximize

$$P = 3x_1 - 2x_2 + x_3 - Ma_1 - Ma_2 - Ma_3$$

subject to the constraints:

$$x_1 - 2x_2 + x_3 - s_1 + a_1 = 5$$

$$x_1 + 3x_2 - 4x_3 - s_2 + a_2 = 10$$

$$2x_1 + 4x_2 + 5x_3 + s_3 = 20$$

$$-3x_1 + x_2 + x_3 + a_3 = 15$$

Key steps for solving a problem using the big M method

- Now that we have learned the procedure for finding the modified problem for a linear programming problem, we will turn our attention to the procedure for actually solving such problems. The procedure is called the Big M Method.

Big M Method: solving the problem

- 1. Form the preliminary simplex tableau for the modified problem.
- 2. Use row operations to eliminate the M's in the bottom row of the preliminary simplex tableau in the columns corresponding to the artificial variables. The resulting tableau is the initial simplex tableau.
- 3. Solve the modified problem by applying the simplex method to the initial simplex tableau found in the second step.

Big M method: continued:

- 4. Relate the optimal solution of the modified problem to the original problem.
  - A) if the modified problem has no optimal solution, the original problem has no optimal solution.
  - B) if all artificial variables are 0 in the optimal solution to the modified problem, delete the artificial variables to find an optimal solution to the original problem
  - C) if any artificial variables are nonzero in the optimal solution, the original problem has no optimal solution.

An example to illustrate the Big M method:

- Maximize

$$P = x_1 + 4x_2 + 2x_3$$

Subject to

$$x_2 + x_3 \leq 4$$

$$x_1 - x_3 = 6$$

$$x_1 - x_2 - x_3 \geq 1$$

Solution:

- Form the preliminary simplex tableau for the modified problem: Introduce slack variables, artificial variables and variable M.

$$x_2 + x_3 + s_1 = 4$$

$$x_1 - x_3 + a_1 = 6$$

$$x_1 - x_2 - x_3 - s_2 + a_2 = 1$$

$$-x_1 - 4x_2 - 2x_3 + Ma_1 + Ma_2 + P = 0$$

Solution:

- Use row operations to eliminate M's in the bottom row of the preliminary simplex tableau.
  - (-M) R2 + R4 = R4
  - (-M)R3 + R4 = R4

Solution:

- Solve the modified problem by applying the simplex method:

The basic variables are s_1, a_1, a_2, P

The basic solution is feasible:

Solution:

- Use the following operations to solve the problem:

$$-1R_2 + R_3 \rightarrow R_3$$

$$(2M+1)R_2 + R_4 \rightarrow R_4$$

$$3R_1 + R_4 \rightarrow R_4$$

$$MR_3 + R_4 \rightarrow R_4$$

$$(1)R_1 + R_4 \rightarrow R_4$$

Solution:

- .

$$
\begin{array}{ccccccccc}
x_1 & x_2 & x_3 & s_1 & a_1 & s_2 & a_2 & P & \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 4 \\
1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 6 \\
0 & -1 & 0 & 0 & -1 & -1 & 1 & 0 & -5 \\
0 & 0 & 1 & 4 & (M+1) & 0 & M & 1 & 22
\end{array}
$$

Solution:

$$x_2 = 4$$

$$x_1 = 6$$

$$P = 22$$

$$x_3 = 0$$