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Question 1:

To produce an equation which computes S_{n+1}^2 from only $(n, S_n^2, \bar{x}_n, \bar{x}_{n+1}, x_{n+1})$, starting from (2):

$$S_{n+1}^2 = \frac{1}{n+1}\sum_{k=1}^{n+1}(x_k - \bar{x}_{n+1})^2$$

$$\therefore S_{n+1}^2 = \left(\frac{1}{n+1}\sum_{k=1}^{n+1} x_k^2\right) - \bar{x}_{n+1}^2$$

$$\therefore S_{n+1}^2 = \frac{1}{n+1}\left(x_{n+1}^2 + \sum_{k=1}^{n} x_k^2\right) - \bar{x}_{n+1}^2 \quad (3)$$

We also have:

$$S_n^2 = \left(\frac{1}{n}\sum_{k=1}^{n} x_k^2\right) - \bar{x}_n{}^2$$

$$\therefore n(S_n^2 + \bar{x}_n{}^2) = \sum_{k=1}^{n} x_k^2 \qquad (4)$$

By substitute (4) into (3), we have:

$$S_{n+1}^2 = \frac{1}{n+1}(x_{n+1}^2 + n(S_n^2 + \bar{x}_n{}^2)) - \bar{x}_{n+1}^2$$

Which is the result that we look for.

Question 2:

```
lcVariance :: [Float] -> Float
lcVariance [] = undefined
lcVariance [x] = 0
lcVariance (x:xs) = (1/nplus1) * (x^2 + n * (lcVariance xs + avg xs ^2)) - avg (x:xs) ^2
    where
        avg :: [Float] -> Float
        avg ys = sum ys / fromIntegral(length ys)
        nplus1 :: Float
        nplus1 = fromIntegral(length (x:xs))
        n :: Float
        n = fromIntegral(length xs)
```

Question 3:

```
trVariance _ _ variance [] = variance
trVariance n xbar variance (x:xs) = trVariance (n+1) avgnew varianceNew xs
    where
        avgnew :: Float
        avgnew = (x+n*xbar)/(n+1)
        varianceNew :: Float
        varianceNew =
            (1/(n+1)) * (x^2 + n * (variance + xbar ^2)) - avgnew ^2
```

Question 4:

```
variance :: [Float] -> Float
variance [] = 0
variance xs = trVariance 0.0 0.0 0.0 xs
```

Question 5:

The iteration invariance of trVariance is:

trVariance n xbar variance [] = variance
trVariance n xbar variance (x:xs) =
$\frac{1}{n+length(xs)}\left(sum[i^2 \mid i \leftarrow (xs)] + n \times (\text{variance } + xbar^2)\right) - (\frac{sum(xs) + n \times xbar}{length(xs)+n})^2$

Question 6:

Prove that the invariance always holds:

Let P(xs) <=> trVariance n xbar variance [] = variance (1)

Otherwise: trVariance n xbar variance (x:xs) =

$$\frac{1}{n+length(xs)}\left(sum[i^2 \mid i \leftarrow (xs)] + n \times (\text{variance } + xbar^2)\right) - (\frac{sum(xs) + n \times xbar}{length(xs)+n})^2 \quad (2)$$

If xs = [] then

LHS(1) = trVariance n xbar variance [] = variance = RHS(1)

Otherwise:

LHS(2) = trVariance n xbar variance []

$$= \frac{1}{n+length(xs)}\left(sum[i^2 \mid i \leftarrow []] + n \times (\text{variance } + xbar^2)\right) - (\frac{sum(xs) + n \times xbar}{length(xs)+n})^2$$

(invariance on RHS)

$$= \frac{1}{n+0}\left(0 + n \times (\text{variance } + xbar^2)\right) - (\frac{0 + n \times xbar}{0+n})^2$$

$= \text{variance } + xbar^2 - (xbar)^2$

= variance = RHS (2)

Thus, P([]).

LHS(3) = trVariance n xbar variance (x:xs)

$$= \frac{1}{n+length(x{:}xs)}\left(sum[i^2 \mid i \leftarrow (x{:}xs)] + n \times (\text{variance } + xbar^2)\right) - (\frac{sum(x{:}xs) + n \times xbar}{length(x{:}xs)+n})^2$$

(invariance on RHS)

$$= \frac{1}{(n+1)+length(xs)}\left(sum[i^2 \mid i \leftarrow (xs)] + x^2 + n \times (\text{variance } + xbar^2)\right) - (\frac{sum(xs)+x + n \times xbar}{length(xs)+(n+1)})^2$$

$$= \text{trVariance } (n+1)\ (\frac{(x + n \times xbar)}{(n+1)})\ \left(\frac{1}{(n+1)}\left(x^2 + n \times (\text{variance } + xbar^2)\right) - \left(\frac{x + n \times xbar}{(n+1)}\right)^2\right)\ (xs)$$

= trVariance (n+1) avgnew $\left(\frac{1}{(n+1)}\left(x^2 + n \times (\text{variance } + xbar^2)\right) - \left(\frac{x + n \times xbar}{(n+1)}\right)^2\right)$

(xs) (substitute ‘avgnew’ as ($\frac{(x + n \times xbar)}{(n+1)}$))

= RHS(3)

Therefore, the iteration invariant for trVariance holds.

Question 7:

The bound value (BV) for trVariance is

BV (trVariance 0 0 0 xs) = length(xs)

Question 8:

Bound value (BV) is non-negative since length x = 0 for any lis xs.

Let P(xs) = trVariance n xbar variance (xs)

And P((x:xs)) = trVariance n xbar variance (x:xs)

BV(P(x:xs))

= length (x:xs)

= 1 + length (xs)

Which is greater than length (xs)

Therefore, BV(P(x:xs)) > BV(P(xs)). Which means the BV is monotonically decreasing and the function will terminate.

Question 9:

1) The variance of a list will equal the variance of the inverse of that list:

```
quickCheck (\xs -> variance xs == variance (reverse (xs)))
```

2) The variance of a list will be equal to the variance of that list if each element in the list increased by n (+n)

```
quickCheck (\xs n -> variance xs == variance (map (+n) xs))
```

Question 10:

Let **P(n) ⇔ plus Zero n = n** for any n ∈ Nat

Base case:

It is true that **plus Zero Zero = Zero (equivalent to 0 + 0 = 0)** because **plus m Zero = m**

Which is our base case P(0).

Induction Hypothesis: Assume P(n)

Induction

If **n** is in **P(n)** then **Succ(n)** will be **n+1** in **P(n+1).**

Notice that **plus Zero (Succ(n)) = Succ(plus Zero n)**

whereas **plus Zero n** is our assumption.

Thus $\boldsymbol{P(n) \Rightarrow P(n+1)}$. Alternatively: Thus $\boldsymbol{P(n+1)}$

Conclusion: By the PMI ∀n ∈ Nat; P(n)

Question 11:

Let **P(n) <=> plus m n = plus n m**

For n, m :: Nat and presume **plus n Succ(m) = plus Succ(n) m**

Base case:

We have **P(0) <=> plus m Zero = plus Zero m** (substitute n with Zero).

LHS:

plus m Zero	
= m	line 3
= plus Zero m	proved in Question 10.

Thus, P(0).

Inductive hypothesis: assume **plus n m = plus m n**, consider **P(n +1) <=> plus m (Succ n) = plus (Succ n) m**. We have:

LHS:

plus m (Succ n)	
= Succ (plus m n)	line 4
= Succ (plus n m)	Inductive hypothesis
= plus n (Succ m)	line 4
= plus (Succ n) m	presumed

Therefore, P(n+1). Thus P(n) follows from the principle mathematical of induction.