COMP3400
Assignment 2 Written

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Tail Recursion

Question 1. [2 MARKS]

Produce an equation which computes S^2_{n+1} from *only* $(n, S^2_n, \bar{x}_n, \bar{x}_{n+1}, x_{n+1})$

Answer: We have

$$S^2_n = \left(\frac{1}{n}\sum_{k=1}^{n} x_k^2\right) - \bar{x}_n^2 \tag{1}$$

$$\implies n(S^2_n + \bar{x}^2_n) = \sum_{k=1}^{n} x_k^2 \tag{2}$$

$$\implies (n+1)(S^2_{n+1} + \bar{x}^2_{n+1}) = \sum_{k=1}^{n+1} x_k^2 \tag{3}$$

Taking Equation (3) – Equation (2) yields

$$(n+1)(S^2_{n+1} + \bar{x}^2_{n+1}) - n(S^2_n + \bar{x}^2_n) = x^2_{n+1}$$

and rearranging gives

$$S^2_{n+1} = \frac{x^2_{n+1} + n(S^2_n + \bar{x}^2_n)}{n+1} - \bar{x}^2_{n+1}$$

1 mark for obtaining an expression for S^2_{n+1}

1 mark for showing work

Question 2. [2 MARKS]

```
lcVariance :: [Float] -> Float
lcVariance (x:[]) = 0
lcVariance (x:xs) = (x**2 + n*(var’ + xb**2))/(n+1) - xb’**2
  where
    xb   = (/) (sum xs) n
    xb’  = (/) (sum (x:xs)) (n+1)
    var’ = lcVariance xs
    n    = (fromIntegral $ length xs)
```

Note. Solution *must* be linear recursive in order to receive *any* marks.

1 mark for base case

1 mark for recursive step

Question 3. [4 MARKS]

```
trVariance :: Float -> Float -> Float -> [Float] -> Float
trVariance _ s _ [] = s
trVariance n s xb (x:xs) = trVariance (n+1) s’ xb’ xs
  where
    s’  = (x**2 + n*(s + xb**2)) / (n+1) - xb’**2
    xb’ = (x + n*xb)/(n+1)
```

Note. Solution *must* be tail recursive in order to receive *any* marks.

1 mark for base case

1 mark for updating variance

1 mark for updating mean

0.5 mark for updating n

0.5 mark for recursing on tail of input

Question 4. [1 MARK]

```
7 variance :: [Float] -> Float
8 variance [] = undefined
9 variance xs = trVariance 0 0 0 xs
```

Note: Students *do not* need line 8.

1 mark for line 9

Question 5. [1 MARK]

$$\begin{aligned}&\texttt{Invariant}(\texttt{trVariance n s xb xs})\\&= \left(\frac{\texttt{n}\cdot\texttt{s} + \texttt{n}\cdot\texttt{xb}^2 + \texttt{sqsum xs}}{\texttt{n} + \texttt{length xs}}\right) - \left(\frac{\texttt{n}\cdot\texttt{xb} + \texttt{sum xs}}{\texttt{n} + \texttt{length xs}}\right)^2\end{aligned}$$

where `sqsum = sum $ map (**2)`.

1 mark for line 9 (or anything equivalent)

Notice this implies

$$\begin{aligned}&\texttt{trVariance 0 0 0 xs}\\&= \left(\frac{0\cdot\texttt{s} + 0\cdot\texttt{xb}^2 + \texttt{sqsum xs}}{0 + \texttt{length xs}}\right) - \left(\frac{0\cdot\texttt{xb} + \texttt{sum xs}}{0 + \texttt{length xs}}\right)^2 && \text{By invariant}\\&= \frac{\texttt{sqsum xs}}{\texttt{length xs}} - \left(\frac{\texttt{sum xs}}{\texttt{length xs}}\right)^2\\&= \texttt{variance xs} && \text{By (1)}\end{aligned}$$

If student's invariant doesn't have this property it is wrong.

Question 6. [5 MARKS]

Prove `trVariance` satisfies your iteration invariant.

$$\begin{aligned}&\texttt{LHS}(\texttt{line 2})\\&= \texttt{trVariance n s xb []}\\&= \left(\frac{\texttt{n}\cdot\texttt{s} + \texttt{n}\cdot\texttt{xb}^2 + \texttt{sqsum []}}{\texttt{n} + \texttt{length []}}\right) - \left(\frac{\texttt{n}\cdot\texttt{xb} + \texttt{sum []}}{\texttt{n} + \texttt{length []}}\right)^2 && \text{By invariant}\\&= \left(\frac{\texttt{n}\cdot\texttt{s} + \texttt{n}\cdot\texttt{xb}^2 + 0}{\texttt{n} + 0}\right) - \left(\frac{\texttt{n}\cdot\texttt{xb} + 0}{\texttt{n} + 0}\right)^2\\&= \texttt{s} + \texttt{xb}^2 - \texttt{xb}^2\\&= \texttt{s}\\&= \texttt{RHS}(\texttt{line 2}).\end{aligned}$$

2 marks

Let

$$\texttt{s}' = \frac{\texttt{x}^2 + \texttt{n}\cdot(\texttt{s} + \texttt{xb}^2)}{\texttt{n}+1} - \texttt{xb}'^2 \implies (\texttt{n}+1)\cdot\texttt{s}' = \texttt{x}^2 + \texttt{n}\cdot(\texttt{s}+\texttt{xb}^2) - (\texttt{n}+1)\cdot\texttt{xb}'^2$$

and

$$\texttt{xb}' = \frac{\texttt{x} + \texttt{n}\cdot\texttt{xb}}{\texttt{n}+1} \implies (\texttt{n}+1)\cdot\texttt{xb}' = \texttt{x} + \texttt{n}\cdot\texttt{xb}$$

in

$$\begin{aligned}
&\texttt{RHS}(\texttt{line 3}) \\
&\quad = \texttt{trVariance}\ (\texttt{n}+1)\ \texttt{s}'\ \texttt{xb}'\ \texttt{xs} \\
&\quad = \left(\frac{(\texttt{n}+1)\cdot\texttt{s}' + (\texttt{n}+1)\cdot\texttt{xb}'^2 + \texttt{sqsum xs}}{(\texttt{n}+1)+\texttt{length xs}}\right) - \left(\frac{(\texttt{n}+1)\cdot\texttt{xb}' + \texttt{sum xs}}{(\texttt{n}+1)+\texttt{length xs}}\right)^2
\end{aligned}$$

By invariant

$$\begin{aligned}
&= \left(\frac{\texttt{x}^2 + \texttt{n}\cdot(\texttt{s}+\texttt{xb}^2) - (\texttt{n}+1)\cdot\texttt{xb}'^2 + (\texttt{n}+1)\cdot\texttt{xb}'^2 + \texttt{sqsum xs}}{(\texttt{n}+1)+\texttt{length xs}}\right) \\
&\quad - \left(\frac{\texttt{x}+\texttt{n}\cdot\texttt{xb}+\texttt{sum xs}}{(\texttt{n}+1)+\texttt{length xs}}\right)^2
\end{aligned}$$

Substitution by definition of $\texttt{s}'$ and $\texttt{xb}'$

$$= \left(\frac{\texttt{n}\cdot(\texttt{s}+\texttt{xb}^2) + \texttt{sqsum}\ (\texttt{x}:\texttt{xs})}{\texttt{n}+\texttt{length}\ (\texttt{x}:\texttt{xs})}\right) - \left(\frac{\texttt{n}\cdot\texttt{xb} + \texttt{sum}\ (\texttt{x}:\texttt{xs})}{\texttt{n}+\texttt{length}\ (\texttt{x}:\texttt{xs})}\right)^2$$

Simplification and by definition of sum, sqsum, and length

$$= \left(\frac{\texttt{n}\cdot\texttt{s} + \texttt{n}\cdot\texttt{xb}^2 + \texttt{sqsum}\ (\texttt{x}:\texttt{xs})}{\texttt{n}+\texttt{length}\ (\texttt{x}:\texttt{xs})}\right) - \left(\frac{\texttt{n}\cdot\texttt{xb} + \texttt{sum}\ (\texttt{x}:\texttt{xs})}{\texttt{n}+\texttt{length}\ (\texttt{x}:\texttt{xs})}\right)^2$$

$$= \texttt{trVariance n s xb}\ (\texttt{x}:\texttt{xs})$$

By invariant

$$= \texttt{LHS}(\texttt{line 3})$$

2 marks

Therefore `trVariance` satisfies the iteration invariant.

1 mark for justifying lines

Question 7. [1 MARK]

```
BV(trVariance n s xb xs) = length xs
```

1 mark

Question 8. [2 MARKS]

By definition `length xs` is always nonnegative.

$$\begin{aligned}
&\texttt{BV}(\texttt{trVariance n s xb}\ (\texttt{x}:\texttt{xs})) \\
&\quad = \texttt{length}\ (\texttt{x}:\texttt{xs}) \\
&\quad = 1 + \texttt{length xs} && \text{By definition}
\end{aligned}$$

$$
\begin{aligned}
&> \texttt{length xs} \\
&= \texttt{BV}(\texttt{trVariance}\ (\texttt{n}+1)\ \texttt{s}'\ \texttt{xb}'\ \texttt{xs})
\end{aligned}
$$

Thus the bound value is strictly descending and always nonnegative. 2 mark

Question 9. [2 MARKS]

For n an integer...

```
variance xs = variance $ map (+n) variance
variance xs = n^2 * variance $ map (*n) variance
variance xs = variance $ reverse xs
```

Note: The solution doesn't have to be a valid quick check. It only needs to adequately communicate what property is being checked.

1 mark for Quickcheck 1

1 mark for Quickcheck 2

Induction

Consider the following definitions for implementing addition on natural numbers.

```
1 data Nat = Zero | Succ Nat deriving Show

2 plus :: Nat -> Nat -> Nat
3 plus m Zero = m
4 plus m (Succ(n)) = Succ (plus m n)
```

Question 10. [7 MARKS]

Using induction prove:

$$\texttt{plus Zero n = n}$$

for any `n :: Nat`.

Answer: Let `P :: Nat -> Bool` where   1 mark for defining P(n)

$$\texttt{P(n)} \iff \texttt{plus Zero n = n}$$

Base case: `plus Zero Zero = Zero` by line 3 and therefore `P(Zero)`.

1 mark for base case

Assume `P(n)` as an induction hypothesis and notice this implies   1 mark for IH

$$\begin{aligned} &\texttt{plus Zero (Succ n)} && \\ &\quad = \texttt{Succ (plus Zero n)} && \text{apply line 4} \\ &\quad = \texttt{Succ n} && \text{By IH} \end{aligned}$$

and therefore `P(Succ n)`.   2 marks for induction

By the PSI/PMI we have `P(n)` for all `n :: Nat`.   1 mark for conclusion

1 marks for justifying lines

Question 11. [8 MARKS]

Using induction prove that `plus` is commutative, that is,

```
plus m n = plus n m
```

for any `n,m :: Nat`.

Answer: Let `m :: nat` be arbitrary and `P :: Nat -> Bool` where

$$\texttt{P(n)} \iff \texttt{plus m n = plus n m}$$

1 mark for defining P(n)

Base case: `plus m Zero = m` by line 3. `plus Zero m = m` by Question 10. Therefore `P(Zero)`.

1 mark for defining P(n)

Assume `P(n)` as an induction hypothesis and notice this implies

1 mark for IH

$$\begin{aligned}
&\texttt{plus m (Succ n)} \\
&\quad = \texttt{Succ (plus m n)} && \text{Apply line 4} \\
&\quad = \texttt{Succ (plus n m)} && \text{By IH} \\
&\quad = \texttt{plus n (Succ m)} && \text{Unapply line 4} \\
&\quad = \texttt{plus (Succ n) m} && \text{Given}
\end{aligned}$$

and therefore `P(succ n)`.

3 marks for induction

By the PSI/PMI we have `P(n)` for all `n :: Nat`.

1 mark for conclusion

1 marks for justifying lines