Math 18 | Final Examination v. A | Name: ______________
March 22, 2017 | (Total Points: *50*) | PID: ______________

Instructions

1. **No calculators, tablets, phones, or other electronic devices are allowed during this exam.**
2. **You may use one handwritten page of notes, but no books or other assistance during this exam.**
3. **Read each question carefully and answer each question completely.**
4. **Show all of your work. No credit will be given for unsupported answers, even if correct.**
5. **Write your Name at the top of each page.**

(2 points) 0. Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

(6 points) 1. Consider the following matrix equation $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -3 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Determine the solution set of the matrix equation $A\mathbf{x} = \mathbf{b}$ and, if appropriate, write it in parametric form.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & 3 & -3 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

$\Rightarrow$ no solution.

(b) Determine the solution set of the corresponding homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$ and, if appropriate, write it in parametric form.

From part a),

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$.

$$\Rightarrow \begin{bmatrix} -3t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

(6 points) 2. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Find A^{-1}, the inverse of A.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Find the matrix X such that $AX = A^T$, the transpose of A.

$$AX = A^T.$$

$$\Rightarrow A^{-1}AX = A^{-1}A^T.$$

$$X = A^{-1}A^T.$$

$$= \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 & -1 \\ -1 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}.$$

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(6 points) 3. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$.

(a) Compute AB. Is AB an invertible matrix? Justify your answer.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}.$$

Since $\det AB = -3+4 = 1 \neq 0$

AB is invertible.

(b) Compute BA. Is BA an invertible matrix? Justify your answer.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & -1 & 0 \end{bmatrix}.$$

$$\det\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & -1 & 0 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + 0 + \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix}$$

$$= 2 + 0 + -2.$$

$$= 0.$$

$\Rightarrow \det(\cancel{AB}\ BA) = 0 \Rightarrow \cancel{AB}\ BA$ is not invertible.

(6 points) 4. The matrices $A = \begin{bmatrix} 3 & -1 & 1 & -6 \\ 2 & 1 & 9 & 1 \\ -3 & 2 & 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are row equivalent.

pivots

(a) Find a basis for $\mathrm{Col}(A)$, the column space of A.

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

(b) Find a basis for $\mathrm{Nul}(A)$, the null space of A.

$$\vec{x} = \begin{bmatrix} -2x_3 + x_4 \\ -5x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -3 \\ 1 \\ \cancel{0}1 \end{bmatrix}.$$

$$\left\{ \begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(c) Find a basis for $\mathrm{Col}\left(A^T\right)^{\perp}$, the orthogonal complement of the column space of A^T. Be sure to explain how you know that it is a basis for $\mathrm{Col}\left(A^T\right)^{\perp}$.

Since $\mathrm{Col}(A^T)^{\perp} = \mathrm{Nul}(A)$.

$$\Rightarrow \left\{ \begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$ basis for $\mathrm{Col}(A^T)^{\perp}$.

(6 points) 5. Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\mathcal{B}_W = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p\}$, and let $\mathcal{B}_{W^\perp} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q\}$ be an orthogonal basis for $W^\perp$, the orthogonal complement of W.

(a) Explain why $\mathcal{S} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q\}$ is an orthogonal set.

$w_i \cdot w_j = 0$ for $i \neq j$ since B_W orthogonal basis.

$v_i \cdot v_j = 0$ for $i \neq j$ since $B_{W^\perp}$ orthogonal basis.

$w_i \cdot v_j = 0$ by definition of orthogonal complement.

(b) Explain why the set $\mathcal{S}$ spans $\mathbb{R}^n$.

For any vector $\vec{u} \in \mathbb{R}^n$, we can decompose

$$\vec{u} = \vec{u}_1 + \vec{u}_2$$

where $\vec{u}_1 \in W$ and $\vec{u}_2 \in W^\perp$.

$\Rightarrow \vec{u}_1 \in \text{span}\{w_1, \ldots, w_p\}$ and $\vec{u}_2 \in \text{span}\{v_1, \ldots, v_q\}$.

$\Rightarrow \vec{u} \in$ ~~span~~ $\text{span}(S) \Rightarrow S$ spans $\mathbb{R}^n$.

(c) Explain why $\mathcal{S}$ is linearly independent.

Because all vectors in S are orthogonal.

(d) Explain why $\dim(W) + \dim(W^\perp) = n$.

S spans $\mathbb{R}^n \Rightarrow \dim S = n$.

$\Rightarrow \dim(W) + \dim W^\perp = n$.

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(6 points) 6. The set $\mathcal{B} = \{1,\ 1+2t,\ 1+2t+4t^2\}$ is a basis for $\mathbb{P}_2$, the vector space of polynomials of degree at most two. The polynomial $\mathbf{p} = 1+4t^2$. Find $[\mathbf{p}]_{\mathcal{B}}$, the coordinate vector for $\mathbf{p}$ with respect to the basis $\mathcal{B}$.

First, we need to write p as a linear combination of vectors in $\mathcal{B}$.

$$1+4t^2 = c_1(1) + c_2(1+2t) + c_3(1+2t+4t^2).$$

$$4t^2+1\ \cancel{1+4} = 4c_3t^2 + (2c_2+2c_3)t + c_1+c_3.$$

$$\Rightarrow \begin{cases} 4c_3 = 4. \\ 2c_2 + 2c_3 = 0. \\ c_1 + c_3 = \cancel{0}\,1. \end{cases} \qquad \begin{cases} c_3 = 1. \\ c_2 + c_3 = 0 \\ c_1 + c_3 = 1 \end{cases} \qquad \begin{matrix} c_3 = 1 \\ c_2 = -1 \\ c_1 = 0. \end{matrix}$$

$$[1+4t^2]_{\mathcal{B}} = \left[-1(1+2t) + 1(1+2t+4t^2)\right]_{\mathcal{B}}.$$

$$= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

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(6 points) 7. Consider the matrix $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix}$.

(a) Determine the eigenvalues of A. (Note: One of the eigenvalues of A is 0.)

$$\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 0 & 2 \\ 0 & 4-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)\begin{vmatrix} 4-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} + 2\begin{vmatrix} 0 & 4-\lambda \\ 2 & 0 \end{vmatrix}.$$

$$= (2-\lambda)(4-\lambda)(2-\lambda) + 2(-2)(4-\lambda).$$

$$= (4-\lambda)\left[(2-\lambda)^2 - 4\right].$$

$$= (4-\lambda)(-\lambda)(4-\lambda)$$

$$\Rightarrow \quad \lambda = 4. \text{ and } \lambda = 0.$$

(b) Find a matrix P that diagonalizes A. That is, find P so that $P^{-1}AP = D$, where D is a diagonal matrix.

Find eigenbasis:

$$\lambda = 4 \Rightarrow (A-4I)\vec{x} = 0 \Rightarrow \begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix}\vec{x} = 0.$$

$$\Rightarrow \left\{ \begin{bmatrix} +1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

$$\lambda = 0 \Rightarrow (A-0I)\vec{x} = 0 \quad \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix}\vec{x} = 0.$$

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}}_{P^{-1}}.$$

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(6 points) 8. Let $A = \begin{array}{c} \begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{array} \\ \begin{bmatrix} -1 & 2 & 10 \\ 2 & 1 & 10 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix} \end{array}$. $\longrightarrow \begin{bmatrix} -1 & 2 & 10 \\ 0 & 5 & 30 \\ 0 & 4 & 10 \\ 0 & 3 & 20 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 2 & 10 \\ 0 & 5 & 30 \\ 0 & 4 & 10 \\ 0 & 0 & 1 \end{bmatrix}$

(a) Find an orthonormal basis for Col(A), the column space of A.

$$\vec{u}_1 = \vec{a}_1 \Rightarrow \hat{u}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{u}_2 = \vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{u}_1}{\|\vec{u}_1\|^2}\vec{u}_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} - \frac{-2+2+2-2}{\|\vec{u}_1\|^2}\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} \Rightarrow \hat{u}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}.$$

$$\vec{u}_3 = \vec{a}_3 - \begin{bmatrix} 10 \\ 10 \\ 0 \\ 0 \end{bmatrix} - \frac{10}{10}\begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{30}{10}\begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 10+1-6 \\ 10-2-3 \\ 0-1-6 \\ 0-2+3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ -7 \\ +1 \end{bmatrix}, \quad \hat{u}_3 = \frac{1}{10}\begin{bmatrix} 5 \\ 5 \\ -7 \\ 1 \end{bmatrix}$$

(b) Find an orthogonal matrix Q and an upper triangular matrix R such that $QR = A$.

$$A = \begin{bmatrix} -1/\sqrt{10} & 2/\sqrt{10} & 5/10 \\ 2/\sqrt{10} & 1/\sqrt{10} & 5/10 \\ 1/\sqrt{10} & 2/\sqrt{10} & -7/10 \\ 2/\sqrt{10} & -1/\sqrt{10} & 1/10 \end{bmatrix}\begin{bmatrix} \sqrt{10} & 0 & \sqrt{10} \\ 0 & \sqrt{10} & 3\sqrt{10} \\ 0 & 0 & 10 \end{bmatrix}.$$

Final ExamvS
Math 18, Section A
June 14, 2017
Time Limit: 180 Minutes

Name (Print): ____________________

PID: ____________________

- **DO NOT begin working, or even open this packet, until instructed to do so.**
- You should be in your assigned seat, unless instructed otherwise by Ed or one of the TAs.
- Enter all requested information on the top of this page, and put your name on the top of every page, in case the pages become separated.
- You may use a two-sided page of notes on this exam.
- You may **not** use your books, additional notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. Unless otherwise directed in the statement of the problem, a correct answer, unsupported by calculations, explanation, or algebraic work will receive little or no credit. An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Question	Points
1	10
2	10
3	10
4	10
5	14
6	6
7	10
8	10
Total:	80

DO NOT turn this page until instructed to do so

1. Let $A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 3 \\ -1 & -2 & -3 & -4 \\ 2 & 4 & 6 & -3 \end{bmatrix}$. A is row-equivalent to $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) (4 points) Find a basis for the nullspace of A.

x_2 and x_3 are free.

$\Rightarrow x_1 = -2x_2 - 3x_3$ and $x_4 = 0$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

$\Rightarrow \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ form a basis for $\mathrm{Nul}(A)$.

(b) (2 points) Solve the equation $A\vec{x} = (2, 4, -2, 4)$. Express your answer in parametric vector form. Hint: the right hand side of that equation is one of the columns of A.

We can find a particular solution: $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ since $A \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \\ 4 \end{bmatrix}$.

$\therefore$ The solution of $A\vec{x} = (2,4,-2,4)$ is

$$\underbrace{t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{solution of } A\vec{x}=0} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{particular solution.}}$$

(c) (4 points) Find a basis for the column space of A.

Since the first and ~~th~~ ~~third~~ fourth columns are pivotal.

$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -4 \\ -3 \end{bmatrix} \right\}$ form a basis for $\mathrm{Col}\,A$.

(Tricky)

2. (10 points) Let V be the vector space of 3×3 matrices, $A \in V$ be an invertible matrix, and $H \subseteq V$ be the set of all matrices B such that AB is a diagonal matrix. It turns out that H is a subspace. Find $\dim H$.

$H = \{ B$ such that $AB = D$ ✓ diagonal $\}$.

$B = A^{-1}D$.

Since any diagonal matrix can be expressed as

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = d_1 \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D_1} + d_2 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D_2} + d_3 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D_3}$$

these (matrix) vectors are linearly independant

$\Rightarrow$ For any $B \in H$, we can write B as

$B = d_1(A^{-1}D_1) + d_2 A^{-1}D_2 + d_3 A^{-1}D_3$

~~wher~~ for some scalar d_1, d_2, d_3.

$\therefore$ $H = \text{span}\{ A^{-1}D_1, A^{-1}D_2, A^{-1}D_3 \}$

$\therefore$ $\dim H = 3$.

3. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$.

(a) (5 points) Find A^{-1}.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 2 & 3 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{R_3 - 2R_1 \\ R_4 - R_1}]{R_2 - R_1} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 6 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} 6 & -1 & -1 & -2 \\ -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

(b) (5 points) Find $\det A$.

From the second step,

$$\det A = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1 \cdot 1 \cdot 1 \cdot 1 = 1.$$

4. (10 points) Let $\mathbb{P}$ be the vector space of polynomials in x and $H = \operatorname{span}\{x, x^2\} \subset \mathbb{P}$. Let $\mathcal{B} = \{x, x^2 + x\}$; this is a basis for H. Let $T : H \to H$ be the linear transformation that sends $p(x)$ to $p(3x) + p(x)$. For example, $T(x^2 - x) = (3x)^2 - (3x) + x^2 - x = 10x^2 - 4x$. Find a matrix M satisfying

$$M[p(x)]_{\mathcal{B}} = [T(p(x))]_{\mathcal{B}}$$

$$[a_1x + a_2x^2]_{\mathcal{B}} = [(a_1-a_2)x^2 + a_2(x^2+x)]_{\mathcal{B}} = \begin{bmatrix} a_1-a_2 \\ a_2 \end{bmatrix}.$$

$$[T(a_1x + a_2x^2)]_{\mathcal{B}} = [3a_1x + 9a_2x^2 + a_1x + a_2x^2]_{\mathcal{B}} = [4a_1x + 10a_2x^2]_{\mathcal{B}}$$

$$= [(4a_1 - 10a_2)x + 10a_2(x^2+x)]_{\mathcal{B}}$$

$$\Rightarrow [T(a_1x + a_2x^2)]_{\mathcal{B}} = \begin{bmatrix} 4a_1 - 10a_2 \\ 10a_2 \end{bmatrix}.$$

Find M such that $M\begin{bmatrix} a_1-a_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4a_1 - 10a_2 \\ 10a_2 \end{bmatrix}.$

$$M\vec{e}_1 = M\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \quad M\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 10 \end{bmatrix} \Rightarrow M = \begin{bmatrix} 4 & -6 \\ 0 & 10 \end{bmatrix}.$$

5. Let $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$.

(a) (10 points) Find an eigenbasis for A.

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = (3-\lambda)(-\lambda)+2 = 0.$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda-2)(\lambda-1) = 0.$$

$\lambda = 2$ and $\lambda = 1$.

$$(A - 2I)\vec{x} = 0$$

$$\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}\vec{x} = 0$$

$$\vec{x} = t\begin{bmatrix} -2 \\ +1 \end{bmatrix}$$

span $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ is an eigenbasis for $\lambda = 2$

$$(A - I)\vec{x} = 0$$

$$\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\vec{x} = 0$$

$$\vec{x} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

span $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is an eigenbasis for $\lambda = 1$.

(b) (4 points) Find an invertible matrix B such that $B^{-1}AB$ is a diagonal matrix.

$$A = \underbrace{\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}}_{P^{-1}}$$

$$\therefore\ P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\therefore\ B = P = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}.$$

6. (a) (3 points) Find two 3×3 matrices A and B that satisfy all three of the following conditions:
   - The eigenvalues of A are 5 and 9.
   - The eigenvalues of B are 5 and 9 as well.
   - The characteristic polynomials of A and B are different.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

$$\det(A - \lambda I) = (5-\lambda)^2(9-\lambda)$$

$$\det(B - \lambda I) = (5-\lambda)(9-\lambda)^2.$$

(b) (3 points) Find a 3×3 matrix C that satisfies both of the following conditions:
   - The only eigenvalue of C is 4
   - The eigenspace of C with eigenvalue 4 is 2-dimensional.

$$C = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}. \Rightarrow C \text{ only has eigenvalue } \lambda = 4.$$

$$C - 4I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \dim \operatorname{Nul}(C - 4I) = 2$$

$\Rightarrow$ the eigenspace has dimension 2.

7. (10 points) Let

$$\vec{b}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

Find an **orthogonal** matrix A satifying both of the following conditions:

- $A\vec{e}_1 \in \operatorname{span}\{\vec{b}_1\}$
- $A\vec{e}_2 \in \operatorname{span}\{\vec{b}_1, \vec{b}_2\}$

Find $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]$ such that $\vec{a}_1 = A\vec{e}_1 \in \operatorname{span}\{\vec{b}_1\}$.
$\vec{a}_2 = A\vec{e}_2 \in \operatorname{span}\{\vec{b}_1, \vec{b}_2\}$.

$\vec{a}_1, \vec{a}_2, \vec{a}_3$ are orthonormal.

$$\vec{u}_1 = \vec{b}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \vec{a}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$\vec{u}_2 = \vec{b}_2 - \frac{b_2 \cdot \vec{u}_1}{\|\vec{u}_1\|} \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$\Rightarrow \vec{a}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

Now, find $\vec{a}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\vec{a}_3 \cdot \vec{a}_1 = \vec{a}_3 \cdot \vec{a}_2 = 0$.

$\Rightarrow$ $y = 0$ and $\frac{x}{\sqrt{2}} + \frac{z}{\sqrt{2}} = 0$.

Take $z = 1/\sqrt{2}$ and $x = -1/\sqrt{2}$.

$$\vec{a}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

$$\Rightarrow A = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

8. (10 points) Let

$$Q = \begin{array}{cc} \vec{q}_1 & \vec{q}_2 \\ \end{array}$$

$$Q = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}, \qquad \vec{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Let A be some 3×2 matrix with linearly independant columns $\vec{b}_1, \vec{b}_2$, so $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ is a basis for $\operatorname{Col} A$, and suppose $A = QR$ is a QR-decomposition of A. Let $\hat{y}$ be the orthogonal projection of $\vec{y}$ onto $\operatorname{Col} A$. Find the coordinate vector $[\hat{y}]_{\mathcal{B}}$. You may answer in terms of R.

Let $\vec{q}_1$ and $\vec{q}_2$ be columns of Q. $\Rightarrow$ $\operatorname{Col}(A) = \operatorname{span}\{\vec{q}_1, \vec{q}_2\}$.

$$\hat{y} = \operatorname{proj}_{\operatorname{Col}(A)}(y) = (\vec{y} \cdot \vec{q}_1)\vec{q}_1 + (\vec{y} \cdot \vec{q}_2)\vec{q}_2.$$

$$\hat{y} = QQ^T\vec{y}.$$

Since $A = [\vec{b}_1, \vec{b}_2] = QR. \Rightarrow Q = AR^{-1}$.

$$\hat{y} = AR^{-1}Q^T\vec{y}.$$

$$\Rightarrow [\hat{y}]_{\mathcal{B}} = R^{-1}Q^T\vec{y} = R^{-1}\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= R^{-1}\begin{bmatrix} 6/\sqrt{3} \\ 1/\sqrt{2} \end{bmatrix}$$

Final Exam vQ
Math 18, Section A
June 14, 2017
Time Limit: 180 Minutes

Name (Print): ______________________

PID: ______________________

- **DO NOT begin working, or even open this packet, until instructed to do so.**
- You should be in your assigned seat, unless instructed otherwise by Ed or one of the TAs.
- Enter all requested information on the top of this page, and put your name on the top of every page, in case the pages become separated.
- You may use a two-sided page of notes on this exam.
- You may **not** use your books, additional notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** Unless otherwise directed in the statement of the problem, a correct answer, unsupported by calculations, explanation, or algebraic work will receive little or no credit. An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Question	Points
1	10
2	8
3	10
4	8
5	10
6	14
7	10
8	10
Total:	80

DO NOT turn this page until instructed to do so

1. (10 points) Let

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \qquad \vec{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Let W be the set of all vectors $\vec{w} \in \mathbb{R}^3$ such that $A\vec{w} \in \operatorname{span}\{\vec{y}\}$. W is a subspace of $\mathbb{R}^3$ (you do not need to prove this). Find a basis for W.

Let's solve $A\vec{w} = \vec{y}$,

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 1 & 1 & 3 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 0 & 2 & 5 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 5/2 & 3/2 \end{array}\right]$$

$$\therefore \quad \cancel{x_1} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1/2 - \frac{1}{2}w_3 \\ 3/2 - \frac{5}{2}w_3 \\ w_3 \end{bmatrix} = w_3 \begin{bmatrix} -1/2 \\ -5/2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 3/2 \\ 0 \end{bmatrix}$$

Thus,

$$A\vec{w} = t\vec{y} \quad \text{if} \quad \vec{w} = t\left(s\begin{bmatrix} -1/2 \\ -5/2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 3/2 \\ 0 \end{bmatrix}\right).$$

$$\therefore \qquad = ts\begin{bmatrix} -1/2 \\ -5/2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1/2 \\ 3/2 \\ 0 \end{bmatrix}.$$

$$\therefore \quad W = \operatorname{span}\left\{\begin{bmatrix} -1/2 \\ -5/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 3/2 \\ 0 \end{bmatrix}\right\},$$

basis.

2. (8 points) Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \det \begin{bmatrix} 2 & x_1 + x_2 & 0 \\ -1 & x_3 & 0 \\ 2 & 3 & 2 \end{bmatrix}$$

(You do not need to prove that T is linear). Find the matrix corresponding to T.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = 2(-1)^{3+3} \det \begin{bmatrix} 2 & x_1 + x_2 \\ -1 & x_3 \end{bmatrix}$$

$$= 2(2x_3 + x_1 + x_2).$$

$$= 4x_3 + 2x_1 + 2x_2 = 2x_1 + 2x_2 + 4x_3$$

$T(\vec{e}_1) = 2 \qquad T(\vec{e}_2) = 2. \qquad T(\vec{e}_3) = 4.$

$$\Rightarrow \begin{bmatrix} 2 & 2 & 4 \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. Let $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$. $\mathcal{E}$ is a basis for $\mathbb{R}^3$, and so is $\mathcal{B}$.

(a) (4 points) Find a basis $\mathcal{C}$ such that $A = \underset{\mathcal{E} \leftarrow \mathcal{C}}{P}$. (Recall that $\underset{\mathcal{E} \leftarrow \mathcal{C}}{P}$ is the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{E}$).

(b) (6 points) Find a basis $\mathcal{D}$ such that $A = \underset{\mathcal{B} \leftarrow \mathcal{D}}{P}$.

4. (8 points) Let V be the vector space of 2×2 matrices, and let $T : V \to \mathbb{R}^5$ be some linear transformation. Let k be the dimension of the range of T. Suppose that

$$T\left(\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This is not enough information to determine k. What is the biggest that k could be? And what is the smallest that k could be?

5. Let V be a vector space with two bases $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$. Let $T : V \to V$ be a linear transformation. Let M be the matrix of T relative to $\mathcal{B}$ (also known as the $\mathcal{B}$-matrix of T). Suppose

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 2 & 0 & -1 \end{bmatrix} \qquad \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

(Recall that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.)

(a) (6 points) Find $T(\vec{c}_2)$.

(b) (4 points) Find a matrix A with $A[\vec{v}]_{\mathcal{B}} = [T(\vec{v})]_{\mathcal{C}}$ for any $\vec{v} \in V$.

6. (a) (5 points) Find the eigenvalues of $\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$.

$$\det(A-\lambda I) = \det\begin{bmatrix} -\lambda & 3 \\ 1 & 2-\lambda \end{bmatrix} = \lambda^2 - 2\lambda - 3 = 0. \Rightarrow (\lambda-3)(\lambda+1)=0$$

$\lambda = 3$ and $\lambda = -1$.

(b) (5 points) Find the eigenspace of $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ corresponding to the eigenvalue 2.

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{bmatrix} -3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

$$\text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(c) (4 points) Let B be a 6×6 matrix with characteristic polynomial $(2-\lambda)^2(5-\lambda)^4$ and let I be the 6×6 identity matrix. If B is not similar to a diagonal matrix, what can you conclude about $\dim \operatorname{Nul}(B-2I)$ and $\dim \operatorname{Nul}(B-5I)$?

$\dim \operatorname{Nul}(B-2I) < 2$ or $\dim \operatorname{Nul}(B-5I) < 4$.

since the eigenspaces of 2 and 5 need not to span $\mathbb{R}^6$

7. (10 points) Let $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and let $W = \text{span}\{\vec{x}\} \subset \mathbb{R}^3$. Find an orthogonal basis for $W^{\perp}$.

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in W^{\perp}$ if $\vec{v} \cdot \vec{x} = 0$

$2v_1 + 3v_2 + v_3 = 0.$

$\begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0.$

$\begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3/2 & 1/2 \end{bmatrix}.$

$$\begin{bmatrix} -\frac{3}{2} x_2 - \frac{1}{2} x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -3/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}.$$

$$\Rightarrow W^{\perp} = \text{span}\left\{ \underbrace{\begin{bmatrix} -3/2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{b}_1}, \underbrace{\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{b}_2} \right\}.$$

Gram-Schmidt:

$\vec{u}_1 = \vec{b}_1 = \begin{bmatrix} -3/2 \\ 1 \\ 0 \end{bmatrix}$

$$\vec{u}_2 = \vec{b}_2 - \frac{\vec{b}_2 \cdot \vec{u}_1}{\|\vec{u}_1\|^2} \vec{u}_1 = \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} - \frac{\frac{3}{4}}{\frac{9}{4}+1} \begin{bmatrix} -3/2 \\ 1 \\ 0 \end{bmatrix}.$$

$$= \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{13} \begin{bmatrix} -3/2 \\ 1 \\ 0 \end{bmatrix}.$$

8. (10 points) Let A be a matrix with the following QR-factorization:

$$A = \begin{bmatrix} 0 & \sqrt{1/3} \\ 1/\sqrt{2} & -\sqrt{1/3} \\ 1/\sqrt{2} & \sqrt{1/3} \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$$

Find the distance from $(3,1,1)$ to $\operatorname{Col} A$.

Let $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

$$\hat{v} = \operatorname{proj}_{Col(A)} \vec{v} = \left(\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + \left(\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

$$= \frac{2}{\sqrt{2}} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + \frac{3}{\sqrt{3}} \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

$$\text{distance} = \|\vec{v} - \hat{v}\| = \left\| \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\| = \sqrt{4+1+1} = \sqrt{6}.$$