Midterm 1 — Winter 2015

1) a) Consider augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 1 & 3 & 2 \\ 1 & -3 & -1 & 3 \end{array}\right] \xrightarrow[R_3-R_1]{} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & -1 & -3 & -2 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & +0 & 8 & 9 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{and} \quad \begin{cases} x_1 + 8x_3 = 9 \\ x_2 + 3x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1 = 9 - 8x_3 \\ x_2 = 2 - 3x_3 \end{cases}$$

pivot $\Rightarrow x_3$ is free variable

Let $x_3 = s$. The solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9-8s \\ 2-3s \\ s \end{bmatrix} = s\begin{bmatrix} -8 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 9 \\ 2 \\ 0 \end{bmatrix} \quad \text{for } s \in \mathbb{R}.$$

b) The solution set of the corresponding homogeneous system is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -8 \\ -3 \\ 1 \end{bmatrix} \quad \text{for } s \in \mathbb{R}.$$

2) $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 1 & 7 \\ 1 & -1 & 1 \end{bmatrix}$. The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

a) The augmented matrix of $A\vec{x} = \vec{0}$ is $[\vec{A} \mid \vec{0}]$. whose RREF is

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \text{free variable is } x_3.$$

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_3 \\ x_2 = -x_3 \end{cases}$$

Let $x_3 = s$. The solution set of $A\vec{x} = \vec{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s \\ -s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \text{ for } s \in R.$$

b) The question is equivalent to "Is there any vector $\vec{b}$ such that the system $A\vec{x} = \vec{b}$ is inconsistent?". From the reduced row echelon form of A, we know that if we pick $\vec{b}$ such that the last column of the REF of the augmented matrix $[A \mid \vec{b}]$ is pivotal, then $A\vec{x} = \vec{b}$ is inconsistent. One candidate for such $\vec{b}$ is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Check:

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 3 & 1 & 7 & 0 \\ 1 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & -8 & -8 & 0 \\ 0 & 4 & 4 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & -8 & -8 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right] \checkmark$$

3) Let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix} \right\}$

a) S is <u>linearly independent</u> if and only if the equation

$$\begin{bmatrix} 1 & -2 & -3 \\ 2 & 1 & 4 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has <u>only</u> the trivial solution $\vec{x} = \vec{0}$.

Consider

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 2 & 1 & 4 & 0 \\ -2 & 2 & 2 & 0 \end{array}\right] \xrightarrow[R_3 + 2R_1]{R_2 - 2R_1} \left[\begin{array}{cccc} 1 & -2 & -3 & 0 \\ 0 & 5 & 10 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow$ $\uparrow$ pivot $\quad$ $\uparrow$ free variable.

$\Rightarrow$ the system has ∞ many solution.

$\therefore$ S is <u>linearly dependent</u>.

b) S does not span $\mathbb{R}^3$ as these 3 vectors are linearly dependent.

For example, take $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, the system

$$\begin{bmatrix} 1 & -2 & -3 \\ 2 & 1 & 4 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ has not solution.

$\Rightarrow$ $\vec{b} \notin$ in the span of S.

4) We need to find $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.

- $T(\vec{e}_1) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\2\end{bmatrix}$.

- $T(\vec{e}_2) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$

$$= \begin{bmatrix}0\\2\\0\end{bmatrix} - \begin{bmatrix}0\\0\\2\end{bmatrix} = \begin{bmatrix}0\\2\\-2\end{bmatrix}.$$

- $\vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix}$.

$\Rightarrow T(\vec{e}_3) = T\left(\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$

$$= T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$$

$$= \begin{bmatrix}2\\2\\0\end{bmatrix} - \begin{bmatrix}0\\0\\2\end{bmatrix} - \begin{bmatrix}0\\2\\-2\end{bmatrix}$$

$$= \begin{bmatrix}2\\-2\\0\end{bmatrix}.$$

$\therefore$ The standard matrix of T is

$$\begin{array}{c} \begin{array}{ccc} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{array} \\ \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 0 \end{bmatrix} \end{array}.$$