Midterm 1 - Winter 2018

1) a) False
   b) True.
   c) False
   d) True.

2) a) The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & -1 & -1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ x_3 is free. $\Rightarrow \begin{cases} x_1 = 2 - x_3 \\ x_2 = -1 - 2x_3 \end{cases}$

Let $x_3 = t$. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2-t \\ -1-2t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

b) No! Since the <u>last row</u> of A does not have a pivot.

c) Yes! They are linearly independent as they are not a scalar multiple of each other.

3) Consider $\left[\begin{array}{cc|c} 1 & -1 & h \\ 1 & h & 4h \\ 2 & -2 & 4 \end{array}\right]$

a) Find REF:

$\xrightarrow[R_3-2R_1]{R_2-R_1} \left[\begin{array}{cc|c} 1 & -1 & h \\ 0 & h+1 & 3h \\ 0 & 0 & 4-2h \end{array}\right]$

the system is consistent if $4-2h=0$. or $h=2$.

when $h=2$, we have

$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow$ $x_1 = 4$ and $x_2 = 2$.

the solution is $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

b) Consider $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & h & 0 \\ 2 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & h+1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

the system has infinitely many solutions if x_2 is a free variable. , i.e. if $h+1=0$

$\Rightarrow$ $h=-1$.

when $h=-1$,

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\Rightarrow$ $x_1 = x_2$. Let $x_2 = s$

the general solution is

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $s \in \mathbb{R}$.

4) a) Suppose not, i.e., $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent. Then there exist non-trivial scalar x_1, x_2, x_3 such that

(*) $$x_1\vec{u} + x_2\vec{v} + x_3\vec{w} = \vec{0}.$$

If $x_3 = 0$, then $\vec{u}$ and $\vec{v}$ are linearly dependent
$\Rightarrow$ This is a contradiction.

If $x_3 \neq 0$, then

(*) $\Rightarrow$ $$x_3\vec{w} = -x_1\vec{u} - x_2\vec{v}.$$

$$\vec{w} = -\frac{x_1}{x_3}\vec{u} - \frac{x_2}{x_3}\vec{v}.$$

$\Rightarrow$ $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$
which is a contradiction.

b) (*) $T(\vec{u}) + T(\vec{v}) = T(2\vec{u} + \vec{w})$

T is not one-to-one, This is because from (*), and T is linear, we obtain

$$T(\vec{u}) + T(\vec{v}) = 2T(\vec{u}) + T(\vec{w}).$$

$$T(\vec{v}) = T(\vec{u}) + T(\vec{w}).$$

$$T(\vec{v}) = T(\vec{u} + \vec{w}).$$

That means two different inputs can have the same output.