HW01

Reading: Review what you learned from your Linear Algebra classes, the lecture note, and Section 12.8.5 and 12.8.6 in *Foundations of Data Science* by Blum, Hopcroft, and Kannan. Review vector norms, matrix norms, orthogonality, projections, and eigenvalues.

1. (a) Let M be the matrix of data points

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix}.$$

   What are M^TM and MM^T?

   (b) Prove that if A is any matrix, then A^TA and AA^T are *symmetric*. (Recall that a matrix S is symmetric if $S = S^T$.)

2. Recall that a matrix $A \in \mathbb{R}^{m\times n}, m \geq n$, is said to have full rank if its columns are linearly independent, i.e., for $\boldsymbol{a}_j$ the jth column of A, $c_1\boldsymbol{a}_1 + \ldots + c_n\boldsymbol{a}_n = 0 \Longrightarrow c_1 = \ldots = c_n = 0$. Show that A has full rank if and only if no two distinct vectors are mapped to the same vector.

3. Sketch the unit circle $\{\boldsymbol{x}, \|\boldsymbol{x}\|_p = 1\}$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ for $p = 1, 2$, and ∞.

4. (a) Write the definition of the vector norm $\|\boldsymbol{x}\|_2$.

   (b) Show that if Q is an orthogonal matrix, then $\|Q\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$.

   (c) Let $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$ and

$$Q = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad \text{(this is a Hadamard matrix).}$$

   Without calculating $Q\boldsymbol{x}$ directly, what is the value of $\|Q\boldsymbol{x}\|_2$?

5. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^m$, the matrix $A = I + \boldsymbol{u}\boldsymbol{v}^T$ is know as a *rank-one perturbation of the identity.* Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha\boldsymbol{u}\boldsymbol{v}^T$ for some scalar α, and give an expression for α. For what $\boldsymbol{u}$ and $\boldsymbol{v}$ is A singular? If it is singular, what is $\mathrm{Null}(A)$?

6. Given $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$, show that if $E = \boldsymbol{u}\boldsymbol{v}^T$, then $\|E\|_2 = \|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2$. Is the same true for the Frobenius norm, i.e., $\|E\|_F = \|\boldsymbol{u}\|_F\|\boldsymbol{v}\|_F$? Prove it or give a counterexample.

7. Consider the matrix

$$A = \begin{bmatrix} -2 & 3 & 2 \\ -4 & 5 & 1 \\ 1 & -2 & 4 \end{bmatrix}.$$

   What are the ℓ^1, ℓ^2, ℓ^∞, and Frobenius norms of A?

8. Given $A \in \mathbb{R}^{m\times n}$ with $m \geq n$, show that $A^T A$ is nonsingular if and only if A has full rank.

9. What is the vector $\boldsymbol{x} \in \mathbb{R}^2$ that achieves the maximum ℓ^1-norm subject to $\|\boldsymbol{x}\|_2 = 1$?

10. Given $A \in \mathbb{R}^{m\times p}$ and $B \in \mathbb{R}^{p\times n}$, show the following.

    (a) $\|AB\|_2 \leq \|A\|_2 \|B\|_2$.

    (b) $\|AB\|_F^2 \leq \|A\|_F^2 \|B\|_F^2$.