MATH 20E, MIDTERM 2 SOLUTIONS
(VERSION C)

1. Note that traversing the boundary counterclockwise corresponds to the same orientation used in Green's theorem. Application of the theorem yields

$$\begin{aligned}\int_{\partial S} y\,dx - x^2\,dy &= \iint_S \left(\frac{\partial}{\partial x}\left(-x^2\right) - \frac{\partial}{\partial y}(y)\right) dxdy \\ &= \int_{-1}^{1}\int_{-1}^{1} (-2x-1)\,dxdy \\ &= -4.\end{aligned}$$

2. The parametrization is given by $\mathbf{\Phi}(u,v) = (u^2, u\cos(v), u\sin(v))$ with $0 \le u \le 3$ and $0 \le v \le 2\pi$.

(a)

$$\begin{aligned}\mathbf{T}_u(u,v) &= (2u, \cos(v), \sin(v)) \\ \mathbf{T}_v(u,v) &= (0, -u\sin(v), u\cos(v))\end{aligned}$$

(b)

$$\begin{aligned}(\mathbf{T}_u \times \mathbf{T}_v) &= (u\cos^2(v) + u\sin^2(v), -2u^2\cos(v), -2u^2\sin(v)) \\ &= (u, -2u^2\cos(v), -2u^2\sin(v)).\end{aligned}$$

Therefore,

$$\|\mathbf{T}_u \times \mathbf{T}_v\|^2 = u^2 + (2u^2)^2 \implies \|\mathbf{T}_u \times \mathbf{T}_v\| = u\sqrt{1+4u^2}.$$

The unit normal is then given by

$$\hat{\mathbf{n}} = \frac{(\mathbf{T}_u \times \mathbf{T}_v)}{\|\mathbf{T}_u \times \mathbf{T}_v\|} = \left(\frac{1}{\sqrt{1+4u^2}}, -\frac{2u\cos(v)}{\sqrt{1+4u^2}}, -\frac{2u\sin(v)}{\sqrt{1+4u^2}}\right)$$

(c) The area of the surface is given by

$$\begin{aligned}A(S) &= \iint_S \|\mathbf{T}_u \times \mathbf{T}_v\|\,dudv \\ &= \int_0^{2\pi}\int_0^3 u\sqrt{1+4u^2}\,dudv \\ &= \frac{\pi}{6}\left(37\sqrt{37}-1\right).\end{aligned}$$

3. (a) Let $\mathbf{c}(t) = (t, t^2)$ with $t \in [-1, 2]$.

(b)

$$\begin{aligned}\int_{\mathbf{c}} \frac{y}{x} ds &= \int_{\mathbf{c}} \frac{y(t)}{x(t)}\,\|\mathbf{c}'(t)\|\,dt \\ &= \int_{-1}^{2} t\sqrt{1+4t^2}\,dt \\ &= \frac{1}{12}\left(17\sqrt{17} - 5\sqrt{5}\right)\end{aligned}$$

(c) The original intent of this problem was to test the Fundamental Theorem of Line Integrals.

$$\int_{\mathbf{c}} (\nabla f) \cdot d\mathbf{s} = \int_{t_1}^{t_2} \left(\frac{d}{dt} f(\mathbf{c}(t)) \right) dt = f(\mathbf{c}(t_2)) - f(\mathbf{c}(t_1)),$$

so that for $f(x,y) = \frac{e^x}{x^2+y^2}$, we would have

$$\int_{\mathbf{c}} (\nabla f) \cdot d\mathbf{s} = f(2,4) - f(-1,1) = \frac{e^2}{20} - \frac{e^{-1}}{2}.$$

Remark. However, the function is not defined at the point $(0,0)$ (at the origin, the function is infinity), and hence it is not a $\mathcal{C}^1$ function on $\mathbb{R}^2$. Therefore, the theorem cannot be applied. The line integral, as written, does not exist because $\mathbf{c}(t)$ passes through a point (the origin) that is not in the domain of $f(x,y)$. It was not at all the original intent of the problem to test these subtleties. Since the function is $\mathcal{C}^1$ on its domain, $\mathbb{R} \times \mathbb{R} \setminus \{(0,0)\}$, the theorem could be applied to other any curve with these two endpoints that does not pass through the origin. In lieu of this, students who applied the fundamental theorem still received full marks.

4. A parametrization of the upper half of a unit sphere is given by

$$\mathbf{\Phi}(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi).$$

Then

$$\begin{aligned}
\text{Flux through } S &= \iint_S \mathbf{F} \cdot d\mathbf{S} \\
&= \iint_S \mathbf{F}(x(\theta,\phi), y(\theta,\phi), z(\theta,\phi)) \cdot \hat{\mathbf{n}} \, \|\mathbf{T}_\theta \times \mathbf{T}_\phi\| \, d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/2} \left(0, 0, \tfrac{1}{2}\cos\phi\right) \cdot (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi) \sin\phi \, d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/2} \tfrac{1}{2}\cos^2\phi \sin\phi \, d\phi d\theta \\
&= \pi \int_0^1 u^2 \, du \\
&= \frac{\pi}{3}.
\end{aligned}$$