Name: Key PID: ____________

TA: ____________ Sec. No: ______ Sec. Time: ______

Math 18
Exam 2 v. 0: Practice
March 2, 2018

Turn off and put away your cell phone and other electronic devices.
You may use one hand-written standard sheet of notes, but calculators, notes, and other assistance are prohibited during this exam.
Read each question carefully, and answer each question completely.
Show all of your work; no credit will be given for unsupported answers.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.

1. Compute the determinant

$$\begin{vmatrix} -2 & -1 & -1 & 0 \\ 4 & 3 & 3 & 2 \\ -4 & -6 & -7 & 0 \\ -1 & -5 & -9 & 0 \end{vmatrix}$$

We'll use cofactor expansion for the 4th column: The desired determinant is equal to:

$$2\cdot(-1)^{2+4} \det\begin{bmatrix} -2 & -1 & -1 \\ -4 & -6 & -7 \\ -1 & -5 & -9 \end{bmatrix}$$

$$= 2\left(-2\begin{vmatrix} -6 & -7 \\ -5 & -9 \end{vmatrix} + \begin{vmatrix} -4 & -7 \\ -1 & -9 \end{vmatrix} - \begin{vmatrix} -4 & -6 \\ -1 & -5 \end{vmatrix}\right).$$

$$= 2\left[-2(54-35) + (36-7) - (20-6)\right].$$

$$= 2(-2\cdot 19 + 29 - 14)$$

$$= 2(-23)$$

$$= -46.$$

2. Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \right\}$.

(a) Explain why $\mathcal{B}$ is a basis of $\mathbb{R}^3$.

$$\begin{bmatrix} 1 & -2 & 2 \\ 2 & -5 & 5 \\ 1 & 2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 1 \\ 0 & 4 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ these vectors are linearly independent, and the matrix formed by them is invertible $\Rightarrow$ they form a basis for $\mathbb{R}^3$.

(b) Let $\mathbf{x} = \begin{pmatrix} -3 \\ -8 \\ -2 \end{pmatrix}$. Find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

We need to find x_1, x_2, and x_3 such that

$$\begin{bmatrix} -3 \\ -8 \\ -2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -5 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -5 & 5 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 2 & -5 & 5 & -8 \\ 1 & 2 & -1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 0 & -1 & 1 & -2 \\ 0 & 4 & -3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -7 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -7 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -7 \end{array}\right].$$

$\Rightarrow \quad x_1 = 1 \quad x_2 = -5, \quad x_3 = -7.$

$$\therefore \left[\begin{pmatrix} -3 \\ -8 \\ -2 \end{pmatrix} \right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -5 \\ -7 \end{bmatrix}.$$

3. The matrix

$$\mathbf{A} = \begin{pmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_6 \\ | & | & & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & 4 & -2 & -4 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 5 & 0 & -1 \\ 2 & -2 & 3 & -6 & 2 & 6 \end{pmatrix}$$

wrong matrix. But it's still ok! Suppose we only know RREF(A)

is row equivalent to

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Let H be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and K be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{a}_4$, $\mathbf{a}_5$, $\mathbf{a}_6$. Find a basis for each of the following subspaces.

(a) Col $\mathbf{A}$, the column space of $\mathbf{A}$.

pivots are columns 1, 2, 3, and 5.

$\Rightarrow$ $\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_5\}$ form a basis for Col(A).

(b) Nul $\mathbf{A}$, the null space of $\mathbf{A}$.

We can use B to find Nul(A). Free variables are x_4 and x_6.

Solution of $A\vec{x} = 0$ is

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 0 \\ -4 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

these two vectors form a basis for Nul(A)

(c) $H + K$, the sum of subspaces H and K.

has not learned this yet.

(d) $H \cap K$, the intersection of H and K.

4. Let $\mathbf{A}$ be a 7×10 matrix. We also know that $\text{Nul}\,\mathbf{A}$ can be written as the linear span of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$, but we do not know if $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ are linearly independent or not.

(a) Find all possible value(s) of $\dim \text{Nul}\,\mathbf{A}$, the dimension of the null space of $\mathbf{A}$.

$\text{Nul}(A) = \text{span}\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\}. \Rightarrow \dim \text{Nul}(A) \leq 4.$

Since A is of size 7×10, A has no more than 7 pivots.

$\Rightarrow \text{rank}(A) \leq 7. \Rightarrow 10 - \text{nullity}(A) \leq 7.$

$\text{nullity}(A) \geq 3.$

$\Rightarrow \dim \text{Nul}(A) = 3$ or 4.

(b) Find all possible value(s) of $\text{rank}\,\mathbf{A}$, the rank of $\mathbf{A}$.

From part a), we know that $\text{rank}(A) \leq 7$,

Since $\dim \text{Nul}(A) \leq 4$,

$10 - \text{rank}(A) \leq 4.$

$\text{rank}(A) \geq 6.$

$\Rightarrow \text{rank}(A) = 6$ or 7.

Midterm 2vS
Math 18, Section A
May 19, 2017
Time Limit: 50 Minutes

Name (Print): ______________________

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Question	Points
1	10
2	10
3	10
4	10
Total:	40

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1. Let $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ and let a, b, c, d be some numbers.

(a) (7 points) Find A^{-1}.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{1/2 R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & \cancel{1/2}\ 1 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 3R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 3/2 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1 & 3/2 & 0 & -3 \\ 0 & 1/2 & 0 & 0 \\ 0 & -1/2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(b) (3 points) Solve the equation $A\vec{x} = (a, b, c, d)$. You may answer in terms of a, b, c and d.

Since A is invertible,

$$\vec{x} = A^{-1} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a + 3/2 b - 3d \\ \frac{1}{2} b \\ -1/2 b + d \\ c \end{bmatrix}.$$

2. (10 points) Let $\mathbb{P}_2$ be the vector space of polynomials with degree at most 2, and let $\mathcal{B} = \{1+x, 1+x^2, x+x^2\}$. $\mathcal{B}$ is a basis for $\mathbb{P}_2$ (and you don't need to prove this). Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy $T\left(\left[a_0 + a_1 x + a_2 x^2\right]_{\mathcal{B}}\right) = (a_0, a_1, a_2)$. Find the matrix corresponding to T.

We need to find $T(\vec{e}_1)$, $T(\vec{e}_2)$, $T(\vec{e}_3)$.

We need to find a polynomial $a_0 + a_1 x + a_2 x^2$ such that

$$[a_0 + a_1 x + a_2 x^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \quad [1+x^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \quad T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = T\left([1+x]_{\mathcal{B}}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

similarly, $[1+x^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = T\left([1+x^2]_{\mathcal{B}}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

and $[x+x^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\Rightarrow \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = T\left([x+x^2]_{\mathcal{B}}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

$\therefore$ The standard matrix of T is.

$$\left[T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3)\right] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

3. (10 points) Let A be a matrix which is row-equivalent to $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and let $T : \mathbb{R}^3 \to \mathbb{R}^7$ be a linear transformation such that $T(\vec{x}) = \vec{0}$ if and only if $\vec{x} = A\vec{y}$ for some $\vec{y} \in \mathbb{R}^4$. Find the dimension of the range of T.

Since $T(\vec{x}) = \vec{0} \iff \vec{x} = A\vec{y}$ for some $\vec{y} \in \mathbb{R}^4$,

$$\text{Nul}(T) = \text{Col}(A).$$

$$\Rightarrow \dim \text{Nul}(T) = \dim \text{Col}(A).$$

since ~~A~~ RREF(A) has two pivots, $\dim \text{Col}(A) = 2$.

$$\Rightarrow \dim \text{Nul}(T) = 2.$$

Since,

$$\dim \text{Nul}(T) + \dim \text{Range}(T) = 3.$$

$$2 + \dim \text{Range}(T) = 3$$

$$\dim \text{Range}(T) = 1.$$

4. (a) (5 points) Let P be a parallelopiped in $\mathbb{R}^3$. Suppose that one of its vertices is $(0,1,0)$, and the three vertices adjacent to that one are $(0,1,2)$, $(1,1,1)$ and $(1,0,-1)$. Find the volume of P.

Volume will not be on the midterm.

(b) (5 points) Let A and B be invertible matrices with $\det A = 4$ and $\det B = 2$. Find $\det(A^2B^{-1})$.

$$\det(A^2B^{-1}) = \det(A^2)\det(B^{-1}).$$
$$= \det(A \cdot A)\,\frac{1}{\det(B)}$$
$$= \det(A)\det(A) \cdot \frac{1}{2}$$
$$= 4^2 \cdot \frac{1}{2}$$
$$= 8.$$

Midterm 2vP
Math 18, Section A
May 19, 2017
Time Limit: 50 Minutes

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2	10
3	10
4	10
Total:	40

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1. (10 points) Let I be the 3×3 identity matrix and let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Find a matrix M satisfying $MA = B + I$.

Since $\det A = 1$, A is invertible.

$$\Rightarrow \quad MAA^{-1} = (B+I)A^{-1}.$$

$$M = (B+I)A^{-1}.$$

$\Rightarrow$ we need to find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{matrix}}_{A^{-1}}$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$M = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \end{bmatrix}.$$

[redacted]

2. (10 points) Let $T : \mathbb{R}^{1903} \to \mathbb{R}^4$ be a linear transformation with range

$$\operatorname{Range} T = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Find $\dim \operatorname{Nul} T$.

$\dim \operatorname{Nul}(T) + \dim \operatorname{range}(T) = 1903.$

we can find $\dim \operatorname{range}(T)$ by

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow \ \uparrow$
pivots.

$\Rightarrow \dim \operatorname{range}(T) = 2.$

$\therefore \dim \operatorname{Nul}(T) = 1901.$

[redacted]

3. (10 points) Let $\mathbb{P}$ be the space of polynomials in x (with the usual addition and scalar multiplication) and let $H = \text{span}\{x, x^2, x^5\} \subset \mathbb{P}$. Then $\mathcal{B} = \{x, x^2, x^5\}$ is a basis for H, and $\mathcal{C} = \{x + x^5, x^2, x\}$ is also a basis for H. Find a matrix A satisfying the following condition:

   - For any $p(x) \in H$,

$$A[p(x)]_{\mathcal{B}} = [p(x)]_{\mathcal{C}}$$

This question is "change of bases" (Section 4.7).
We have not learned it yet!

4. (10 points) Suppose that S is a region of $\mathbb{R}^3$ whose volume is 3. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 4 & 6 \\ 0 & 5 & 3 \end{bmatrix}$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation $T(\vec{x}) = A^{2017}\vec{x}$. Find the volume of $T(S)$.

We will not cover volume for our midterm!