Problem 3: We want to evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where the vector field $\vec{F}$ is complicated.

$\Rightarrow$ solving directly may be challenging.

$\Rightarrow$ Gauss' divergence thm.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_W \nabla \cdot \vec{F} \, dV = \iiint_W 2 \, dV = 2 \underbrace{\iiint_W dV}_{\substack{\text{Vol of a sphere} \\ \text{of radius } a. \\ = \frac{4}{3}\pi a^3.}} = \frac{8\pi a}{3}$$

In fact, we can solve $\iiint_W dV$ fast by using spherical coordinates.

$$x = \rho \sin\phi \cos\theta \qquad 0 \le \phi \le \pi$$
$$y = \rho \sin\phi \sin\theta \qquad 0 \le \theta \le 2\pi$$
$$z = \rho \cos\phi \qquad 0 \le \rho \le a$$

Jacobian $= -\rho^2 \sin\phi$

$$\iiint_W dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} |-\rho^2 \sin\phi| \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \int_0^a \rho^2 \, d\rho \int_0^{\pi} \sin\phi \, d\phi$$

$$= 2\pi \left(\frac{\rho^3}{3} \Big|_0^a \right) \left(-\cos\phi \right) \Big|_0^{\pi}$$

$$= \frac{4\pi a^3}{3}.$$