Midterm 2B MATH 20E, LECTURE A00, SPRING 2019

NAME: Key PID:

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(This exam is worth 40 points)

Problem 0.(1 points.) Follows the instructions on this exam and any additional instructions given during the exam.

NAME:

Problem 1.(9 points.) *Use Green's theorem* to evaluate $\int_C -xy\,dx + y\,dy$, where C is the boundary of the region between the curves $y = x$ and $y = x^3$, $0 \le x \le 1$, and is oriented in the counterclockwise direction.

$$\int_C \underbrace{-xy}_{P}\,dx + \underbrace{y}_{Q}\,dy = \int_0^1\int_{x^3}^{x}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dy\,dx$$

$$= \int_0^1\int_{x^3}^{x}(0 + x)\,dy\,dx$$

$$= \int_0^1\int_{x^3}^{x} x\,dy\,dx$$

$$= \int_0^1 xy\Big|_{y=x^3}^{x}\,dx$$

$$= \int_0^1 x^2 - x^4\,dx.$$

$$= \frac{x^3}{3} - \frac{x^5}{5}\Big|_0^1$$

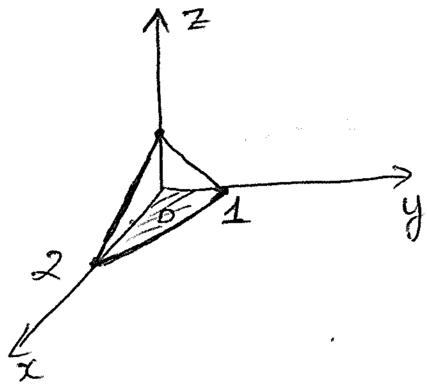
$$= \frac{1}{3} - \frac{1}{5}$$

$$= \frac{2}{15}$$

See Example 4, p. 397

NAME:

Problem 2.(10 points.) Let S be the triangle with vertices $(0,0,1),(2,0,0)$, and $(0,1,0)$. Compute $\int\int_S x\,dS$.



First, find the equation of S:

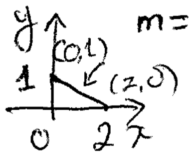
$$(0,0,1)-(2,0,0) = (-2,0,1)$$

and $(0,0,1)-(0,1,0) = (0,-1,1)$

are vectors on S.

The normal vector to S is:

$$(-2,0,1)\times(0,-1,1) = \begin{vmatrix} i & j & k \\ -2 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = (1,2,\cancel{0}\,2)$$



$m = -\frac{1}{2}$.

$y = -\frac{1}{2}(x-0)+1$.

$y = 1-\frac{x}{2}$

Therefore, $\vec{n} = \frac{1}{3}(1,2,2)$.

$$\Rightarrow \vec{n}\cdot\vec{k} = \frac{1}{3}(1,2,2)\cdot(0,0,1) = \frac{2}{3} \Rightarrow \cos\theta = \frac{2}{3}.$$

$$\iint_S x\,dS = \iint_D x\frac{1}{\cos\theta}\,dx\,dy = \frac{3}{2}\int_0^2\int_0^{1-\frac{x}{2}} x\,dy\,dx$$

$$= \frac{3}{2}\int_0^2 xy\Big|_{y=0}^{1-\frac{x}{2}} dx$$

$$= \frac{3}{2}\int_0^2 x\left(1-\frac{x}{2}\right)dx$$

$$= \frac{3}{2}\int_0^2 x-\frac{x^2}{2}\,dx$$

$$= \frac{3}{2}\left(\frac{x^2}{2}-\frac{x^3}{6}\right)\Big|_0^2$$

$$= \frac{3}{2}\left(\frac{4}{2}-\frac{8}{6}\right)$$

$$= \cancel{\frac{4}{3}}\ 1$$

Note that ~~the~~ another approach is to think of S as a graph:

$$\frac{1}{3}(x-0)+\frac{2}{3}(y-0)+\frac{2}{3}(z-1) = 0.$$

$$z = -\frac{1}{2}x - \cancel{2}y + \cancel{1}\,1 \Rightarrow \Phi(x,y) = \left(x, y, -\frac{x}{2}-y+1\right).$$

NAME:

Problem 3.(10 points.) Let $\vec{c}(t) = (1, -t^2, \cos t), 0 \leq t \leq \pi$. Evaluate

$$\int_{\vec{c}} \cos z \, dx - y^2 \, dy + xz \, dz.$$

$$= \int_0^{\pi} \left(\cos(\cos t) \cdot 0 - (-t^2)^2(-2t) + 1 \cdot \cos t \, (-\sin t) \right) dt.$$

$$= \int_0^{\pi} 2t^5 - (\cos t)(\sin t) \, dt.$$

$$= \frac{2t^6}{6} + \frac{\cos^2 t}{2} \Big|_{t=0}^{\pi}$$

$$= \frac{\pi^6}{3} + \frac{\cos^2(\pi)}{2} - 0 - \frac{\cos^2(0)}{2}$$

$$= \frac{\pi^6}{3}$$

NAME:

Problem 4.(10 points.) Consider the solid hemisphere formed by taking the portion of the unit ball with $y \geq 0$. Let S be the *surface* of this region and be oriented by the outward-pointing normal (so that S is a hemisphere, together with a flat base in the xz-plane). Find the flux of the vector field $\vec{F}(x,y,z) = (z, 1, -x)$ out of the surface S. That is, compute $\int\int_S \vec{F} \cdot d\vec{S}$.
(You might find the following identity useful: $\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}$.)

Note that S has two parts: S_1 the hemisphere
S_2 the flat base in xz-plane

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}.$$ outward pointing normal.

7 pts.

For S_1: we take the parametrization

$$\Phi(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$ $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \pi$.

$$T_\theta = (-\sin\theta \sin\phi, \cos\theta \sin\phi, \cos\phi)$$
$$T_\phi = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$$
$$T_\theta \times T_\phi = -\sin\phi(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$ points inward.

$\to$ consider $T_\phi \times T_\theta$.

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \int_0^\pi \int_0^\pi + \sin(\theta) \sin^2\phi \, d\phi \, d\theta$$
$$= \int_0^\pi \sin^2\phi \, d\phi \int_0^\pi \sin\theta \, d\theta.$$
$$= \left(\int_0^\pi \frac{1-\cos(2\phi)}{2} d\phi\right) \cdot \left(-\cos\theta \Big|_{\theta=0}^{\pi}\right)$$
$$= \left(\frac{\phi + \sin(2\phi)/2}{2}\Big|_{\phi=0}^{\pi}\right)(+1+1)$$
$$\boxed{= \pi.}$$

2 pts

For S_2; we can parametrize it as $\Phi(r,\theta) = (r\cos\theta, 0, r\sin\theta)$
$0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

$T_r = (\cos\theta, 0, \sin\theta)$ $T_\theta = (-r\sin\theta, 0, r\cos\theta)$.

$T_r \times T_\theta = (0, -r, 0) \Leftarrow$ point outward.

$$\iint_{S_2} \vec{F} \cdot d\vec{S}_2 = \int_0^1 \int_0^{2\pi} (r\sin\theta, 1, -r\cos\theta) \cdot (0,-r,0) \, d\theta \, dr = -\int_0^1 \int_0^{2\pi} r \, d\theta \, dr$$
$$\boxed{= -\pi}$$

1pt. $\Rightarrow$ $$\iint_S \vec{F} \cdot d\vec{S} = \pi - \pi = \boxed{0.}$$

Midterm 2A

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$P = xy$, $Q = y$.

$$\int_C xy\,dx + y\,dy = \int_0^1 \int_{x^3}^{x} (0 - x)\,dy\,dx.$$

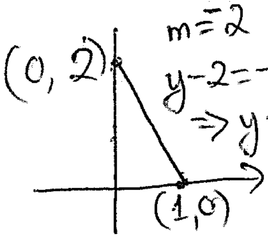
$$= \dots = -\frac{2}{15}.$$

NAME:

Problem 2.(10 points.) Let S be the triangle with vertices $(0, 0, 1)$, $(1, 0, 0)$, and $(0, 2, 0)$. Compute $\int\int_S x\,dS$.

$$S:\quad z = -x - \frac{y}{2} + 1.$$

$$\Phi(x, y) = (x, y, \underbrace{-x - \frac{y}{2} + 1}_{g(x,y)}).$$



$m = 2$

$y - 2 = -2(x)$

$\Rightarrow y = -2x + 2$

$$\iint_S x\,dS = \int_0^1 \int_0^{-2x+2} x\sqrt{(+1)^2 + \left(\frac{1}{2}\right)^2 + 1}\;dy\,dx$$

$$= \sqrt{1 + \frac{1}{4} + 1}\int_0^1 \int_0^{-2x+2} x\,dy\,dx$$

$$= \frac{3}{2}\int_0^1 xy\Big|_{y=0}^{-2x+2}\,dx$$

$$= \frac{3}{2}\int_0^1 -2x^2 + 2x\,dx$$

$$= \frac{3}{2}\left(-\frac{2}{3}x^3 + x^2\right)\Big|_0^1$$

$$= \frac{3}{2}\left(-\frac{2}{3} + 1\right)$$

$$= \frac{1}{2}.$$

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Problem 3.(10 points.) Let $\vec{c}(t) = (1, -t^2, \cos t), 0 \le t \le \pi$. Evaluate

$$\int_{\vec{c}} \sin z \, dx - y^2 \, dy + 3xz \, dz.$$

$$= \int_0^{\pi} \sin(\cos t) \cdot 0 - (-t^2)^2(-2t) + 3\cos t(-\sin t) \, dt.$$

$$= \int_0^{\pi} 2t^5 - 3\cos t \sin t \, dt.$$

$$= \frac{\pi^6}{3}.$$

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Similar approach and result as version B.