MIDTERM 2A, MATH 152, WINTER 2019 TOTAL POINTS: 25

Name: **PID:**

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(This exam is worth 25 points)

Problem 0.(1 point.) Follows the instructions on this exam and any additional instructions given during the exam.

Name:

Problem 1.(6 points.) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Find the SVD of A.

(b) Run the power method starting from $x^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for $k = 3$ steps.

a) $B = A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Find eigenvalues and eigenvectors of B:

$(2-\lambda)^2 - 1 = 0 \Rightarrow \lambda_1 = 3$ and $\lambda_2 = 1$.

For $\lambda_1 = 3$, $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; For $\lambda_2 = 1$, $\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ +1 \end{bmatrix}$.

$\vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ $\quad \vec{u}_2 = \frac{1}{\sqrt{1}}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 2/\sqrt{6} & 0 \\ 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

b) $x^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow x^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$x^{(2)} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

$x^{(3)} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \end{bmatrix}$.

$\Rightarrow \frac{x^{(3)}}{\|x^{(3)}\|_2} = \frac{1}{\sqrt{14^2+13^2}}\begin{bmatrix} 14 \\ 13 \end{bmatrix}$.

Name:

Problem 2.(6 points.) Suppose that a matrix A has the following SVD

$$A = [u_1\ u_2\ u_3]\Sigma \begin{bmatrix} -v_1^T- \\ -v_2^T- \\ -v_3^T- \end{bmatrix} = \begin{bmatrix} .13 & .02 & -.01 \\ .41 & .07 & -.03 \\ .55 & .09 & .-0.4 \\ .68 & .11 & -.05 \\ .15 & -.59 & .65 \\ .07 & -.73 & -.67 \\ .07 & -.29 & .32 \end{bmatrix} \begin{bmatrix} 12.4 & 0 & 0 \\ 0 & 9.5 & 0 \\ 0 & 0 & 1.3 \end{bmatrix} \begin{bmatrix} .56 & .59 & .56 & .09 & .09 \\ .12 & -.02 & .12 & -.69 & -.69 \\ .40 & -.80 & .40 & .09 & .09 \end{bmatrix}.$$

Let $\sigma_1 = 12.4, \sigma_2 = 9.5$, and $\sigma_3 = 1.3$ be the singular values of A. Let $A_2 = \sum_{i=1}^{2} \sigma_i u_i v_i^T$.

(a) Express $\|A_2\|_F^2$ and $\|A - A_2\|_2^2$ in term of singular values of A. (You need not to simplify.)

(b) What is the best rank-1 approximation matrix to A (in Frobenius norm)?

a) $\|A_2\|_F^2 = 12.4^2 + 9.5^2.$

$\|A - A_2\|_2^2 = \| \sigma_3 u_3 v_3^T \|_2^2 = \sigma_3^2 = 1.3^2.$

b) It's $A_1 = \sigma_1 u_1 v_1^T.$

Name:

Problem 3.(6 points.) Describe the process of estimating F_0 (or counting distinct elements in a data stream.) that you learned in class.

Consider a hash function $h: [m] \to [0,1]$. $x_1, \dots, x_k$

and suppose that there are k distinct elements (which we want to estimate). Suppose $h(a_1), \dots, h(a_n)$ are independent uniform over $[0,1]$.

$$Y = \min(h(x_1), \dots, h(x_k)).$$

$$\Rightarrow \mathbb{E}[Y] = \frac{1}{k+1}.$$

Then we can show. $\mathbb{P}(|Y - \mathbb{E}[Y]| \geq \varepsilon \mathbb{E}[Y]) \leq \frac{1}{\varepsilon^2}$.

for $\varepsilon \in (0,1)$.

$\Rightarrow$ need to improve $\Rightarrow$ mean of estimator.

Consider t independent trials of Y, says $Y_1, \dots, Y_t$. and let $Z = \frac{Y_1 + \dots + Y_t}{t}$

$$\Rightarrow \mathbb{E}[Z] = \mathbb{E}[Y] = \frac{1}{k+1}.$$

can show that

$$\Rightarrow \mathbb{P}(|Z - \mathbb{E}[Z]| \geq \varepsilon \mathbb{E}[Z]) \leq \frac{1}{t\varepsilon^2}.$$

Name:

Problem 4.(6 points.) Suppose there is a random variable X taking values in $[0,1]$. Note that we don't know the distribution of X. How can you estimate $\mathbb{E}[X]$ up to an error 0.1 and with probability at least 90%? In this case, how many samples do you need to take? Explain your answer clearly. (You need not to simplify.)

Let $\cancel{X}$ $Y = \frac{X_1 + \dots + X_n}{n}$, where X_i's are independent copies of X.

$$\mathbb{E}[Y] = \frac{\mathbb{E}[X_1] + \dots + \mathbb{E}[X_n]}{n} = \mathbb{E}[X].$$

By Chernoff's bound:

$$\mathbb{P}(|Y - \mathbb{E}[Y]| \geq \varepsilon) \leq 2e^{-2\varepsilon^2 n} = \delta$$

Given $\varepsilon = 0.1$ and $\delta = 1 - 0.9 = 0.1$,

we need to take $n = \frac{1}{2\varepsilon^2} \ln\left(\frac{2}{\delta}\right)$ samples

and use Y to estimate $\mathbb{E}[X]$. $= \frac{1}{2(0.1)^2} \ln\left(\frac{2}{0.1}\right)$.

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(a) Find the SVD of A.

(b) Run the power method starting from $x^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $k = 3$ steps.

a) $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow \lambda_1 = 3$ and $\lambda_2 = 1$.

$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{u}_2 = \cancel{1}\begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$= \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ $\quad = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

$$A = \begin{bmatrix} 2/\sqrt{6} & 0 \\ -1/\sqrt{6} & -1/\sqrt{2} \\ -1/\sqrt{6} & +1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

b) $x^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$x^{(2)} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$x^{(3)} = \begin{bmatrix} 13 \\ 14 \end{bmatrix} \Rightarrow \frac{x^{(3)}}{\|x^{(3)}\|_2} = \frac{1}{\sqrt{13^2 + 14^2}}\begin{bmatrix} 13 \\ 14 \end{bmatrix}$.

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Problem 2.(6 points.) Suppose that a matrix A has the following SVD

$$A = [u_1\ u_2\ u_3]\Sigma \begin{bmatrix} -v_1^T- \\ -v_2^T- \\ -v_3^T- \end{bmatrix} = \begin{bmatrix} .13 & .02 & -.01 \\ .41 & .07 & -.03 \\ .55 & .09 & .-0.4 \\ .68 & .11 & -.05 \\ .15 & -.59 & .65 \\ .07 & -.73 & -.67 \\ .07 & -.29 & .32 \end{bmatrix} \begin{bmatrix} 12.4 & 0 & 0 \\ 0 & 9.5 & 0 \\ 0 & 0 & 1.3 \end{bmatrix} \begin{bmatrix} .56 & .59 & .56 & .09 & .09 \\ .12 & -.02 & .12 & -.69 & -.69 \\ .40 & -.80 & .40 & .09 & .09 \end{bmatrix}.$$

Let $\sigma_1 = 12.4, \sigma_2 = 9.5$, and $\sigma_3 = 1.3$ be the singular values of A. Let $A_2 = \sum_{i=1}^{2} \sigma_i u_i v_i^T$.

(a) Express $\|A_2\|_2^2$ and $\|A - A_2\|_F^2$ in term of singular values of A. (You need not to simplify.)

(b) What is the best rank-1 approximation matrix to A (in Frobenius norm)?

a) $\|A_2\|_2^2 = \sigma_1^2 = 12.4^2.$

$\|A - A_2\|_F^2 = \sigma_3^2 = 1.3^2.$

b) $A_1 = \sigma_1 u_1 v_1^T.$

Name:

Problem 3.(6 points.) Describe the process of estimating F_0 (or counting distinct elements in a data stream.) that you learned in class.

same as ~~part~~ Ver A.

Name:

Problem 4.(6 points.) Suppose there is a random variable X taking values in $[0,1]$. Note that we don't know the distribution of X. How can you estimate $\mathbb{E}[X]$ up to an error 0.15 and with probability at least 95%? In this case, how many samples do you need to take? Explain your answer clearly. (You need not to simplify.)

$$\varepsilon = 0.15$$

$$\delta = 0.05.$$

$$\Rightarrow \quad n = \frac{1}{2(0.15)^2} \ln\left(\frac{2}{0.05}\right).$$