MIDTERM 2A, MATH 152, SPRING 2019 TOTAL POINTS: 25

Name: Key **PID:**

- Print your *NAME* on every page and write your PID in the space provided above.
- Show all of your work in the spaces provided. No credit will be given for unsupported answers, even if correct.
- Supporting work for a problem must be on the page containing that problem. No scratch paper will be accepted.
- No calculators, tables, phones, or other electronic devices are allowed during this exam. You may use your double-sided handwritten notes, but no books or other assistance.

DO NOT TURN PAGE UNTIL INSTRUCTED TO DO SO
(This exam is worth 25 points)

Problem 0.(1 point.) Follows the instructions on this exam and any additional instructions given during the exam.

Name:

Problem 1.(6 points.) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Determine the SVD of the matrix A.

0.5 pt $B = A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

1 pt $\left\{ \right.$ Find eigenvalues and eigenvectors of B:

$$\det \begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)^2 - 2^2 = 0.$$

$$-\lambda(4-\lambda) = 0.$$

$$\lambda_1 = 4 \quad \text{and} \quad \lambda_2 = 0 \Rightarrow \sigma_1 = \sqrt{4} = 2 \text{ and } \sigma_2 = 0.$$

1 pt $\left\{ \right.$ For $\lambda_1 = 4$, $\begin{bmatrix} 2-4 & 2 \\ 2 & 2-4 \end{bmatrix} \vec{x} = 0 \Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \vec{x} = 0.$

$x_1 = x_2$

$$\therefore \vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

1 pt $\left\{ \right.$ For $\lambda_2 = 0$, $\begin{bmatrix} 2-0 & 2 \\ 2 & 2-0 \end{bmatrix} \vec{x} = 0 \Rightarrow x_1 = -x_2.$

$$\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

1 pt $\left\{ \right.$ $Av_1 = \sigma_1 u_1 \Rightarrow u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

1 pt $\left\{ \right.$ $Av_2 = \sigma_2 u_2 \Rightarrow Av_2 = 0.$

we can pick u_2 by selecting any nonzero vector of unit norm and orthogonal to u_1. A candidate is $u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

0.5 pt. $$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Ver B: $$A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Name:

Problem 2.(6 points.) Given the SVD of a matrix A as

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

a) (1 point) Find $\|A\|_2$, the matrix 2-norm of A.

b) (3 points) List the singular values, left singular vectors, and right singular vectors of A.

c) (2 points) Draw a careful, labeled picture of the unit ball in $\mathbb{R}^2$ and its image under A, together with the singular vectors, with the coordinates of their vertices marked.

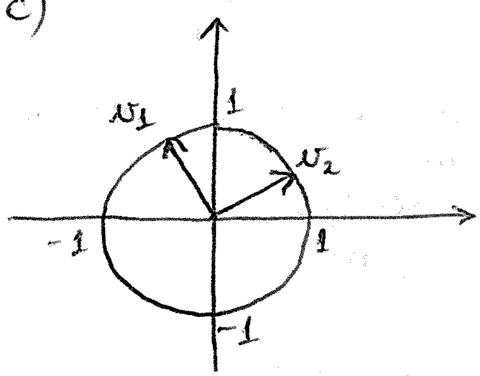
a) $\|A\|_2 = \sigma_1 = 10\sqrt{2}$.

b) $\sigma_1 = 10\sqrt{2}$, $\sigma_2 = 5\sqrt{2}$. ← singular values

left singular vectors: $u_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ and $u_2 = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$

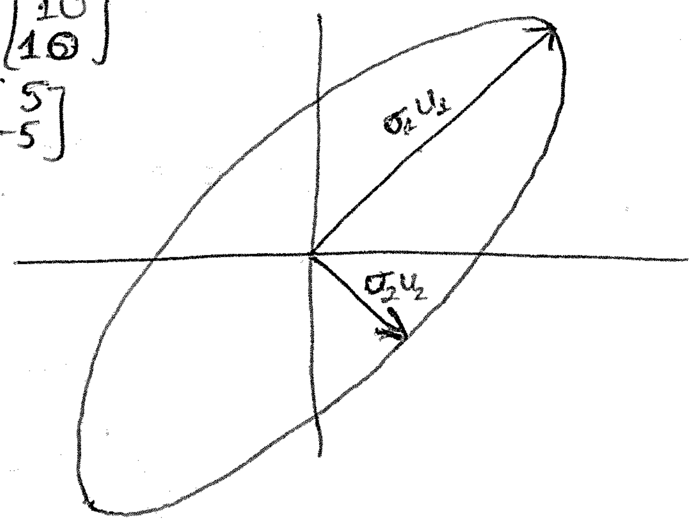
right singular vectors: $v_1 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$.

c)



$\sigma_1 u_1 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

$\sigma_2 u_2 = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$

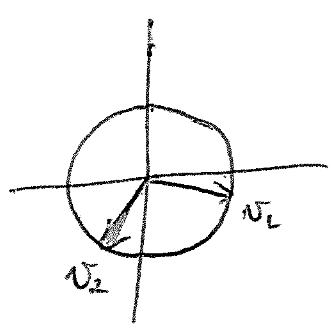


Ver B: a) $\sigma_1 = 10\sqrt{2}$.

b) $\sigma_1 = 10\sqrt{2}$, $\sigma_2 = 5\sqrt{2}$.

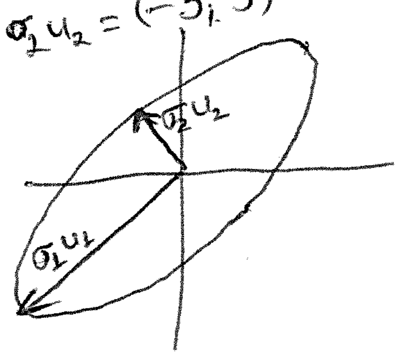
$u_1 = \begin{bmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$ $u_2 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$

$v_1 = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$ $v_2 = \begin{bmatrix} -4/5 \\ -3/5 \end{bmatrix}$.



$\sigma_1 u_1 = (-10, -10)$

$\sigma_2 u_2 = (-5, 5)$



Name:

Problem 3.(6 points.) Let $X_1, \ldots, X_n$ be independent uniform random variables over $[0,1]$ and $Y = \min(X_1, \ldots, X_n)$.

a) (2 points) Show that $\mathbb{P}(Y \geq y) = (1-y)^n$ for $0 \leq y \leq 1$.

b) (4 points) Suppose that $\mathbb{E}[Y] = \frac{1}{n+1}$ and $\mathrm{Var}[Y] \leq \mathbb{E}[Y]^2$. Let $Y_1, \ldots, Y_m$ be independent copies of Y. (That is, these Y_i's have the same distribution as Y and are independent.) Let $Z = \frac{Y_1 + \ldots + Y_m}{m}$. Show that for $\epsilon > 0$,

$$\mathbb{P}\left(\left|Z - \frac{1}{n+1}\right| \geq \frac{\epsilon}{n+1}\right) \leq \frac{1}{m\epsilon^2}.$$

2 pts

a) $\mathbb{P}(Y \geq y) = \mathbb{P}(\{X_1 \geq y\} \cap \{X_2 \geq y\} \cap \ldots \cap \{X_n \geq y\})$

$= \mathbb{P}(X_1 \geq y)\,\mathbb{P}(X_2 \geq y) \cdots \mathbb{P}(X_n \geq y)$ since X_i's are independent.

$= (1-y)(1-y) \cdots (1-y)$

$= (1-y)^n$.

b) Since $Z = \frac{Y_1 + \ldots + Y_m}{m}$

1pt $\quad \mathbb{E}[Z] = \mathbb{E}\left[\frac{Y_1 + \ldots + Y_m}{m}\right] = \frac{\mathbb{E}[Y_1] + \ldots + \mathbb{E}[Y_m]}{m} = \frac{m\mathbb{E}[Y]}{m} = \mathbb{E}[Y]$

1 pt $\quad \mathrm{Var}[Z] = \mathrm{Var}\left[\frac{Y_1 + \ldots + Y_m}{m}\right] = \frac{1}{m^2}\left(\mathrm{Var}[Y_1] + \ldots + \mathrm{Var}[Y_m]\right)$

$= \frac{\mathrm{Var}[Y]}{m}$.

2pts

By Chebyshev's inequality:

$$\mathbb{P}(|Z - \underbrace{\mathbb{E}[Z]}_{= \frac{1}{n+1}}| \geq \varepsilon\,\mathbb{E}[Z]) \leq \frac{\mathrm{Var}[Z]}{\varepsilon^2\,\mathbb{E}[Z]^2}$$

$$= \frac{\mathrm{Var}[Y]}{m\varepsilon^2\,\mathbb{E}[Y]^2}$$

$$\leq \frac{1}{m\varepsilon^2}.$$

Name:

Problem 4.(6 points.) Let $X_1, \ldots, X_{100}$ be independent random variables such that for some $t > 0$,

$$\mathbb{P}(X_i > t) \leq \frac{1}{2^i} \qquad \text{for } i = 1, \ldots, 100.$$

Let $M = \max(X_1, \ldots, X_n)$. Show that $\mathbb{P}(M \leq t) \geq 2^{-100}$.
(Note that $\frac{1}{2} + \ldots + \frac{1}{2^{100}} = 1 - \frac{1}{2^{100}}$.)

Recall $\mathbb{P}(M \leq t) = 1 - \mathbb{P}(M > t)$.

$$\mathbb{P}(M > t) = \mathbb{P}(\{X_1 > t\} \cup \{X_2 > t\} \cup \ldots \cup \{X_{100} > t\})$$

union bound

$$\leq \mathbb{P}(X_1 > t) + \mathbb{P}(X_2 > t) + \ldots + \mathbb{P}(X_{100} > t)$$

$$\leq \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{100}}$$

$$= 1 - \frac{1}{2^{100}}.$$

Hence,

$$\mathbb{P}(M \leq t) = 1 - \mathbb{P}(M > t) \geq 1 - (1 - 2^{-100}) = 2^{-100}.$$

Another approach:

$$\mathbb{P}(M \leq t) = \mathbb{P}(\{X_1 \leq t\} \cap \{X_2 \leq t\} \cap \ldots \cap \{X_{100} \leq t\})$$

$$= \mathbb{P}(X_1 \leq t)\, \mathbb{P}(X_2 \leq t) \ldots \mathbb{P}(X_{100} \leq t)$$

since X_i's are independent.

$$= (1 - \mathbb{P}(X_1 > t))(1 - \mathbb{P}(X_2 > t)) \ldots (1 - \mathbb{P}(X_{100} > t)).$$

$$\geq \underbrace{\left(1 - \frac{1}{2}\right)}_{\geq \frac{1}{2}} \underbrace{\left(1 - \frac{1}{2^2}\right)}_{\geq \frac{1}{2}} \ldots \underbrace{\left(1 - \frac{1}{2^{100}}\right)}_{\geq \frac{1}{2}}$$

$$\geq \left(\frac{1}{2}\right)^{100} = 2^{-100}.$$