MIDTERM 1A, MATH 152, WINTER 2019 TOTAL POINTS: 25

Name: **PID:**

- Print your *NAME* on every page and write your PID in the space provided above.
- Show all of your work in the spaces provided. No credit will be given for unsupported answers, even if correct.
- Supporting work for a problem must be on the page containing that problem. No scratch paper will be accepted.
- No calculators, tables, phones, or other electronic devices are allowed during this exam. You may use your double-sided handwritten notes, but no books or other assistance.

DO NOT TURN PAGE UNTIL INSTRUCTED TO DO SO

(This exam is worth 25 points)

Problem 0.(1 point.) Follows the instructions on this exam and any additional instructions given during the exam.

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$ $\text{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2.$

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Problem 1.(6 points.) Suppose we have a standard six-sided dice, and we roll it 100 times. Let X_i be the number on the face of the dice for roll i. Let X be the sum of the dice rolls, i.e., $X = \sum_{i=1}^{n} X_i$. $n \to 100$

a) (3 points) Find $\mathbb{E}[X_i], \mathbb{E}[X]$, and $\text{Var}[X]$.

b) (3 points) Use Chebyshev's inequality to bound $\mathbb{P}(|X - 350| \geq 60)$.

a) $\mathbb{E}[X_i] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{1}{6} \cdot \frac{6(7)}{2} = 3.5$

$\mathbb{E}[X] = \sum_i \mathbb{E}[X_i] = 100(3.5) = 350.$

$\text{Var}[X_i] = \cancel{1\text{—}3}.\ \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2$

$= (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2)\frac{1}{6} - \left(\frac{7}{2}\right)^2$

$= \frac{6(7)(13)}{6} \cdot \frac{1}{6} - \left(\frac{7}{2}\right)^2.$

$= \frac{91}{6} - \frac{49}{4}.$

$= \frac{182 - 147}{12} = \frac{35}{12}.$

$\text{Var}[X] = 100\left(\frac{35}{12}\right)$

b) $\mathbb{P}(|X - 350| \geq 60) \leq \frac{\text{Var}[X]}{60^2} = \frac{100(35/12)}{60^2}.$

Ver B: $\mathbb{P}(|X - 350| \geq 70) \leq \frac{100(35/12)}{70^2}.$

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Problem 2.(6 points.) Let $X_1, \dots, X_n$ be independent random variables where

$$X_i = \begin{cases} 1, & \text{with probability } 1/3 \\ 0, & \text{with probability } 1/3 \\ -1, & \text{with probability } 1/3. \end{cases}$$

Let $X = \sum_{i=1}^{n} X_i$. Bound $\mathbb{P}(|X - \mathbb{E}[X]| \geq \sqrt{n})$ by using the following inequalities.

a) (3 points) Chebyshev's inequality.

b) (3 points) Chernoff's inequality.

a) $\mathbb{E}[X_i] = 1\left(\frac{1}{3}\right) + 0\left(\frac{1}{3}\right) - 1\left(\frac{1}{3}\right) = 0.$

$\mathbb{E}[X] = 0.$

$\mathbb{E}[X_i^2] = 1^2\left(\frac{1}{3}\right) + 0^2\left(\frac{1}{3}\right) + (-1)^2 \cdot \frac{1}{3} = \frac{2}{3}.$

$\text{Var}[X_i] = \mathbb{E}[X_i^2] - \mathbb{E}[X]^2 = \frac{2}{3}.$

$\text{Var}[X] = \frac{2n}{3}.$

~~Chyb~~ Chebyshev's inequality:

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq \sqrt{n}) \leq \frac{\text{Var}[X]}{n} = \frac{2n/3}{n} = \frac{2}{3}.$$

b) Chernoff's inequality.

$$\mathbb{P}(X \geq \sqrt{n}) \leq \max\left(e^{-n/4\left(\frac{2n}{3}\right)}, e^{-\sqrt{n}/2}\right) = e^{-\sqrt{n}/2}$$

Note: Some students may get different constants. It's ok!

Ver B: $\mathbb{P}(X \geq 2\sqrt{n}) \leq \max\left(e^{-4n/4\left(\frac{2n}{3}\right)}, e^{-\sqrt{n}}\right) = e^{-\sqrt{n}}$

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Problem 3.(6 points.)

a) (2 points) Given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, show that if $A = \vec{u}\vec{v}^T$, then $\|A\|_2 = \|\vec{u}\|_2\|\vec{v}\|_2$.

b) (4 points) Given $\vec{u} = (1, -1, 1)^T$ and let $M = \vec{u}\vec{u}^T$. Find 2-norm and 1-norm of M. (Hint: you can use Part a)

a) $\|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sup_{x \neq 0} \frac{\|uv^Tx\|_2}{\|x\|_2} = \sup_{x \neq 0} \|u\|_2 \frac{|\langle v, x\rangle|}{\|x\|_2}$

By Cauchy-~~Stwart~~ Schwartz inequality:

$|\langle v, x\rangle| \leq \|v\|_2 \|x\|_2$.

$\Rightarrow (*)$ $\|A\|_2 \leq \|u\|_2 \|v\|_2$.

Take $x = \frac{v}{\|v\|_2}$, then since

$\frac{\|Ax\|_2}{\|x\|_2} \leq \|A\|_2$.

$\Rightarrow$ $\|A \frac{v}{\|v\|_2}\|_2 \leq \|A\|_2$

$\frac{\|uv^Tv\|_2}{\|v\|_2} \leq \|A\|_2$.

$(**)$ $\|u\|_2 \|v\|_2 \leq \|A\|_2$.

From $(*)$ and $(**)$, $\|A\|_2 = \|u\|_2 \|v\|_2$.

b) $\|M\|_2 = \|u\|_2^2 = 3$.

$M = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$\|M\|_1 = 3$.

Ver B: $\|M\|_2 = 2$

$M = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$\|M\|_1 = 2$.

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Problem 4.(6 points.) Given an $n \times m$ matrix A with entries in $\{0,1\}$ and a vector $\vec{b} \in \mathbb{R}^m$, i.e.,

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}, \text{ and } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

a) (3 points) Write a formula for $\|A\vec{b}\|_\infty$.

b) (3 points) Suppose that b_j's are independent random variables, where $b_j = -1$ with probability 1/2 and 1 else. Let $Z_1 = \sum_{j=1}^m a_{1j} b_j$. Use Chernoff's bound inequality to bound

$$\mathbb{P}(|Z_1| \geq \sqrt{4m \ln n})$$

a) $\|A\vec{b}\|_\infty = \max_{1 \le i \le n} \left| \sum_{j=1}^m a_{ij} b_j \right|$

b) Let $X_j = a_{1j} b_j$ then $\mathbb{E}[X_j] = 0$ and $|X_j| \leq 1$.

$\Rightarrow \mathbb{E}[Z_1] = 0$

$Var[X_j] = Var[a_{1j} b_j] = a_{1j}^2 \, Var[b_j] = a_{1j}^2 (\mathbb{E}[b_j^2] - \mathbb{E}[b_j]^2)$

$= a_{1j}^2 \cdot 1.$

$= a_{1j}$

since $a_{1j} \in \{0,1\}$, $a_{1j}^2 = a_{1j}$.

$\Rightarrow Var[Z] = \sum_{j=1}^m a_{1j}$.

$\mathbb{P}(Z_1 \geq \sqrt{4m \ln n}) \leq \max\left(e^{-4m \ln n / 4 \sum_{j=1}^m a_{1j}}, e^{-\sqrt{4m \ln n}/2}\right)$.