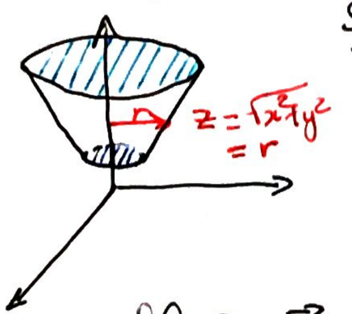
Problem 2: $\vec{F} = \left(\frac{1}{3}x^3 z,\ x+z,\ \frac{1}{2}y^2z^2\right)$

Find $\iint_S \vec{F}\cdot d\vec{S}$ inward. $\quad z = \sqrt{x^2+y^2} \qquad 1 \le z \le 2.$



Sol. If you want to solve it directly, it may be challenging as S has 3 parts: the cone, and two caps.

$\Rightarrow$ Try Gauss' divergence thm:

$$\iint_S \vec{F}\cdot d\vec{S} = \underset{\text{inward pointing}}{-}\iiint_W \nabla\cdot\vec{F}\,dV = -\iiint_W (x^2z + y^2z)\,dV$$

$$= -\iiint_W (x^2+y^2)\,z\,dV$$

Now, how to ~~sho~~ solve the triple integral?

$\Rightarrow$ Try change of variables as we can use polar coordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$$\boxed{\begin{aligned} 0 &\le \theta \le 2\pi \\ 1 &\le z \le 2 \\ 0 &\le r \le z \end{aligned}}$$

Jacobian:

$$\frac{\partial(x,y,z)}{\partial(\theta,r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\sin\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

then,

$$-\iiint_W (x^2+y^2)\,z\,dV = -\int_0^{2\pi}\int_1^{2}\int_0^{z} r^2 z \cdot r\,dr\,dz\,d\theta$$

$$= -2\pi\int_1^2 z\cdot\frac{r^4}{4}\Big|_{r=0}^{z}\,dz = -\frac{2\pi}{4}\int_1^2 z^5\,dz = -\frac{\pi}{2}\,\frac{2^6-1}{6}$$