Midterm 1A

MATH 20E, LECTURE A00, SPRING 2019

NAME:

PID:

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DO NOT TURN PAGE UNTIL INSTRUCTED TO DO SO

(This exam is worth 40 points)

Problem 0.(1 points.) Follows the instructions on this exam and any additional instructions given during the exam.

NAME:

Problem 1.(9 points.) Evaluate the path integral of $f(x, y) = 2x$ along the path $\vec{c}(t) = (t, t^2)$ where $0 \leq t \leq 1$.

$\vec{c}(t) = (t, t^2) \Rightarrow \vec{c}'(t) = (1, 2t)$. and $\|\vec{c}'(t)\| = \sqrt{1^2 + (2t)^2} = \sqrt{1+4t^2}$

Since $x(t) = t$ along the path $\vec{c}(t)$,

$f(x(t), y(t)) = 2t$.

the path integral is

$$\int_{\vec{c}} f\,ds = \int_0^1 2t \|\vec{c}'(t)\|\,dt$$

$$= \int_0^1 2t\sqrt{1+4t^2}\,dt.$$

~~let $u = 2t$, $du = 2dt$.~~

let $u = 1 + 4t^2$.

then $du = 8t\,dt$.

$\frac{du}{8} = t\,dt$.

$$= \int_1^5 2\sqrt{u}\,\frac{du}{8}$$

$$= \frac{1}{4}\int_1^5 \sqrt{u}\,du.$$

$$= \frac{1}{4}\left(\frac{2}{3}u^{3/2}\Big|_1^5\right)$$

$$= \frac{1}{6}\left(5^{3/2} - 1\right).$$

NAME:

Problem 2.(10 points.) Define $g : \mathbb{R}^2 \to \mathbb{R}^3$ by $g(s,t) = (s^2t, s+2t^2, st)$ and $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x,y,z) = e^{2x-y+z}$. Find $D(f \circ g)(1,1)$.

$$(f\circ g)(1,1) = f(g(1,1)) = f(1,3,1) \cancel{\text{to}}$$

$$D(f\circ g)(1,1) = Df(1,3,1)\, Dg(1,1)$$

$\Rightarrow$ we need to find $Df(1,3,1)$ and $Dg(1,1)$.

$$Df(x,y,z) = \left[\frac{\partial(e^{2x-y+z})}{\partial x} \quad \frac{\partial(e^{2x-y+z})}{\partial y} \quad \frac{\partial(e^{2x-y+z})}{\partial z} \right]$$

$$= \left[2e^{2x-y+z} \quad -e^{2x-y+z} \quad e^{2x-y+z} \right].$$

$$\therefore Df(1,3,1) = [2e^0 \quad -e^0 \quad e^0] = [2 \ -1 \ 1]$$

$$Dg(s,t) = \begin{bmatrix} 2st & s^2 \\ 1+\cancel{2t} & 4t \\ t & s \end{bmatrix} \Rightarrow Dg(1,1) = \begin{bmatrix} 2 & 1 \\ 1 & 4 \\ 1 & 1 \end{bmatrix}$$

$$\therefore D(f\circ g)(1,1) = [2 \ -1 \ 1] \begin{bmatrix} 2 & 1 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} = [4-1+1 \quad 2-4+1]$$

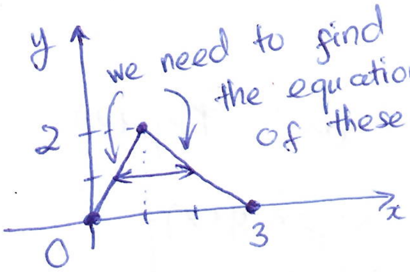
$$= [3 \ -1].$$

NAME:

Problem 3.(10 points.) Evaluate the following integral

$$\int\int_D ye^x \, dA,$$

where D is the triangular region with vertices $(0,0),(1,2)$, and $(3,0)$. (Hint: you may want to use integration by parts.)



we need to find the equations of these two sides.

The first one passes through $(0,0)$ and $(1,2)$

slope $= \frac{2}{1} = 2$.

$y = 2x$. or $x = \frac{y}{2}$

the second one passes through $(1,2)$ and $(3,0)$

slope $= \frac{-2}{2} = -1$.

$y - 0 = -1(x-3)$.

$y = -x+3$. or $x = -y+3$.

$\therefore \; D = \left\{ \begin{array}{l} 0 \le y \le 2 \\ \frac{y}{2} \le x \le -y+3 \end{array} \right\}$.

$$\int_0^2 \int_{y/2}^{-y+3} ye^x \, dx\,dy = \int_0^2 ye^x \Big|_{x=\frac{y}{2}}^{-y+3} dy = \int_0^2 y(e^{-y+3} - ye^{y/2})\,dy.$$

$u = y$ and $dv = (e^{-y+3} - e^{y/2})dy$ integration by parts

$du = dy$ $\quad v = -e^{-y+3} - 2e^{y/2}$

$$= y(-e^{-y+3} - 2e^{y/2})\Big|_{y=0}^{2} - \int_0^2 (-e^{-y+3} - 2e^{y/2})\,dy.$$

$$= -2(e^1 + 2e^1) + (-e^{-y+3})\Big|_{y=0}^{2} + 4e^{y/2}\Big|_{y=0}^{2}$$

$$= -6e + (-e + e^3) + 4e - 4.$$

$$= -3e + e^3 - 4.$$

NAME:

Problem 4.(10 points.) Let $D^* = \{(u,v) : 0 \le u \le 2\pi, 0 \le v \le 1\}$ and let D be the image of D^* under the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(u,v) = (2v\cos(u), v\sin(u))$.

a) (4 points) Describe D.

b) (6 points) Find the area of D.

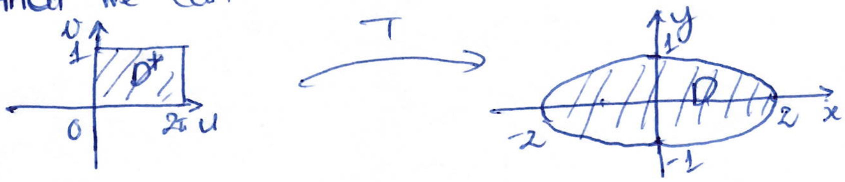
a) D is the set of all points $(x,y) \in \mathbb{R}^2$ which satisfies $x(u,v) = 2v\cos(u)$ and $y(u,v) = v\sin(u)$.

$$\Rightarrow \frac{x^2}{4} + y^2 = \frac{4v^2\cos^2(u)}{4} + v^2\sin^2(u) = v^2\cos^2(u) + v^2\sin^2(u) = v^2.$$

$$\Rightarrow \frac{x^2}{4} + \frac{y^2}{1} = v^2 \le 1.$$

$\therefore$ D is a region bounded (and including) by the eclipse $\frac{x^2}{4} + y^2 \le 1$.

Note that we can sketch D as



b) $\text{Area}(D) = \iint_D dA = \iint_{D^*} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

$x(u,v) = 2v\cos(u)$ $\quad \frac{\partial x}{\partial u} = -2v\sin(u)$ $\quad \frac{\partial x}{\partial v} = 2\cos(u)$

$y(u,v) = v\sin(u)$ $\quad \frac{\partial y}{\partial u} = v\cos(u)$ $\quad \frac{\partial y}{\partial v} = \sin u.$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|-2v\sin^2(u) - 2v\cos^2(u)\right| = |2v| = 2v.$$

$$\therefore \text{Area}(D) = \int_0^1\int_0^{2\pi} 2v\,du\,dv = 2\pi\int_0^1 2v\,dv = 2\pi\, v^2\Big|_0^1 = 2\pi.$$

Midterm 1B

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Problem 1.(9 points.) Evaluate the path integral of $f(x,y,z) = e^{\sqrt{z}}$ along the path $\vec{c}(t) = (1,2,t^2)$ where $0 \le t \le 1$.

$\vec{c}'(t) = (0,0,2t) \Rightarrow \|\vec{c}'(t)\| = \sqrt{(2t)^2} = 2t.$ since $t \geq 0.$

Along path $\vec{c}$, $z(t) = t^2$.

$$\int_{\vec{c}} f\,ds = \int_0^1 e^{\sqrt{t^2}}\, 2t\,dt.$$

$$= 2\int_0^1 te^t\,dt$$

$u = t$ $\quad dv = e^t dt$

$du = dt$ $\quad v = e^t$

$$= 2\left[te^t\Big|_0^1 - \int_0^1 e^t\,dt\right].$$

$$= 2\left[e^1 - e^t\Big|_0^1\right].$$

$$= 2(e^1 - e^1 + 1).$$

$$= 2.$$

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Problem 2.(10 points.) Define $g : \mathbb{R}^2 \to \mathbb{R}^3$ by $g(s,t) = (s^2t, s+2t^2, st)$ and $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x,y,z) = xyz$. Find $D(f \circ g)(1,1)$.

$g(1,1) = (1,3,1)$.

$Df(x,y,z) = [yz \quad xz \quad xy]$

$\Rightarrow Df(1,3,1) = [3 \quad 1 \quad 3]$.

$Dg(1,1) = \begin{bmatrix} 2 & 1 \\ 1 & 4 \\ 1 & 1 \end{bmatrix}$ (see version A).

$$D(f\circ g)(1,1) = Df(1,3,1)\, Dg(1,1)$$
$$= [3 \quad 1 \quad 3]\begin{bmatrix} 2 & 1 \\ 1 & 4 \\ 1 & 1 \end{bmatrix}$$
$$= [6+1+3 \quad 3+4+3]$$
$$= [10 \quad 10].$$

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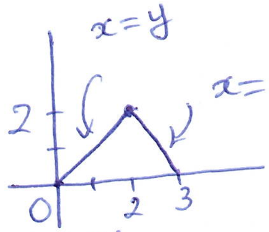
where D is the triangular region with vertices $(0,0)$, $(2,2)$, and $(3,0)$. (Hint: you may want to use integration by parts.)



$$D = \left\{ \begin{array}{l} 0 \le y \le 2 \\ \cancel{3 - \frac{y}{2} \le x \le y} \\ y \le x \le 3 - \frac{y}{2} \end{array} \right\}.$$

$$\int_0^2 \int_{\cancel{3-\frac{y}{2}}\ y}^{3-y/2} ye^x \, dx \, dy = \int_0^2 y e^x \Big|_{x=\cancel{3-\frac{y}{2}}\ y}^{\cancel{y}\ 3-\frac{y}{2}} dy$$

$$= \int_0^2 y\left(-e^y + e^{3-\frac{y}{2}}\right) dy. \qquad u = y \quad du = dy \qquad dv = \left(-e^y + e^{3-\frac{y}{2}}\right)dy. \quad v = \left(-e^y \mp 2e^{3-\frac{y}{2}}\right)$$

$$= y\left(-e^y \mp 2e^{3-\frac{y}{2}}\right)\Big|_0^2 + \int_0^2 \left(e^y + 2e^{3-\frac{y}{2}}\right) dy.$$

$$= -2(e^2 + 2e^2) + \left[e^y - 4e^{3-\frac{y}{2}}\right]_{y=0}^2$$

$$= -6e^2 + \left(e^2 - 4e^2 - 1 + 4e^3\right).$$

$$= -6e^2 + \left(-3e^2 - 1 + 4e^3\right)$$

$$= -9e^2 - 1 + 4e^3.$$

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a) (4 points) Describe D.

b) (6 points) Find the area of D.

a) $x^2 + \frac{y^2}{4} = v^2 \leq 1.$

b) $\text{Area}(D) = \int_0^1 \int_0^{2\pi} 2v \, du \, dv = 2\pi.$