MIDTERM 1A, MATH 152, SPRING 2019 TOTAL POINTS: 25

Name: Key **PID:**

- Print your *NAME* on every page and write your PID in the space provided above.
- Show all of your work in the spaces provided. No credit will be given for unsupported answers, even if correct.
- Supporting work for a problem must be on the page containing that problem. No scratch paper will be accepted.
- No calculators, tables, phones, or other electronic devices are allowed during this exam. You may use your double-sided handwritten notes, but no books or other assistance.

DO NOT TURN PAGE UNTIL INSTRUCTED TO DO SO

(This exam is worth 25 points)

Problem 0.(1 point.) Follows the instructions on this exam and any additional instructions given during the exam.

Name:

Problem 1.(6 points.)

a) (3 points) Given $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{bmatrix}$. Find $\|A\|_\infty$.

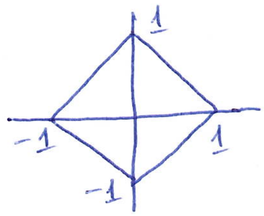
b) (3 points) Sketch the unit circle $\{\vec{x} \in \mathbb{R}^2, \|\vec{x}\|_p = 1\}$ for $p = 1$ and $p = 2$.

a) $\|A\|_\infty = \max\{|1| + |2| + |-1|, |0| + |3| + |-1|, |5| + |-1| + |1|\}$.

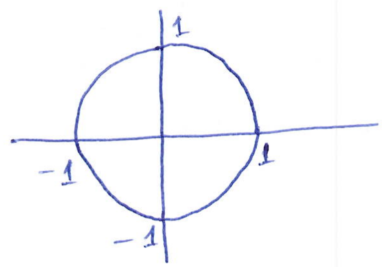
$= \max\{4, 4, 7\}$.

$= 7$.

b) For $p = 1$:

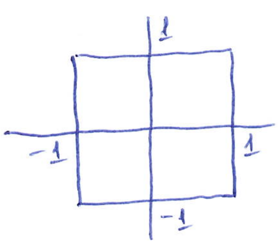


For $p = 2$:

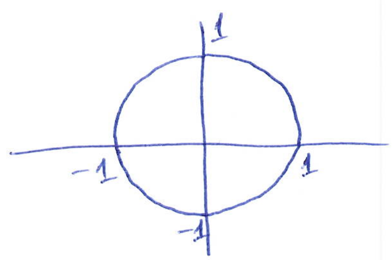


Ver B: a) $\|A\|_1 = \max\{6, 6, 3\} = 6$.

b) $p = \infty$:



$p = 2$



Name:

Problem 2.(6 points.) Let u_1, v_1, u_2, v_2 be vectors in $\mathbb{R}^n$ and $A = u_1 v_1^T + u_2 v_2^T$.

a) (3 points) Suppose that $w \in \mathbb{R}^n$ is orthogonal to v_1 and v_2. Show that w is in the null space of A.

b) (3 points) Suppose that $\|u_1\|_2 = \|v_1\|_2 = 1$ and $\langle u_1, u_2 \rangle = 0$. Show that $\|A^T u_1\|_2 = 1$.

a) We need to show that $Aw = 0$.

$$\begin{aligned} Aw &= (u_1 v_1^T + u_2 v_2^T) w \\ &= u_1 v_1^T w + u_2 v_2^T w \\ &= u_1 \langle v_1, w \rangle + u_2 \langle v_2, w \rangle . \\ &= 0 + 0 \quad \text{since } w \text{ is orthogonal to } v_1 \text{ and } v_2. \\ &= 0. \end{aligned}$$

$\therefore w \in \text{Nul}(A)$.

b)
$$\begin{aligned} A^T u_1 &= (u_1 v_1^T + u_2^T v_2^T)^T u_1 . \\ &= (v_1 u_1^T + v_2 u_2^T) u_1 \\ &= v_1 u_1^T u_1 + v_2 u_2^T u_1 \\ &= v_1 \|u_1\|_2^2 + v_2 \langle u_2, u_1 \rangle . \\ &= v_1 + 0 \quad \text{since } \|u_1\|_2 = 1 \text{ and } \langle u_1, u_2 \rangle = 0 \\ &= v_1 \end{aligned}$$

$\therefore \|A^T u_1\|_2 = \|v_1\|_2 = 1.$

Ver B: Same as version A.

Name:

Problem 3.(6 points.) Let $X_1, \ldots, X_n$ be independent $\{-1, 1\}$-valued random variables. Each X_i takes the value 1 with probability 1/2 and else -1. Let $S = \sum_{i=1}^n X_i$.

a) (3 points) Show that with probability at least 75%, $|S| < 2\sqrt{n}$.

b) (3 points) Suppose we know that for any $a > 0, \mathbb{P}(|S| \geq a) \leq 2e^{-\frac{a^2}{2n}}$. What is the lower bound for $\mathbb{P}(|S| < 3\sqrt{n})$?

a) $S = \sum_{i=1}^{n} X_i$

$E[S] = \sum_{i=1}^{n} E[X_i] = 0$ and $Var[S] = \sum_{i=1}^{n} Var[X_i]$

$= \sum_{i=1}^{n} \left(E[X_i^2] - E[X_i]^2\right).$

$= \sum_{i=1}^{n} 1$

$\therefore \; Var[S] = n$

$P(|S| \leq 2\sqrt{n}) = 1 - P(|S| \geq 2\sqrt{n}).$

The easiest way is to use Chebyshev's bound:

$P(|S| \geq 2\sqrt{n}) \leq \frac{Var[S]}{(2\sqrt{n})^2} = \frac{n}{4n} = \frac{1}{4}.$

$\therefore \; P(|S| < 2\sqrt{n}) \geq 1 - \cancel{P}\,\frac{1}{4} = \frac{3}{4} = 75\%.$

b) $P(|S| \geq a) \leq 2e^{-a^2/2n}$

Take $a = 3\sqrt{n}$, then $P(|S| \geq 3\sqrt{n}) \leq 2e^{-9n/2n} = 2e^{-9/2}.$

$\therefore \; P(|S| < 3\sqrt{n}) \geq 1 - 2e^{-9/2}.$

Ver B. a) $P(|S| \geq \sqrt{10n}) \leq \frac{n}{10n} = \frac{1}{10}.$

$\therefore \; P(|S| < \sqrt{10n}) \geq 1 - \frac{1}{10} = 90\%.$

b) $P(|S| \geq 2\sqrt{n}) \leq 2e^{-4n/2n} = 2e^{-2}$

$P(|S| < 2\sqrt{n}) \geq 1 - 2e^{-2}.$

Name:

Problem 4.(6 points.) Consider a biased coin with probability $p = 1/3$ of landing heads and probability $2/3$ of landing tails. Suppose the coin is flipped some number n of times, and let X_i be a random variable denoting the ith flip, where $X_i = 1$ means heads, and $X_i = 0$ means tails.

a) (4 points) Let $S = X_1 + \ldots + X_n$ be the number of coin flips which come out heads. Use the Chernoff inequality to bound $\mathbb{P}(S \geq \frac{n}{2})$.

b) (2 points) Determine a value for n so that the probability that (equal to or) more than half of the coin flips come out heads is less than 0.01.

a) $\mathbb{E}[S] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \left(1\cdot\frac{1}{3} + 0\cdot\frac{2}{3}\right) = \sum_{i=1}^{n} \frac{1}{3} = \frac{n}{3}.$

$\mathbb{P}\left(S \geq \frac{n}{2}\right) = \mathbb{P}\left(S \geq \left(1+\frac{1}{2}\right)\frac{n}{3}\right) \leq e^{-\frac{n}{3}\left(\frac{1}{2}\right)^2/4} = e^{-n/48}.$

b) Use part a):

$\mathbb{P}\left(S \geq \frac{n}{2}\right) \leq e^{-n/48} \leq 0.01.$

$\Rightarrow$ we need to find n such that

$e^{-n/48} \leq 0.01.$

$\ln e^{-n/48} \leq \ln 0.01.$

$-\frac{n}{48} \leq \ln 0.01.$

$n \geq -\cancel{48}\frac{\ln 0.01}{\cancel{48}} = \frac{96 \ln 10}{\cancel{48}} = \cancel{\frac{\ln 10}{24}}$

Ver B: a) Same as Ver A.

b) $e^{-n/48} \leq 0.02.$

$n \geq -48\frac{\ln 0.02}{\cancel{48}} = \frac{48 \ln 50}{\cancel{48}}.$