7.5 Integrals of Scalar functions over surfaces.

Eg. The mass of a thin ~~set~~ sheet of metal $S = \Phi(D)$ where $D \subset \mathbb{R}^2$ and $\Phi: \mathbb{R}^2 \to \mathbb{R}^3$ where the density of the metal is given by $f(x,y,z)$.

Def: The integral of a scalar function over a surface

$$\iint_S f(x,y,z)\, dS = \iint_D f(\Phi(u,v)) \|T_u \times T_v\|\, du\, dv.$$

$$= \iint_D f(x(u,v), y(u,v), z(u,v)) \sqrt{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(u,v)}\right)^2}\, du\, dv$$

E.g. $f(x,y,z) = \sqrt{x^2 + y^2 + 1}$ (helicoid Example 2 Section 7.4.)

S: $(r\cos\theta, r\sin\theta, \theta)$

D: $0 \le \theta \le 2\pi$, $0 \le r \le 1$

Compute $\iint_S f\, dS$.

$$\iint_S f\, dS = \iint_D f(r\cos\theta, r\sin\theta, \theta) \|T_r \times T_\theta\|\, dr\, d\theta$$

$$= \iint_D \sqrt{r^2+1}\, \|T_r \times T_\theta\|\, dr\, d\theta.$$

$T_r = (\cos\theta, \sin\theta, 0)$

$T_\theta = (-r\sin\theta, r\cos\theta, 1)$

$T_r \times T_\theta = \sin\theta\, \vec{i} - \cos\theta\, \vec{j} + r\vec{k}.$

$\|T_r \times T_\theta\| = \sqrt{r^2+1}$.

$$\Rightarrow \iint_S f\, dS = \int_0^{2\pi}\int_0^1 \sqrt{r^2+1} \cdot \sqrt{r^2+1}\, dr\, d\theta.$$

$$= 2\pi \left(r + \frac{r^3}{3}\right)\Big|_0^1 = \frac{8\pi}{3}.$$

* Surface integrals over graphs of functions:

Suppose S is the graph of a differentiable function $z = g(x,y)$

$\Rightarrow$ we can parametrize it as $(u, v, g(u,v))$

Then $\|T_u \times T_v\| = \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2}$

So $$\iint_S f(x,y,z)\,dS = \iint_D f(u, v, g(u,v))\sqrt{1+\left(\frac{\partial g}{\partial u}\right)^2+\left(\frac{\partial g}{\partial v}\right)^2}\,du\,dv$$

E.g. Compute $\displaystyle\iint_S \frac{x}{\sqrt{4x^2+4y^2+1}}\,dS$

where S is the hyperbolic paraboloid $z = y^2 - x^2$ over the region $-1 \le y \le 1$, $-1 \le x \le 1$.

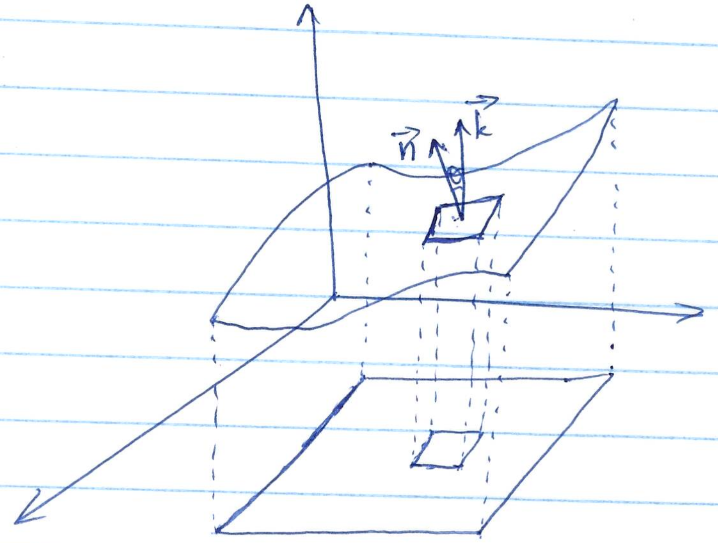
we parametrize $(x, y, y^2 - x^2)$

$$\iint_S f\,dS = \iint_D \frac{x}{\sqrt{4x^2+4y^2+1}}\sqrt{1+(-2x)^2+(2y)^2}\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy = 0.$$

* Integrals over graphs:

$S: (x, y, g(x,y))$ $\qquad$ D: simple region.



here $\vec{n} = \dfrac{\vec{N}}{\|\vec{N}\|}$ is the unit normal and

$\vec{N} = -\dfrac{\partial g}{\partial x}\vec{i} - \dfrac{\partial g}{\partial y}\vec{j} + \vec{k}$ is normal to the surface.

Since $\vec{N}\cdot\vec{k} = \|\vec{N}\|\,\|\vec{k}\| \cos\theta = \|\vec{N}\| \cos\theta$.

then $\cos\theta = \dfrac{\vec{N}}{\|\vec{N}\|}\cdot\vec{k} = \dfrac{\left(-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}\right)\cdot\vec{k}}{\|\vec{N}\|}$

$$= \frac{1}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}$$

So $\displaystyle\iint_S f\, dS = \iint_D f(x,y,g(x,y))\underbrace{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}\, dx\,dy$

$\displaystyle = \iint_D f(x,y,g(x,y))\,\frac{1}{\cos\theta}\,dx\,dy.$

θ depends on x & y.

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~~The total~~

E.g. Compute the mass of the helicoid $S: (r\cos\theta, r\sin\theta, \theta)$ where $0 \le r \le 1$, $0 \le \theta \le 2\pi$ if its mass density is $m(x,y,z) = \sqrt{x^2+y^2}$.

The total mass of ~~a~~ a surface with mass density m is given by

$$M(S) = \iint_S m(x,y,z)\, dS$$

$$= \iint_D \sqrt{r^2}\, \|T_r \times T_\theta\|\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\sqrt{r^2+1}\, dr\, d\theta$$

$u = r^2+1$

$du = 2r\,dr$

$$= 2\pi \cdot \frac{1}{3}(r^2+1)^{3/2}\Big|_0^1$$

$$= 2\pi \cdot \frac{1}{3}\left(2^{3/2}-1\right)$$

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Surface integrals of vector fields.

Def: The surface integral of $\vec{F}$ over Φ:

$$\iint_{\Phi} \vec{F}\cdot d\vec{S} := \iint_{D} \vec{F}\cdot(\vec{T}_u \times \vec{T}_v)\, du\,dv.$$

Example. $\vec{F}(x,y,z) = (x,y,z)$

S: sphere of radius 1.

Find $\iint_S \vec{F}\cdot d\vec{S}$.

Sol. Parametrize S by using spherical coordinates.

$(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$.

Then $\vec{T}_\theta = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$

$\vec{T}_\phi = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$.

$$\vec{T}_\theta \times \vec{T}_\phi = (-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi)$$

$$= (-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi).$$

On the sphere, $\vec{F} = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$.

$$\Rightarrow \vec{F}\cdot(\vec{T}_\theta \times \vec{T}_\phi) = -\cos^2\theta\sin^3\phi - \sin^2\theta\sin^3\phi - \sin\phi\cos^2\phi.$$

$$= -\sin^3\phi - \sin\phi\cos^2\phi$$

$$= -\sin\phi.$$

$$\Rightarrow \iint \vec{F}\cdot d\vec{S} = \iint \vec{F}\cdot(T_\theta \times T_\phi)\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} -\sin\phi\, d\theta\, d\phi$$

$$= -2\pi\int_0^{\pi}\sin\phi\, d\phi = +2\pi\cos\phi\Big|_0^{\pi} = -4\pi.$$

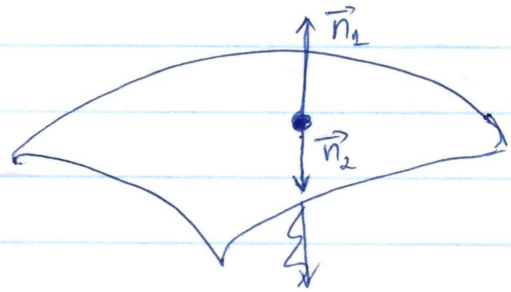
E.g. The volume of water per unit of time flowing "through a surface" S with velocity given by the field $\vec{F} = \iint_{\Phi} \vec{F} \cdot d\vec{S}$

~~Interpret~~

Remark: We implicitly chose an orientation for the surface when we used $\vec{T}_\theta \times \vec{T}_\phi$ instead of $\vec{T}_\phi \times \vec{T}_\theta$.

$\Rightarrow$ orientation?

An oriented surface is one where at each $(x, y, z) \in S$ there are 2 unit normal vectors $\vec{n}_1$ and $\vec{n}_2$ with $\vec{n}_1 = -\vec{n}_2$. and each can be associated with a side of the surface.



Not every surface is orientable: e.g. Mobius strip is not.

E.g. The unit sphere can be given an ~~oriented~~ orientation by selecting $\vec{n} = \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{\cancel{\|(x, y, z)\|}}$.

This is an orientation preserving. (i.e. points outwards.)

But in the previous example with the sphere, our parametrization gave $\vec{T}_\theta \times \vec{T}_\phi = -\vec{n} \sin\phi$ $\quad (0 \le \phi \le \pi \Rightarrow \sin\phi \ge 0)$.

$\Rightarrow$ this parametrization was a orientation reversing.

<u>Thm</u>: Surface integrals are independent of parametrization, provided they are orientation preserving.

$$\boxed{\underline{\text{Thm}}: \iint_S \vec{F}\cdot \underbrace{d\vec{S}}_{(\vec{T}_u\times\vec{T}_v)\,du\,dv} = \iint_S \vec{F}\cdot \underbrace{\vec{n}}_{\frac{\vec{T}_u\times\vec{T}_v}{\|T_u\times T_v\|}}\ \underbrace{dS}_{\|\vec{T}_u\times T_v\|\,du\,dv}.}$$

<u>E.g.</u> Heat flow

If $T(x,y,z)$ is the temperature at (x,y,z),

Then $\nabla T = \frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k}$ is the temperature gradient; $\vec{F} = -k\nabla T$ is a vector field associated with heat flow and $\iint \vec{F}\cdot d\vec{S}$ is the flux (or total rate of heat flow) accross S.

Suppose $T(x,y,z) = x^2+y^2+z^2$. Find the heat flux accross the unit sphere oriented with the outward normal. (use $k=1$).

$\vec{n} = x\vec{i} + y\vec{j} + z\vec{k}$. outward normal.

$\vec{F} = -\nabla T = -2x\vec{i} - 2y\vec{j} - 2z\vec{k}$.

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS$$

$$= -\iint_S (2x\vec{i} + 2y\vec{j} + 2z\vec{k})\cdot(x\vec{i} + y\vec{j} + z\vec{k})\, dS$$

$$= -2\iint_S (x^2+y^2+z^2)\, dS$$

$$= -2\,A(S)$$

$$= -2(4\pi)$$

$$= -8\pi.$$

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* Surface integrals over graphs:

Suppose S is the graph of a function, so

$S: (x, y, g(x,y))$

Additionally, suppose S is oriented with upward pointing normal ~~(i.e. $\vec{k}$ component)~~

$$\vec{n} = \frac{-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}} \quad \text{since } T_x \times T_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial g}{\partial x} \\ 0 & 1 & \frac{\partial g}{\partial y} \end{vmatrix}$$

Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right) \cdot \left(-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}\right) dx\,dy$$

$$\Rightarrow \boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-F_1\frac{\partial g}{\partial x} + -F_2\frac{\partial g}{\partial y} + F_3\right) dx\,dy.}$$

E.g. $z = x^2 + y^2$, $x^2 + y^2 \leq 4$

Suppose $\vec{F} = -y\vec{i} + x\vec{j} + \vec{k}$.

Compute $\iint_S \vec{F} \cdot d\vec{S}$.

S is the graph of a function.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(+y \cdot 2x - x\,2y + 1\right) dx\,dy.$$

$$= \iint_D dx\,dy.$$

$$= 4\pi.$$