Midterm 1A

MATH 18, LECTURE C00, WINTER 2019

NAME: PID:

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- Supporting work for a problem must be on the page containing that problem. No scratch paper will be accepted.
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DO NOT TURN PAGE UNTIL INSTRUCTED TO DO SO

(This exam is worth 40 points)

Problem 0.(1 points.) Follows the instructions on this exam and any additional instructions given during the exam.

NAME:

Problem 1.(10 points.) Let $A = \begin{bmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{bmatrix}$.

a) (5 points) Find the reduced row echelon form of A.

b) (5 points) Describe the solution set of the homogeneous equation $A\vec{x} = \vec{0}$.

a) $A \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 5 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{bmatrix}$

b) $[A | \vec{0}] \rightarrow \left[\begin{array}{ccc|c} 2 & -2 & 4 & 0 \\ 0 & 5 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow[R_2/5]{R_1/2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

x_3 is free.

Let $x_3 = s$. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2s \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad \text{for } s \in \mathbb{R}.$$

NAME:

Problem 2.(10 points.) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$.

a) (5 points) Is it possible to solve $A\vec{x} = \vec{b}$ for *any given* vector $\vec{b}$? Explain your answer.

b) (5 points) Describe the set of all vectors $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ for which $A\vec{x} = \vec{b}$ is consistent.

a) $A \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

It's not possible since the last row does not have a pivot.

b) Consider $[A \mid \vec{b}]$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 1 & 3 & 0 & b_2 \\ 1 & 1 & 2 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 1 & -1 & b_2 - b_1 \\ 0 & -1 & 1 & b_3 - b_1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 1 & -1 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{array}\right]$$

The system is consistent if and only if

$$b_3 + b_2 - 2b_1 = 0$$

NAME:

Problem 3.(10 points.) Given $\vec{v}_1 = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 5 \\ -9 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ h \\ h \end{bmatrix}$.

a) (5 points) Find all values h such that the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span $\mathbb{R}^3$.

b) (5 points) Find all values h such that the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ *do not span* $\mathbb{R}^3$.

$$\begin{bmatrix} 3 & 1 & 1 \\ 3 & 5 & h \\ -3 & -9 & h \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 3 & 1 & 1 \\ 0 & 4 & h-1 \\ 0 & -8 & h+1. \end{array}\right]$$

$$\rightarrow \begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & h-1 \\ 0 & 0 & 3h-1 \end{bmatrix}.$$

a) $3h - 1 \neq 0 \quad \Rightarrow \quad h \neq \frac{1}{3}.$

b) $h = \frac{1}{3}.$

NAME:

Problem 4.(10 points.) Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & -1 & 3 \end{bmatrix}$. Define the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$ for $\vec{x} \in \mathbb{R}^3$.

a) (7 points) Is $\vec{b} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ in the range of T?

b) (3 points) Is T one-to-one?

a) $\vec{b}$ is in range(T) if $A\vec{x} = \vec{b}$ is consistent.

Consider

$$[A \mid \vec{b}\,] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 2 & 1 & 3 & 2 \\ 1 & -1 & 3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & -1 & 1 & -1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -3 \end{array}\right].$$

Since the last column is pivotal, the system is inconsistent $\therefore$ $\vec{b} \notin$ range(T).

b) From part a), the third column of A is not pivotal. Hence, T is <u>not</u> one-to-one.

Midterm 1B

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Problem 0.(1 points.) Follows the instructions on this exam and any additional instructions given during the exam.

NAME:

Problem 1.(10 points.) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & -2 \\ -3 & 5 & -1 \end{bmatrix}$.

a) (5 points) Find the reduced row echelon form of A.

b) (5 points) Describe the solution set of the homogeneous equation $A\vec{x} = \vec{0}$.

a) $A \xrightarrow[R_3+3R_1]{R_2+R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{R_3+2R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

b) $[A \,\vdots\, \vec{0}] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

x_3 is free

Let $x_3 = s$. Then the solution set includes any vector

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s \\ -s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ for $s \in \mathbb{R}$.

NAME:

Problem 2.(10 points.) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & -2 \\ 1 & 3 & 0 \end{bmatrix}$.

a) (5 points) Is it possible to solve $A\vec{x} = \vec{b}$ for *any given* vector $\vec{b}$? Explain your answer.

b) (5 points) Describe the set of all vectors $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ for which $A\vec{x} = \vec{b}$ is consistent.

a) $A \xrightarrow[R_3 - R_1]{R_2 + R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

$\Rightarrow$ it's not possible since the last row does not have a pivot.

b) $[A \vdots \vec{b}] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 1 & -1 & b_2 + b_1 \\ 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right]$

The system is consistent iff $b_3 - b_2 - 2b_1 = 0$.

NAME:

Problem 3.(10 points.) Given $\vec{v}_1 = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 5 \\ -9 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ h \\ h \end{bmatrix}$.

a) (5 points) Find all values h such that the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span $\mathbb{R}^3$.

b) (5 points) Find all values h such that the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ *do not span* $\mathbb{R}^3$.

$$\begin{bmatrix} 3 & 1 & 0 \\ 3 & 5 & h \\ -3 & -9 & h \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 1 & 0 \\ 0 & 4 & h \\ 0 & -8 & h \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 3 & 1 & 0 \\ 0 & 4 & h \\ 0 & 0 & 3h \end{bmatrix}.$$

a) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span $\mathbb{R}^3$ if the last row has a pivot, i.e. $3h \neq 0$ or $h \neq 0$.

b) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ do not span $\mathbb{R}^3$ if $h = 0$.

NAME:

Problem 4.(10 points.) Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & -1 & 3 \end{bmatrix}$. Define the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$ for $\vec{x} \in \mathbb{R}^3$.

a) (7 points) Is $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ in the range of T?

b) (3 points) Is T one-to-one?

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & -1 & 3 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -1 & 1 & -3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -6 \end{array}\right].$$

a) No!

b) No!