HW05

Problem 1. Let's start with 2x2 matrices.

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix} = \begin{bmatrix}1\cdot4+2\cdot2 & 1\cdot3+2\cdot1\\ 3\cdot4+4\cdot2 & 3\cdot3+4\cdot1\end{bmatrix}$$

$$= \begin{bmatrix}1\cdot4 & 1\cdot3\\ 3\cdot4 & 3\cdot3\end{bmatrix} + \begin{bmatrix}2\cdot2 & 2\cdot1\\ 4\cdot2 & 4\cdot1\end{bmatrix}.$$

$$= \begin{bmatrix}1\\ 3\end{bmatrix}\begin{bmatrix}4 & 3\end{bmatrix} + \begin{bmatrix}2\\ 4\end{bmatrix}\begin{bmatrix}2 & 1\end{bmatrix}.$$

For general matrices,

$$AB = \begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}\begin{bmatrix}b_{11} & b_{12} & \cdots & b_{1p}\\ b_{21} & b_{22} & \cdots & b_{2p}\\ \vdots & \vdots & & \vdots\\ b_{n1} & b_{n2} & \cdots & b_{np}\end{bmatrix}$$

$$= \begin{bmatrix}a_{11}b_{11}+a_{12}\cdot b_{21}+\cdots+a_{1n}b_{n1} & \cdots & a_{11}b_{11}+\cdots+a_{1n}b_{np}\\ a_{21}b_{11}+a_{22}\cdot b_{21}+\cdots+a_{2n}b_{n2} & \cdots & a_{21}b_{1p}+\cdots+a_{2n}b_{np}\\ \vdots & & \vdots\\ a_{m1}b_{11}+a_{m2}b_{21}+\cdots+a_{mn}b_{n1} & \cdots & a_{m1}b_{1p}+\cdots+a_{mn}b_{np}\end{bmatrix}$$

$$= \begin{bmatrix}a_{11}\,b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1p}\\ a_{21}\,b_{11} & a_{21}b_{12} & \cdots & a_{21}b_{1p}\\ \vdots & \vdots & & \vdots\\ a_{m1}\,b_{11} & a_{m1}b_{12} & \cdots & a_{m1}b_{1p}\end{bmatrix} + \cdots + \begin{bmatrix}a_{1n}b_{n1} & a_{1n}b_{n2} & \cdots & a_{1n}b_{np}\\ a_{2n}b_{n1} & a_{2n}b_{n2} & \cdots & a_{2n}b_{np}\\ \vdots & \vdots & & \vdots\\ a_{mn}b_{n1} & a_{mn}b_{n2} & \cdots & a_{mn}b_{np}\end{bmatrix}$$

$$= A(:,1)B(1,:) + \cdots + A(:,n)B(n,:)$$

Problem 2.

$$A_{2\times3}B_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}*b_{11} + a_{12}*b_{21} + a_{13}*b_{31} & a_{11}*b_{12}+a_{12}*b_{22}+a_{13}*b_{32} & a_{11}*b_{13}+a_{12}*b_{23}+a_{13}*b_{33} \\ a_{21}*b_{11} + a_{22}*b_{21} + a_{23}*b_{31} & a_{21}*b_{12}+a_{22}*b_{22}+a_{23}*b_{32} & a_{21}*b_{13}+a_{22}*b_{23}+a_{23}*b_{33} \end{bmatrix}$$

There are 18 (*) operations and 12 (+) operations. $\Big\}$ $\Rightarrow$ total operations $= 30$.

Note that, in class we learned that the number of operations we need to multiply two matrices $A_{m\times n} B_{n\times p}$ is $O(mnp)$. This is only when m, n, and p are very large.

Problem 3.

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 5 \\ 0 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 11 \\ 1 & -7 & -10 \end{bmatrix}$$

a) $\mathbb{E}[X] = \frac{1}{3}\cdot 3\,A(:,1)B(1,:) + \frac{1}{3}\cdot 3A(:,2)B(2,:) + \frac{1}{3}\cdot 3A(:,3)B(3,:)$

$= A(:,1)B(1,:) + A(:,2)B(2,:) + A(:,3)B(3,:)$

$= AB.$

$\mathrm{Var}[X] = \sum_{k=1}^{3} 3\,\|A(:,k)\|_2^2\,\|B(k,:)\|_2^2 - \|AB\|_F^2$

$= 3(2)(30) + 3(4)(26) + 3(5)(110) - 308.$

$= 1834$

b) $\|A\|_F^2\,\|B\|_F^2 = (1+1+1+2^2+2^2)(1+5^2+2^2+1+5^2+1^2+9^2)$

$= \cancel{7}\,(11)(138) = 1518.$

b) In class, we take $P_k = \dfrac{\|A(:,k)\|^2}{\|A\|_F^2}$ (length squared sampling technique).

then we get $Var[X] \leq \|A\|_F^2 \|B\|_F^2$.

But ~~h~~ in this exercise, $P_k = \frac{1}{3}$.

and we obtain

$Var[X]$ ~~≥~~ $> \|A\|_F^2 \|B\|_F^2$.

Problem 4. $A = \begin{bmatrix} 1 & -5 & 2 \\ 1 & 0 & 5 \\ 0 & 1 & -9 \end{bmatrix}$.

a) $|A|_1 = |1| + |-5| + |2| + |1| + |5| + |1| + |-9|$

$= 24.$

Note that $|A|_1$ is not $\|A\|_1$ which is $\|A\|_1 = 10$.

b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -9 \end{bmatrix}$.

c) $P_{11} = \frac{1}{24}$, $P_{12} = \frac{5}{24}$, $P_{13} = \frac{2}{24}$,

$P_{21} = \frac{1}{24}$ $P_{22} = 0$ $P_{23} = \frac{5}{24}$

$P_{31} = 0$ $P_{32} = \frac{1}{24}$ $P_{33} = \frac{9}{24}$.

d) See lecture note.

e) $Y = X - A \Rightarrow E[Y] = E[X - A] = E[X] - A = 0$.

$\|Y\|_2 = \|X - A\|_2 \leq \|X\|_2 + \|A\|_2 \leq 24 + \|A\|_2$.

$\|X\|_2 = \| \frac{1}{P_{ij}} A_{i,j} \|_2 = 24 \| \frac{A_{ij}}{|a_{i,j}|} \|_2 \leq 24$