Midterm 2A

MATH 18, LECTURE C00, WINTER 2019

NAME: Key.

PID:

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DO NOT TURN PAGE UNTIL INSTRUCTED TO DO SO
(This exam is worth 40 points)

Problem 0.(1 points.) Follows the instructions on this exam and any additional instructions given during the exam.

NAME:

Problem 1. (9 points.) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -2 & -2 \\ 2 & 1 & 1 & 4 & 5 \\ 1 & 0 & 0 & 3 & 3 \end{bmatrix}$ and its reduced row echelon form $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

pivots.

Give bases and dimensions for the following three bases:

a) (3 points) Col(A)

b) (3 points) Nul(A)

c) (3 points) Row(A)

Ver. B

a) $\left\{ \begin{bmatrix}1\\0\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\1\\3\end{bmatrix} \right\}$.

b) $\begin{bmatrix} -x_4 \\ -x_3 - x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3\begin{bmatrix}0\\-1\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-1\\-1\\0\\1\\0\end{bmatrix}$

c) $[1\ 0\ 0\ -1\ 0]$, $[0\ 1\ 1\ 2\ 0]$, $[0\ 0\ 0\ 0\ 1]$.

a) Since the first, second, and fifth columns of rref(A) are pivots.

~~so~~ $\left\{ \begin{bmatrix}1\\0\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\5\\3\end{bmatrix} \right\}$ form a basis for Col(A).

$\Rightarrow$ dim Col(A) = 3.

b) Since x_3 and x_4 are free variables,

$x_1 = -3x_4$

$x_2 = -x_3 + 2x_4$

$x_5 = 0$.

The vector in Nul(A) is of the form

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}-3x_4\\-x_3+2x_4\\x_3\\x_4\\0\end{bmatrix} = x_3\begin{bmatrix}0\\-1\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-3\\2\\0\\1\\0\end{bmatrix}.$$

$\Rightarrow \left\{ \begin{bmatrix}0\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\2\\0\\1\\0\end{bmatrix} \right\}$ form a basis for Nul(A).

dim Nul(A) = 2.

c) Row(A) = span{[1 0 0 3 0], [0 1 1 -2 0], [0 0 0 0 1]}

these all form basis for Row(A).

dim Row(A) = 3.

NAME:

Problem 2.(10 points.) Let A and B be 3×3 matrices such that $\det(A) = 2$ and $\det(B) = 3$. For each of the following, give its value if you have enough information. If you do not have enough information, say "not enough information".

a) (2.5 points) $\det(A^{-1})$ $= 1/\det A = 1/2$.

b) (2.5 points) $\text{rank}(A)$ $= 3$ since A is invertible (or full rank).

c) (2.5 points) $\det(A + B)$ not enough information.

d) (2.5 points) $\det(A^T B)$ $= \det(A^T)\det(B) = \det(A)\cdot 3 = 2.3 = 6$.

Ver B: a) $\det(B^{-1}) = 1/3$

b) $\text{rank}(B) = 3$.

c) not enough information.

d) $\det(AB^T) = 6$.

NAME:

Problem 3.(10 points.) A linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$ is given by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_1 - x_3 \end{bmatrix}$.

a) (2.5 points) What is the kernel of T (i.e., the space of all vector $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = \vec{0}$).

b) (2.5 points) Find a set of vectors that span the range of T. (They need not be a basis.)

c) (2.5 points) Find a standard matrix A of T.

d) (2.5 points) What is the dimension of the range of T?

a) $T(\vec{x}) = 0 \iff \begin{matrix} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \\ x_1 - x_3 = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 = x_3 \\ x_2 = -x_1 . \end{matrix}$

$\Rightarrow \ker(T) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

b) $T(\vec{x}) = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

$\Rightarrow \text{range}(T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

c) We need to find $T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)$.

$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad T(\vec{e}_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad T(\vec{e}_3) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

$\Rightarrow A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

d) $\dim \text{range}(T) = 3 - \dim \ker(T)$

$= \cancel{1}\ 2$.

Ver B: a) $\ker T = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$. b) $\text{range}(T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

c) $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

d) $\dim \text{range}(T) = 2$.

NAME:

Problem 4.(10 points.) Let $\mathbb{P}_2$ be the space of all polynomials of degree at most 2, and $\mathbb{P}_3$ the space of all polynomials of degree at most 3.

a) (3 points) Given a basis $\mathcal{B} = \{1-x^2, x-x^2, 2-x+x^2\}$ for $\mathbb{P}_2$, find the coordinate vector of the element $3-x^2$ in $\mathbb{P}_2$, relative to $\mathcal{B}$. That is, find $[3-x^2]_{\mathcal{B}}$.

b) (4 points) In the space $\mathbb{P}_3$, are the vectors $\{1+x^3, 1+x-x^2, -x+2x^2-x^3\}$ linearly independent? Explain your answer.

c) (3 points) Do the vectors in Part b) form a basis for $\mathbb{P}_3$? Explain your answer.

a) $\cancel{[3-x^2]}$ Since $3-x^2 = (1-x^2) + (x-x^2) + (2-x+x^2)$

$$[3-x^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

b) Consider a standard basis $\mathcal{B}' = \{1, x, x^2, x^3\}$ of $\mathbb{P}_3$.

$$[1+x^3]_{\mathcal{B}'} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad [1+x-x^2]_{\mathcal{B}'} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \qquad [-x+2x^2-x^3]_{\mathcal{B}'} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since these vectors are linearly independent in $\mathbb{R}^4$, $\{1+x^3, 1+x-x^2, -x+2x^2-x^3\}$ linearly independent.

c) No because there are only 3 vectors!

Ver B: a) $[3-2x+x^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$

b) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Yes!

c) No!

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a) (2.5 points) $\det(B^{-1})$

b) (2.5 points) $\operatorname{rank}(B)$

c) (2.5 points) $\det(A + B)$

d) (2.5 points) $\det(AB^T)$

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