exercise P

15656

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1 P2.1

We know that that we cannot represent translation in 3D space in a 3x3 matrix, since translation is not linear, so we add a 4^{th} dimension $\mathbf{w}$. In this way, the old 3D space become a hyperplane inside a 4D space, with the plane being orthogonal to the newly added axis w. Simply put, translation in 3D space is controlling axis w like a in the direction of x,y,z. After that, the new intersection between w and the hyperplane will be the translated origin of that hyperplane. The last row of the matrix: [0 0 0 1] indicates that when decompose x,y,z into new components vectors , w doesn't contribute at all, so this transformation keep everything on that hyperplane still on that hyperplane.From 4D space we can generalize the idea onto n-D space.
let $\mathbf{A}$ and $\mathbf{B}$ be 2 matrices with size $m * n$

$\bar{A} = \left[\begin{array}{ccc|c} & \mathbf{A} & & \mathbf{a} \\ 0 & \dots & 0 & 1 \end{array}\right]$

$\bar{B} = \left[\begin{array}{ccc|c} & \mathbf{B} & & \mathbf{b} \\ 0 & \dots & 0 & 1 \end{array}\right]$

$\bar{A} * \bar{B} = \left[\begin{array}{ccc|c} & \mathbf{A} & & \mathbf{a} \\ 0 & \dots & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{ccc|c} & \mathbf{B} & & \mathbf{b} \\ 0 & \dots & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} & \mathbf{A} \cdot \mathbf{B} & & \mathbf{A} \cdot \mathbf{b} + \mathbf{a} \\ 0 & \dots & 0 & 1 \end{array}\right]$

as it can be seen, the multiplication of A and B result in a matrix that also has the similar last row , thus composition of these transformations will result in a new matrix which still prevents the points from being flung out of the hyperplane that we do transformation on.

2 P2.3

$$\mathbf{T} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$T \cdot C = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{np} \end{bmatrix} = \begin{bmatrix} \mathbf{T} \cdot \mathbf{c_1} & \mathbf{T} \cdot \mathbf{c_2} & \dots & \mathbf{T} \cdot \mathbf{c_p} \end{bmatrix}$$

$$T \cdot B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix} = \begin{bmatrix} \mathbf{T} \cdot \mathbf{b_1} & \mathbf{T} \cdot \mathbf{b_2} & \dots & \mathbf{T} \cdot \mathbf{b_p} \end{bmatrix}$$

$$T \cdot B + T \cdot C = \begin{bmatrix} \mathbf{T} \cdot \mathbf{b_1} + \mathbf{T} \cdot \mathbf{c_1} & \mathbf{T} \cdot \mathbf{b_2} + \mathbf{T} \cdot \mathbf{c_2} & \dots & \mathbf{T} \cdot \mathbf{b_p} + \mathbf{T} \cdot \mathbf{c_p} \end{bmatrix}$$

$$= \mathbf{T} \cdot \begin{bmatrix} \mathbf{b_1} + \mathbf{c_1} & \mathbf{b_2} + \mathbf{c_2} & \dots & \mathbf{b_p} + \mathbf{c_p} \end{bmatrix}$$

Hence transformation T preserves addition. It can also be proven that T preserves scalar multiplication, hence it's a linear map.

Matrix representation of that transformation is : $\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$

if we assume the order of the basis is $b_{11}, b_{12}, b_{21}, b_{22}$

3 P2.4

Denote x_1 and x_2 two elements of $ker(\mathbf{f})$

Since f is a linear map, we have $f(x_1) + f(x_2) = f(x_1 + x_2) \implies f(x_1 + x_2) = \vec{0}$ hence $x_1 + x_2$ also belong to $ker(\mathbf{f})$, thus it is closed under addition. Similarly, $ker(\mathbf{f}$ is also closed under scalar multiplication Hence, $ker(\mathbf{f})$ is a subspace

$\mathbf{f}$ is still a linear map, the matrix of $\mathbf{f}$ is $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and the nullity of $\mathbf{f}$ is 1