Exercise P3

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1 P3.1

Problem 1 *Prove the following Cauchy-Schwarz inequality* $| \langle v, w \rangle | \leq \|v\|\|w\|$

1.1 Solution

$$\cos\theta = \frac{\langle v, w \rangle}{\|v\|\|w\|}$$

$$\implies |\cos\theta| = \frac{|\langle v, w \rangle|}{\|v\|\|w\|}$$

$$\implies \frac{|\langle v, w \rangle|}{\|v\|\|w\|} \leq 1$$

Hence, $| \langle v, w \rangle | \leq \|v\|\|w\|$

2 P3.2

Problem 2 *Prove the Triangle inequality* $\|v\| - \|w\| \leq \|v + w\| \leq \|v\| + \|w\|$

2.1 Solution

$$(\|v\| - \|w\|)^2 = \langle v, v \rangle + \langle w, w \rangle - 2\|v\|\|w\| \leq \langle v, v \rangle + \langle w, w \rangle + 2\langle v, w \rangle$$
$$= \langle v, v + w \rangle + \langle w + v, w \rangle = \langle v + w, v + w \rangle = \|v + w\|^2$$

Hence $\|v\| - \|w\| \leq \|v + w\|$
Similarly:

$$(\|v\| + \|w\|)^2 = \langle v, v \rangle + \langle w, w \rangle + 2\|v\|\|w\| \geq \langle v, v \rangle + \langle w, w \rangle + 2\langle v, w \rangle$$

$$= \langle v, v + w \rangle + \langle w + v, w \rangle = \langle v + w, v + w \rangle = \|v + w\|^2$$

Hence $\|v\| + \|w\| \geq \|v + w\|$

3 P3.3

Problem 3 *Prove the Parallellogram law* $\|v+w\|^2+\|v-w\|^2 = 2(\|v\|^2+\|w\|^2)$

3.1 Solution

$\|v\|^2+\|w\|^2+2\langle v,w\rangle = \langle v,v\rangle+\langle v,w\rangle+\langle w,w\rangle+\langle w,v\rangle = \langle v+w,v+w\rangle = \|v+w\|^2$

Similarly:

$\|v\|^2+\|w\|^2-2\langle v,w\rangle = \langle v,v\rangle-\langle v,w\rangle+\langle w,w\rangle-\langle w,v\rangle = \langle v-w,v-w\rangle = \|v-w\|^2$

Adding up 2 equations together:

$$\|v+w\|^2+\|v-w\|^2 = 2(\|v\|^2+\|w\|^2)$$

4 P3.4

Problem 4 *Prove the Polarization identity:* $\langle v,w\rangle = \frac{1}{2}\big(\|v+w\|^2-\|v\|^2-\|w\|^2\big)$

4.1 Solution

It is proven that:

$\|v\|^2+\|w\|^2+2\langle v,w\rangle = \langle v,v\rangle+\langle v,w\rangle+\langle w,w\rangle+\langle w,v\rangle = \langle v+w,v+w\rangle = \|v+w\|^2$

Hence

$$2\langle v,w\rangle = \|v+w\|^2-\|v\|^2-\|w\|^2$$
$$\implies \langle v,w\rangle = \frac{1}{2}(\|v+w\|^2-\|v\|^2-\|w\|^2)$$

5 P3.5

Problem 5 *Find the distance from a point* $\mathbf{w} = (w_x,w_y,w_z)^\top$ *to the plane* H_n *normal to* $\mathbf{n} = (a,b,c)^\top$ *and translated from the origin by vector* $\mathbf{p} = (x_0,y_0,z_0)^\top$*. Show that it is the smallest distance to any point in* H_n

5.1 Solution

Given $H_n : \forall v \in V : \langle v-p,n\rangle = 0$
Denote w_H an orthogonal projection of w on H
Hence

$$w-w_H = kn$$
$$\implies w_H = w-kn$$

Also $w_H \in H$

$$\implies \langle w_H-p,n\rangle = 0$$

$$\implies \langle w - kn - p, n\rangle = 0 \implies k = \frac{\langle w - p, n\rangle}{\|n\|^2}$$

Hence the distance is :

$$d = k\|n\| = \frac{\langle w - p, n\rangle}{\|n\|}$$

6 P3.6

Problem 6 *Prove that $w_p = \mathsf{proj}_{V_k}(w)$ is the vector in subspace V_k *closest* to w using 2 approaches:*

6.1 Solution

Denote w_V the orthogonal projection of w on V_k.
Denote w_i a point on a hyperplane V_k

$$\implies w - w_i = (w - w_V) + (w_V - w_i)$$

Since $(w - w_H)$ and $(w_H - w_i)$ are perpendicular $(w_H - w_i \in H_n$ and $w - w_H = kn)$,applying the Pythagorean theorem:

$$\|w - w_i\|^2 = \|w_H - w_i\|^2 + \|w - w_H\|^2 \implies \|w - w_i\| \geq \|w - w_H\|$$

Hence, the shortest distance is the distance between a vector w and its projection in subspace V_k

7 P3.7

Problem 7 *Least squares solution of $A\mathbf{x} = \mathbf{b}$ is $\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{b} - A\mathbf{x}\|$. By column view of matrix multiplication, $A\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{b}$ onto the column space $\mathsf{Col}(A)$ of matrix A. Thus we must have $(\mathbf{b} - A\hat{\mathbf{x}}) \perp \mathsf{Col}(A)$. Write down matrix form for this and solve for $\hat{\mathbf{x}}$, given that A is full rank. Show that the orthogonal projection operator of $\mathbf{b}$ onto $\mathsf{Col}(A)$ is: $P_{\mathsf{Col}(A)} = A(A^\top A)^{-1}A^\top$*

$$\begin{aligned}
&(\mathbf{b} - A\hat{\mathbf{x}}) \perp \mathsf{Col}(A) \\
\iff &A^T(b - A\hat{\mathbf{x}}) = 0 \\
\iff &A^T b = A^T A\hat{\mathbf{x}}(1)
\end{aligned}$$

Since $A^T A$ is square and full rank $(rank(A^T A) = rank(A)$, it is invertible
Hence (1)

$$\begin{aligned}
\iff &\hat{x} = (A^T A)^{-1}A^T b \\
\implies &A(A^T A)^{-1}A^T b = A\hat{x}(2)
\end{aligned}$$

From (2) we can see that b is transformed into the orthogonal projection of b in hyperplane $Col(A)$. Hence, the matrix representation of orthogonal projection of b onto $Col(A)$ is $P_{Col(A)} = A(A^T A)^{-1}A^T$

8 P3.8

Problem 8 *Show that the generalized dot product* $\langle \mathbf{x}, \mathbf{y} \rangle_M = \mathbf{x}^\top M \mathbf{y}$ *with* $M \in R^{n \times n}$ *a symmetric positive definite matrix satisfies all 3 requirements of a proper inner product.*

8.1 Solution

We prove that the dot product satisfies the following 3 requirements:
-Symmetry:

$$\langle y, x \rangle = y^T M x = (x^T M^T y)^T = x^T M y$$

(because M is symmetric)
-Linearity:

$$\langle a\mathbf{x} + b\mathbf{y}, c\mathbf{z} + d\mathbf{t} \rangle$$

$$= (a\mathbf{x} + b\mathbf{y})^T M (c\mathbf{z} + d\mathbf{t})$$

$$= a\mathbf{x}^T M c\mathbf{z} + a\mathbf{x}^T M d\mathbf{t} + b\mathbf{y}^T M c\mathbf{z} + b\mathbf{y}^T M d\mathbf{t}$$

-Positive definiteness:

$$= ac\langle \mathbf{x}, \mathbf{z} \rangle + ad\langle \mathbf{x}, \mathbf{t} \rangle + bc\langle \mathbf{y}, \mathbf{z} \rangle + bd\langle \mathbf{y}, \mathbf{t} \rangle$$

$$\langle x, x \rangle = x^T M x > 0$$

(since M is positive definite)
Hence, the inner product $\langle x, y \rangle = x^T M y$ is a proper inner product.

9 P3.9

Problem 9 *Prove that the null space and the row space of a matrix are orthogonal, i.e. every vector in null space is orthogonal to every vector in row space (zero dot product)*

9.1 Solution

Given a vector $x \in N(A)$, where $rank(A) = r$, we have:

$$Ax = \vec{0}$$

Applying elementary row operation on both size to reduce A to row-echelon form

$$\implies EAx = \vec{0}$$

$$\iff \begin{bmatrix} \langle \mathbf{e_1}^T, x \rangle \\ \langle \mathbf{e_2}^T, x \rangle \\ \vdots \\ \langle \mathbf{e_n}^T, x \rangle \end{bmatrix} = \vec{0}$$

As it can be seen here , x must be perpendicular to every row in EA. However, after reducing A to row echelon form,we can ensure that first r rows form a basis that span $Row(A)$, hence, x must be orthogonal to $Row(A)$ Therefore the null space of A and the row space of A are orthogonal.

10 P3.10

Problem 10 *What is the distance from a vector to a line? Recall from lecture 1, a line through v in direction of u:* $L_1 = \{w \in V : w = v + tu,\ \forall t \in R,\ u, v \in V\}$

10.1 Solution

Denote w_l an orthogonal projection of w on L_1. Hence $w_l = v + t_l u$ .We also have that:

$$\langle w - w_l, u \rangle = 0$$
$$\iff w - v - t_l u, u \langle = 0$$
$$\iff t_l = \frac{\langle w - v, u \rangle}{\|u\|^2}$$

The distance from w to L_1 is :

$$d = \|w - w_l\| = \|w - v - t_1 u\| = \|w\|^2 + \|v\|^2 + t_1^2\|u\|^2 - 2t_1\langle w - v, u \rangle = \|w\|^2 + \|v\|^2 - t_1^2\|u\|^2$$
$$= \|w\|^2 + \|v\|^2 - \frac{\langle w - v, u \rangle^2}{\|u\|^2}$$

11 E3.1

Problem 11 *Let V be an n-dimensional inner product space with pairwise orthogonal subspaces*

$$W_1, \ldots, W_m\,,$$

where $\sum_{i=1}^{m} \dim(W_i) = n$*. Prove that every vector* $v \in V$ *can be represented uniquely as*

$$v = w_1 + \cdots + w_m\,,$$

where $w_i \in W_i$ *for* $i = 1, \ldots, m$*, i.e.,*

$$V = W_1 \oplus \cdots \oplus W_m\,.$$

11.1 Solution

According to the result derived from P3.9, since W_1 and W_2 are orthogonal subspaces:

$$W_{1,2} = W_1 \oplus W_2$$
$$W_{1,2,3} = W_{1,2} \oplus W_3 = W_1 \oplus W_2 \oplus W_3$$

In a similar fashion, we can see that:

$$W_{1,2,3,..,m} = W_1 \oplus W_2 \oplus ... \oplus W_m$$

$$\implies V = W_1 \oplus W_2 \oplus ... \oplus W_m$$

12 E3.3

Problem 12 *Given an orthonormal basis $\mathcal{U} = (u_1, \ldots, u_n)$ of R^n. Let U be the matrix whose distinct columns are the basis vectors in $\mathcal{U}$. Prove again U is a transformation that does not change distance, angle, size of the objects being transformed: geometrically it is a rotation or reflection!*

12.1 Solution

So, we need to prove these equation:
1.U Preserve size:

$$\|v\| = \|Uv\|$$

2.U preserve distance

$$\|v - w\| = \|U(v - w)\|$$

3.U preserve angle

$$\cos(v, w) = \cos(Uv, Uw)$$

So, we first have (1)

$$\iff \langle v, v\rangle = \langle Uv, Uv\rangle$$

$$\iff v^T v = (Uv)^T Uv$$

$$\iff v^T v = v^T U^T Uv$$

However since U is an orthogonal matrix $\implies U^T U = I$, hence we get the desired outcome. Similarly proven for (2)

(3) happens:

$$\iff \frac{\langle v, w\rangle}{\|v\|\|w\|} = \frac{\langle Uv, Uw\rangle}{\|Uv\|\|Uw\|}$$

$$\iff \langle v, w\rangle = \langle Uv, Uw\rangle$$

(since $\|x\| = \|Ux\|$)

$$\iff v^T w = (Uv)^T (Uw)$$

$$\iff v^T w = v^T U^T Uw$$

which is true, since $U^T U = I$