Exercise P4

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1 P4.1

Problem 1 *Consider linear operator* $f : R^2 \to R^2$ *with* $f(v_1) = v_1 + 2v_2$ *and* $f(v_2) = 2v_1 + v_2$ *(1). Verify that* $f(v_1 + v_2) = 3(v_1 + v_2)$ *and* $f(v_1 - v_2) = -(v_1 - v_2)$ *(2)*
Express f *using basis* $B_v = \{v_1, v_2\}$*, then* $B_e = \{e_1 = v_1 + v_2, e_2 = v_1 - v_2\}$
What is the matrix representation of f *in* B_v*? in* B_e*?*

1.1 Solution

Verify that $f(v_1 + v_2) = 3(v_1 + v_2)$ and $f(v_1 - v_2) = -(v_1 - v_2)$:

$$f(v_1 + v_2) = f(v_1) + f(v_2) = v_1 + 2v_2 + 2v_1 + v_2 = 3(v_1 + v_2)$$

$$f(v_1 - v_2) = f(v_1) - f(v_2) = v_1 + 2v_2 - 2v_1 - v_2 = -(v_1 - v_2)$$

What is the matrix representation of f in B_v? in B_e?:
The matrix representation of f in B_v is :

$$A_v = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

The matrix representation of f in B_e is :

$$A_e = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

2 P4.2

Problem 2 *A is a Markov matrix, i.e., each column* $\mathbf{a}_i$ *is a probability vector. Show that if* p_0 *is a probability n-vector then* Ap_0 *is also a probability vector.*

2.1 Solution

$$x = Ap_0 = \left[\ \mathbf{a_1} \mid \mathbf{a_2} \mid \dots \mid \mathbf{a_n}\ \right] \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = p_1\mathbf{a_1} + p_2\mathbf{a_2} + ... + p_n\mathbf{a_n}$$

$$\implies sum(x) = p_1 sum(\mathbf{a_1}) + p_2 sum(\mathbf{a_2}) + ... + p_n sum(\mathbf{a_n})$$

$$\implies sum(x) = p_1 + p_2 + ... + p_n$$

$$\implies sum(x) = 1$$

Furthermore, every elements of $p_k\mathbf{a_k}$ is non-negative, as p_k and a_k are probability vector. Hence, every elements of x is non-negative.

3 P4.3

Problem 3 *Summarize Google's PageRank Algorithm from Gatech interactive book.*

3.1 Solution

First, let's talk a bit about ranking websites in Google.
There are n websites, each websites will give hyperlink to other sites and share its point to those page. So we have a Markov matrix $n \times n$ representing state transition between n states, where M_{ij} represent the probability that state j transit to state i.
So what does this mean in Google Pagerank?.M_{ij} here represent the proportion of score of site j shared for site i
Originally, vector $x_0 = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}$ denotes the score of n website as a probability vector.
After every step :

$$x_{k+1} = Mx_k$$

Thus

$$x_n = M^n x_0$$

According to Perron-Frobenius theorem for stochastic matrix:
from M we can find an eigenvector e_1 with its eigenvalue λ_1 equals to 1.
Also all the absolute of M's eigenvalues except λ_1 are smaller than 1.
so $x_0 = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$
Then $M^n x_0$ will converge to $\alpha_1 e_1$.
However, M is not yet stochastic, since many elements in columns of M are still 0.
2 possible worst-case scenario is
1. Disconnected Internet, where eigenspace of eigenvalue 1 is more-than-1-dimensional.

2. A website not linked to any other website, in which case there will be no eigenvalue 1, so the steady state here will be zeroed out.
To fix this, beside Markov matrix, we will also add a random state change matrix, it will become:

$$A = (1 - p)M + pB$$

where $B = \frac{1}{n} I_n$
p is called damping factor (usually p = 0.15). and this solve the problem of M not being stochastic.
In this model here, there will also be a chance for a surfer to change to a completely random page.

4 P4.4

Problem 4 *Prove again if $\beta = \{v_1, \ldots, v_n\}$ is an eigenbasis of $A_{n\times n}$ with eigenvalues λ_i's, then A is diagonalizable: $A = QDQ^{-1}$ with $D = \mathsf{diag}(\lambda_1, \ldots, \lambda_n)$ and Q a square matrix composed of v_i's as columns*

4.1 Solution

: Since $A_{n\times n}$ has n vectors as a basis, A is full rank and hence,invertible.

$$AQ = A\left[\begin{array}{c|c|c|c} \mathbf{v_1} & \mathbf{v_2} & \ldots & \mathbf{v_n} \end{array}\right] = \left[\begin{array}{c|c|c|c} A\mathbf{v_1} & A\mathbf{v_2} & \ldots & A\mathbf{v_n} \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c} \lambda_1\mathbf{v_1} & \lambda_2\mathbf{v_2} & \ldots & \lambda_n\mathbf{v_n} \end{array}\right] = \left[\begin{array}{c|c|c|c} \mathbf{v_1} & \mathbf{v_2} & \ldots & \mathbf{v_n} \end{array}\right] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} = QD$$

$$\implies AQ = QD$$
$$\implies AQQ^{-1} = QDQ^{-1}$$
$$\implies A = QDQ^{-1}(qed)$$

5 P4.5

Problem 5 *Summarize the problem description and solution to find F_n in Fibonacci's rabbits: $F_0 = 0, F_1 = 1, \ldots, F_n = F_{n-1} + F_{n-2}$.*

5.1 Solution

So , the problem here will be to find an update function $f(x_{n-1}, x_n) = (x_n, x_{n-1}+ x_n$ Represent the function in form of matrix multiplication (with $r_n = \begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix}$):

$$r_{n+1} = Fr_n$$

$$\implies \begin{bmatrix} x_n \\ x_n + x_{n-1} \end{bmatrix} = F \begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix}$$

$$\implies F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Now we find eigenvalue λ of F:

$$det(A - \lambda I) = 0$$

$$\iff -\lambda(1-\lambda) - 1 = 0$$

$$\iff \lambda^2 - \lambda - 1 = 0$$

$$\iff \lambda_1 = \frac{1+\sqrt{5}}{2} \vee \lambda_2 = \frac{1-\sqrt{5}}{2}$$

Then the equivalent basis is

$$v_1 = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$r_n = F^{n-1} r_1$$

However

$$r_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} v_1 - \frac{1}{\sqrt{5}} v_2$$

Hence

$$r_n = F^{n-1} r_1 = \frac{\lambda_1^{n-1}}{\sqrt{5}} v_1 + \frac{\lambda_2^{n-1}}{\sqrt{5}} v_2$$

6 P4.6

Problem 6 *Show that λ is an eigenvalue of a square $n \times n$ matrix A *iff** $det(A - \lambda I_n) = 0$.

6.1 Solution

λ is an eigenvalue of A

$$\iff Ax = \lambda x \quad \text{(x is the corresponding eigenvector)}$$

$$\iff (A - \lambda I_n)x = 0$$

However, $x \neq 0$, hence, $(A - \lambda I_n)$ is not one-to-one, thus not invertible.
Therefore $det(A - \lambda I_n) = 0$
The only if part is solved by going backward from the last line.

7 P4.7-4.9

Problem 7 *Given a symmetric positive definite matrix $A > 0$, i.e., $x^\top Ax > 0\ \forall x \neq 0_n$.*
- P4.7 Show that all its eigenvalues are positive, $\lambda_i > 0\ \forall i$.
*- P4.8 Show that all its eigenvectors form an **orthonormal basis** β.*
- P4.9 Show that there exists a positive definite matrix $A^{\frac{1}{2}} > 0$ s.t. $A^{\frac{1}{2}}A^{\frac{1}{2}} = A$.

7.1 Solution

P4.7 Denote x a random eigenvector of A, we have that:

$$x^T Ax = x^T \lambda x = \lambda x^T x$$

$$\implies \lambda x^T x > 0$$

Since $x^T x > 0 \forall x \neq 0$, $\lambda > 0$
Hence, every eigenvalue of A must be positive.

P4.8 Denote transformation T : V $\longrightarrow$ V , whose matrix representation is A:
Since A is positive definite: $A^* = A^T$
Since A is symmetric: $A^T = A$
So $A * A = A^T A = AA^T = AA*$, thus A is normal.
P4.9 As proven above, we can decompose A :

$$A = Qdiag(\lambda 1, \lambda 2, ..., \lambda n)Q^T$$

with Q being an orthogonal matrix whose columns form an eigenbasis of A

$$A = Qdiag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, ..., \sqrt{\lambda_n})Q^T Qdiag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, ..., \sqrt{\lambda_n})Q^T$$

Denote $A^{\frac{1}{2}} = Qdiag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, ..., \sqrt{\lambda_n})Q^T$, then

$$A = A^{\frac{1}{2}}A^{\frac{1}{2}}$$

Now we need to prove that $A^{\frac{1}{2}}$ is also symmetric positive definite.
Proof for symmetry:

$$A^{\frac{1}{2}T} = (Qdiag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, ..., \sqrt{\lambda_n})Q^T)^T = Qdiag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, ..., \sqrt{\lambda_n})Q^T = A^{\frac{1}{2}}$$

Proof for positive definite :

$$x^T Ax > 0$$

$$\iff x^T Qdiag(\lambda_1, \lambda_2, ..., \lambda_n)Q^T x > 0$$

denote $y = Q^T x$, then

$$y^T diag\left(\lambda_1, \lambda_2, ..., \lambda_n\right) y > 0$$

$$\iff \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2 > 0$$

$$\implies \sqrt{\lambda_1} y_1^2 + \sqrt{\lambda_2} y_2^2 + ... + \sqrt{\lambda_n} y_n^2 > 0$$

Hence,$A^{\frac{1}{2}}$ is positive definite.