exercise P

15656

April 2020

1 P2.1

We know that that we cannot represent translation in 3D space in a 3x3 matrix, since translation is not linear, so we add a 4^{th} dimension $\mathbf{w}$. In this way, the old 3D space become a hyperplane inside a 4D space, with the plane being orthogonal to the newly added axis w. Simply put, translation in 3D space is controlling axis w like a in the direction of x,y,z. After that, the new intersection between w and the hyperplane will be the translated origin of that hyperplane. The last row of the matrix: [0 0 0 1] indicates that when decompose x,y,z into new components vectors , w doesn't contribute at all, so this transformation keep everything on that hyperplane still on that hyperplane.From 4D space we can generalize the idea onto n-D space.
let $\mathbf{A}$ and $\mathbf{B}$ be 2 matrices with size $m * n$

$\bar{A} = \left[\begin{array}{ccc|c} & \mathbf{A} & & \mathbf{a} \\ 0 & \dots & 0 & 1 \end{array}\right]$

$\bar{B} = \left[\begin{array}{ccc|c} & \mathbf{B} & & \mathbf{b} \\ 0 & \dots & 0 & 1 \end{array}\right]$

$\bar{A} * \bar{B} = \left[\begin{array}{ccc|c} & \mathbf{A} & & \mathbf{a} \\ 0 & \dots & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{ccc|c} & \mathbf{B} & & \mathbf{b} \\ 0 & \dots & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} & \mathbf{A} \cdot \mathbf{B} & & \mathbf{A} \cdot \mathbf{b} + \mathbf{a} \\ 0 & \dots & 0 & 1 \end{array}\right]$

as it can be seen, the multiplication of A and B result in a matrix that also has the similar last row , thus composition of these transformations will result in a new matrix which still prevents the points from being flung out of the hyperplane that we do transformation on.

2 P2.3

$\mathbf{T} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

$T{\cdot}C = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{np} \end{bmatrix} = \begin{bmatrix} \mathbf{T} \cdot \mathbf{c_1} & \mathbf{T} \cdot \mathbf{c_2} & \dots & \mathbf{T} \cdot \mathbf{c_p} \end{bmatrix}$

$T{\cdot}B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix} = \begin{bmatrix} \mathbf{T} \cdot \mathbf{b_1} & \mathbf{T} \cdot \mathbf{b_2} & \dots & \mathbf{T} \cdot \mathbf{b_p} \end{bmatrix}$

$T \cdot B + T \cdot C = \begin{bmatrix} \mathbf{T} \cdot \mathbf{b_1} + \mathbf{T} \cdot \mathbf{c_1} & \mathbf{T} \cdot \mathbf{b_2} + \mathbf{T} \cdot \mathbf{c_2} & \dots & \mathbf{T} \cdot \mathbf{b_p} + \mathbf{T} \cdot \mathbf{c_p} \end{bmatrix}$
$= \mathbf{T} \cdot \begin{bmatrix} \mathbf{b_1} + \mathbf{c_1} & \mathbf{b_2} + \mathbf{c_2} & \dots & \mathbf{b_p} + \mathbf{c_p} \end{bmatrix}$
Hence transformation T preserves addition. It can also be proven that T preserves scalar multiplication, hence it's a linear map.

Matrix representation of that transformation is : $\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$

if we assume the order of the basis is $b_{11}, b_{12}, b_{21}, b_{22}$

3 P2.4

Denote x_1 and x_2 two elements of $ker(\mathbf{f})$
Since f is a linear map, we have $f(x_1) + f(x_2) = f(x_1 + x_2) \implies f(x_1 + x_2) = \vec{0}$ hence $x_1 + x_2$ also belong to $ker(\mathbf{f})$, thus it is closed under addition. Similarly, $ker(\mathbf{f}$ is also closed under scalar multiplication Hence, $ker(\mathbf{f})$ is a subspace

$\mathbf{f}$ is still a linear map, the matrix of $\mathbf{f}$ is $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and the nullity of $\mathbf{f}$ is 1

4 P2.5

Denote

$$S = \mathsf{diag}(\lambda_1, \dots, \lambda_n) \in R^{n \times n}$$

Then

$$y = Sx = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1\lambda_1 \\ x_2\lambda_2 \\ \vdots \\ x_n\lambda_n \end{bmatrix}$$

As it can be seen clearly,each element x_i of the coordinate x is scaled by a factor λ_i

5 P2.6

Given 2 matrices $A_{m\times n}, B_{n\times p}$
In column view, each column in B is transformed by matrix A

$$\mathbf{AB} = \mathbf{A}\left[\begin{array}{c|c|c|c} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{array}\right] = \left[\begin{array}{c|c|c|c} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \dots & \mathbf{Ab}_p \end{array}\right]$$

In row view, B transforms each row in A

$$\mathbf{AB} = \left[\begin{array}{c} \mathbf{a}_1^T \\ \hline \mathbf{a}_2^T \\ \hline \vdots \\ \hline \mathbf{a}_m^T \end{array}\right] \mathbf{B} = \left[\begin{array}{c} \mathbf{a}_1^T\mathbf{B} \\ \hline \mathbf{a}_2^T\mathbf{B} \\ \hline \vdots \\ \hline \mathbf{a}_m^T\mathbf{B} \end{array}\right]$$

6 P2.7

Matrix representing rotation around z axis an angle θ:

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix representing translation in (x,y) plane an amount of $(p_x, p_y, 0)$:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix representing reflection around the origin :

$$\mathbf{C} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the combined transformation is :

$$\mathbf{CBA} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\cos\theta & -\sin\theta & -p_x \\ \sin\theta & -\cos\theta & -p_y \\ 0 & 0 & 1 \end{bmatrix}$$

7 P2.8

$$A_{n\times k} = \left[\begin{array}{ccccc} \multicolumn{5}{c}{\mathbf{I_k}} \\ \hline 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{array}\right]$$

8 P2.9

Denote a transformation $A : V \to W$
Given a vector $x \in N(A)$

$$\implies Ax = 0$$
$$\implies A^T Ax = 0$$
$$\implies x \in N(A^T A)$$

Hence $N(A) \subseteq N(A^T A)$
Similarly let y be a vector such that: $y \in N(A^T A)$

$$\implies A^T Ay = 0$$
$$\implies y^T A^T Ay = 0$$
$$\implies (Ay)^T Ay = 0$$
$$\implies Ay = 0$$
$$\implies y \in N(A)$$

Hence $N(A^T A) \subseteq N(A)$
Therefore

$$N(A^T A) = N(A)$$
$$\implies nullity(A^T A) = nullity(A)$$
$$\implies dim(V) - nullity(A^T A) = dim(V) - nullity(A)$$
$$\implies rank(A^T A) = rank(A)$$

Similarly, $rank(AA^T) = rank(A)$

9 P2.10

Given a square and full rank matrix A, it is proven that A is invertible, hence $AA^{-1} = A^{-1}A = I$
1.Since A,B,C are square and we can do the operation ABC, A,B,C must have the same shape.

$$ABCC^{-1}B^{-1}A^{-1} = ABIB^{-1}A^{-1} = AIA^{-1} = I$$

2.

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

10 P2.11

Denote $\beta_c \beta_r$ respectively basises of $range(A_{m\times n}), range(A^T)$, hence $Span(\beta_c) = Col(A), span(\beta_r) = Row(A)$ and $rank(A) = |\beta c| = p$,$rank(A^T) = |\beta r| = q$
Let $B^c_{m\times p}, B^r n \times q$ matrices with their columns being vectors of β_c, β_r
We have $A = B^c W_{p\times n}$
In row view, each row of A is a linear combination of the row of W, whose number of rows is p
$\implies$ The basis of A^T is also made of p rows of W.
$\implies \ q \leq p \leq n$
In a similar fashion, $A^T = B^r W_{q\times m}$ and $p \leq q \leq m$
Hence $p = q = min(m, n)$