Report

Chaos control and schedule of shuttle buses

Vũ Trung Thành - 20220066

Abstract

This report analyzes a shuttle bus system, demonstrating how deterministic chaos arises from the interaction of passenger loading and bus headways. We utilize a nonlinear map with a loading parameter (passenger flow) and a speedup parameter (control mechanism). Replicating Nagatani's simulations, we show the system transitions from regular to periodic to chaotic schedules as passenger load increases. Crucially, we demonstrate that adjusting the speedup parameter can suppress these chaotic fluctuations, restoring predictable service. The report also discusses the model's parameter sensitivity and robustness, offering insights into complex transportation dynamics.

1 Introduction

The phenomenon of chaos, where randomness and unpredictability emerge from purely deterministic rules, is not confined to abstract mathematics but is frequently observed in everyday systems. Urban traffic flow, for instance, is a classic example where the interactions of individual agents following simple rules can lead to complex collective behaviors like traffic jams and phantom bottlenecks. Recently, the tools of nonlinear dynamics and statistical mechanics have been increasingly applied to understand and manage these transportation challenges.

A common and often frustrating experience for commuters is the irregular arrival of public transport, such as shuttle buses. One might observe long periods with no bus, followed by the arrival of several buses in a "bunch". This irregularity can be attributed not only to stochastic factors but also to the inherent dynamics of the system itself. This report investigates a model, originally proposed by Takashi Nagatani [1], that explores this very issue in a simplified shuttle bus system operating between an origin and a destination.

The fundamental mechanism driving this irregularity is a feedback loop. When a bus arrives at a stop with a large number of waiting passengers, the boarding process takes a significant amount of time, causing the leading bus to slow down. This delay increases the time headway between this bus and the one that preceded it. Consequently, the next bus in the sequence arrives to find fewer waiting passengers, allowing it to maintain or even reduce its headway and eventually overtake the slower bus. Even without external random influences, this deterministic process can generate fully chaotic behavior.

The central objective of this report is to investigate whether this inherent chaos can be managed to ensure a regular and reliable service. The proposed control mechanism is to allow buses to increase their speed to compensate for delays incurred during passenger loading.

Following a structured modeling approach, the first step is to clearly define the system's components, the rules governing them, and the central question to be answered. These are detailed below.

Variables:
$t_i(m)$: The arrival time of bus i at the origin on trip m
$B_i(m)$: The number of passengers boarding bus i at the origin on trip m
$V_i(m)$: The mean speed of bus i during trip m
$h_i(m)$: The time headway between bus i and the bus arrived just before it
$\Delta t_i(m)$: The tour time of bus i for trip m
γ: The time it takes for one passenger to board a bus
η: The time it takes for one passenger to alight from a bus
μ: The rate at which new passengers arrive at the origin
s_i: The degree of speedup for bus i
L: The distance between the origin and the destination
$\Gamma \equiv \mu(\gamma + \eta)$: The dimensionless loading parameter
$S_i \equiv s_i \mu(\gamma + \eta) 2L/V_0^2$: The dimensionless speedup parameter for bus i

Assumptions:
System has M buses, single origin-destination
Buses pass freely
Bus capacity is sufficiently large
$B_i(m) = \mu(t_i(m) - t_{i'}(m'))$
$t_i(m+1) = t_i(m) + (\gamma + \eta)B_i(m) + \frac{2L}{V_i(m)}$
$V_i(m) = V_0 + s_i(\gamma + \eta)B_i(m)$
$t_n, v_n, B_n, \ldots \geq 0$
$\Gamma, S_i > 0 \in \mathbb{R}$

Objective:
To investigate how the bus dynamics are affected by the loading parameter (Γ) and the speedup parameter (S_i)
To determine if the chaotic behavior can be controlled

Figure 1: Key variables, assumptions, and objectives of the shuttle bus system model

2 Modeling approach

To capture the evolution of the shuttle bus system, the chosen modeling approach is a discrete-time nonlinear map. This type of dynamical system describes how the state of a system at one distinct time step evolves to the state at the next time step [2]. This approach is particularly well-suited for shuttle bus operations for several key reasons.

First, the system's behavior is naturally event-driven and sequential. Key events, such as a bus completing a tour and arriving at the origin, occur at discrete moments in time, making a discrete-time formulation more natural than a continuous one. The state of the system can be effectively defined by the arrival times of the buses at the origin, and the model's objective is to find a function that maps the set of current arrival times to the next set of arrival times.

Second, the model is a specific case of a dynamical system where the future state depends only on the current state, not on the sequence of past states that led to it. This memory-less property is a defining characteristic of a map-based model and significantly simplifies the analysis while still capturing the essential physics of the problem.

Third, the relationships governing the system are inherently nonlinear. The travel time of a bus, for instance, is inversely proportional to its speed. The speed, in turn, is adjusted based on passenger-induced delays. This inverse relationship, combined with the feedback loop of passenger accumulation, prevents the system from being described by simple linear equations and is the primary source of the complex dynamics, including chaos. A nonlinear map is therefore necessary to accurately represent these crucial feedback mechanisms. By simplifying the complex reality of bus operations into a concise mathematical map, this approach allows for a focused investigation of the feedback loops that generate chaotic patterns.

3 The mathematical model

The model describes a system of M buses shuttling between an origin and a destination. Passengers arrive at the origin, board a bus, travel to the destination to alight, and the bus returns to the origin to begin the cycle anew. Buses are permitted to pass each other freely. The model's dynamics are captured by a set of equations that define the arrival time of each bus on successive trips.

Let $t_i(m)$ be the arrival time of bus i at the origin for its m-th trip. The arrival time for the next trip, $t_i(m+1)$, is the sum of the current arrival time, the total time spent stationary for passenger loading and unloading, and the time spent moving between the origin and destination:

$$t_i(m+1) = t_i(m) + (\gamma + \eta)B_i(m) + \frac{2L}{V_i(m)}$$

Here, $B_i(m)$ is the number of passengers boarding bus i on trip m, γ and η are the times required for a single passenger to board and alight, respectively, L is the one-way distance between the origin and destination, and $V_i(m)$ is the mean speed of bus i during trip m.

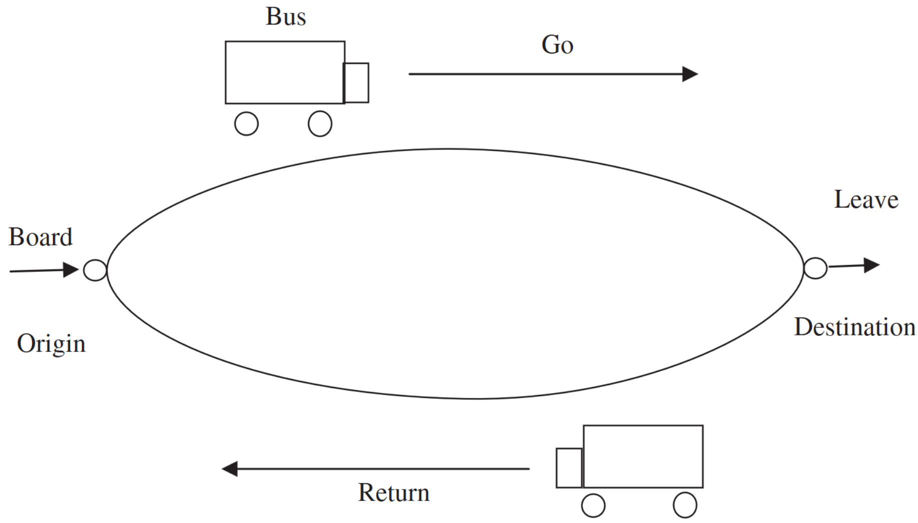

Figure 2: Schematic illustration of the shuttle bus system

The model is built upon two critical assumptions that create the feedback loop:

1. **Passenger Loading:** The number of passengers boarding a bus, $B_i(m)$, is assumed to be equal to the number of passengers who have accumulated at the origin since the departure of the *previous* bus. If new passengers arrive at a constant rate μ, this number is proportional to the time headway between bus i and the bus that arrived just before it, denoted as bus i'. This headway is $t_i(m) - t_{i'}(m')$, where $t_{i'}(m')$ is the arrival time of the preceding bus.

$$B_i(m) = \mu(t_i(m) - t_{i'}(m'))$$

2. **Speed Control:** The bus speed, $V_i(m)$, is not constant. It consists of a base speed V_0 and a corrective component that allows the bus to speed up to recover time lost to passenger loading. This speedup is assumed to be proportional to the total stopping time, with s_i being the speedup coefficient for bus i.

$$V_i(m) = V_0 + s_i(\gamma + \eta)B_i(m)$$

Substituting these assumptions into the primary equation yields a recurrence relation for the arrival times. To simplify this equation and identify the key controlling parameters, the model is non-dimensionalized by dividing time by the characteristic travel time $2L/V_0$. This results in the final nonlinear map for the dimensionless arrival time $T_i(m) \equiv t_i(m)V_0/2L$:

$$T_i(m+1) = T_i(m) + \Gamma(T_i(m) - T_{i'}(m')) + \frac{1}{1 + S_i(T_i(m) - T_{i'}(m'))}$$

The dynamics of the entire system are governed by two dimensionless parameters:

- The **loading parameter**, $\Gamma \equiv \mu(\gamma+\eta)$, which combines the passenger arrival rate and the time per passenger for boarding and alighting. It represents the overall passenger-induced load on the system.

- The **speedup parameter**, $S_i \equiv s_i\mu(\gamma+\eta)2L/V_0^2$, which represents the effectiveness of bus i's ability to compensate for delays.

The system's evolution is found by iterating this map simultaneously for all buses.

4 Simulation and Analysis

The mathematical model, being a system of discrete-time nonlinear maps, does not lend itself to a direct analytical solution. Therefore, its complex dynamics are explored through numerical simulation. The process involves iterating the core map equation over thousands of successive trips to observe the long-term behavior of the system.

4.1 Solving algorithm

The simulation is designed as an event-based system where the primary "event" is the arrival of a bus at the origin. To efficiently manage the sequence of these events, especially since buses can overtake one another, a min-heap (or priority queue) data structure is employed. This ensures that the next event processed is always the one that occurs earliest in time, regardless of which bus it corresponds to.

The simulation begins with a set of initial conditions for the bus arrival times. It then iteratively calculates the next arrival time for each bus based on its current state and its headway relative to the bus that arrived immediately prior. This process requires maintaining a history of all arrival events to accurately determine the correct headway for each calculation.

A crucial aspect of simulating dynamical systems is to distinguish between initial transient behavior and the system's long-term, stable dynamics. To achieve this, a "burn-in" period is implemented. The simulation is run for a large number of initial trips (e.g., 900) which are discarded. Data for analysis, such as headways and tour times, are only recorded for subsequent trips once the system is assumed to have settled onto its characteristic attractor.

The core logic of the simulation algorithm is captured in Algorithm 1.

Algorithm 1 Shuttle bus system simulation

```
function SIMULATE_BUSES(Γ, S, initial_times, num_trips)
    events = MinHeap()
    recorded_data = { headways: [], tour_times: [] }
    for each bus i do
        arrival = initial_times[i]
        events.PUSH(arrival, i, 0)
    end for
    trips_recorded = 0
    last_arrival = initial_times[0]
    while trips_recorded < num_trips do
        (time, i, m) = events.POP()
        headway = time - last_arrival
        last_arrival = time
        if m > burnin then
            Record headway
            Increment trips_recorded
        end if
        next_time = time + Γ· headway + 1/(1+S_i·headway)
        tour_time = next_time - time
        if m > burnin then
            Record tour_time
        end if
        events.PUSH(next_time, i, m+1)
    end while
    return recorded_data
end function
```

5 Simulation result

The numerical simulations successfully replicate the key findings presented in Nagatani's original work, providing a detailed picture of the system's transition from order to chaos.

The relationship between the bus headway and the loading parameter Γ reveals the system's core dynamics. As shown in the bifurcation diagrams (Figure 3, panel a-d), when no speedup is applied ($S_1 = S_2 = 0$), the system's motion is regular only for very small values of Γ. As the loading increases, the headway values rapidly spread out into a wide, seemingly random band, indicating chaotic motion. However, when a speedup factor is introduced (e.g., $S_1 = S_2 = 0.2$), the system maintains a stable, regular motion over a much larger range of Γ. This clearly demonstrates that the speedup mechanism is effective at suppressing the system's chaotic tendencies.

A closer inspection of these diagrams (Figure 4, panel a-d) highlights the intricate route to chaos, particularly when the speedup parameters are asymmetric (e.g., $S_1 = 0.5, S_2 = 0.2$). In this case, the system does not transition directly to chaos but first undergoes a series of period-adding bifurcations, where the number of possible states for the headway increases in discrete steps. Within the chaotic

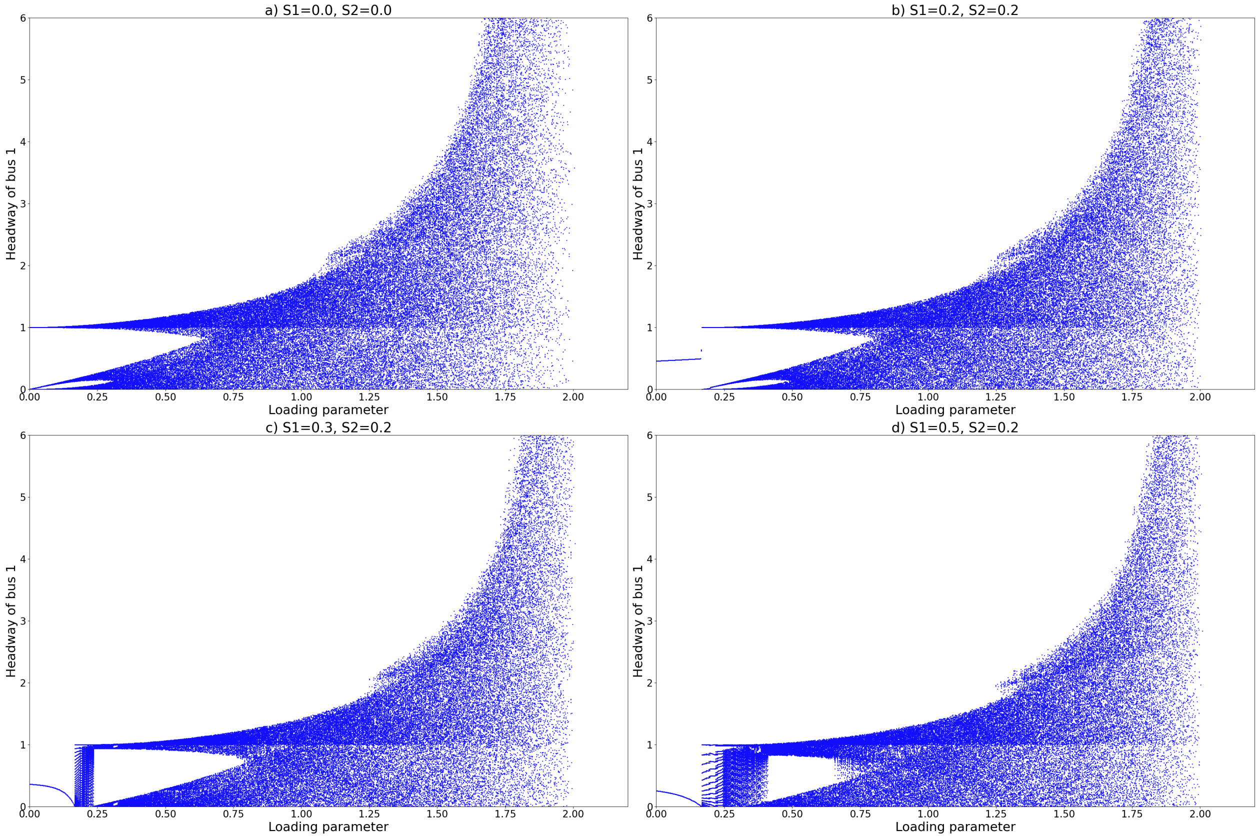

Figure 3: Plots of the time headway $H_1(m)$ of bus 1 against loading parameter Γ from sufficiently large trip m = 900 - 1000

regions, these plots also reveal complex fractal-like patterns, a classic signature of deterministic chaos.

The behavior of the bus tour times directly mirrors that of the headways. In the regular regime, tour times are constant, while in the chaotic regime, they fluctuate across a wide range of values (Figure 5).

To confirm the deterministic nature of this chaos, return maps are generated by plotting the headway of a bus on one trip against its headway on the next, $H_1(m+1)$ vs. $H_1(m)$. For periodic motion, these maps consist of a finite set of discrete points. For chaotic motion, however, the points trace a well-defined, one-dimensional curve known as a strange attractor (Figure 7). The existence of this underlying structure proves that the system's behavior, while unpredictable, is not random but is governed by the deterministic rules of the model.

For practical application, the most crucial result is the phase diagram, which maps the system's behavior across the parameter space of loading (Γ) and speedup (S) (Figure 9). This diagram clearly illustrates a boundary separating the region of regular, predictable motion from the region of periodic or chaotic motion. It shows that for any given passenger load Γ, there is a corresponding speedup value S that can be applied to push the system back into the regular regime.

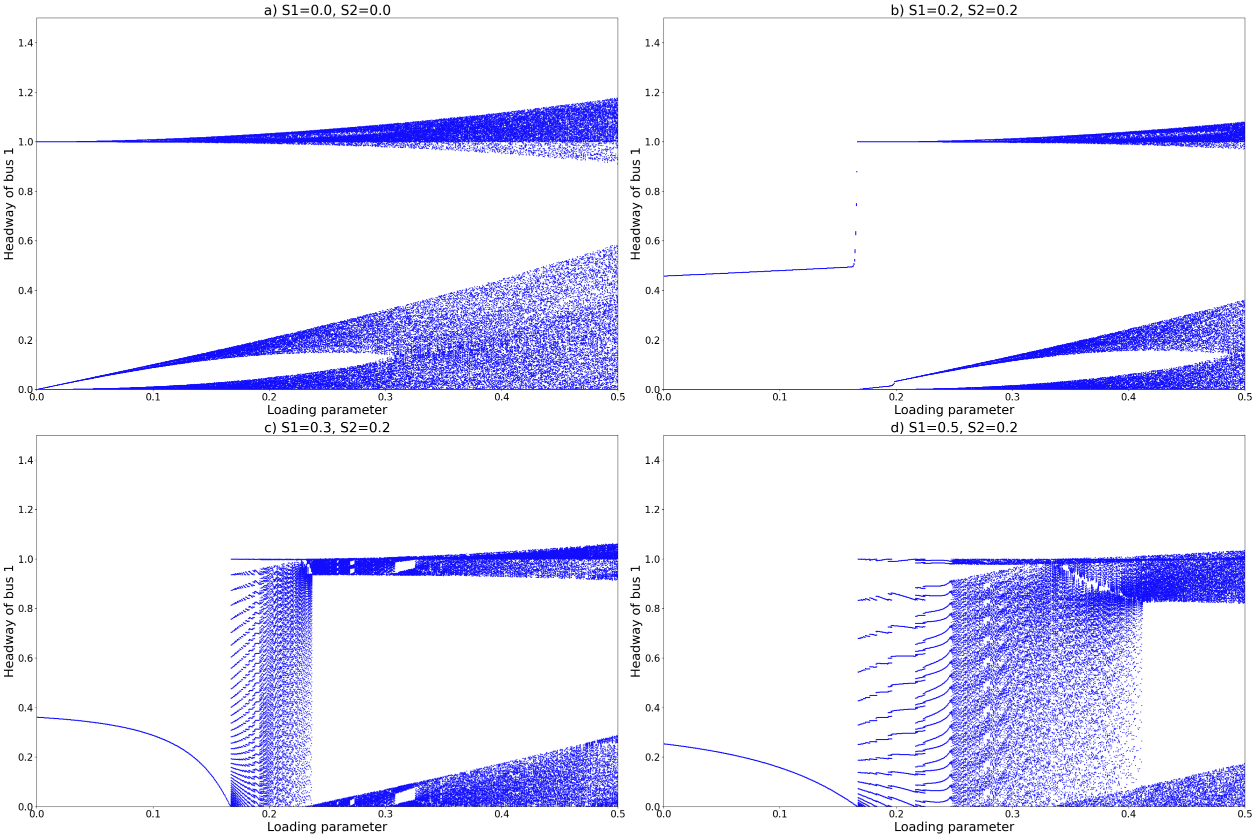

Figure 4: Enlargements of Figure 3 for $0 < \Gamma < 0.5$

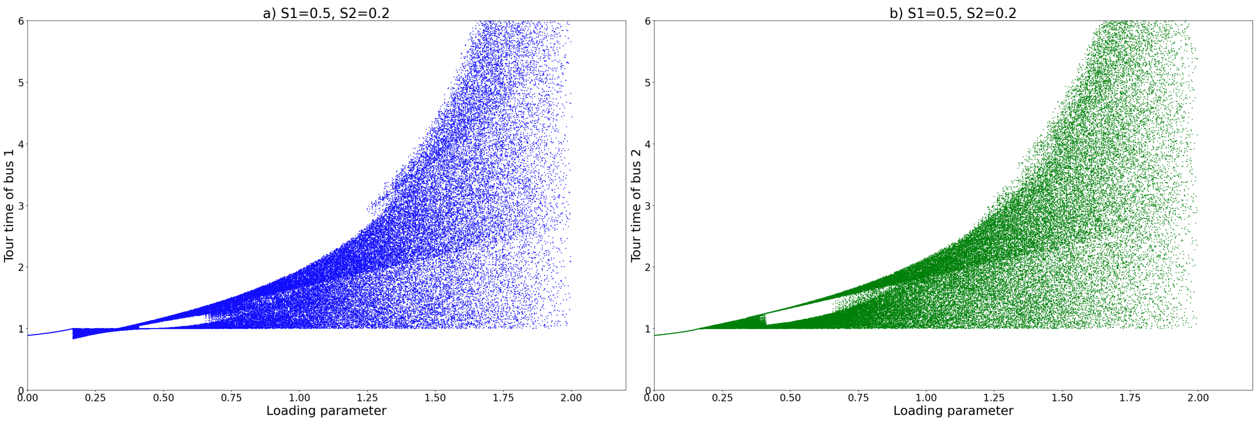

Figure 5: Plots of tour times of buses 1 and 2 against loading parameter Γ from sufficiently large trip m = 900-1000 for case of $S_1 = 0.5, S_2 = 0.2$

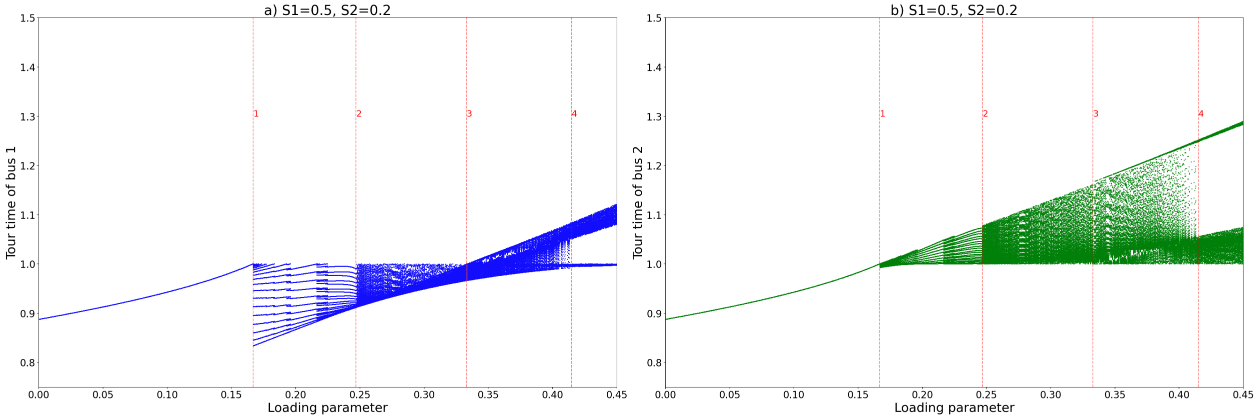

Figure 6: Enlargements of Figure 5 for $0 < \Gamma < 0.45$

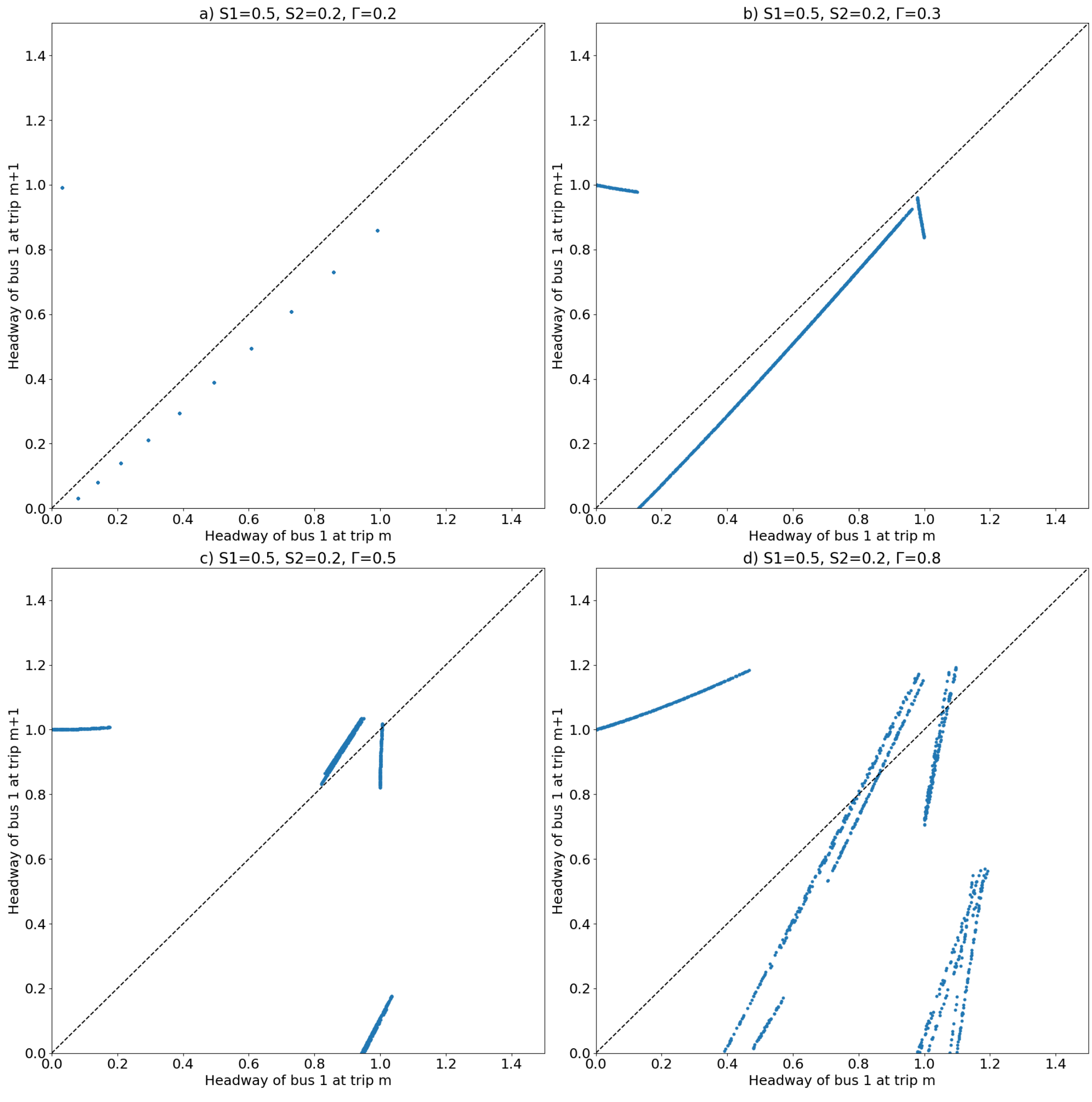

Figure 7: Plots of $H_1(m+1)$ against $H_1(m)$ from m = 1000 to 2000 for speedup parameters $S_1 = 0.5, S_2 = 0.2$

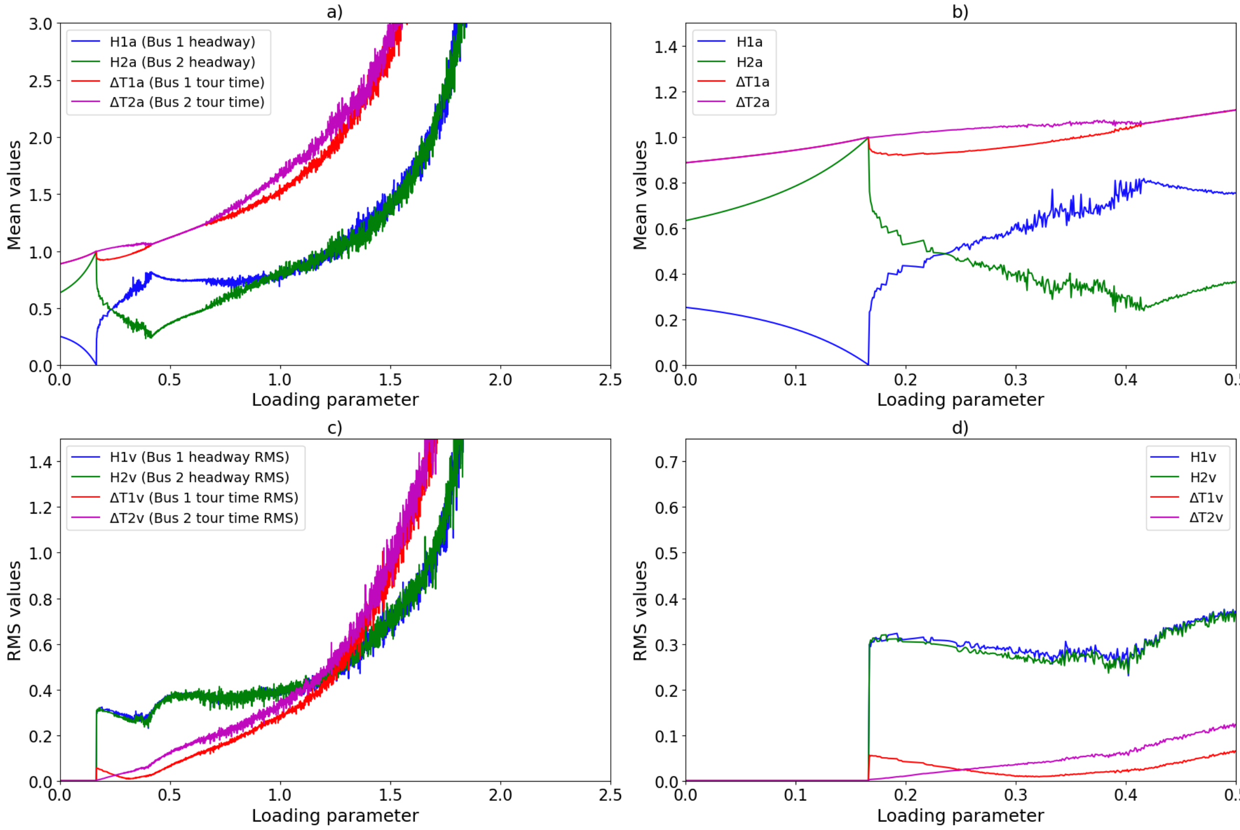

Figure 8: Plots of mean and rms headway $H_{1\text{mean}}$, $H_{2\text{mean}}$, $H_{1\text{rms}}$, $H_{2\text{rms}}$ and tour times $\Delta T_{1\text{mean}}$, $\Delta T_{2\text{mean}}$, $\Delta T_{1\text{rms}}$, $\Delta T_{2\text{rms}}$ against loading parameter Γ for speedup parameters $S_1 = 0.5$ and $S_2 = 0.2$

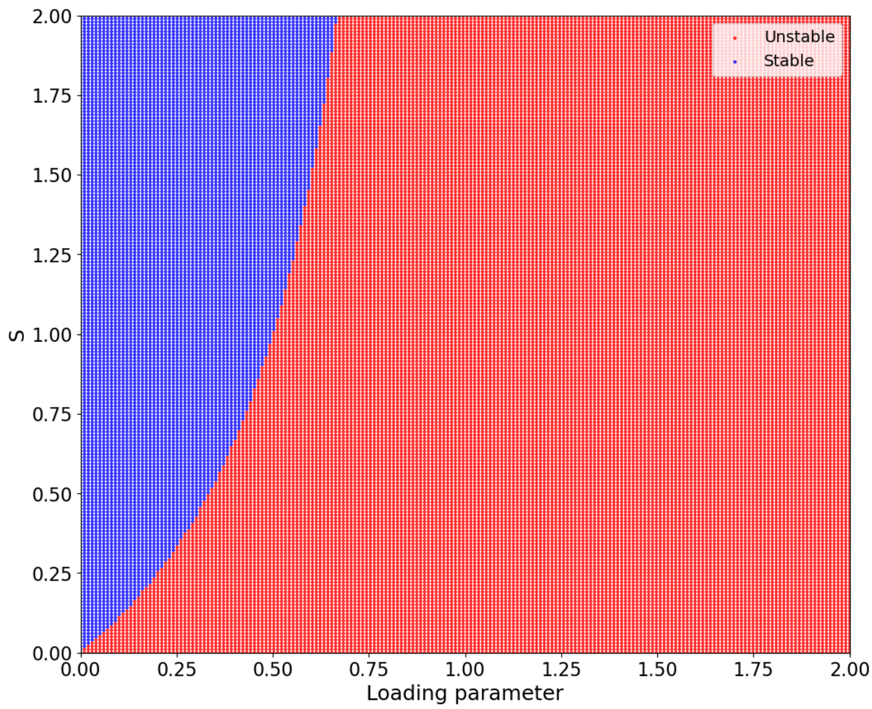

Figure 9: Phase diagram (region map) for the regular and periodic (or chaotic) motions in phase space (Γ, S_1) of loading and speedup parameters for $S_1 = S_2$

6 Discussion

A deeper analysis of the model requires an examination of its sensitivity to parameter variations and the robustness of its conclusions to changes in its underlying assumptions.

6.1 Sensitivity analysis

The loading parameter Γ acts as a critical bifurcation parameter, controlling the transitions between different dynamical regimes. The bifurcation diagrams (Figures 3 and 4) show that the system's state is highly sensitive to Γ. For low values, the system has a single, stable state (regular motion). As Γ crosses a certain threshold, the system bifurcates, and the stable state gives way to periodic oscillations, which in turn devolve into chaos through further bifurcations. This extreme sensitivity, where an infinitesimal change in a parameter can lead to a qualitative change in the long-term behavior (e.g., from periodic to chaotic), is a defining feature of chaotic systems.

The model is also highly sensitive to the speedup parameters, which function as the primary control mechanism. As shown in the phase diagram (Figure 9), the threshold for the onset of chaos is a direct function of S. Increasing S enhances the system's stability, pushing the transition to chaos to higher loading levels. Furthermore, the system is sensitive to disparities between the speedup parameters. When $S_1 \neq S_2$, the route to chaos becomes significantly more complex, featuring intricate periodic windows within the chaotic domain. This suggests that for optimal control, not only is the magnitude of the speedup important, but so is its uniform application across the fleet.

6.2 Robustness of the model

Robustness assesses how well a model's core conclusions hold if its simplifying assumptions are relaxed or altered to be more realistic.

The model assumes that a bus has unlimited capacity and can always board all waiting passengers. This is a significant simplification. In reality, a full bus would leave passengers behind, who would then be added to the queue for the next bus. This would introduce a new saturation effect and another layer of feedback into the model. While this would certainly alter the specific quantitative results (e.g., the exact location of the transition points), it is unlikely to eliminate the fundamental chaotic behavior. The core mechanism - delay caused by passenger interaction leading to headway variability - would persist. The model's conclusion that this feedback can lead to chaos is therefore likely robust.

The model assumes a constant, deterministic rate of passenger arrivals, μ. Real-world passenger arrivals are stochastic. Introducing randomness would transform the model from a purely deterministic system into a stochastic one. The sharp, well-defined bifurcations and strange attractors seen in the plots would likely become "blurred" or "fuzzy." However, the underlying deterministic skeleton would still govern the general dynamics. The chaotic attractors would likely persist as regions in the state space that the system preferentially occupies. Therefore, the main conclusion - that the system has an inherent tendency towards irregularity that can be managed by speedup - remains robust even in the presence of noise.

The model employs simple linear relationships for passenger accumulation and speedup response. The assumption of linearity can be relaxed without necessarily changing the qualitative conclusions. For example, if the speedup function $V(B_i(m))$ were a more complex nonlinear function, as long as it remained an increasing function of the number of passengers (and thus the delay), the fundamental control principle would hold. The phase portrait of the system would be distorted, but the essential topological features - like the existence of stable and unstable regions - would likely be preserved. The model's core finding is robust because it depends on the general feedback structure, not the precise functional form of the interactions.

7 Conclusion

This re-examination of Nagatani's model provides a clear and compelling illustration of how complex, chaotic behavior can arise from a simple, deterministic transportation system. The study demonstrates that the irregular arrival patterns and bunching of shuttle buses can be explained by an inherent feedback loop connecting passenger loading delays to bus headways, without needing to invoke external random factors.

The simulation results conclusively show that as the passenger load increases, the system undergoes dynamic transitions from regular motion to periodic oscillations and finally to deterministic chaos. These transitions, and the complex structures they create, were successfully reproduced, confirming the validity of the original findings.

The most significant contribution of the work is the demonstration of effective chaos control. By introducing a speedup parameter that allows buses to compensate for delays, the chaotic fluctuations can be suppressed, and a regular, predictable service can be restored. The phase diagram constructed from the simulations serves as a powerful and practical tool, providing a clear map for bus operators to determine the necessary level of speedup to ensure schedule stability under varying passenger loads.

Ultimately, this analysis underscores the critical importance of understanding nonlinear dynamics in the management of real-world systems. It shows that by identifying the underlying mechanisms that drive complexity, it becomes possible to design targeted and effective control strategies. The stability of a shuttle bus schedule, as this study emphasizes, depends on a finely tuned balance between passenger demand and the operational capability to respond to it, offering a valuable lesson for the design of more robust and reliable transportation services.

References

[1] Takashi Nagatani. Chaos control and schedule of shuttle buses. *Physica A: Statistical Mechanics and its Applications*, 371(2):683–691, 2006. ISSN 0378-4371. doi: https://doi.org/10.1016/j.physa.2006.04.056. URL https://www.sciencedirect.com/science/article/pii/S0378437106004754.

[2] Mark M. Meerschaert. Chapter 4 - Introduction to Dynamic Models. In *Mathematical Modeling (Fourth Edition)*, pages 115–137. Academic Press, Boston, 2013. ISBN 978-0-12-386912-8. doi: https://doi.org/10.1016/B978-0-12-386912-8.50004-X. URL https://www.sciencedirect.com/science/article/pii/B978012386912850004X.