Show that

1. $\lim_{z \to z_0} \mathrm{Re}(z) = \mathrm{Re}(z_0)$

   Let $\epsilon > 0$.

   Let $0 < \delta < \epsilon$.

   Then, for any z, z_0 such that, $|z - z_0| < \delta$, we have the following:

$$|\mathrm{Re}(z) - \mathrm{Re}(z_0)| = |\mathrm{Re}(z - z_0)| \leq |z - z_0| < \delta < \epsilon$$

   So, $\lim_{z \to z_0} \mathrm{Re}(z) = \mathrm{Re}(z_0)$

2. $\lim_{z \to z_0} \bar{z} = \bar{z}_0$

   Let $\epsilon > 0$.

   Let $0 < \delta = \epsilon$.

   Then, for any z, z_0 such that, $|z - z_0| < \delta$, we have the following:

$$|\bar{z} - \bar{z}_0| = |\overline{z - z_0}| = |z - z_0| < \delta = \epsilon$$

   So, $\lim_{z \to z_0} \bar{z} = \bar{z}_0$.

3. $\lim_{z \to 0} \frac{\bar{z}^2}{z} = 0$

   Let $\epsilon > 0$.

   Let $\delta < \frac{\epsilon}{2}$.

   For any real number a, b, we have that $(a - b)^2 = a^2 + b^2 - 2ab \geq 0$, or $a^2 + b^2 \geq 2ab$.

   For any $z = x + iy$ such that $|z| < \delta$, we have the following:

$$\begin{aligned}
|\frac{\bar{z}^2}{z}| &= |\frac{x^2 - y^2 - 2ixy}{x + iy}| \\
&= |x - iy - \frac{2ixy}{x + iy}| \\
&\leq |\bar{z}| + |\frac{2ixy}{x + iy}| \\
&= |z| + \frac{|2xy|}{|z|} \\
&\leq |z| + \frac{|x^2 + y^2|}{|z|} = |z| + \frac{|z|^2}{|z|} = 2|z| < 2\delta < \epsilon.
\end{aligned}$$

   So, $\lim_{z \to 0} \frac{\bar{z}^2}{z} = 0$.

4. If a function $f(z)$ is continuous and nonzero at z_0, then there exists some neighborhood of z_0 such that $f(z) \neq 0$ for all points in said neighborhood.

   Let $\epsilon = \frac{|f(z_0)|}{2} > 0$.

   Note that, for such ϵ, there exists $\delta > 0$ such that, if $|z - z_0| < \delta$,

$$\begin{aligned} |f(z_0)| &= |f(z_0) - f(z) + f(z)| \\ &\leq |f(z_0) - f(z)| + |f(z)| \\ &< \epsilon + |f(z)| \end{aligned}$$

   So, $|f(z)| > |f(z_0)| - \epsilon = \epsilon$, or $f(z) \neq 0$. $\square$