1. Use the definition of continuity to directly prove that $f : \mathbb{R} \to \mathbb{R}; x \mapsto x^2$ is continuous.

Let $\epsilon > 0$, $c \in \mathbb{R}$ be given.

Let $\delta > 0$ such that $\delta < \sqrt{\epsilon + c^2} - c$.

It follows, $\delta - \sqrt{\epsilon + c^2} + c < 0$

$\Leftrightarrow (\delta + c - \sqrt{\epsilon + c^2})(\delta + c + \sqrt{\epsilon + c^2}) < 0$ since both δ and $c + \sqrt{\epsilon + c^2}$ are positive

$\Leftrightarrow (\delta + c)^2 - (\epsilon + c^2) < 0$

$\Leftrightarrow \delta^2 + 2c\delta - \epsilon < 0$

$\Leftrightarrow \delta(\delta + 2c) < \epsilon$

Note that, with $|x - c| < \delta$, we have $|x + c| < \delta + 2c$, so $|x^2 - c^2| = |x - c||x + c| < \delta(\delta + 2c) < \epsilon$.

Thus, $f : \mathbb{R} \to \mathbb{R}; x \mapsto x^2$ is continuous.$\square$

2. Define

$$f : \mathbb{R} \to \mathbb{R}; x \mapsto \begin{cases} x & x \in \mathbb{Q} \\ \\ x^2 & else \end{cases}$$

Prove that f is continuous at 1 and discontinuous at 2.

(1) Let $\{x_n\}$ be any convergent sequence of rational number such that $\{x_n\} \to 1$

Since $\lim_{n\to\infty} f(\{x_n\}) = 1$ and $f(1) = 1$, f is continuous at 1.

(2) Let $\{x_n\}$ be any convergent sequence of irrational number such that $\{x_n\} \to 2$

Since $\lim_{n\to\infty} f(\{x_n\}) = 4$, but $f(2) = 2$, f is not continuous at 2.$\square$

3. Give examples of functions $f, g : \mathbb{R} \to \mathbb{R}$ so that $h(x) \coloneqq f(x) + g(x)$ is

continuous but f and g are not continuous.

Let $f : \mathbb{R} \to \mathbb{R}; x \mapsto \frac{1}{x}$ and $h : \mathbb{R} \to \mathbb{R}; x \mapsto 1 - \frac{1}{x}$. Then $h(x)$ will be $h : \mathbb{R} \to \mathbb{R}; x \mapsto 1$.

Note that, f, g are not continuous at $x = 0$, and h is continuous in $\mathbb{R}$. $\square$

4. Suppose $f : S \to \mathbb{R}$ is continuous. Let A be any nonempty subset of S. Prove that $f|_A$ is also continuous.

Let $a \in A \subset S$. Since f is continuous in S, f is continuous at $x = a$. Thus, $f|_A$ is continuous.$\square$

5. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) = 0$ and suppose $f : \mathbb{R} \to \mathbb{R}$ is such that $|f(x) - f(y)| \leq g(x - y)$ for all x and y. Show that f is continuous.

Let $\epsilon > 0$ and $c \in \mathbb{R}$ be given.

Since g is continuous at $x = 0$, $\exists \delta > 0$ such that $|(x - c) - 0| < \delta$, $|g(x - c) - g(0)| = |g(x - c)| < \epsilon$.

Since $|f(x) - f(y)| \leq g(x - y)$ for all x and y, it is also true with $y = c$, which is $|f(x) - f(c)| \leq g(x - c)$.

Finally, with $|x - c| < \delta$, we have $|f(x) - f(c)| \leq g(x - c) \leq |g(x - c)| < \epsilon$. Thus f is continuous.$\square$