Let m be the Lebesgue measure as described in class. Let f and g be simple functions, and $c \in \mathbb{R}$.

1. Prove that $f + g$ is a simple function
2. Prove that cf is a simple function
3. Prove that $\int f + gdm = \int fdm + \int gdm$
4. Prove that $\int cfdm = c \int fdm$
5. Prove that if $f \leq g$ on $\mathbb{X}$, then $\int fdm \leq \int gdm$.

Proof:

1. Let E and F be the sequences of measurable sets for f and g.

   Define $G = \{E_i \cap F_j | E_i \in E, F_j \in F\}$. Then G contains all mesurable sets by both f and g. Define $\alpha_{G_n} = \alpha_{E_i} + \alpha_{F_j}$ and $\chi_{G_n} = \max(\chi_{G_n \setminus E_i}, \chi_{G_n \setminus F_j})$ where $G \ni G_n = E_i \cap F_j$.

   Then $f + g = \sum \alpha_{G_n} \chi_{G_n}$ is simple.

2. $cf = c \sum \alpha_{E_n} \chi_{E_n} = \sum c\alpha_{E_n} \chi_{E_n}$, and $c, \alpha_{E_n} \in \mathbb{R}$, so it is simple.

3. Define $\mu(G_n) = \mu(G_n \cap E_i)$ where $G_n \subseteq E_i$.

   Then $\int f + gdm = \sum \alpha_{G_n} \mu(G_n) = \sum \alpha_{E_i} \mu(G_n) + \sum \alpha_{F_j} \mu(G_n) = \int fdm + \int gdm$, since $\bigcup G_n = \bigcup E_i = \bigcup F_j$.

4. $\int cfdm = \int \sum c\alpha_{E_i} \chi_{E_i} dm = \sum c\alpha_{E_i} \mu(E_i) = c \sum \alpha_{E_i} \mu(E_i) = c \int fdm$.

5. Rewrite simple functions f and g with G. Since $f \leq g$ on $\mathbb{X}$, for all $x \in \mathbb{X}$, $f(x) \leq g(x)$, and by using G_n, we have $\int fdm \leq \int gdm$.

$\square$