1. Show that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) \coloneqq sin(\frac{1}{x})$ is not uniformly continuous.

Let $\epsilon = 2$.

Let $x_n = \frac{1}{\frac{\pi}{2}+2n\pi}$ and $y_n = \frac{1}{\frac{3\pi}{2}+2n\pi}$.

Consider $|f(x) - f(y)| = |\sin(\frac{\pi}{2} + 2n\pi) - \sin(\frac{3\pi}{2} + 2n\pi)| = |\sin\frac{\pi}{2} - \sin\frac{3\pi}{2}| = 2 \geq \epsilon$,

while $|x_n - y_n| = |\frac{1}{\frac{\pi}{2}+2n\pi} - \frac{1}{\frac{3\pi}{2}+2n\pi}| = |\frac{2}{\pi+2n\pi} - \frac{2}{3\pi+2n\pi}| = \frac{2}{\pi}|\frac{1}{4n+1} - \frac{1}{4n+3}| = \frac{2}{\pi}|\frac{2}{(4n+1)(4n+3)}|$.

Since $|\frac{2}{(4n+1)(4n+3)}|$ can be made arbitrarily small, f is uniformly continuous. $\square$

2. Let $f : S \to \mathbb{R}$ be uniformly continuous. Let $A \subset S$. Prove that $f|_A$ is uniformly continuous.

Let $\epsilon > 0$ be given.

Since f is uniformly continuous on S, for some $\delta > 0$, $\forall x, y \in S$ that if $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$.

Then $\forall x, y \in A \subset S$ that if $|x - y| < \delta, |f(x) - f(y)| < \epsilon$. So $f|_A$ is uniformly continuous. $\square$

3. Let A, B be intervals. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be uniformly continuous such that $f(x) = g(x)$ for all $x \in A \cap B$. Define the function

$$h(x) = \begin{cases} f(x) & x \in A \\ \\ g(x) & x \in B \setminus A \end{cases}$$

Prove that h is uniformly continuous.

Let $x, y \in A \cup B$.

Let $\epsilon > 0$ be given.

Case 1: x and y both in A or B.

Then, for either f or g, $\exists \delta > 0$ such that if $|x - y| < \delta$, $|h(x) - h(y)|$ will be equal to either $|f(x) - f(y)| < \epsilon$ or $|g(x) - g(y)| < \epsilon$, since both are uniformly continuous.

Case 2: x and y are in different set.

Assume that $A \cap B \neq \emptyset$.

Let $z \in A \cap B$.

Since f is uniformly continuous, $\exists \delta_1 > 0$ such that $\forall x, z \in A$ such that if $|x - z| < \delta_1, |f(x) - f(z)| < \frac{\epsilon}{2}$.

Since g is uniformly continuous, $\exists \delta_2 > 0$ such that $\forall y, z \in B$ such that if $|z - y| < \delta_2$, $|g(z) - g(y)| < \frac{\epsilon}{2}$.

Then, $\exists \delta_1 + \delta_2 > 0$ such that

$$|x - y| = |x - z + z - y| \leq |x - z| + |z - y| < \delta_1 + \delta_2,$$

we have

$$\begin{aligned} |h(x) - h(y)| &= |f(x) - f(z) + g(z) - g(y)| \\ &\leq |f(x) - f(z)| + |g(z) - g(y)| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

So h is uniformly continuous. $\square$

In case $A \cap B = \emptyset$.

Consider $A = [1, 2)$ and $B = [2, 3]$.

Let $f : A \to \mathbb{R}; x \mapsto x$ and $g : B \to \mathbb{R}; x \mapsto x^2$.

For any $\epsilon > 0$, let $\delta = \epsilon > 0$. Then for all x and y such that $|x - y| < \delta$, $|f(x) - f(y)| = |x - y| < \delta = \epsilon$. So, f is uniformly continuous.

Since g is a polynomial, it is continuous on closed interval $[2, 3]$, the g is uniformly continuous.

We have $1 \leq f < 2$ and $4 \leq g \leq 9$, so $2 < |f - g| < 8$.

Let $\epsilon = 1$, then $\forall x \in A, y \in B, |h(x) - h(y)| = |f(x) - g(y)| > \epsilon$, so h is not uniformly continuous.

4. Suppose $f : S \to \mathbb{R}$ and $g : [0, \infty) \to [0, \infty)$ are functions, g is continuous at 0, $g(0) = 0$, and whenever x, y are in S, we have $|f(x) - f(y)| \leq g(|x - y|)$. Prove that f is uniformly continuous.

Let $\epsilon > 0$ be given.

Since g is continuous at $x = 0$, $\exists \delta > 0$ such that $|x - y| - 0 < \delta$, $|g(|x - y|) - g(0)| = g(|x - y|) < \epsilon$.

For all $x, y \in S$ that $|x - y| < \delta$, we have $|f(x) - f(y)| \leq |g(|x - y|)| < \epsilon$. Thus f is uniformly continuous.$\square$