Let $m$ be the Lebesgue measure as described in class.

1. Compute $\int x^2\ dm$ over the interval $[0,1]$.

2. Compute $\int f\ dm$ over the interval $[0,1]$, where $f(x) = \begin{cases} x^2 & x \in \mathbb{Q}^c \\ 0 & x \in \mathbb{Q} \end{cases}$.

3. Compute $\int g\ dm$ over the interval $[0,1]$, where $g(x) = \begin{cases} x^2 & x \in \mathbb{R} \setminus (\mathbb{Q} \cup \mathcal{C}) \\ \infty & x \in \mathbb{Q} \\ 0 & x \in \mathcal{C} \setminus \mathbb{Q} \end{cases}$, where

   $\mathcal{C}$ is the Cantor Set.

Proof:

1. Let $E_{0,1} = [0,1]$, and let the function $S_0 = 0 \times \chi_{[0,1]}$, and $\int S_0\ dm = 0$

   Let $E_{1,0} = [0, \frac{1}{2})$ and $E_{1,1} = [\frac{1}{2}, 1]$, and let the function $S_1 = 0 \times \chi_{E_{1,0}} + \frac{1}{2^2} \times \chi_{E_{1,1}}$, and $\int S_1 dm = \frac{1}{8}$.

   Let $E_{n,i} = [\frac{i}{2^n}, \frac{i+1}{2^n})$, and let the function $S_n = \sum\limits_{i=0}^{2^n-1} \frac{i^2}{4^n} \chi_{E_{n,i}}$.

   $$\begin{aligned}
   \text{Then } \int S_n\ dm &= \sum_{i=0}^{2^n-1} \frac{i^2}{4^n}\ \mu(E_{n,i}) \\
   &= \sum_{i=0}^{2^n-1} \frac{i^2}{4^n} \frac{1}{2^n} \\
   &= \frac{1}{8^n} \sum_{i=0}^{2^n-1} i^2 \\
   &= \frac{2^n(2^n-1)(2^{n+1}-1)}{6 \times 8^n} \\
   &= \frac{2^{2n+1} - 2^{n+1} - 2^n + 1}{6 \times 4^n} \\
   &= \frac{2 \times 4^n}{6 \times 4^n} - \frac{2}{6 \times 2^n} - \frac{1}{6 \times 2^n} + \frac{1}{6 \times 4^n} \\
   &= \frac{1}{3} - \frac{2}{6 \times 2^n} - \frac{1}{6 \times 2^n} + \frac{1}{6 \times 4^n}
   \end{aligned}$$

   Since, for all $n$, $S_n \leq x^2$, so $\int x^2 dm = \lim\limits_{n \to \infty} \int S_n\ dm = \frac{1}{3}$.

2. Consider the same function $S_n$ as part 1, but $E_{n,i} = [\frac{i}{2^n}, \frac{i+1}{2^n}) \cap \mathbb{Q}^c$.

   With the same calculation, $\int S_n\ dm = \frac{1}{3} - \frac{2}{6 \times 2^n} - \frac{1}{6 \times 2^n} + \frac{1}{6 \times 4^n}$

   Since for all $n$, $S_n \leq f$, so $\int f\ dm = \lim\limits_{n \to \infty} \int S_n\ dm = \frac{1}{3}$.

3. Consider the same function $S_n$ as part 1, but $E_{n,i} = [\frac{i}{2^n}, \frac{i+1}{2^n}) \setminus (\mathbb{Q} \cap \mathcal{C})$.

   Since for all $n$, $S_n \leq g$, so $\int g\ dm = \lim\limits_{n \to \infty} \int S_n\ dm = \frac{1}{3}$.