Suppose a_k is a decreasing sequence such that $a_k \geq 0$ for all $k \in \mathbb{N}$. Prove that the series $\Sigma_{k=1}^{\infty} a_k$ converges if and only if the series $\Sigma_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Note that since $a_k \geq 0$, both series are monotone. So we only have to show that a series is bounded if and only if the other is bounded. Let the partial sum series of Σa_k and $\Sigma 2^k a_{2^k}$ be $T_m = \Sigma_{m=1}^{m} a_m$ and $S_m = \Sigma_{m=0}^{m} 2^m a_{2^m}$.

Suppose $\Sigma\ a_k$ is bounded, so T_m is also bounded. Note that, for any $k \in \mathbb{N}$,

$$
\begin{aligned}
T_{2^k} &= a_1 + a_2 + a_3 + \cdots + a_{2^k} \\
&= a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\
&\geq a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} = \frac{1}{2}(S_k + a_1)
\end{aligned}
$$

So, $2T_{2^k} - a_1 \geq S_k$, and since T_{2^k} is bounded, S_k and $\Sigma_{k=0}^{\infty} 2^k a_{2^k}$ is bounded and converges by Monotone Convergence Theorem.

Suppose $\Sigma\ 2^k a_{2^k}$ is bounded, so S_m is also bounded. Note that, for any $k \in \mathbb{N}$,

$$
\begin{aligned}
S_k &= a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} \\
&\geq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\
&= T_{2^{k+1}-1}
\end{aligned}
$$

So, for any $m \in \mathbb{N}$ such that $m \leq 2^{k+1} - 1$, we have that $S_k \geq T_{2^{k+1}-1} \geq T_m$, and since S_k is bounded, T_m and $\Sigma_{k=1}^{\infty} a_k$ is bounded and converges by Monotone Convergence Theorem. $\square$