1. Prove by induction that $\sum\limits_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

2. Use 1 and the same approach we did in class for $f(x) = x$ on $[0, 1]$ to verify that $g(x) = x^2$ is indeed Riemann Integrable on $[0, 1]$, and compute the integral.

Proof:

1. Base case: $n = 1$: $1^2 = \frac{1\times2\times3}{6}$.

   Induction hypothesis: let $1^2 + \ldots k^2 = \frac{k(k+1)(2k+1)}{6}$.

   Consider $1^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

   $= \frac{(k+1)(2k^2+k+6(k+1))}{6} = \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$.

   Thus, $\sum\limits_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

2. Consider the partition $P = \{0, \frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{2^n-1}{2^n}, 1\}$.

   For each k, $m_k = \inf(x^2 | x \in [\frac{k-1}{2^n}, \frac{k}{2^n}]) = \frac{(k-1)^2}{4^n}$ and $M_k = \sup(x^2 | x \in [\frac{k-1}{2^n}, \frac{k}{2^n}]) = \frac{k^2}{4^n}$.

   Then, $L(f, P) = \sum\limits_{k=1}^{2^n} m_k(\frac{k}{2^n} - \frac{k-1}{2^n})$

   $$= \sum_{k=1}^{2^n} \frac{(k-1)^2}{4^n} \times \frac{1}{2^n}$$

   $$= \frac{2^n(2^n-1)(2^{n+1}-1)}{6\times8^n}$$

   $$= \frac{2^{2n+1}-2^{n+1}-2^n+1}{6\times4^n}$$

   $$= \frac{2\times4^n}{6\times4^n} - \frac{2}{6\times2^n} - \frac{1}{6\times2^n} + \frac{1}{6\times4^n}$$

   $$= \frac{1}{3} - \frac{2}{6\times2^n} - \frac{1}{6\times2^n} + \frac{1}{6\times4^n}$$

   As $L(f) = \sup\{L(f, P) | P \textit{ is a partition}\}$, $L(f) = \frac{1}{3}$.

Similarly, $U(f,P) = \sum_{k=1}^{2^n} M_k(\frac{k}{2^n} - \frac{k-1}{2^n})$

$$= \sum_{k=1}^{2^n} \frac{k^2}{4^n} \times \frac{1}{2^n}$$

$$= \frac{2^n(2^n+1)(2^{n+1}+1)}{6\times 8^n}$$

$$= \frac{2^{2n+1}+2^{n+1}+2^n+1}{6\times 4^n}$$

$$= \frac{2\times 4^n}{6\times 4^n} + \frac{2}{6\times 2^n} + \frac{1}{6\times 2^n} + \frac{1}{6\times 4^n}$$

$$= \frac{1}{3} + \frac{2}{6\times 2^n} + \frac{1}{6\times 2^n} + \frac{1}{6\times 4^n}$$

As $U(f) = \inf\{U(f,P) | P \text{ } is \text{ } a \text{ } partition\}$, $U(f) = \frac{1}{3}$.

Since $U(f) = L(f) = \frac{1}{3}$, $\int_0^1 x^2\, dx$ exists and $\int_0^1 x^2\, dx = \frac{1}{3}$. $\square$