1.

Prove that if $\{x_n\}$ is a convergent sequence, $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n^k = (\lim_{n\to\infty} x_n)^k$$

Let $\lim_{n\to\infty} x_n = a$. We WTS that $(\lim_{n\to\infty} x_n)^k = a^k$ by induction.

Base case: $k = 1$, then we have $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} x_n^1 = \lim_{n\to\infty} x_n = a^1$, which is true.

Induction hypothesis: Suppose that $\lim_{n\to\infty} x_n^k = a^k$.

We will prove that $\lim_{n\to\infty} x_n^{k+1} = a^{k+1}$.

Note that $x_n^{k+1} = x_n^k \times x_n$, and both sequences $\{x_n\}$ and $\{x_n^k\}$ converge, so:

$$\lim_{n\to\infty} x_n^{k+1} = \lim_{n\to\infty} x_n^k \times \lim_{n\to\infty} x_n = a^k \times a = a^{k+1}$$

So, $\lim_{n\to\infty} x_n^{k+1} = a^{k+1}$. $\square$

2.

Prove or disprove: if $\{x_n\}_{n=1}^{\infty}$ is a sequence such that $\{x_n^2\}_{n=1}^{\infty}$ converges, then $\{x_n\}_{n=1}^{\infty}$ converges.

Counterexample: let $x_n = (-1)^n$, then $\{x_n^2\}_{n=1}^{\infty} = \{1^n\}_{n=1}^{\infty} = \{1, 1, 1, \dots\}$, and this sequence converges to 1. Meanwhile, $\{x_n\}_{n=1}^{\infty} = \{-1, 1, -1, \dots\}$, and this sequence diverges. $\square$

3.

Let $\{a_n\}, \{b_n\}$ be sequences.

(a) Suppose $\{a_n\}$ is bounded and $\{b_n\}$ converges to 0. Show that $\{a_nb_n\}$ converges to 0.

Since $\{a_n\}$ is bounded, $\exists M \in \mathbb{R}$, such that $\forall n \in \mathbb{N}, |x_n| \leq M$.

Since $\{b_n\}$ converges to 0, $\forall \epsilon > 0, \exists N \in \mathbb{N}$, such that $\forall n \geq N, |b_n| < \frac{\epsilon}{M}$.

Note that, given $\epsilon > 0$ and $n \geq N$, $|a_n b_n| = |a_n||b_n| < M \times \frac{\epsilon}{M} = \epsilon$, thus $\{a_n b_n\}$ converges to 0. $\square$

(b) Find an example where $\{a_n\}$ is unbounded, $\{b_n\}$ converges to 0, and $\{a_n b_n\}$ diverges.

It is clearly that the sequence $\{a_n\}$ where $a_n = n^2$ is unbounded.

The sequence $\{b_n\}$ where $b_n = \frac{1}{n}$ converges to 0,

Note that, then the sequence $\{a_n b_n\}$ will be defined as $\{n^2 \times \frac{1}{n}\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$, and this sequence is unbounded. $\square$

(c) Find an example where $\{a_n\}$ is bounded, $\{b_n\}$ converges to some $x \neq 0$, and $\{a_n b_n\}$ is not convergent.

Let $\epsilon > 0$ be given. Let the sequence $\{a_n\}$ defined by $a_n = (-1)^n$. Clearly, this sequence is bounded since we have that $\forall n \in \mathbb{N}, a_n < 2$.

Let the sequence $\{b_n\}$ defined by $b_n = 1$. This sequence converges to 1, since $\forall n > 10$, we have $|b_n - 1| = 0 < \epsilon$.

Finally, the sequence $\{a_n b_n\}$ will be just $\{(-1)^n\}_{n=1}^{\infty}$, and this sequence diverges. $\square$

4.

Prove that $\lim_{n\to\infty}(n^2+1)^{\frac{1}{n}} = 1$.

First, we want to prove that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

Let $\epsilon > 0$ be given.

Let N be the smallest integer larger than $\frac{2}{\epsilon^2} + 1$, then $\forall n > N$, we have

$$n > \frac{2}{\epsilon^2} + 1$$

$$\Leftrightarrow n-1 > \frac{2}{\epsilon^2}$$

$$\Leftrightarrow \frac{1}{n-1} < \frac{\epsilon^2}{2}$$

$$\Leftrightarrow \frac{2}{n-1} < \epsilon^2$$

$$\Leftrightarrow n < \frac{n(n-1)}{2}\epsilon^2$$

From binomial theorem, we have that

$$(\epsilon+1)^n = \binom{n}{0}\epsilon^n + \binom{n}{1}\epsilon^{n-1} + \cdots + \binom{n}{n-2}\epsilon^2 + \binom{n}{n-1}\epsilon + 1 > \binom{n}{n-2}\epsilon^2 = \frac{n(n+1)}{2}\epsilon^2$$

From above, we have that

$$n < \frac{n(n-1)}{2}\epsilon^2 < (\epsilon+1)^n$$

$$\Leftrightarrow n^{\frac{1}{n}} < \epsilon + 1$$

$$\Leftrightarrow |n^{\frac{1}{n}} - 1| < \epsilon$$

So, $\forall n > N$, we have $|n^{\frac{1}{n}} - 1| < \epsilon$, so $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

Note that

$$(n^2)^{\frac{1}{n}} < (n^2+1)^{\frac{1}{n}} < (n^2+2n+1)^{\frac{1}{n}}$$

$$\Leftrightarrow \lim_{n\to\infty} n^{\frac{2}{n}} < \lim_{n\to\infty}(n^2+1)^{\frac{1}{n}} < \lim_{n\to\infty}(n+1)^{\frac{2}{n}}$$

From $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$, we have that $\lim_{n\to\infty} n^{\frac{2}{n}} = \lim_{n\to\infty}(n+1)^{\frac{2}{n}} = 1$, then by Squeeze lemma, we have that $\lim_{n\to\infty}(n^2+1)^{\frac{1}{n}} = 1$. $\square$