Let $\mathcal{E} \subseteq \mathbb{R}$ be compact, and $\mathcal{E} \subseteq \mathcal{D}$, where $\mathcal{D}$ is the domain of a function f. Suppose f is continuous. Then $f(\mathcal{E}) = \{f(x) : x \in \mathcal{E}\}$ is compact.

Proof:

Let $\{a_n\}$ be a sequence in $f(\mathcal{E})$, and let $\{b_n\}$ be the sequence in $\mathcal{E}$ such that, for any n, $f(b_n) = a_n$. Since $\mathcal{E}$ is compact, there exists a convergent subsequence, $\{b_{n_i}\}$, of $\{b_n\}$, such that $\{b_{n_i}\} \to b \in \mathcal{E}$. Then, since f is continuous, $f(b_{n_i})$, which is a subsequence of $f(a_n)$ converges to $f(b) \in f(\mathcal{E})$. Thus, $f(\mathcal{E})$ is compact. $\square$

[Extreme Value Theorem]Let $\mathcal{E} \subseteq \mathbb{R}$ be compact, and $\mathcal{E} \subseteq \mathcal{D}$, where $\mathcal{D}$ is the domain of a function f. Suppose f is continuous. Then f attains its maximum and minimum on $\mathcal{E}$, i.e., there is an $e \in \mathcal{E}$ such that $f(e) \geq f(x)$ for all $x \in \mathcal{E}$, and likewise for '$\leq$'.

Proof:

From the theorem above, we have $f(\mathcal{E})$ is compact, then $f(\mathcal{E})$ is bounded.

Let $A = \inf(f(e) : e \in \mathcal{E})$, then there exists a sequence $\{x_n\}$ in $\mathcal{E}$ such that $\lim_{n\to\infty}\{f(x_n)\} \to A$. Since $\mathcal{E}$ is compact, there exists a convergent subsequence $\{x_{n_i}\} \to x \in \mathcal{E}$ of $\{x_n\}$.

Then, $f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty}\{f(x_n)\} = A$. So, $f(x) \leq f(e)$ for all $e \in \mathcal{E}$, or f attains its minimum on $\mathcal{E}$.

Similar argument can be made to sup, so f attains its maximum on $\mathcal{E}$. $\square$