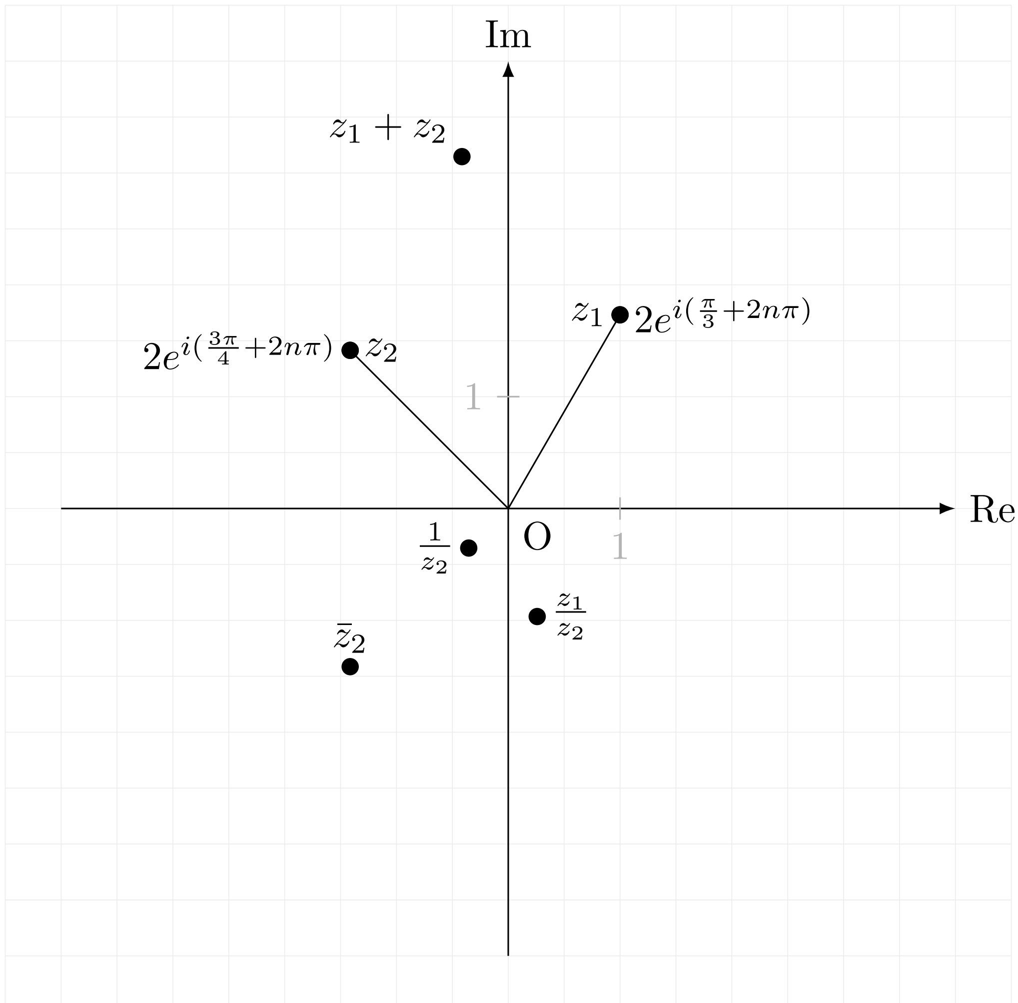


Figure 1: $\mathbb{C}$

1. $z_1 = 1 + i\sqrt{3}$ and $z_2 = -\sqrt{2} + i\sqrt{2}$

   (a) Plotting $z_1$ and $z_2$ into the plane

   (b) $z_1 + z_2 = (1 - \sqrt{2}) + i(\sqrt{2} + \sqrt{3})$

   (c) $\bar{z}_2 = -\sqrt{2} - i\sqrt{2}$

   (d) $\frac{1}{z_2} = \frac{\bar{z}_2}{|z_2|} = \frac{-\sqrt{2}-i\sqrt{2}}{4}$

(e) $z_1 = 2\exp(i(\frac{\pi}{3} + 2n\pi))$ and $z_2 = 2\exp(i(\frac{3\pi}{4} + 2n\pi))$ for $n \in \mathbb{N}$

(f) $\frac{z_1}{z_2} = \frac{1+i\sqrt{3}}{-\sqrt{2}+i\sqrt{2}} = \frac{(1+i\sqrt{3})(-\sqrt{2}-i\sqrt{2})}{4} = \frac{(\sqrt{6}-\sqrt{2})-i(\sqrt{2}+\sqrt{6})}{4}$

2. $z = \sqrt{3} - i = 2\exp(i(\frac{7\pi}{6} + 2n\pi))$ for $n \in \mathbb{N}$, so the three roots of $z$ are

- $c_0 = \sqrt[3]{2}\exp(i(\frac{7\pi}{18}))$
- $c_1 = \sqrt[3]{2}\exp(i(\frac{7\pi}{18} + \frac{2\pi}{3}))$
- $c_2 = \sqrt[3]{2}\exp(i(\frac{7\pi}{18} + \frac{4\pi}{3}))$

3. $|Im(z^2+2\bar{z}+3)| \leq |z^2+2\bar{z}+3| \leq |z^2|+2|\bar{z}|+3 = |z|^2+2|z|+3 \leq 2^2+2\times 2+3 = 11$

4. (a) $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta) = r^{-3}\cos(-3\theta) - ir^{-3}\sin(3\theta)$

So, $u(r,\theta) = r^{-3}\cos(-3\theta)$, and $v(r,\theta) = r^{-3}\sin(-3\theta)$.

Taking the partial derivatives yields,

$$ru_r = v_\theta = -3r^{-3}\cos(-3\theta)$$

and

$$u_\theta = -rv_r = -3r^{-4}\sin(-3\theta)$$

which exists for all $r \neq 0$ and any $\theta$.

So $f$ is differentiable on $\mathbb{C} \setminus \{0\}$
and $f'(r,\theta) = e^{-i\theta}(u_r + iv_r) = -e^{-i\theta}(3r^{-4}\cos(-3\theta) + i3r^{-4}\sin(-3\theta))$.

(b) $f(z) = \sin(\bar{z})$

Consider, for $x \in \mathbb{R}$, $\cosh(x) = \frac{e^x+e^{-x}}{2}$, so $\cosh(x)' = \frac{e^x-e^{-x}}{2} = \sinh(x)$. Also, $\sinh(x) = \frac{e^x-e^{-x}}{2}$, so $\sinh(x)' = \frac{e^x+e^{-x}}{2} = \cosh(x)$.

Also, $\sinh(x) = 0$ when $x = 0$ and, for all $x \in \mathbb{R}$, $\cosh(x) \neq 0$.

For $z = x + iy$,

$$\begin{aligned} f(z) = \sin(\bar{z}) &= \sin(x)\cosh(-y) + i\cos(x)\sinh(-y) \\ &= \sin(x)\cosh(y) - i\cos(x)\sinh(y) \end{aligned}$$

The components are $u(x,y) = \sin(x)\cosh(y)$ and $v(x,y) = -\cos(x)\sinh(y)$.

Taking the partial derivatives yields,

$$u_x = -v_y = \cos(x)\cosh(y)$$

and

$$v_x = u_y = \sin(x)\sinh(y)$$

For the Cauchy-Riemann to be satisfied, both equations must equal to 0.

Since $\cosh(y)$ cannot be 0, $\cos(x) = 0$. So $x = \frac{\pi}{2} + n\pi$ for $n \in \mathbb{N}$. When $x$ as above, $\sin(x) = 1$, so $\sinh(y) = 0$, or $y = 0$.

So $f$ is differentiable in all real numbers $\frac{\pi}{2} + 2n\pi$ for $n \in \mathbb{N}$,

and $f'(z) = u_x + iv_x = \cos(x)\cosh(y) + i\sin(x)\sinh(y)$.

5. (a) The set $\mathbb{C} \setminus \{0\}$ is open, and $f$ has derivatives everywhere in that set, so $f$ is analytic on $\mathbb{C} \setminus \{0\}$.

   (b) For any point described above, any neighborhoods around that point will contain some complex numbers, any of which $f$ does not have derivatives at. So $f$ is nowhere analytic in $\mathbb{C}$.

6. The principal value of $(1 + i\sqrt{3})^{4i}$,

$$\begin{aligned}
(1 + i\sqrt{3})^{4i} &= \exp(4i \ \mathrm{Log}(1 + i\sqrt{3})) \\
&= \exp\left(4i \ \left(\ln(2) + i\frac{\pi}{3}\right)\right) \\
&= \exp\left(-\frac{4\pi}{3} + i\ln(16)\right) \\
&= \frac{\cos(\ln 16)}{e^{\frac{4\pi}{3}}} + i\frac{\sin(\ln 16)}{e^{\frac{4\pi}{3}}}
\end{aligned}$$

7. The contour's parameterization: $z(t) = 2e^{it}$ for $\frac{\pi}{3} \leq t \leq \frac{3\pi}{4}$

   Then

$$\begin{aligned}
\int_{\mathcal{C}} 3z^2 - 2z + 1dz &= \int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} f(z(t))z'(t)dt \\
&= \int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} (12e^{2it} - 4e^{it} + 1)(2ie^{it})dt \\
&= \int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} 24ie^{3it} - 8ie^{2it} + 2ie^{it}dt \\
&= 8e^{3it} - 4e^{2it} + 2e^{it}\Big|_{\frac{\pi}{3}}^{\frac{3\pi}{4}} \\
&= 3\sqrt{2} + 5 + i(5\sqrt{2} + 4 + \sqrt{3})
\end{aligned}$$

8. (a) $f(z) = \log z$, and the parameterization $z(t) = 2e^{it}$ for $\frac{\pi}{2} < t < \frac{5\pi}{2}$.

   Consider $f(z(t)) = \log(2e^{it}) = \ln 2 + it$, so

$$\begin{aligned}
f(z(t))z'(t) &= (\ln 2 + it)2ie^{it} \\
&= 2i(\ln 2 + it)(\cos t + i\sin t) \\
&= (-2t\cos t - \ln 4 \sin t) + i(\ln 4 \cos t - 2t \sin t)
\end{aligned}$$

   Since $\lim_{t \to \frac{\pi}{2}^+} f(z(t))z'(t)$ exists and equals to $-\ln 4 - i\pi$, it is piecewise continuous on $[\frac{\pi}{2}, \frac{5\pi}{2}]$ and the integral exists.

   Finally,

$$\begin{aligned}
\int_{\mathcal{C}} f(z)dz &= \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} (-2t\cos t - \ln 4 \sin t) + i(\ln 4 \cos t - 2t \sin t)dt \\
&= -2t\sin t - 2\cos t + \ln 4 \cos t + i(\ln 4 \sin t + 2t\cos t - 2\sin t)\Big|_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \\
&= -4\pi
\end{aligned}$$

(b) $e^z - 2z + 1$ is the sum of three entire functions, so it is entire.

So $\int_{\mathcal{C}} f(z)dz = 0$ by Cauchy-Goursat Theorem.

(c)

$$\begin{aligned}
\int_{\mathcal{C}} \frac{z+1}{z^2-3z}dz &= \int_{\mathcal{C}} \frac{1}{z} + \frac{4}{3}\left(\frac{1}{z-3} + \frac{1}{z}\right) dz \\
&= \int_{\mathcal{C}} \frac{1}{z}dz + \frac{4}{3}\int_{\mathcal{C}} \frac{1}{z-3}dz + \frac{4}{3}\int_{\mathcal{C}} \frac{1}{z}dz \\
&= 2\pi i + \frac{4}{3} \times 0 + \frac{4}{3}2\pi i \\
&= \frac{14}{3}\pi i
\end{aligned}$$

The first and third integral is due to Cauchy's integral formula, and the second is due to Cauchy-Goursat theorem.

(d) $z^3 - i$ is a polynomial, thus is entire. Then, by Cauchy's integral formula,

$$\int_{\mathcal{C}} \frac{z^3-i}{(z-i)^3}dz = \frac{2\pi i}{2!} \times (z^3-i)''(i) = -6\pi$$