Let $\mathbb{X} \subseteq \mathbb{R}$, $\mathcal{M}$ a σ-algebra, and μ is a measure on $(\mathbb{X}, \mathcal{M})$. Prove the following:

1. If $E, F \in \mathcal{M}$, and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
2. If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$, then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$
3. If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$, then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$
4. If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ and $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$, then $\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$

Proof:

1. Note that $\mu(F) = \mu(E \cup E^c) = \mu(E) + \mu(E^c) \geq \mu(E)$.
2. It suffices to show the property holds with E_1 and E_2. Note that, $E_1 \cap E_2 \subseteq E_1 \cup E_2$, so $\mu(E_1 \cap E_2) \leq \mu(E_1 \cup E_2)$

   Note that, $\mu(E_1 \cup E_2)$
   $$\begin{aligned} &= \mu(E_1 \setminus \{E_1 \cap E_2\}) + \mu(E_2 \setminus \{E_1 \cap E_2\}) + \mu(E_1 \cap E_2) \\ &\leq \mu(E_1 \setminus \{E_1 \cap E_2\}) + \mu(E_2 \setminus \{E_1 \cap E_2\}) + \mu(\{E_1 \cup E_2\}) \\ &\leq \mu(E_1) + \mu(E_2). \end{aligned}$$
3. Let $A = \lim_{j\to\infty} \mu(E_j)$. Then $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $A - \mu(E_n) < \epsilon$.

   Since, for all $i \geq 1$, $E_i \subseteq E_{i+1}$, $\mu(E_n) = \mu(\bigcup_{i=1}^{n} E_i)$.

   Also, by similar reason, $\mu(\bigcup^{\infty} E_i) \geq \mu(\bigcup_{i=1}^{n} E_i)$.

   Finally, $\epsilon > A - \mu(E_n) = A - \mu(\bigcup^{n} E_i) \geq \mu(\bigcup^{\infty} E_i)$, so $A = \mu(\bigcup^{\infty} E_i)$.
4. From part 3, for each i, let F_i such that $E_i^c = F_i$.

   Then $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$, and

   $$\begin{aligned} \mu(\bigcap_{j=1}^{\infty} F_j) &= \mu(\mathbb{X}) - \mu((\bigcap_{j=1}^{\infty} F_j)^c) \\ &= \mu(\mathbb{X}) - \mu(\bigcup_{j=1}^{\infty} E_j) \\ &= \mu(\mathbb{X}) - \lim_{j\to\infty} \mu(E_j) \\ &= \lim_{j\to\infty} \mu(E_j^c) \\ &= \lim_{j\to\infty} \mu(F_j). \ \square \end{aligned}$$