Suppose that f_n converges to f uniformly on some $D \subseteq \mathbb{R}$. For each of the following statements, either prove it true or find a counter example disproving the claim.

1. If each f_n is uniformly continuous, then f is uniformly continuous.

2. If each f_n is bounded, then f is bounded.

3. If each f_n has a finite number of discontinuities, then f has a finite number of discontinuities.

Proof:

1. Let $\epsilon > 0$.

   Since f_n converges to f uniformly, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in D$, $|f_n(x) - f(x)| < \frac{\epsilon}{3}$.

   Since f_n is uniformly continuous, there exist $\delta > 0$ such that for all $x, y \in D$ such that $|x - y| < \delta$, $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$ for all n.

   Note that, for δ as above, for all $x, y \in D$ such that $|x-y| < \delta$ and $n > N$,

   $$
   \begin{aligned}
   |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\
   &< |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
   &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon.
   \end{aligned}
   $$

   So f is uniformly continuous.

2. Let $\epsilon > 0$.

   Since f_n converges to f uniformly, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in D$, $|f(x) - f_n(x)| < \epsilon$.

Since f_n is bounded, for each n, there exists $M_n > 0$ such that

$|f_n(x)| \leq M_n$ for all $x \in D$.

Then, for $n \geq N$, we have that

$|f(x)| = |f(x) - f_n(x) + f_n(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < M_n + \epsilon.$

So f is bounded.

3. Let $\epsilon > 0$.