1. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} y_n = 0$. Suppose that for all $k \in \mathbb{N}$ and for all $m \geq k$, we have

$$|x_m - x_k| \leq y_k$$

Show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy.

Proof:

Let $\epsilon > 0$ be given

Since $\lim_{n\to\infty} y_n = 0$, we have that $\exists N \in \mathbb{N}$ such that $\forall k > N$, we have

$$|y_k| < \epsilon$$

Observe that:

$$|x_m - x_k| \leq y_k \leq |y_k| < \epsilon$$

So, $\{x_n\}_{n=1}^{\infty}$ is Cauchy. $\square$

3. Prove or disprove: A Cauchy sequence can have an unbounded subsequence.

Proof:

Let $\{x_n\}$ be a Cauchy sequence. So $\{x_n\}$ is bounded, so $\exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, x_n \leq M$. So any subsequence $\{x_{n_k}\}$ will be bounded by M. So, there is not a subsequence of $\{x_n\}$ that is unbounded. $\square$

2. Suppose $|x_n - x_k| \leq \frac{n}{k^2}$ for all n and k. Show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy.

Proof:

Let $\epsilon > 0$ be given.

Let $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Then $\forall n \geq N$, and any $k \in \mathbb{N}$, we have the following:

$$\epsilon > \frac{1}{n} = \frac{n}{n^2} > \frac{n}{(n+k)^2} \tag{1}$$

Since $|x_n - x_k| \leq \frac{n}{k^2}$ for all n and k, it is also true when $k = n + k$, then

$$(1) \Leftrightarrow \epsilon > \frac{n}{(n+k)^2} > |x_n - x_{n+k}|$$

Since n and $n + k$ is any number larger than N, we have that $\{x_n\}_{n=1}^{\infty}$ is Cauchy. $\square$