1. $\cosh(x)$ and $\sinh(x)$ for $x \in \mathbb{R}$

   $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$

2. $\sin(iy) = \frac{e^{-y} - e^{y}}{2i} = i \sinh(y)$

3. $\cos(iy) = \frac{e^{-y} + e^{y}}{2} = \cosh(y)$

4. $\sin(z) = \sin(x + iy)$

   $$= \sin(x)\sin(iy) + \cos(x)\cos(iy)$$

   $$= i\sin(x)\sinh(y) + \cos(x)\cosh(y)$$

   $$= \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

5. $\cos(z) = \cos(x + iy)$

   $$= \cos(x)\cos(iy) - \sin(x)\sin(iy)$$

   $$= \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

6. $\cosh(y)^2 - \sinh(y)^2 = \frac{e^{2y} + e^{-2y} + 2}{4} - \frac{e^{2y} + e^{-2y} - 2}{4} = 1.$

7. $|\sin(z)|^2 = \sin(x)^2\cosh(y)^2 + \cos(x)^2\sinh(y)^2$

   $$= \sin(x)^2\sinh(x)^2 + \sin(x)^2 + \cos(x)^2\sinh(y)^2$$

   $$= \sin(x)^2 + \sinh(y)^2$$

8. $|\cos(z)|^2 = \cos(x)^2\cosh(y)^2 + \sin(x)^2\sinh(y)^2$

   $$= \cos(x)^2\sinh(x)^2 + \cos(x)^2 + \sin(x)^2\sinh(y)^2$$

   $$= \sinh(x)^2 + \cos(x)^2$$

9. Let $\sin(z)$ has zeroes at $z_0 = x_0 + iy_0$. Then $|\sin(z_0)|^2 = \sin(x_0)^2 + \sinh(y_0)^2 = 0$, or $\sin(x_0) = \sinh(y_0) = 0$. This happens if and only if $x_0 = \pi + k\pi$ for $k \in \mathbb{Z}$ and $y_0 = 0$. Thus $z_0 = \pi + k\pi$ for $k \in \mathbb{Z}$, same zeroes as real function $\sin(x)$.

   Let $\cos(z)$ has zeroes at $z_0 = x_0 + iy_0$. Then $|\cos(z_0)|^2 = \cos(x_0)^2 + \sinh(y_0)^2 = 0$, or $\cos(x_0) = \sinh(y_0) = 0$. This happens if and only if $x_0 = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$ and $y_0 = 0$. Thus $z_0 = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$, same zeroes as real function $\cos(x)$.

10. $\tan(z) = \frac{\sin(z)}{\cos(z)}$, $\cot(z) = \frac{\cos(z)}{\sin(z)}$, $\csc(z) = \frac{1}{\sin(z)}$, $\sec(z) = \frac{1}{\cos(z)}$

    Let $f(z) = \sin(z) = u(x,y) + iv(x,y) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$.

    Then $u_x = v_y = \cos(x)\cosh(y)$ and $u_y = -v_x = \sin(x)\sinh(y)$ for all x, y, $\sin(z)$ is entire in $\mathbb{C}$, and $\frac{d}{dz}\sin(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y) = \cos(z)$. Similarly, $\cos(z)$ is entire in $\mathbb{C}$, and $\frac{d}{dz}\cos(z) = -\sin(z)$.

    Then, by theorem (5) in Section 20, all four functions are analytic everywhere except in the zeroes of the denominator. Specifically, $\tan(z)$ and $\sec(z)$ are analytic everywhere except for $z_0 = \frac{\pi}{2} + k\pi$, where $\cot(z)$ and $\csc(z)$ are analytic everywhere except for $z_0 = \pi + k\pi$ for $k \in \mathbb{Z}$.

    And, by the rule of differentiation,

    $\frac{d}{dz}\tan(z) = \frac{\cos(z)\cos(z)+\sin(z)\sin(z)}{\cos(z)^2} = \sec(z)^2$

    $\frac{d}{dz}\cot(z) = -\csc(z)^2$

    $\frac{d}{dz}\csc(z) = \frac{-\cos(z)}{\sin(z)^2} = -\cot(z)\csc(z)$

    $\frac{d}{dz}\sec(z) = \frac{\sin(z)}{\cos(z)^2} = \tan(z)\sec(z)$