For each of the following, decide if the stated function μ does indeed define a measure on the provided $\mathbb{X}$. Explain your answer.

1. $\mathbb{X} \subseteq \mathbb{R}$ nonempty, $\mathcal{M}$ a σ-algebra. $\mu : \mathcal{M} \to [0, \infty]$ by $\mu(E) =$the number of elements in E for all $E \in \mathcal{M}$.

2. $\mathbb{X} \subseteq \mathbb{R}$ nonempty, $x_0 \in \mathbb{X}$, $\mathcal{M}$ a σ-algebra. $\mu : \mathcal{M} \to [0, \infty]$ by $\mu(E) = \begin{cases} 1 & x_0 \in E \\ 0 & x_0 E \end{cases}$ for all $E \in \mathcal{M}$.

3. $\mathbb{X} \subseteq \mathbb{R}$ uncountable, $\mathcal{M}$ the σ-algebra $\mathcal{M} = \{E \subseteq \mathbb{X} : E \text{ is countable or } E^c \text{ is countable}\}$. $\mu : \mathcal{M} \to [0, \infty]$ by $\mu(E) = \begin{cases} 0 & E \text{ is countable} \\ 1 & E^c \text{ is countable} \end{cases}$ for all $E \in \mathcal{M}$.

4. $\mathbb{X} \subseteq \mathbb{R}$ infinite, $\mathcal{M} = \mathcal{P}(\mathbb{X})$. $\mu : \mathcal{M} \to [0, \infty]$ by $\mu(E) = \begin{cases} 0 & E \text{ finite} \\ \infty & E \text{ infinite} \end{cases}$ for all $E \in \mathcal{M}$.

Proof

1. Yes. First $\mu(\emptyset) = 0$. Disjoint sets do not have same elements, so the number of elements of the union of them will equal to the sum of the number of elements, including infinitely many.

2. Yes. First $\mu(\emptyset) = 0$. For any element x_0, among all disjoint sets, x_0 is inside at most one of them. So the measurement of the union will be either 1, if one of them contains x_0, or 0, if none of them contains x_0. And that is equal to either 1 or 0 in both cases.

3. Yes. First $\mu(\emptyset) = 0$. For 2 countable sets, the measurement of the union is 0. The sum of measurement is 0.

   For 2 uncountable sets, the measure of the union is 1. The sum of measurement is 1.

   For a countable and an uncountable, the measure of the union is 1. The sum of measurement is 1.

4. No. For $n \in \mathbb{N}$, let $E_n = \{n\}$. The union of E_n is infinite. Its measurement is ∞, and the sum of each E_n is 0.

$\square$