(Alternating series test) Let $\{a_n\}$ be a sequence such that:

1. $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \dots$

2. $a_n \to 0$

then the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Proof:

Let $\epsilon > 0$.

Since the series a_n is a nonincreasing series and converges to 0, $\forall n \in \mathbb{N}, a_n \geq 0$ and there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $|a_n| < \epsilon$.

Define the partial sum $S_n \coloneqq a_1 - a_2 + a_3 \cdots + (-1)^{n+1}a_n$. We will show that, for any $m > n \geq N$, we have $|S_m - S_n| \leq a_{n+1} \leq |a_{n+1}| < \epsilon$.

If $m - n$ is even, we have the following:

$$
\begin{aligned}
|S_m - S_n| &= \pm(S_m - S_n) \\
&= a_{n+1} - a_{n+2} + \ \dots \ + a_{m-1} - a_m \\
&= a_{n+1} - (a_{n+2} - a_{n+3}) - \ \dots \ - (a_{m-2} - a_{m-1}) - a_m \\
&\leq a_{n+1} < \epsilon.
\end{aligned}
$$

If $m - n$ is odd, we have the following:

$$
\begin{aligned}
|S_m - S_n| &= \pm(S_m - S_n) \\
&= a_{n+1} - a_{n+2} + \ \dots \ - a_{m-1} + a_m \\
&= a_{n+1} - (a_{n+2} - a_{n+3}) - \ \dots \ - (a_{m-1} - a_m) \\
&\leq a_{n+1} < \epsilon.
\end{aligned}
$$

So the sequence of partial sum is Cauchy convergent, thus the alternating series converges. $\square$

(Generalized Comparison Test) Let $\{a_n\}$ and $\{b_n\}$ be sequences such that there exists an $N \in \mathbb{N}$ with $0 \leq a_n \leq b_n$ for all $n \geq N$. Then

1. If $\sum_{k=1}^{\infty} b_k$ converges, then so does $\sum_{k=1}^{\infty} a_k$
2. If $\sum_{k=1}^{\infty} a_k$ diverges, then so does $\sum_{k=1}^{\infty} b_k$

Proof:

1. If $\sum_{k=1}^{\infty} b_k$ converges, $\sum_{k=N}^{\infty} b_k$ converges, and by Comparison Test, $\sum_{k=N}^{\infty} a_k$ converges, then so does $\sum_{k=1}^{\infty} a_k$.
2. Proof by contrapositive of above. $\square$

(Ratio Test) Let $\{a_n\}$ be a sequence such that

1. $a_n \neq 0$
2. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$

Then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Proof:

Let $\epsilon = \frac{1-r}{2} > 0$.

Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$, with the ϵ above, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have the following:

$$||\tfrac{a_{n+1}}{a_n}| - r| < \epsilon = \tfrac{1-r}{2}$$

$$\Leftrightarrow |\tfrac{a_{n+1}}{a_n}| - r \leq ||\tfrac{a_{n+1}}{a_n}| - r| < \tfrac{1-r}{2}$$

$$\Leftrightarrow |\tfrac{a_{n+1}}{a_n}| < \tfrac{1-r}{2} + r = \tfrac{r+1}{2}$$

Let $r_0 = \frac{r+1}{2} < 1$, so $|a_{n+1}| < r_0|a_n|$. Observe that, for any $k \in \mathbb{N}$, $|a_{n+k}| < r_0\ |a_{n+k-1}| < r_0^2\ |a_{n+k-2}| < \cdots < r_0^k\ |a_n|$. Then the series $\sum_{n=N}^{\infty} |a_n| < \sum_{n=N}^{\infty} r_0^n |a_n|$, which is a geometric series and $r_0 < 1$, so it converges, and by Comparison Test, $\sum_{n=N}^{\infty} |a_n|$ converges. Thus $\sum_{n=1}^{\infty} |a_n|$ converges and $\sum_{n=1}^{\infty} a_n$ converges absolutely. $\square$