1. Prove that bounded $f(x)$ is Riemann Integrable on $[a, b]$ if and only if for all $\epsilon > 0$, there exists a partition P_ϵ of $[a, b]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.

2. Prove that if $f(x)$ is continuous on $[a, b]$, then $f(x)$ is Riemann Integrable.

Proof:

1. Let $\epsilon > 0$.

   Assume there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.

   For any partition P, $U(f) \leq U(f, P)$ and $L(f, P) \leq L(f)$,

   $U(f) - L(f) \leq U(f, P) - L(f, P) < \epsilon$, so $U(f) = L(f)$, so f is Riemann Integrable.

   Assume f is Riemann Integrable, so $U(f) = L(f)$. From the definition of inf, there exists a partition P_1 such that $U(f, P_1) - U(f) < \frac{\epsilon}{2}$. Similarly, there exists a partition P_2 such that $L(f) - L(f, P_2) < \frac{\epsilon}{2}$.

   Let $P = P_1 \cup P_2$, so P is a refinement of both P_1 and P_2. Then

   $U(f, P) - U(f) < U(f, P_1) - U(f) < \frac{\epsilon}{2}$ and

   $L(f) - L(f, P) < L(f) - L(f, P_2) < \frac{\epsilon}{2}$.

   Finally, we have that $\epsilon > U(f, P) - U(f) + L(f) - L(f, P)$

   $$or\ \epsilon > U(f, P) - L(f, P). \ \square$$

2. Let $\epsilon > 0$.

   f is continuous over a bounded interval, so f is uniformly continuous over $[a, b]$, then there exists $\delta > 0$ such that for any $x, y \in [a, b]$,

$|a-b| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{b-a}$.

Let P_n be a partition that for any $k \in [1, n], x_k - x_{k-1} < \delta$. Then

$$\begin{aligned}
U(f, P_n) - L(f, P_n) &= \sum_{k=1}^{n} M_k(x_k - x_{k-1}) - \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\
&< \tfrac{\epsilon}{b-a} \sum_{k=1}^{n} (x_k - x_{k-1}) \\
&< \tfrac{\epsilon}{b-a}(b-a) \\
&< \epsilon
\end{aligned}$$

So from Part 1, f is Riemann Integrable. $\square$