Find a power series, or disprove, such that:

1. Converges for all values $x \in \mathbb{R}$.

2. Diverges for all values $x \in \mathbb{R}$.

3. Converges absolutely for all $x \in [-1, 1]$ and diverges otherwise.

4. Converges conditionally at $x = -1$ and converges absolutely at $x = 1$

5. Converges conditionally at both $x = -1$ and $x = 1$.

Proof:

1. Yes. Let $a_i = 0$. Then $a_i x^i = 0$ for all $x \in \mathbb{R}$, so the series converges.

2. No. Let $x = 0$. Then $a_i x^i = 0$ for all i, so the series converges.

3. Yes. Let $a_n = \frac{1}{n^2}$. For $x = \pm 1$, the series becomes $\sum |\frac{1}{n^2}| = \sum \frac{1}{n^2}$, which is absolutely convergent. Since the power series converges at $x = 1$, it converges absolutely on $(-1, 1)$, therefore on $[-1, 1]$ (Theorem 6.5.1.)

   For $x > 1$, let $x = 1 + \epsilon$ for some $\epsilon > 0$. Since, as $n \to \infty$, $\sqrt[n]{n} \to 1$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $1 + \epsilon > \sqrt[n]{n}$, and the series becomes $\sum a_n x^n = \sum \frac{(1+\epsilon)^n}{n^2} > \sum \frac{n}{n^2} = \sum \frac{1}{n}$. So by Comparison Test, it diverges.

   For $x < -1$, let $x = -(1 + \epsilon)$ for some $\epsilon > 0$. The series becomes $\sum a_n x^n = \sum \frac{(-1-\epsilon)^n}{n^2} > \sum \frac{-n}{n^2} > \sum \frac{-1}{n}$. So by Comparison Test, it diverges.

4. No. Since the series converges absolutely at $x = 1$, the series $\sum |a_n|$ converges. Since the series converges conditionally at $x = -1$, the series $\sum |a_n(-1)^n| = \sum |a_n|$ diverges, which is contradictory.

5. Yes. Let $x_0 = 0$. For $n \geq 1$, let

$$a_n = \begin{cases} 0 & n \textit{ is odd} \\ \frac{(-1)^{\frac{n}{2}+1}}{\frac{n}{2}} & n \textit{ is even} \end{cases}$$

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At $x = \pm 1$, the series becomes $\sum a_n x^n = a_2 + a_4 + a_6 + \cdots = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$, which converges conditionally.

□