Problem 1: Determine how many different lottery tickets can be made?

(a) A ticket of selecting six numbers in [16] where repitition is allows and order matters.

(b) A ticket of selecting five numbers in [25] where order does not matter.

(c) A ticket of selecting four distinct numbers in [18] and order matters.

Problem 2: Binary strings are words whose characters are either 0 or 1. For each $k, n \in \mathbb{N}$, with $k \leq n$, how many binary strings of length n have exactly k ones? How many have exactly k zeros?

Problem 3: How many $n-$digit palindromes can be created using $1 - 9$ and repitition is allowed?

Problem 4: How many subsets of [20] that have:

(a) have the smallest number be 4 and the largest be 15?

(b) contains no even numbers?

(c) have size 10 and do not contain any number larger than 17?

Problem 5: Enumerate and count all $4-$lists $(\ell_1, \ell_2, \ell_3, \ell_4)$ such that $\ell_1, \ell_2, \ell_3, \ell_4 \in \mathbb{Z}$ and $1 \leq \ell_1 < \ell_2 < \ell_3 < \ell_4 \leq 6$.

Problem 6: How many 6-letter words do not simultaneously begin and end with a vowel?

Problem 7: How many 4-permutations of [10] have maximum element equal to 6? How many have the maximum element of at most 6?

Problem 8: Find the number of $4-$list of the form (x_1, x_2, x_3, x_4) such that $x_i \in \mathbb{N}$ for each integer $1 \leq x_i \leq 4$ such that $x_1 + x_2 + x_3 + 4x_4 = 5$

Problem 9: How many binary strings of length n have at least one 0 and at least one 1 simultaneously?

Problem 10: How many nonempty subsets of [10] have the product of their elements even?

Problem 11: In $n \in \mathbb{Z}^+$ how many are in the less-than relation on $[n]$? How many are in the less-than-or-equal-to relation?

Problem 12: Show that the number of subsets of $[n]$ is equal to the number of binary strings of length n.

Problem 13: Suppose $n \in \mathbb{N}$. Show that the number of even-size subset equals to the number of odd-size subsets.

Problem 14: Define $(n)^k$ and show that $\left(\!\binom{n}{k}\!\right) = \frac{(n)^k}{k!}$.

Problem 15: Suppose an ice-cream shop makes three different types of ice-cream cones. How many different ways are there to buy 9 cones and eat one of them?

Problem 16: Suppose $A \subseteq \mathbb{R}$ is a finite set and $n, k \in \mathbb{Z}^+$

(a) Prove that the number of solutions to $a_1 \leq a_2 \leq \cdots \leq a_k$ with $a_i \in A$ for each $i \in [k]$ is $\left(\!\binom{|A|}{k}\!\right)$.

(b) Determine the number of solutions to $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n$ where a_i is an odd integer in $[n]$ for each $i \in [k]$.

Problem 17: Suppose that $n \in \mathbb{Z}^+$ Use the sum principle to prove that the total number of compositions of n is 2^{n-1}.

Problem 18: Determine the number of ordered integer solutions to $x_1+x_2+x_3+x_4 = 26$ such that $1 \leq x_4 \leq 3$ and $x_i \geq i$ for each $i \in [3]$.

Problem 19: Suppose $k, n \in \mathbb{Z}^+$ with $m \leq n$ and $k \leq \lfloor \frac{m-n}{2} \rfloor$. In P_n, how many ways can we color k non-adjacent nodes red and at most m nodes green?

Problem 20: Suppose n is a positive integer. Determine the maximum and mimimum possible size of an equivalence relation on $[n]$?

Problem 21: Suppose $n \geq 3$ is an integer. Determine the number of ways to seat n people at a round table where two arrangements are considered equivalent whenever everyone at the table has the asme exact same set of neighbors.

Problem 22: How many ways are there to seat five professors and five students around a round table with 10 chairs such that the seating alternates student-professors?

Problem 23: Use the equivalence principle to prove the formula $(n)_k = \frac{n!}{(n-k)!}$

Problem 24: How many ways can we split a group of 10 people into two groups of size 3 and one group of size 4?

Bonus: Suppose $j, k \in \mathbb{Z}+$ with $n = j \cdot k$. Use the combinatorial proof to show that

$$\frac{1}{j!}\prod_{i=0}^{j-1}\binom{n-ik}{k} = \frac{n!}{(k!)^j j!}$$

Hints: try to justify the left hand side, and then generalize the argument for arbitrary value of j and k.

Problem 25: Suppose $n \in \mathbb{Z}^+$ and let $S \in \binom{[2n]}{n+1}$. Prove that there exist two element in S whose sum is $2n+1$.

Problem 26: Consider any five points in the plane that have integer coordinates.

(a) Prove that there are two points such that the midpoint of the line segment joining those two points also has integer coordinates.

(b) Show the counterexample for 4 points.

Problem 27: Show a counterexample of n^2 sequence of distinct real numbers can only have at most n increasing or decreasing subsequence.

Problem 28: Let $n \in \mathbb{Z}^+$ and $k \in \mathbb{N}$. Suppose exactly k nodes in P_n are colored blue. Prove that there exists a blue node and a non-blue node that are at least $\lceil \frac{n-k}{k+1} \rceil$ steps away from each other.

Problem 29: Consider the possible functions $f : [7] \to [9]$.

(a) How many have $f(2) = 4$ and $f(3) \neq 8$?

(b) How many have $f(3) \neq 8$ and are one-to-one?

(c) How many have $f(i)$ even for all $i \in [7]$ and are one-to-one?

(d) How many have $f(i)$ even for all $i \in [7]$?

Problem 30: Let $n \in \mathbb{Z}^+$. Determine the number of surjective functions $f : [n] \to [4]$

Problem 31: Suppose $n, j, k \in \mathbb{Z}^+$ with $k, j \leq n$. Give a combinatorial proof that

$$(n)_k = (n-j)_{(k-j)} \cdot (n)_j$$

Problem 32: Suppose k and n are integers with $1 \leq k \leq n$. Give combinatorial proof that

$$(n)_k = \sum_{n=k}^{k} k(j-1)_{k-1}$$

Problem 33: Prove that, for $k, n \in \mathbb{Z}^+$, $\left(\!\binom{n}{k}\!\right) = \left(\!\binom{n-1}{k}\!\right) + \left(\!\binom{n}{k-1}\!\right)$

Problem 34: Prove that, for $n \geq 1$, $3^n = \sum\limits_{k=0}^{n} \binom{n}{k} 2^{n-k}$

Problem 35: Prove that, for $m, n \in \mathbb{Z}^+$ and $m \leq n$, $\sum\limits_{j=m}^{n} \binom{j}{m} = \binom{n+1}{m+1}$.

Problem 36: Prove that the number of even-sized subset equals the number of odd-sized subset.

Bonus: Prove that, for $n \in \mathbb{Z}^+$ and $x, y \in \mathbb{R}$,

$$(x+y)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k (y)_{n-k}$$

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Problem 38: Consider the letters in the wod INVISIBILITIES

(a) How many distinct ways are there to arrange 14 letters in the word?

(b) How many distinct ways are there if no tow I's are allowed to be adjacent?

(c) How many distinct arrangements are there if the longest number of consecutive I's in a row is 4?

Problem 39: Let $n \in \mathbb{Z}^+$ and consider the sequence of non-negative integers $x_1, x_2, \ldots, x_n$. If there are some people are placed into teams so that for each $i \in [n]$, there are exactly x_i teams of size i, then the number of ways to create the teams is

$$\frac{n!}{\prod\limits_{i=1}^{n} (i!)^{x_i} x_i!}$$

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Problem 40: Determine $\lim\limits_{k \to \infty} \frac{1}{k} \binom{-1}{k}$.

Problem 41: Prove that $\sum\limits_{k=m}^{n} \binom{n}{k} \binom{k}{m} = \binom{n}{m} 2^{n-m}$.

Problem 42: Let A be a 10 element set. How many equivalence relations are there on A? How many have exactly 8 equivalence classes?

Problem 43: How many onto functions form [9] to [7] have only one element mapped to 7?

Problem 44: Give a formula for the nubmer of ways to place n distinguishable boxes so that at least one box stays empty.

Problem 45: Give a combinatorial proof that for integer $k, n \geq 1$, $S(n,k) = \sum^{n-1} i = 0\binom{n-1}{i}S(i, k-1)$.

Problem 46: Use the face that $P(n,k) = P(n-1,k-1) + P(n-k,k)$ for integers n, k to prove that for all integer $n \geq 3$, $P(n,k) = \lfloor \frac{n}{2} \rfloor$.

Problem 47: For each $n \in \mathbb{Z}^+$, let $P(n)$ be the nubmer of partitions of n.

(a) Give a combinatorial proof that $P(n) = \sum_{k=1}^{n} P(n,k)$

(b) Gibve a bijective proof that $P(n) = P(2n, n)$ for each $n \in \mathbb{N}$.

Problem 48: Let z_i be a partitions of n into k parts in non-increasing order. Show how to compute the conjugate of the partitions in terms of z_i

Problem 49: Suppose $P(n)$ is odd for $n \in \mathbb{Z}^+$. Show that at least one partitions of n is self-conjugate.

Problem 50: How many integer in [100] are not divisible by 4, 6, and 7?

Problem 51: Suppose that in an inclusion-exclusion problem, there exits a functions f such that for any subset $J \subseteq P$ with $|J| = j$, $N_{\geq}(J) = f(j)$. Prove that

$$N_{=}(\varnothing) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j f(j)$$

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Problem 52: Use inclusion-exclusion to prove that the number of partitions of an $n-$set into k parts is

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (k-j)^n$$

Problem 53: Derive an identity for $\binom{n}{k}$ via inclusion-exclusion by counting the $k-$multisets of $[n]$ is which each element of $[n]$ appears at most once.

Problem 54: Suppose $n, k \in \mathbb{Z}^+$ and suppose j is an integer with $0 \leq j \leq k$,

(a) How many $k-$lists taken from n have exactly j entries that are n?

(b) Use part (a) to partition the $k-$lists taken from $[n]$ and get the formula for n^k.

(c) Prove the formula using the binomial theorem

Problem 55: Prove that, for all $n, k \in \mathbb{Z}^+$, $\sum_{j=1}^{n} \left(\!\binom{j}{k-1}\!\right) = \left(\!\binom{n}{k}\!\right)$.

Problem 56: Prove that, for all $n, k \in \mathbb{Z}^+$, $\binom{kn}{2} = k\binom{n}{2} + n^2\binom{k}{2}$.

Problem 57: Prove the following identities:

(a) $\binom{20}{8}\binom{8}{5}\binom{5}{3} = \binom{20}{3}\binom{17}{2}\binom{15}{4}$

(b) For all positive integer $n \geq k \geq j$, $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$