1. Let $w(t)$ be a piecewise continuous complex-valued function defined over an interval $a \leq t \leq b$. Prove that $\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|dt$.

   Note that,

$$\begin{aligned}\int_a^b |w(t)|dt &\geq \max\left(\int_a^b w(t)dt,\ \int_a^b -w(t)dt\right)\\ &= \max\left(\int_a^b w(t)dt,\ -\int_a^b w(t)dt\right) = \left|\int_a^b w(t)dt\right|\end{aligned}$$

2. Let $\mathcal{C}$ be a contour with length L and $f(z)$ continuous on $\mathcal{C}$. Let M be a real number such that $|f(z)| \leq M$ on all point on $\mathcal{C}$. Prove that $\left|\int_{\mathcal{C}} f(z)dz\right| \leq ML$.

   For $a \leq x \leq b$, let $w(t)$ be a parameterization of $\mathcal{C}$.

   Rewrite $L = \int_a^b |w'(t)|dt$, then,

$$\begin{aligned}\left|\int_{\mathcal{C}} f(z)dz\right| &= \left|\int_a^b f(w(t))w'(t)dt\right|\\ &\leq \int_a^b |f(w(t))w'(t)|dt\\ &= \int_a^b |f(w(t))||w'(t)|dt\\ &\leq \int_a^b M|w'(t)|dt = M\int_a^b |w'(t)|dt = ML\end{aligned}$$

3. Let f be a function analytic inside and on a positively oriented circle $\mathcal{C}_R$, centered at z_0 with radius R. Let M_R denote the maximum value of $|f(z)|$ on $\mathcal{C}_R$. Prove that $|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n}$ for $n \in \mathbb{N}$.

   For any z inside $\mathcal{C}_R$, $|z - z_0| \leq R$. Using the Cauchy integration formula, we have that

$$\begin{aligned}|f^{(n)}(z_0)| &= \left|\frac{n!}{2\pi i}\int_{\mathcal{C}_R} \frac{f(z)}{(z-z_0)^{n+1}}dz\right|\\ &\leq \frac{n!}{2\pi}\int_{\mathcal{C}_R} \frac{|f(z)|}{|(z-z_0)^{n+1}|}dz\\ &\leq \frac{n!M_R}{2\pi}\int_{\mathcal{C}_R} |z-z_0|^{-n-1}dz\end{aligned}$$

   The integral $\int_{\mathcal{C}_R} |z-z_0|^{-n-1}dz$ is of the function $g(z) = |z-z_0|^{-n-1}$ over $\mathcal{C}_R$, which has the length $L = 2\pi R$, so from part 2, $\int_{\mathcal{C}_R} |z-z_0|^{-n-1}dz \leq 2\pi R R^{-n-1} = 2\pi R^{-n}$.

   Put this back in, we have $|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n}$. $\square$