Prove that if $f : [a,b] \to \mathbb{R}$ is continuous and $\int_a^x f dx = 0$ for all $x \in [a,b]$, then $f(x) = 0$ everywhere on $[a,b]$. Find an example to show that the implication is not true if continuity is not in the hypothesis.

Proof:

Let F be a function such that $F' = f$. By Fundamental Theorem of Calculus, for all $x \in [a,b]$, $\int_a^x f dx = 0 = F(x) - F(a)$. Then, for all $x \in [a,b]$, $F(x) = c$ for some $c \in \mathbb{R}$.

For any $x \in \mathbb{R}$, $f(c) = F'(c) = \lim\limits_{x \to} \frac{F(x)-F(c)}{x-c} = \lim\limits_{x \to c} 0 = 0$. Thus $f(x) = 0$ everywhere on $[a,b]$.

Conterexample: let

$$f(x) = \begin{cases} 1 & x = \frac{1}{2} \\ \\ 0 & else \end{cases}$$

For all $x \in [0,1]$, $\int_0^x f dx = 0$ and f is not continuous. And $f(x)$ does not equal to 0 everywhere on $[0,1]$. $\square$