

1. Prove by induction that  $\sum_{i=1}^n i^2 = 1 + 4 + 9 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$
2. Use 1 and the same approach we did in class for  $f(x) = x$  on  $[0, 1]$  to verify that  $g(x) = x^2$  is indeed Riemann Integrable on  $[0, 1]$ , and compute the integral.

Proof:

1. Base case:  $n = 1$ :  $1^2 = \frac{1 \times 2 \times 3}{6}$ .

Induction hypothesis: let  $1^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$ .

$$\begin{aligned} \text{Consider } 1^2 + \dots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{(k+1)(2k^2+k+6(k+1))}{6} = \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2(k+1)+1)}{6}. \end{aligned}$$

$$\text{Thus, } \sum_{i=1}^n i^2 = 1 + 4 + 9 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. Consider the partition  $P = \{0, \frac{1}{2^n}, \frac{2}{2^n}, \dots, \frac{2^n-1}{2^n}, 1\}$ .

For each  $k$ ,  $m_k = \inf(x^2 | x \in [\frac{k-1}{2^n}, \frac{k}{2^n}]) = \frac{(k-1)^2}{4^n}$  and  $M_k = \sup(x^2 | x \in [\frac{k-1}{2^n}, \frac{k}{2^n}]) = \frac{k^2}{4^n}$ .

$$\begin{aligned} \text{Then, } L(f, P) &= \sum_{k=1}^{2^n} m_k \left( \frac{k}{2^n} - \frac{k-1}{2^n} \right) \\ &= \sum_{k=1}^{2^n} \frac{(k-1)^2}{4^n} \times \frac{1}{2^n} \\ &= \frac{2^n(2^n-1)(2^{n+1}-1)}{6 \times 8^n} \\ &= \frac{2^{2n+1} - 2^{n+1} - 2^n + 1}{6 \times 4^n} \\ &= \frac{2 \times 4^n}{6 \times 4^n} - \frac{2}{6 \times 2^n} - \frac{1}{6 \times 2^n} + \frac{1}{6 \times 4^n} \\ &= \frac{1}{3} - \frac{2}{6 \times 2^n} - \frac{1}{6 \times 2^n} + \frac{1}{6 \times 4^n} \end{aligned}$$

As  $L(f) = \sup\{L(f, P) | P \text{ is a partition}\}$ ,  $L(f) = \frac{1}{3}$ .

$$\begin{aligned}
\text{Similarly, } U(f, P) &= \sum_{k=1}^{2^n} M_k \left( \frac{k}{2^n} - \frac{k-1}{2^n} \right) \\
&= \sum_{k=1}^{2^n} \frac{k^2}{4^n} \times \frac{1}{2^n} \\
&= \frac{2^n(2^n+1)(2^{n+1}+1)}{6 \times 8^n} \\
&= \frac{2^{2n+1} + 2^{n+1} + 2^n + 1}{6 \times 4^n} \\
&= \frac{2 \times 4^n}{6 \times 4^n} + \frac{2}{6 \times 2^n} + \frac{1}{6 \times 2^n} + \frac{1}{6 \times 4^n} \\
&= \frac{1}{3} + \frac{2}{6 \times 2^n} + \frac{1}{6 \times 2^n} + \frac{1}{6 \times 4^n}
\end{aligned}$$

As  $U(f) = \inf\{U(f, P) | P \text{ is a partition}\}$ ,  $U(f) = \frac{1}{3}$ .

Since  $U(f) = L(f) = \frac{1}{3}$ ,  $\int_0^1 x^2 dx$  exists and  $\int_0^1 x^2 dx = \frac{1}{3}$ .  $\square$