Let f and g be Riemann Integrable functions on the interval $[a, b]$, and $k \in \mathbb{R}$. Prove the following:

1. The function $f + g$ is Riemann Integrable on $[a, b]$, and $\int_a^b (f+g)dx = \int_a^b f dx + \int_a^b g dx$.

2. The function kf is Riemann Integrable on $[a, b]$, and $\int_a^b (kf)dx = k\int_a^b f dx$.

3. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f dx \leq \int_a^b g dx$.

Proof:

1. Let P be a partition.

   For each k, let $m_k = \inf\{f(x)|x \in [x_{k-1}, x_k]\}$ and $n_k = \inf\{g(x)|x \in [x_{k-1}, x_k]\}$. Then, for all $x \in [x_{k-1}, x_k]$,

   $$m_k \leq f(x)$$

   $$\Leftrightarrow m_k + g(x) \leq f(x) + g(x)$$

   $$\Leftrightarrow m_k + g(x) \leq \inf\{f(x) + g(x)\}$$

   $$\Leftrightarrow \inf\{m_k + g(x)\} \leq m_k + g(x) \leq \inf\{f(x) + g(x)\}$$

   $$\Leftrightarrow m_k + n_k \leq \inf\{f(x) + g(x)\}$$

   $$\Leftrightarrow \sum(m_k + n_k)(x_k - x_{k-1}) \leq \sum \inf\{f(x) + g(x)\}(x_k - x_{k-1})$$

   $$\Leftrightarrow L(f, P) + L(g, P) \leq L(f + g, P)$$

   Similarly, we have $U(f + g, P) \leq U(f, P) + U(g, P)$, so the inequality becomes $L(f, P) + L(g, P) \leq L(f + g, P) \leq U(f + g, P) \leq U(f, P) + U(g, P)$.

Since P is arbitrary and f, g are Riemann Integrable, we have

$L(f) + L(g) = L(f + g) = U(f + g) = U(f) + U(g)$, so $f + g$ is Riemann Integrable. And, $\int_a^b f dx + \int_a^b g dx = U(f) + U(g) = U(f+g) = \int_a^b (f+g) dx$.

2. Let $\epsilon > 0$.

   Assume $k = 0$, then $kf(x) = 0$, and it is Riemann Integrable, and $\int_a^b (kf) dx = 0 = k \int_a^b f dx$

   Assume $k < 0$. Let P be the partition such that $U(f, P) - L(f, P) < \frac{\epsilon}{|k|}$. For each i, for all $x \in [x_{i-1}, x_i]$, $m_i \leq f(x)$, so $km_i \geq kf(x)$, so $km_i \geq \sup\{kf(x)\}$. Similarly, $kM_i \leq \inf\{kf(x)\}$. Then,

   $U(kf, P) - L(kf, P) \leq k(m_i - M_i) < -k \frac{\epsilon}{|k|} < \epsilon$.

   Assume $k > 0$. Let P be the partition such that $U(f, P) - L(f, P) < \frac{\epsilon}{k}$. For each i, for all $x \in [x_{i-1}, x_i]$, $m_i \leq f(x)$, so $km_i \leq kf(x)$, so $km_i \leq \inf\{kf(x)\}$. Similarly, $kM_i \geq \sup\{kf(x)\}$. Then,

   $U(kf, P) - L(kf, P) \leq k(M_i - m_i) < k \frac{\epsilon}{k} < \epsilon$.

   WLOG, let $k > 0$. For any partition P, $kU(f) = k \inf\{U(f, P)\} = \inf\{kU(f, P)\} = \inf\{U(kf, P)\} = U(kf)$. Similarly, $kL(f) = L(kf)$. So $\int_a^b (kf) dx = U(kf) = kU(f) = k \int_a^b f dx$.

3. Let P be an arbitrary partition. For any i, for all $x \in [x_{i-1}, x_i]$,

   $f(x) \leq g(x)$, so $\inf(f(x)) \leq f(x) \leq \inf(g(x)) \leq g(x)$.

   So, $L(f, P) \leq L(g, P)$, which results in

   $L(f, P) \leq L(f) = \sup\{L(f, P)\} \leq L(g, P) \leq \sup\{L(g, P)\} = L(g)$.

   Thus, $\int_a^b f dx \leq \int_a^b g dx$. $\Box$.