1. Let $\mathcal{C}$ be a category. Prove that $\mathcal{C}(-, c)$ is a functor.

For each object d in $\mathcal{C}$, there is an object $\mathcal{C}(d, c)$, the set of all morphisms between from d to c in $\mathcal{C}$, in **Set**.

For each morphism $f : x \to y$ in $\mathcal{C}$, there is a morphism
$f^* : \mathcal{C}(y, c) \to \mathcal{C}(x, c); g \mapsto fg$ in **Set**.

For any composable pair $f : x \to y$ and $g : y \to z$ in category $\mathcal{C}$, the composition is $fg : x \to z$.

Let $h \in \mathcal{C}(z, c)$.

Note that the morphism $(fg)^*$ in **Set** from $\mathcal{C}(z, c)$ to $\mathcal{C}(x, c)$ maps h to $fgh \in \mathcal{C}(x, c)$. Also, the morphism $g^* f^*$ in **Set**, from $\mathcal{C}(z, c)$, through $\mathcal{C}(y, c)$, to $\mathcal{C}(x, c)$, maps h to $gh \in \mathcal{C}(y, c)$ then to $fgh \in \mathcal{C}(x, c)$. So $(fg)^* = g^* f^*$.

For any object x in $\mathcal{C}$ with the morphism $k : x \to c$, $\mathcal{C}(-, c)$ maps the identity morphism 1_x to $1_x^* : \mathcal{C}(x, c) \to \mathcal{C}(x, c);\ k \mapsto 1_x k = k$. So 1_x^* is the identity morphism of $\mathcal{C}(x, c)$.

So, $\mathcal{C}(-, c)$ is a contravariant functor. $\square$

2. Given functors $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{E} \to \mathcal{C}$, there is a category, called the comma category $F \downarrow G$.

(a) Prove : $F \downarrow G \to \mathcal{E}$ is a functor.

For each object (d, e, f) in $F \downarrow G$, there is a object e in $\mathcal{E}$.

For each morphism (h, k) in $F \downarrow G$, there is a morphism k in $\mathcal{E}$.

For any composable morphism (h, k) and (h', k') in $F \downarrow G$,
$[(h, k)(h', k')]^{cod} = (hh', kk')^{cod} = kk' = (h, k)^{cod}\ (h', k')^{cod}$.

For any identity morphism $(1_d,\ 1_e)$, $(1_d,\ 1_e)^{cod} = 1_e$, which is the identity morphism of object e in $\mathcal{E}$.

So : $F \downarrow G \to \mathcal{E}$ is a functor.

(b) Construnct a canonical natural transformation $\alpha : \text{dom}F \implies \text{cod}G$.

For each object (d, e, f) in $F \downarrow G$, define the component $\alpha_{(d,e,f)} \coloneqq f$. We WTS α forms a natural transformation.

For any object (d, e, f) in $F \downarrow G$, $(d, e, f)\text{dom}F = [(d, e, f)\text{dom}]F = dF$. Similarly, $(d, e, f)\text{cod}G = [(d, e, f)cod]G = eG$. Since both dF and eG are objects of $\mathcal{C}$, we can define the component of α at (d, e, f) to be just the arrow $f : dF \to eG$.

For any morphism $(h, k) : (d, e, f) \to (d', e', f')$, note that $\alpha_{(d,e,f)}(h, k)^{\text{cod}G} = f[(h, k)^{\text{cod}}]^G = fk^G$. Also, $(h, k)^{\text{dom}F}\alpha_{(d',e',f')} = [(h, k)^{\text{dom}}]^F f' = h^F f'$. Finally, since $fk^G = h^F f'$ by the commutivity of morphisms in $F \downarrow G$, we have $\alpha_{(d,e,f)}(h, k)^{\text{cod}G} = (h, k)^{\text{dom}F}\alpha_{(d',e',f')}$.

So α forms a natural transformation. $\square$