1. Consider the morphism $f : x \to y$. Show that if there exists a pair of morphisms $g, h : y \to x$ so that $fg = 1_x$ and $hf = 1_y$, then $g = h$ and f is an isomorphism.

Let $y_0 \in y$.

Note that: $(hf)g(y_0) = h(fg)(y_0)$

$\Leftrightarrow g(hf(y_0)) = (fg)(h(y_0))$

$\Leftrightarrow g(y_0) = h(y_0)$

$\Leftrightarrow g = h$

Since $g = h$, the initial conditions become $fg = 1_x$ and $gf = 1_y$, so f is an isomorphism. $\square$

2. For any category $\mathcal{C}$ and any object $c \in \mathcal{C}$, show that there is a category $\mathcal{C}/c$ whose objects are morphisms with codomain c and morphism from $f : x \to c$ to $g : y \to c$ is a map $h : x \to y$ so that $f = gh$.

Let f, g, h be objects in $c/\mathcal{C}$ and $f : x \to c,\ g : y \to c,\ h : z \to c$.

Let k, l be morphisms in $c/\mathcal{C}$ and $k : x \to y,\ l : y \to z$.

Since the set of morphisms in $c/\mathcal{C}$ is a subset of morphisms in $\mathcal{C}$, it is associative.

To compose k and l, by substituting $g = lh$ into $kg = f$, it becomes $k(lh) = f$. By associativity, it follows that $(kl)h = f$, so kl is a morphisms in $\mathcal{C}/c$, and $kl : x \to z$.

For any object $f : x \to c$, the identity morphism of f is the morphism $1_f : x \to x$. Also, in $\mathcal{C}$, the identity morphism of object x is the morphism $1_x : x \to x$. So $1_f = 1_x$.

So for any morphism $k : f \to g; x \mapsto y$, the composition morphism $1_f k = 1_x k = k$. Similarly, the composition morphism $k 1_g = k 1_y = k$.

So $\mathcal{C}/c$ forms a category. $\square$