Suppose that f_n converges to f pointwise on some $D \subseteq \mathbb{R}$. For each of the following statements, find a counter example disproving the claim.

1. If each f_n is uniformly continuous, then f is uniformly continuous.
2. If each f_n is bounded, then f is bounded.
3. If each f_n has a finite number of discontinuities, then f has a finite number of discontinuities.

Proof:

1. Define $f_n(x) = x^n$ and $D = [0,1]$. For all n, $f_n(x)$ is continuous over the closed interval $[0,1]$, so $f_n(x)$ is uniformly continuous.

   Note that $f_n(x)$ converges pointwise to

   $$f(x) = \begin{cases} 1 & x = 1 \\ 0 & else \end{cases}$$

   which is not continuous.

2. Define

   $$f_n(x) = \begin{cases} n & x > n \\ x & -n \leq x \leq n \\ -n & x < -n \end{cases}$$

   Define $f(x) = x$. Note that f_n converges to f pointwise by choosing $N \geq |x_0|$. Finally, for each n, f_n is bounded by n, and f is unbounded.

3. Define

$$f_n(x) = \begin{cases} 1 & x = n \\ \\ 0 & else \end{cases}$$

Define $f(x) = 0$. Note that, for each n, $f_n(x)$ has exactly one discontinuities, and $f_n(x)$ converges to $f(x)$ pointwise by choosing $N \geq x_0$. Finally, $f(x)$ does not have any discontinuities.