1. Show that the set of nondecreasing functions forms a monoid of functions on $\mathbb{R}$.

Let f and g be nondecreasing functions on $\mathbb{R}$.

Let $x, y \in \mathbb{R}$ and $x > y$.

Since f is a nondecreasing function, $f(x) \geq f(y)$. Since g is a nondecreasing function, $g(f(x)) \geq g(f(y))$. Thus $g \circ f$ is a nondecreasing function.

Define the function $e : x \mapsto x$. Note that e is a nondecreasing function, and $(f \circ e)(x) = f(e(x)) = f(x) = e(f(x)) = (e \circ f)(x)$.

So the set of nondecreasing function forms a monoid of function on $\mathbb{R}$. $\square$

2. Show that the set $C(\mathbb{R})$ of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ forms a monoid of functions on $\mathbb{R}$.

Let f and g be continuous functions on $\mathbb{R}$.

Let $x \in \mathbb{R}$.

Since g is a continuous function on $\mathbb{R}$, g is defined and continuous at x. Since f is a continuous function on $\mathbb{R}$, f is defined and continuous at $g(x)$. So $f \circ g$ is a continuous function on $\mathbb{R}$.

Define the function $e : x \mapsto x$. Note that e is a continuous function, and $(f \circ e)(x) = f(e(x)) = f(x) = e(f(x)) = (e \circ f)(x)$.

So the set $C(\mathbb{R})$ forms a monoid of function on $\mathbb{R}$. $\square$

3. Let $f : X \to Y$ be a function with nonempty domain X. Show that there is a function $g : Y \to X$ such that $f = f \circ g \circ f$.

For each $y \in \text{range}(f)$, let $x \in f^{-1}(\{y\})$. Define the function $g : Y \to X : y \mapsto x$.

For each $y \in Y \setminus \text{range}(f)$, let $x_0 \in X$. Define the function

$g : Y \to X : y \mapsto x_0$.

Then g is a well-defined function, and for any $x \in X$ and $f(x) = y \in Y$,

$(f \circ g \circ f)(x) = f(g(y)) = f(x)$. $\square$