1. $|Inn(\text{G})| = 1$ if and only if G is Abelian.

   Suppose $|Inn(\text{G})| = 1$. We know that $\phi_e \in (Inn(\text{G}))$, so for any $g \in G$, $\phi_g = \phi_e$. This means that, for any $x \in G$, $\phi_g(x) = gxg^{-1} = \phi_e(x) = x$. Reduce further yields $gx = xg$ for any g and x in G. So G is Abelian.

   Suppose G is Abelian. Let $g \in G$. Consider $\phi_g \in Inn(G)$. For any $x \in G$, $\phi_g(x) = gxg^{-1} = gg^{-1}x = x$. So, for any $g \in G$, ϕ_g is the identity innner isomorphism. Thus $|Inn(G)| = 1$.

2. Show that a group with more than one subgroup of order 5 must have order of at least 25.

   Let H and K be subgroups of order 5 of G. Consider $H \cap K$. For any element in $H \cap K$, that element belongs to both H and K. Since H and K are groups, they both contains the inverse of that element, and by extension, $H \cap K$ also contains that inverse. We also know that $e \in H \cap K$. So $H \cap K$ is a subgroup of H and K.

   By Lagrange's theorem, $|H \cap K|$ is a divisor of 5. If $|H \cap K| = 5$, they are the same group, so $|H \cap K| = 1$. Finally, any element of HK is an element of G because of closure, $|G| \geq |HK| = \frac{|H||K|}{|H \cap K|} = 25$.

3. Let $G = U(16), H = \{1, 15\}, K = \{1, 9\}$. Determine with proof whether $H \cong K$ and $G/H \cong G/K$.

   Consider the mapping $f : H \to K$ such that $1 \overset{f}{\mapsto} 1$ and $15 \overset{f}{\mapsto} 9$. Clearly f is well-defined and bijective. And

$$\begin{aligned} f(1 \times 1) &= f(1) = 1 = f(1) \times f(1) \\ f(1 \times 15) &= f(15) = 9 = f(1) \times f(15) \\ f(15 \times 15) &= f(1) = 1 = f(15) \times f(15) \end{aligned}$$

   So, $H \cong K$.

   Assume that $G/H \cong G/K$, then there exists a function $f : G/H \to G/K$ that is bijective and operation-preserving.

   Consider $1 \times \{1, 15\}$ and $3 \times \{1, 15\}$ of G/H.

   Note that,

$$\begin{aligned} (1 \times \{1, 15\})^2 &= 1 \times \{1, 15\} = \{1, 15\} \\ (3 \times \{1, 15\})^2 &= 9 \times \{1, 15\} = \{7, 9\} \end{aligned}$$

   The set G/K contains all of the following:

$$\begin{aligned} 1 \times \{1, 9\} &= \{1, 9\} = 9 \times \{1, 9\} \\ 3 \times \{1, 9\} &= \{3, 11\} = 11 \times \{1, 9\} \\ 5 \times \{1, 9\} &= \{5, 13\} = 13 \times \{1, 9\} \\ 7 \times \{1, 9\} &= \{7, 15\} = 15 \times \{1, 9\} \end{aligned}$$

   Note that, $(1 \times \{1, 9\})^2 = (3 \times \{1, 9\})^2 = (5 \times \{1, 9\})^2 = (7 \times \{1, 9\})^2 = \{1, 9\}$. Then, since f is bijective and operation-preserving, $\forall x, y \in G/H$, $x^2 = y^2$, which is a contradiction.

   So, $G/H \ncong G/K$.

4. (Easy)(Chapter 10, Exercise 8)

Let $sgn(\sigma)$ be a function such that,

$$sgn : S_n \to (\{-1, 1\}, \times); sgn(\sigma) \mapsto \begin{cases} 1 & \sigma \text{ is even} \\ -1 & \sigma \text{ is odd} \end{cases}$$

Show that sgn is a homomorphism, and find $Ker(sgn)$.

Clearly, $(\{1, -1\}, \times)$ is a group. For any 2 permuation in S_n, one of the following must holds:

$$sgn(even \times even) = sgn(even) = 1 = 1 \times 1 = sgn(even) \times sgn(even)$$
$$sgn(odd \times even) = sgn(odd) = -1 = -1 \times 1 = sgn(ood) \times sgn(even)$$
$$sgn(odd \times odd) = sgn(even) = 1 = -1 \times -1 = sgn(odd) \times sgn(odd)$$

So, sgn is a homomorphism. Since 1 is the identity, the $Ker(sgn)$ is A_n.

5. (Moderate)(Chaper 7, Exercise 48)

Let G be a group and $|G| = pqr$ where p, q, r are prime numbers. Let H and K be subgroup of G such that $|H| = pq$ and $|K| = qr$. Show that $|H \cap K| = q$.

Proof:

Since $H \cap K$ is a subgroup of H and K (see Problem 2), $|H \cap K|$ is a common divisor of $|H|$ and $|K|$. Since p, q, r are prime number, common divisors of pq and qr are 1 and q.

Note that, similar to Problem 2, $pqr = |G| \geq |HK| = \frac{|H||K|}{|H \cap K|} = \frac{pq^2r}{|H \cap K|}$,
so $|H \cap K| \geq \frac{pq^2r}{pqr} = q$.

So $|H \cap K| = q$.

6. (Moderate)(Chapter 6, Exercise 66)

   Show that $(\mathbb{Q} \setminus \{0\}, \times)$ and $(\mathbb{Q}, +)$ are not isomorphic.

   Proof:

   Suppose there exists $f : (\mathbb{Q} \setminus \{0\}, \times) \to (\mathbb{Q}, +)$ such that f is bijective and operation-preserving.

   Let $p_0 \in \mathbb{Q} \setminus \{0\}$ such that $f(p_0) = 0$. For any $x \in (\mathbb{Q} \setminus \{0\}, \times)$,
   $f(p_0 \times x) = f(p_0) + f(x) = f(x)$. Since f is injective, $p_0 \times x = x$, so $p_0 = 1$.

   Consider $f(-1 \times -1) = f(-1) + f(-1)$. The left hand side $f(-1 \times -1) = f(1) = 0$, while the right hand side is $2f(-1)$, so $f(-1) = 0$, which is a contradiction since f is injective.

   So $(\mathbb{Q} \setminus \{0\}, \times)$ is not isomorphic to $(\mathbb{Q}, +)$.

7. (Moderate)(Chapter 9, Exercise 9)

   Show that a subgroup of index 2 is normal.

   Proof:

   Let G be a group and H is a subgroup of index 2.

   Let $g \in G$. If $g \in H$, $gH = H = Hg$, so H is normal.

   If $g \notin H$, gH and H are the two disjoint left coset of H in G. So, $gH = G - H$. Similar argument can be made for disjoint right coset, so $Hg = G - H$.
   Thus, $gH = G - H = Hg$.

   In either case, H is normal.

8. (Hard)(MIT Putnam Seminar Fall 2018, Abstract Algebra, Problem 17)

   Show that a group of order $4n+2$ for $n \in \mathbb{N}$ has a proper normal subgroup.

   Proof:

   Let G be a group of order $4n+2$, $n \geq 1$. By Cayley's Theorem, G is isomorphic to a subgroup of S_{4n+2}.

   Let $f : G \to \bar{G}$ such that $\bar{G}$ is a subgroup of S_{4n+2} and f satisfies the isomorphism properties.

   By Cauchy's Theorem, there exists an element $g \in G$ such that $|g| = 2$.

   Note that, $f(g)$ is a permuation in S_{4n+2} and can be written as the product of disjoint cycles. Also, since $|g| = 2$, $g = g^{-1}$, thus $f(g) = f(g)^{-1}$.

   So $|f(g)| = 2$. Then $f(g)f(g)^{-1} = e$, and since e can be written as product of $4n+2$ disjoint cycles, and $f(g)$ and $f(g)^{-1}$ have the same length, $f(g)$ can be written as the product of $\frac{4n+2}{2} = 2n+1$ disjoint cycles. Thus, $f(g)$ is odd.

   In $f(G)$, for any even permuation, the product of $f(g)$ with that cycle is odd. So the number of odd and even permuation is the same and equal to $2n+1$.

   Clearly, the set of all even permuation in this group is a proper subgroup by One-step Subgroup test. For any permuation, its coset with this subgroup is either exactly this subgroup (when the permutation is even) or the set of odd permutations. So the index of this subgroup is 2, and by Problem 7, it is normal.

   Thus, its pre-image in G is proper and normal.

   $\square$

Problem 4 is too easy because it only requires an easy definition and there is not too much to prove it.

Problem 5 is moderate because it tests deeper understanding of the definition and some properties of subgroups and their order, while not being too hard and out-of-reach.

Problem 6 is moderate because it is using a contradiction proof, and how to derive that contradiction requires creative choosing of elements and specific propery of isomorphism.

Problem 7 is moderate because it requires more than one properties of coset but still doable if understanding those properties.

Problem 8 is hard because it requires multiples theorems and properties to solve. Also, we did not cover any specific properties or theorem that guarantee the existence of a normal subgroup, so it can be hard to know where to start. Without distinctly being able to associate the original group with the permutation group and Problem 7, it is hard.