1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $|f(x) - f(y)| \leq |x - y|^2$ for all $x, y \in \mathbb{R}$. Show that $f(x) = C$ for some constant C.

Let $c \in \mathbb{R}$.

Note that: $\frac{|f(x)-f(c)|}{|x-c|} \leq \frac{|x-c|^2}{|x-c|} = |x - c|$, then

$\lim_{x\to c} \frac{|f(x)-f(c)|}{|x-c|} \leq \lim_{x\to c} |x - c| = 0$. So, $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c) = 0$ for all $c \in \mathbb{R}$. So f is differentiable on $\mathbb{R}$.

Let $a, b \in \mathbb{R}$ such that $a < b$, then the interval $[a, b] \in \mathbb{R}$. So, by Mean Value Theorem, $\exists n \in (a, b)$ such that $f(b) - f(a) = f'(n)(b - a)$. Since $f'(n) = 0$ for all $n \in \mathbb{R}$, this follows $f(a) = f(b) = C$ for all $a, b \in \mathbb{R}$ and some constant C. $\square$

2. Suppose $f : S \to \mathbb{R}$ is a differentiable function and f' is bounded. Prove that f is a Lipschitz continuous function.

Let $x, y \in \mathbb{R}$ and $x < y$.

Then the interval $[x, y] \in \mathbb{R}$ and by Mean Value Theorem, $\exists c \in (x, y)$ such that $|\frac{f(y)-f(x)}{y-x}| = |f'(c)|$. Since f' is bounded, $\forall c \in \mathbb{R}, |f'(c)| \leq M$ for some $M \in \mathbb{R}$. Finally, we have $|\frac{f(y)-f(x)}{y-x}| = |f'(c)| \leq M$, thus $|f(y)-f(x)| \leq M|y-x|$ for all x, y in $\mathbb{R}$. So f is a Lipschitz continuous function.$\square$

3. Suppose $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ are differentiable functions such that $f'(x) = g'(x)$ for all $x \in (a, b)$. Prove that there exists a constant C such that $f(x) = g(x) + C$.

Let $h : (a, b) \to \mathbb{R}$ and $h(x) \coloneqq f(x) - g(x)$. Since both f and g are differentiable, h is also differentiable and $h'(x) = f'(x) - g'(x) = 0$ for all $x \in (a, b)$.

Then by Proposition 4.2.6, $h(x) = C$ for some $C \in \mathbb{R}$, thus $f(x) - g(x) = C$.

$\square$

4. Suppose $a, b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f'(x) = a$ for all $x \in \mathbb{R}$, and $f(0) = b$. Find f and prove its uniqueness.

Let $x, y \in \mathbb{R}$ and $x < y$.

Then the interval $[x, y] \in \mathbb{R}$ and by Mean Value Theorem, $\exists c \in (x, y)$ such that $\frac{f(x)-f(y)}{x-y} = f'(c) = a$. This follows $f(y) - f(x) = a(y - x)$. Since this holds for all $x, y \in \mathbb{R}$, it is also true for $x = 0$. Then the equation becomes $f(y) - b = ay$, or $f(y) = ay + b$.

Let h be a differentiable function satisfies all conditions for f, and $h(x) \neq ax + b$. Note that for all $x \in \mathbb{R}$, $f'(x) = h'(x) = a$, then by Problem 3, $f(x) = h(x) + C$ for some constant C. Substituting $x = 0$ shows that $C = 0$, so $f(x)$ is unique. $\square$