

Statistical Modeling 2

Exercise 1

January 23, 2017

1 Bayesian inference in simple conjugate families

A

$$\begin{aligned}
 p(w \mid x_1, \dots, x_N) &\propto p(x_1, \dots, x_N \mid w)p(w) && \text{(Bayes rule)} \\
 &\propto \prod_{i=1}^N p(x_i \mid w) w^{a-1} (1-w)^{b-1} && \text{(independence)} \\
 &\propto w^s (1-w)^{N-s} w^{a-1} (1-w)^{b-1} && \text{(let } s = \sum_{i=1}^N x_i) \\
 &= w^{s+a-1} (1-w)^{N-s+b-1} \\
 &\propto \text{Beta}(s+a, N-s+b)
 \end{aligned}$$

B

Let $f(x_1, x_2) = (y_1, y_2) = (x_1/(x_1 + x_2), x_1 + x_2)$, we have:

$$f^{-1}(y_1, y_2) = (x_1, x_2) = (y_1 y_2, y_2 - y_1 y_2)$$

We then calculate the Jacobian of f^{-1} :

$$\begin{aligned}
 \partial x_1 / \partial y_1 &= y_2 \\
 \partial x_1 / \partial y_2 &= y_1 \\
 \partial x_2 / \partial y_1 &= -y_2 \\
 \partial x_2 / \partial y_2 &= 1 - y_1
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 |J(f^{-1})| &= \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1 - y_1 \end{vmatrix} \\
 &= y_2(1 - y_1) + y_1 y_2 \\
 &= y_2
 \end{aligned}$$

Let p_X be the joint density of (x_1, x_2) . We have the joint density of y_1 and y_2 :

$$\begin{aligned}
p(y_1, y_2) &= p_X(f^{-1}(y_1, y_2)) |J(f^{-1}(y_1, y_2))| \\
&= \text{Ga}(y_1 y_2; a_1, 1) \text{Ga}(y_2 - y_1 y_2; a_2, 1) y_2 \\
&= \frac{(y_1 y_2)^{a_1-1} \exp(-y_1 y_2)}{\Gamma(a_1)} \frac{((1 - y_1) y_2)^{a_2-1} \exp(y_1 y_2 - y_2)}{\Gamma(a_2)} y_2 \\
&= \frac{y_1^{a_1-1} y_2^{a_1+a_2-1} (1 - y_1)^{a_2-1} \exp(-y_2)}{\Gamma(a_1) \Gamma(a_2)}
\end{aligned}$$

The marginals are:

$$\begin{aligned}
p(y_1) &= \int_0^\infty p(y_1, y_2) dy_2 \\
&= \frac{y_1^{a_1-1} (1 - y_1)^{a_2-1}}{\Gamma(a_1) \Gamma(a_2)} \int_0^\infty y_2^{a_1+a_2-1} \exp(-y_2) dy_2 \\
&= \frac{y_1^{a_1-1} (1 - y_1)^{a_2-1} \Gamma(a_1 + a_2)}{\Gamma(a_1) \Gamma(a_2)}
\end{aligned}$$

and

$$\begin{aligned}
p(y_2) &= \int_0^\infty p(y_1, y_2) dy_1 \\
&= \frac{y_2^{a_1+a_2-1} \exp(-y_2)}{\Gamma(a_1) \Gamma(a_2)} \int_0^\infty y_1^{a_1-1} (1 - y_1)^{a_2-1} dy_1 \\
&= \frac{y_2^{a_1+a_2-1} \exp(-y_2)}{\Gamma(a_1) \Gamma(a_2)} \text{Beta}(a_1, a_2)
\end{aligned}$$

C

The posterior is:

$$\begin{aligned}
p(\theta | x_1, \dots, x_N) &\propto p(x_1, \dots, x_N | \theta) p(\theta) \\
&= \prod_{i=1}^N \text{N}(x_i; \theta, \sigma^2) \text{N}(\theta; m, v) \\
&\propto \prod_{i=1}^N \exp\left(-\frac{(x_i - \theta)^2}{2\sigma^2}\right) \exp\left(-\frac{(\theta - m)^2}{2v}\right) \\
&= \prod_{i=1}^N \exp\left(-\frac{x_i^2 - 2x_i\theta + \theta^2}{2\sigma^2}\right) \exp\left(-\frac{\theta^2 - 2\theta m + m^2}{2v}\right) \\
&= \exp\left(-\frac{-\sum_i x_i^2 + 2\sum_i x_i\theta - N\theta^2}{2\sigma^2}\right) \exp\left(-\frac{-\theta^2 + 2\theta m - m^2}{2v}\right) \\
&= \exp\left(-\theta^2 \left(\frac{N}{2\sigma^2} + \frac{1}{2v}\right) + \theta \left(\frac{\sum_i x_i}{\sigma^2} + \frac{m}{v}\right) - \frac{\sum_i x_i^2}{2\sigma^2} - \frac{m^2}{2v}\right)
\end{aligned}$$

We then complete the square by setting the posterior to:

$$\begin{aligned}
&= \exp \left[-a (\theta^2 - 2b\theta + b^2) \right] \\
&= \exp \left[-a (\theta - b)^2 \right] \\
&= \exp \left[-\frac{(\theta - b)^2}{2(1/(2a))} \right]
\end{aligned}$$

We calculate a, b by matching coefficients:

$$\begin{aligned}
a &= \frac{N}{2\sigma^2} + \frac{1}{2v} = \frac{Nv + \sigma^2}{2\sigma^2 v} \\
2ab &= \frac{\sum_i x_i}{\sigma^2} + \frac{m}{v} \\
\Rightarrow b &= \frac{v \sum_i x_i + m\sigma^2}{v\sigma^2} \frac{\sigma^2 v}{Nv + \sigma^2} \\
&= \frac{v \sum_i x_i + m\sigma^2}{Nv + \sigma^2}
\end{aligned}$$

The posterior is then:

$$\begin{aligned}
&\text{N}(b, 1/(2a)) \\
&= \text{N} \left(\frac{v \sum_i x_i + m\sigma^2}{Nv + \sigma^2}, \frac{\sigma^2 v}{Nv + \sigma^2} \right)
\end{aligned}$$

D

$$\begin{aligned}
p(\omega \mid x_1, \dots, x_N) &\propto \prod_{i=1}^N p(x_i \mid \theta, \omega) p(\omega) \\
&\propto \prod_{i=1}^N \omega^{1/2} \exp \left[-\frac{\omega}{2} (x_i - \theta)^2 \right] \frac{b^a}{\Gamma(a)} \omega^{a-1} \exp(-b\omega) \\
&\propto \omega^{N/2+a-1} \exp \left[-\omega \left(b + \frac{\sum_i (x_i - \theta)^2}{2} \right) \right] \\
&\propto \text{Ga} \left(a + \frac{N}{2}, b + \frac{\sum_i (x_i - \theta)^2}{2} \right)
\end{aligned}$$

We have the posterior of the variance:

$$p(\sigma^2 \mid x_1, \dots, x_N) = \text{IG} \left(a + \frac{N}{2}, b + \frac{\sum_i (x_i - \theta)^2}{2} \right)$$

E

The posterior is:

$$\begin{aligned}
p(\theta|x_1, \dots, x_N) &\propto p(x_1, \dots, x_N|\theta)p(\theta) \\
&= \prod_{i=1}^N N(x_i; \theta, \sigma_i^2) N(\theta; m, v) \\
&\propto \prod_{i=1}^N \exp\left(-\frac{(x_i - \theta)^2}{2\sigma_i^2}\right) \exp\left(-\frac{(\theta - m)^2}{2v}\right) \\
&= \exp\left(-\sum_{i=1}^n \frac{(x_i - \theta)^2}{2\sigma_i^2} - \frac{(\theta - m)^2}{2v}\right) \\
&= \exp\left[-\frac{1}{2}\left(\sum_{i=1}^n \frac{(x_i - \theta)^2}{\sigma_i^2} + \frac{(\theta - m)^2}{v}\right)\right] \\
&= \exp\left[-\frac{1}{2}\left(\sum_{i=1}^n \frac{x_i^2}{\sigma_i^2} + \sum_{i=1}^n \frac{-2x_i\theta}{\sigma_i^2} + \sum_{i=1}^n \frac{\theta^2}{\sigma_i^2} + \frac{\theta^2 - 2\theta m + m^2}{v}\right)\right] \\
&= \exp\left\{-\frac{1}{2}\left[\theta^2\left(\sum_{i=1}^n \frac{1}{\sigma_i^2} + \frac{1}{v}\right) - 2\theta\left(\sum_{i=1}^n \frac{x_i}{\sigma_i^2} + \frac{m}{v}\right) + \sum_{i=1}^n \frac{x_i^2}{\sigma_i^2} + \frac{m^2}{v}\right]\right\} \\
\text{set } &= \exp\left\{-\frac{1}{2}\left[a(\theta^2 - 2\theta b + b^2)\right]\right\} \\
&= \exp\left\{-\frac{1}{2}\left[\frac{(\theta - b)^2}{1/a}\right]\right\}
\end{aligned}$$

Matching the coefficients, we have:

$$\begin{aligned}
a &= \sum_{i=1}^n \frac{1}{\sigma_i^2} + \frac{1}{v} \\
b &= \left(\sum_{i=1}^N \frac{x_i}{\sigma_i^2} + \frac{m}{v}\right) / \left(\sum_{i=1}^n \frac{1}{\sigma_i^2} + \frac{1}{v}\right)
\end{aligned}$$

The posterior is:

$$N(b, 1/a)$$

F

$$\begin{aligned}
p(x) &= \int_0^\infty p(x \mid \sigma^2) p(\sigma^2) d\sigma^2 \\
&= \int_0^\infty p(x \mid \omega) p(\omega) d\omega \\
&= \int_0^\infty \left(\frac{\omega}{2\pi}\right)^{1/2} \exp\left(-\frac{\omega}{2}x^2\right) \frac{b^a}{\Gamma(a)} \omega^{a-1} \exp(-b\omega) d\omega \\
&\propto \int_0^\infty \omega^{1/2+a-1} \exp\left(-\omega\left(\frac{x^2}{2} + b\right)\right) d\omega \\
&= \frac{\Gamma(a+1/2)}{(b+x^2/2)^{a+1/2}} \\
&\propto \frac{\Gamma(\frac{2a+1}{2})}{(1+\frac{x^2/2}{b})^{a+1/2}}
\end{aligned}$$

2 The multivariate normal distribution

A

$$\begin{aligned}
\text{cov}(x) &= E\{(x - \mu)(x - \mu)^T\} \\
&= E\{xx^T - x\mu^T - \mu x^T + \mu\mu^T\} \\
&= E(xx^T) - E(x)\mu^T - \mu E(x)^T + \mu\mu^T \\
&= E(xx^T) - \mu\mu^T
\end{aligned}$$

We have:

$$E(Ax + b) = AE(x) + b = A\mu + b$$

then

$$\begin{aligned}
\text{cov}(Ax + b) &= E\{[(Ax + b) - (A\mu + b)][(Ax + b) - (A\mu + b)]^T\} \\
&= E\{(Ax - A\mu)(Ax - A\mu)^T\} \\
&= E\{A(x - \mu)(x - \mu)^T A^T\} \\
&= AE\{(x - \mu)(x - \mu)^T\}A^T \\
&= A\text{cov}(x)A^T
\end{aligned}$$

B

$$\begin{aligned} p(z) &= \prod_{i=1}^p p(z_i) \\ &= \frac{1}{(\sqrt{2\pi})^p} \exp\left(-\sum_{i=1}^p \frac{z_i^2}{2}\right) \\ &= \frac{1}{(\sqrt{2\pi})^p} \exp\left(-\frac{z^T z}{2}\right) \end{aligned}$$

The MGF of z is:

$$\begin{aligned} E(\exp(t^T z)) &= E\left[\exp\left(\sum_{i=1}^p t_i z_i\right)\right] \\ &= E\left[\prod_{i=1}^p \exp(t_i z_i)\right] \\ &= \prod_{i=1}^p E[\exp(t_i z_i)] \\ &= \prod_{i=1}^p \exp(t_i^2/2) \\ &= \exp\left[\sum_{i=1}^p t_i^2/2\right] \\ &= \exp(t^T t/2) \end{aligned}$$

C