Statistical Modeling 2
Exercise 1

January 22, 2017

1 Bayesian inference in simple conjugate families

A

$$
\begin{aligned}
p(w \mid x_1, \ldots, x_N) &\propto p(x_1, \ldots, x_N \mid w)p(w) && \text{(Bayes rule)} \\
&\propto \prod_{i=1}^{N} p(x_i|w) w^{a-1}(1-w)^{b-1} && \text{(indepence)} \\
&\propto w^s (1-w)^{N-s} w^{a-1}(1-w)^{b-1} && (\text{let } s = \textstyle\sum_{i=1}^{N} x_i) \\
&= w^{s+a-1}(1-w)^{N-s+b-1} \\
&\propto \mathrm{Beta}(s+a, N-s+b)
\end{aligned}
$$

B

Let $f(x_1, x_2) = (y1, y2) = (x_1/(x_1 + x_2), x_1 + x_2)$, we have:

$$f^{-1}(y1, y2) = (x_1, x_2) = (y_1 y_2, y_2 - y_1 y_2)$$

We then calculate the Jacobian of f^{-1}:

$$
\begin{aligned}
\partial x_1/\partial y_1 &= y_2 \\
\partial x_1/\partial y_2 &= y_1 \\
\partial x_2/\partial y_1 &= -y_2 \\
\partial x_2/\partial y_2 &= 1 - y_1
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|J(f^{-1})| &= \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{vmatrix} \\
&= y_2(1-y_1) + y_1 y_2 \\
&= y_2
\end{aligned}
$$

Let p_X be the joint density of (x_1, x_2). We have the joint density of y_1 and y_2:

$$
\begin{aligned}
p(y_1, y_2) &= p_X(f^{-1}(y_1, y_2))|J(f^{-1}(y_1, y_2))| \\
&= \mathrm{Ga}(y_1 y_2; a_1, 1)\mathrm{Ga}(y2 - y_1 y_2; a_2, 1) y_2 \\
&= \frac{(y_1 y_2)^{a_1 - 1} \exp(-y_1 y_2)}{\Gamma(a_1)} \frac{((1 - y_1) y_2)^{a_2 - 1} \exp(y_1 y_2 - y_2)}{\Gamma(a_2)} y_2 \\
&= \frac{y_1^{a_1 - 1} y_2^{a_1 + a_2 - 1} (1 - y_1)^{a_2 - 1} \exp(-y_2)}{\Gamma(a_1)\Gamma(a_2)}
\end{aligned}
$$

The marginals are:

$$
\begin{aligned}
p(y_1) &= \int_0^\infty p(y_1, y_2) dy_2 \\
&= \frac{y_1^{a_1 - 1}(1 - y_1)^{a_2 - 1}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty y_2^{a_1 + a_2 - 1} \exp(-y_2) dy_2 \\
&= \frac{y_1^{a_1 - 1}(1 - y_1)^{a_2 - 1}\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)}
\end{aligned}
$$

and

$$
\begin{aligned}
p(y_2) &= \int_0^\infty p(y_1, y_2) dy_1 \\
&= \frac{y_2^{a_1 + a_2 - 1} \exp(-y_2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty y_1^{a_1 - 1}(1 - y_1)^{a_2 - 1} \\
&= \frac{y_2^{a_1 + a_2 - 1} \exp(-y_2)}{\Gamma(a_1)\Gamma(a_2)} \mathrm{Beta}(a_1, a_2)
\end{aligned}
$$

C

The posterior is:

$$
\begin{aligned}
p(\theta | x_1, \ldots, x_N) &\propto p(x_1, \ldots, x_N | \theta) p(\theta) \\
&= \prod_{i=1}^N \mathrm{N}(x_i; \theta, \sigma^2) \mathrm{N}(\theta; m, v) \\
&\propto \prod_{i=1}^N \exp\left(-\frac{(x_i - \theta)^2}{2\sigma^2}\right) \exp\left(-\frac{(\theta - m)^2}{2v}\right) \\
&= \prod_{i=1}^N \exp\left(-\frac{x_i^2 - 2x_i\theta + \theta^2}{2\sigma^2}\right) \exp\left(-\frac{\theta^2 - 2\theta m + m^2}{2v}\right) \\
&= \exp\left(\frac{-\sum_i x_i^2 + 2\sum_i x_i \theta - N\theta^2}{2\sigma^2}\right) \exp\left(\frac{-\theta^2 + 2\theta m - m^2}{2v}\right) \\
&= \exp\left(-\theta^2 \left(\frac{N}{2\sigma^2} + \frac{1}{2v}\right) + \theta\left(\frac{\sum_i x_i}{\sigma^2} + \frac{m}{v}\right) - \frac{\sum_i x_i^2}{2\sigma^2} - \frac{m^2}{2v}\right)
\end{aligned}
$$

We then complete the square:

D

$$
\begin{aligned}
p(\omega \mid x_1, \ldots, x_N) &\propto \prod_{i=1}^{N} p(x_i \mid \theta, \omega) p(\omega) \\
&\propto \prod_{i=1}^{N} \omega^{1/2} \exp\left[-\frac{\omega}{2}(x_i - \theta)^2\right] \frac{b^a}{\Gamma(a)} \omega^{a-1} \exp(-b\omega) \\
&\propto \omega^{N/2+a-1} \exp\left[-\omega\left(b + \frac{\sum_i (x_i - \theta)^2}{2}\right)\right] \\
&\propto \mathrm{Ga}\left(a + \frac{N}{2}, b + \frac{\sum_i (x_i - \theta)^2}{2}\right)
\end{aligned}
$$

We have the posterior of the variance:

$$
p(\sigma^2 \mid x_1, \ldots, x_N) = \mathrm{IG}\left(a + \frac{N}{2}, b + \frac{\sum_i (x_i - \theta)^2}{2}\right)
$$

E

F

$$
\begin{aligned}
p(x) &= \int_0^\infty p(x \mid \sigma^2) p(\sigma^2) d\sigma^2 \\
&= \int_0^\infty p(x \mid \omega) p(\omega) d\omega \\
&= \int_0^\infty \left(\frac{\omega}{2\pi}\right)^{1/2} \exp\left(-\frac{\omega}{2} x^2\right) \frac{b^a}{\Gamma(a)} \omega^{a-1} \exp(-b\omega) d\omega \\
&\propto \int_0^\infty \omega^{1/2+a-1} \exp\left(-\omega\left(\frac{x^2}{2} + b\right)\right) d\omega \\
&= \frac{\Gamma(a + 1/2)}{(b + x^2/2)^{a+1/2}} \\
&\propto \frac{\Gamma(\frac{2a+1}{2})}{(1 + \frac{x^2/2}{b})^{a+1/2}}
\end{aligned}
$$