Homework 2

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1 Proof that

a. Gaussian distribution is normalized

$$p = (x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$$

If this equation is normalized, we have to show that:

$$\int_{-\infty}^{\infty} exp\left(-\frac{1}{2\sigma^2}x^2\right) dx = \sqrt{2\pi\sigma^2}(1)$$

Let:

$$I = \int_{-\infty}^{\infty} exp\left(-\frac{1}{2\sigma^2}x^2\right) dx$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp\left(-\frac{1}{2\sigma^2}x^2 - \frac{1}{2\sigma^2}y^2\right) dxdy(2)$$

We make transformation from (x, y) to (r, θ), which is defined by:

$$x = rcos(\theta)$$

$$y = rsin(\theta)$$

We have $\cos^2(\theta) + \sin^2(\theta) = 1$, so that $x^2 + y^2 = r^2$ (3)
From equation (2) and (3), we have:

$$I^2 = \int_0^{2\pi}\int_0^{\infty} exp\left(-\frac{r^2}{2\sigma^2}\right) rdrd\theta$$

$$I^2 = 2\pi\int_0^{\infty} exp\left(-\frac{r^2}{2\sigma^2}\right) rdr$$

We use $r^2 = u$, so that:

$$I^2 = 2\pi\int_0^{\infty} exp\left(-\frac{u}{2\sigma^2}\right) \frac{1}{2}du$$

$$I^2 = \pi \left[exp\left(-\frac{u}{2\sigma^2}(-2\sigma^2) \right) \right]$$

$$I^2 = 2\pi\sigma^2$$

Thus:

$$I = \sqrt{2\pi\sigma^2}(4)$$

From (1) and (4), we finally prove that Gaussian is normalized.

b. Expectation of Gaussian distribution is μ (mean)

From Gaussian distribution, we have:

$$p(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right)$$

The definition of the expected value shows that:

$$E[X] = \int_{-\infty}^{\infty} xp(x|\mu, \sigma^2)dx$$

$$E[X] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} xexp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right) dx$$

Let $t = \frac{x-\mu}{\sqrt{2}\sigma} \rightarrow x = \sqrt{2}\sigma t + \mu$, $dt = \frac{1}{\sqrt{2}\sigma}dx \rightarrow \sqrt{2}\sigma dt = dx$

$$E[X] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma t + \mu)exp(-t^2)\sqrt{2}\sigma dt$$

$$E[X] = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\sqrt{2}\sigma texp(-t^2) + (\mu exp(-t^2) \right) dt$$

$$E[X] = \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\sigma \int_{-\infty}^{\infty} texp(-t^2)dt + \mu \int_{-\infty}^{\infty} exp(-t^2)dt \right)$$

$$E[X] = \frac{1}{\sqrt{\pi}} \left(\left[\sqrt{2}\sigma \frac{-1}{2} exp(-t^2) \right] + \mu\sqrt{\pi} \right)$$

$$E[X] = \frac{\mu\sqrt{\pi}}{\sqrt{\pi}}$$

$$E[X] = \mu$$

So that the expectation of Gaussian distribution is μ

c. Variance of Gaussian distribution is σ^2 (variance)

From Gaussian distribution, we have:

$$p(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right)$$

The definition of variance value shows that:

$$Var[X] = \int_{-\infty}^{\infty} (x-\mu)^2 p(x|\mu,\sigma^2)$$

$$Var[X] = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$Var[X] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu)^2 exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Let $t = \frac{x-\mu}{\sqrt{2}\sigma} \to x = \sqrt{2}\sigma t + \mu$, $dt = \frac{1}{\sqrt{2}\sigma}dx \to \sqrt{2}\sigma dt = dx$

$$Var[X] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma t)^2 exp(-t)^2 \sqrt{2}\sigma dt$$

$$Var[X] = \frac{\sqrt{2}\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sigma^2 t^2 exp(-t)^2 dt$$

$$Var[X] = \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 exp(-t)^2 dt$$

$$Var[X] = \frac{2\sigma^2}{\sqrt{\pi}} \left(\left[\frac{-t}{2} exp(-t)^2 \right] + \int_{-\infty}^{\infty} \frac{1}{2} exp(-t)^2 dt \right)$$

$$Var[X] = \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \int_{-\infty}^{\infty} exp(-t)^2 dt$$

$$Var[X] = \frac{\sigma^2}{\sqrt{\pi}} \sqrt{\pi}$$

$$Var[X] = \sigma^2$$

So that the variance of Gaussian distribution is σ^2

d. Multivariate Gaussian distribution is normalized

2 Calculate

a. The conditional of Gaussian distribution

Multivariate normal vector $Y \sim N(\mu, \Sigma)$, consider μ and Y into:

$$\mu = \begin{bmatrix} \mu 1 \\ \mu 2 \end{bmatrix}$$

$$Y = \begin{bmatrix} y1 \\ y2 \end{bmatrix}$$

With a similar partition of Σ into:

$$\Sigma = \begin{bmatrix} \Sigma 11 & \Sigma 12 \\ \Sigma 21 & \Sigma 22 \end{bmatrix}$$

$$\rightarrow A = \Sigma^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

Σ is symmetric so Σ_{aa} and Σ_{bb} are symmetric while $\Sigma_{ab} = \Sigma_{ba}^T$
We are looking for conditional distribution $p(x_a|x_b)$
We have:

$$-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) = -\frac{1}{2}(x-\mu)^T A(x-\mu)$$

$$= -\frac{1}{2}(x_a-\mu_a)^T A_{aa}(x_a-\mu_a) - \frac{1}{2}(x_a-\mu_a)^T A_{ab}(x_b-\mu_b) - \frac{1}{2}(x_b-\mu_b)^T A_{ba}(x_a-\mu_a) - \frac{1}{2}(x_b-\mu_b)^T A_{bb}(x_b-\mu_b)$$

$$= -\frac{1}{2}x_a^T A_{aa}^{-1} x_a + x_a^T(A_{aa}\mu_a - A_{ab}(x_b-\mu_b)) + const$$

It is quadratic form of x_a hence conditional distribution $p(x_a|x_b)$ will be Gaussian, because this distribution is characterized by its mean and its variance. Compare with Gaussian distribution

$$\Delta^2 = -\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu + const$$

$$\Sigma_{a|b} = A_{aa}^{-1}$$

$$\mu_{a|b} = \Sigma a|b(A_{aa}\mu_a - A_{ab}(x_b-\mu_b)) = \mu_a - A_{aa}^{-}1A_{ab}(x_b-\mu_b)$$

By using Schur complement:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} M & -MBD^{-1} \\ -D^{-1}CMD^{-1} & D^{-1}CMBD^{-1}, M = (A-BD^{-1}C) \end{pmatrix}$$

$$\Rightarrow A_{aa} = (\Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba})^{-1}$$

$$A_{ab} = -(\Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba})^{-1}\Sigma_{ab}\Sigma_{bb}^{-1}$$

As a result

$$\mu_{a|b} = \mu_a + \Sigma_{ab}\Sigma_{bb}^{-1}(x_b-\mu_b)$$

$$\Sigma_{a|b} = \Sigma_{aa} - \Sigma ab\Sigma_{bb}^{-1}\Sigma_{ba}$$

$$\Rightarrow p(x_a|x_b) = N(x_{a|b}|\mu_{a|b}, \Sigma_{a|b})$$

b. The marginal of Gaussian distribution

The marginal distribution given by

$$p(x_a) = \int p(x_a, x_b)dx_b$$

We need to integrate out x_b by looking the quandratic form related x_b

$$-\frac{1}{2}x_b^T A_{bb} x_b + x_b^T m = -\frac{1}{2}(x_b - A_{bb}^{-1}m)^T A_{bb}(x_b - A_{bb}^{-1}m) + \frac{1}{2}m^T A_{bb}^{-1} m$$

with $m = A_{bb}\mu_b - A_{ba}(x_a - \mu_a)$ We can integrate over unnormalized Gaussian

$$\int exp\left(-\frac{1}{2}(x_b - A_{bb}^{-1}m)^T A_{bb}(x_b - A_{bb}^{-1}m)\right) dx_b$$

The remaining term

$$-\frac{1}{2}x_a^T(A_{aa} - A_{ab}A_{bb}^{-1}A_{ba})x_a + x_a^T(A_{aa} - AabA_{bb}^{-1}A_{ba})^{-1}\mu_a + const$$

Similarly, we have

$$E[x_a] = \mu_a$$
$$cov[x_a] = \Sigma_{aa}$$
$$\Rightarrow p(x_a) = N(x_a|\mu_a, \Sigma_{aa})$$