Homework 2

Thanh Hue

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1 Multivariate Gaussian distribution

1. The form of Multivariate Gaussian distribution is:

$$p(x \mid \mu, \sigma^2) = \frac{1}{(2\pi)^{\frac{D}{2}} |\Sigma|^{\frac{1}{2}}} e^{\frac{-1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$$

μ is a D-dimensional mean vector
Σ is a D x D covariance matrix
$|\Sigma|$ denotes the determinant of σ

Set:

$$\Delta^2 = \frac{-1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)$$

=

$$\frac{-1}{2}(x^T - \mu^T)\Sigma^{-1}(x-\mu)$$

=

$$\frac{-1}{2}x^T \Sigma^{-1} x + \frac{1}{2}x^T \Sigma^{-1} \mu + \frac{1}{2}\mu^T \Sigma^{-1} x - \frac{1}{2}\mu^T \Sigma^{-1} \mu$$

We have:

$$a^T b \,\epsilon\, R$$

So:

$$a^T b = (a^T b)^T = b^T (a^T)^T = b^T a$$

$\Rightarrow$

$$x^T \mu = \mu^T x$$

$\Rightarrow$

$$\Delta^2 = \frac{-1}{2}x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu + const$$

It is a quadratic form of Gaussian distribution.
Proof: Σ: symmetric, then Σ^{-1} is symmetric
We have:

$$\Sigma \Sigma^{-1} = I$$

$$I = I^T$$
$$\Sigma\Sigma^{-1} = (\Sigma\Sigma^{-1})^T$$
$$\Sigma\Sigma^{-1} = (\Sigma^{-1})^T\Sigma^T$$
$$\Sigma^{-1}\Sigma\Sigma^{-1} = \Sigma^{-1}(\Sigma^{-1})^T\Sigma$$
$$\Sigma^{-1} = (\Sigma^{-1})^T$$

Then Σ^{-1} is symmetric

2. Consider eigenvalues and eigenvector of Σ

$$\Sigma u_i = \lambda_i u_i \ , i = 1, ..., D$$

Because Σ is a real, symmetric matrix, its eigenvalues will be real and its eigenvector form an orthnormal set.

Proof: $\Sigma = \sum_{i=i}^{D} \lambda_i u_i u_i^T$ **then** $\Sigma^{-1} = \sum_{i=1}^{D} \frac{1}{\lambda_i} u_i u_i^T$

Let $\Sigma = \sum_{i=1}^{D} \lambda_i u_i u_i^T = PDP^T$
P is $D \times D$ matrix with eigenvector as its columns
D is a diagonal matrix
Because P is a orthogonal matrix $P^{-1} = P^T$

$$\Sigma^{-1} = (PDP^T)^{-1} = (P^T)^{-1}D^{-1}P^{-1} = PD^{-1}P^T = \sum_{i=1}^{D} \frac{1}{\lambda_i} u_i u_i^T$$