Homework 1

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1 Exercise 1

P(Hansen) = 0.05 ⇒ P(noHansen) = 0.95

$$P(+) = P(+|Hansen) * P(Hansen) + P(+|noHansen) * P(noHansen)$$

=

$$0.98 * 0.05 + 0.03 * 0.95$$

=

$$0.0775$$

$$P(Hansen|+) = \frac{P(+|Hansen).P(Hansen)}{P(+)}$$

=

$$\frac{0.98 * 0.05}{0.0775}$$
$$\approx\ 0.6323$$

2 Exercise 2

2.1 Proof the distributions are normalized

The format of univariate normal distribution is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

To proof the distribution is normalized, we need show that:

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = 1$$

Let

$$Z = \frac{x-\mu}{\sigma}$$

So

$$\sigma Z = x - \mu$$

$$\sigma dZ = dx$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(Z)^2}{2}} \sigma dZ$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-(Z)^2}{2}} dZ$$

We must show that:

$$\int_{-\infty}^{\infty} e^{\frac{-(Z)^2}{2}} dZ = \sqrt{2\pi}$$

Let

$$I = \int_{-\infty}^{\infty} e^{\frac{-(Z)^2}{2}} dZ$$

$$I^2 = \int_{-\infty}^{\infty} e^{\frac{-(Z)^2}{2}} dZ \int_{-\infty}^{\infty} e^{\frac{-(Y)^2}{2}} dY$$

$=$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{-(Z^2+Y^2)}{2}} dZdY$$

Let $Z = rcos\theta, Y = rsin\theta \Rightarrow dZdY = rd\theta dr$ Then:

$$I^2 = \int_{0}^{\infty} \int_{0}^{2\pi} e^{\frac{-(r^2)}{2}} rd\theta dr$$

$=$

$$2\pi \int_{0}^{2\pi} re^{\frac{-(r^2)}{2}} dr$$

$=$

$$-2\pi e^{\frac{-(r^2)}{2}} |_0^{2\pi}$$

$=$

$$2\pi \ (e^{+\infty} = 0)$$

$\Rightarrow I = \sqrt{2\pi}$

Thus:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-(Z)^2}{2}} dZ = 1$$

$$or \int_{-\infty}^{\infty} f(x)dx = 1$$

2.2 Mean

$$E(x) = xf(x)$$

So

$$E(x) = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Let

$$Z = \frac{x-\mu}{\sigma}$$

⇒

$$x = \sigma Z + \mu$$

$$\sigma dZ = dx$$

$$E(x) = \int_{-\infty}^{\infty} (\sigma Z + \mu) \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(Z)^2}{2}} \sigma dZ$$

$$E(x) = \int_{-\infty}^{\infty} \sigma Z \frac{1}{\sqrt{2\pi}} e^{\frac{-(Z)^2}{2}} dZ + \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi}} e^{\frac{-(Z)^2}{2}} dZ$$

We see:

$$E(Z) = \int_{-\infty}^{\infty} \sigma Z \frac{1}{\sqrt{2\pi}} e^{\frac{-(Z)^2}{2}} dZ$$

=

$$\frac{-\sigma}{\sqrt{2\pi}} e^{\frac{-(Z)^2}{2}} |_{-\infty}^{+\infty}$$

=

$$0$$

⇒

$$E(x) = \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi}} e^{\frac{-(Z)^2}{2}} dZ$$

$$E(x) = \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-(Z)^2}{2}} dZ$$

But we have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-(Z)^2}{2}} dZ = 1$$

So

$$E(x) = \mu$$

2.3 Standard Deviation

$$Var(Z) = E(Z^2) - (E(Z))^2$$

=

$$E(Z^2) - 0$$

=

$$E(Z^2)$$

=

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z^2 e^{\frac{-(Z)^2}{2}} dZ$$

Let u = Z, $dv = Ze^{\frac{-(Z)^2}{2}} dZ \Rightarrow v = -e^{\frac{-(Z)^2}{2}}$

$$Var(Z) = \frac{1}{\sqrt{2\pi}}(-Ze^{\frac{-(Z)^2}{2}}|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} -e^{\frac{-(Z)^2}{2}} dZ)$$

=

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(Z)^2}{2}} dZ$$

=

$$1$$

$$Z = \frac{x - \mu}{\sigma}$$

$\Rightarrow$

$$x = \sigma Z + \mu$$

$\Rightarrow$

$$Var(x) = \sigma^2 Var(Z) + Var(\mu)$$

$\Rightarrow$

$$Var(x) = \sigma^2 * 1 + 0$$

$\Rightarrow$

$$Var(x) = \sigma^2$$