Slides adapted from Robert Peter Matthew

EECS C106A/206A

Discussion #1: Rotations

Agenda

- Introductions
- Logistics
- Rotations
  - Coordinate frames
  - Rotation Matrices
  - Axis-Angle Rotations

Introductions

Logistics

- Fill out the **Pre-Course Survey**!
  - https://tinyurl.com/berkeleyrobotics2018

- Upcoming:
  - Homework 1 due **9/6**
  - Midterm 1 on **9/27**

Logistics

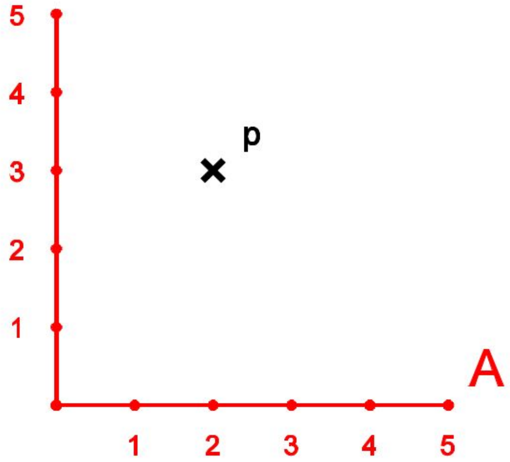
- Office Hours
  - Starting next week!
  - Tuesdays & Thursdays @ 11:30 - 12:30, Locations TBA
  - By appointment: brentyi@berkeley.edu

- Discussions
  - Will not comprehensively cover lecture topics
  - Attendance very much not required

Coordinate Frames

Coordinate Frames

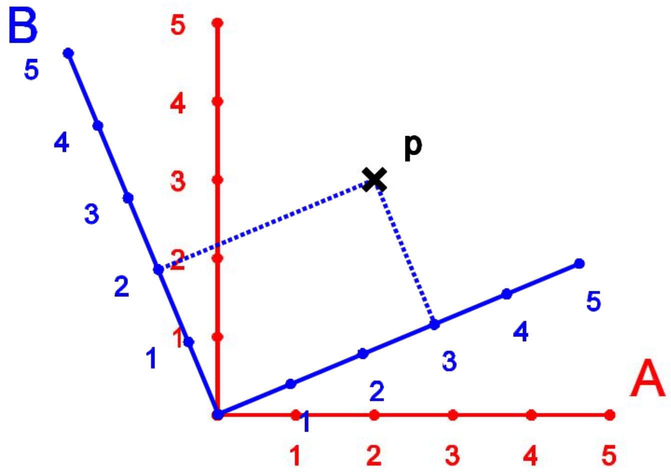
Points must be described relative to a coordinate frame.



$$p_A = \begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Coordinate Frames

A single point can be described relative to multiple coordinate frames!



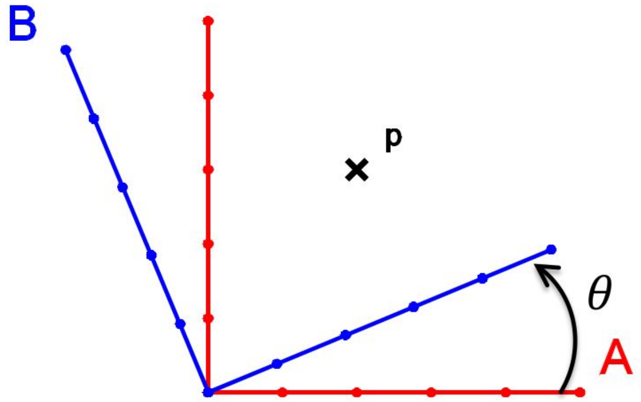
$$p_A = \begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$p_B = \begin{bmatrix} x_B \\ y_B \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Rotation Matrices

Rotations

Rotation matrices can translate between coordinates in different frames.



$$p_A = R_{AB} p_B$$

What's the rotation matrix when theta = 0?

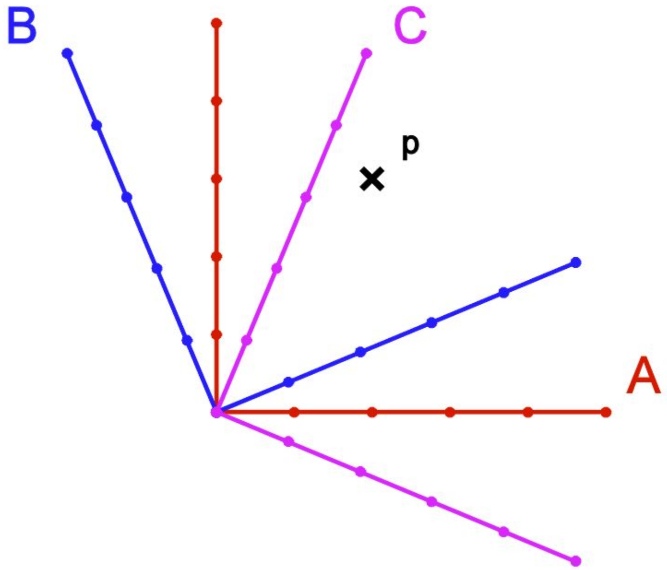
$$p_A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} p_B$$

Multiple Rotations

Given R_{AB} and R_{BC}, how do we compute R_{AC} ?

$$p_A = R_{AB} p_B$$
$$p_B = R_{BC} p_C$$
$$p_A = R_{AC} p_C$$

$$R_{AC} = R_{AB} R_{BC}$$



Properties of Rotations

Rotation matrices have two key properties:

- Orthonormality
  - $RR^T = R^T R = I$
- Determinant = 1
  - $det(R) = +1$

Properties of Rotations

We can define this set of matrices mathematically as:

$$SO(n) = \{R \in \mathbb{R}^{n\times n} \mid RR^T = I_n, det(R) = 1\}$$

...the members of which have the properties:

1. **Closure:** $R_1, R_2 \in SO(n) \implies R_1 R_2 \in SO(n)$
2. **Identity:** $R \in SO(n) \implies R = RI_n = I_n R$
3. **Inverse:** $R \in SO(n) \implies RR^{-1} = R^{-1}R = I_n$
4. **Associativity:** $R_1, R_2, R_3 \in SO(n) \implies (R_1 R_2)R_3 = R_1(R_2 R_3)$

Axis-Angle Rotations

Rodrigues' Formula

Given an axis $\boldsymbol{\omega}$ and an angle $\boldsymbol{\theta}$, we can calculate a corresponding rotation matrix $\boldsymbol{R}$:

$$R = \mathbb{I}_3 + \frac{\widehat{\omega}}{\|\omega\|_2} sin(\theta) + \frac{\widehat{\omega}^2}{\|\omega\|_2^2}\left(1 - cos(\theta)\right)$$

$$\widehat{\omega} = \widehat{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Euler Angles

Rotations about the standard Euclidean bases can be written:

$$R_X(\theta_X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta_X) & -sin(\theta_X) \\ 0 & sin(\theta_X) & cos(\theta_X) \end{bmatrix}$$

$$R_Y(\theta_Y) = \begin{bmatrix} cos(\theta_Y) & 0 & sin(\theta_Y) \\ 0 & 1 & 0 \\ -sin(\theta_Y) & 0 & cos(\theta_Y) \end{bmatrix}$$

$$R_Z(\theta_Z) = \begin{bmatrix} cos(\theta_Z) & -sin(\theta_Z) & 0 \\ sin(\theta_Z) & cos(\theta_Z) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler Angles

Euler angles are an alternative way to describe rotations, based on the standard Euclidean bases (X, Y, Z)

Often given in "Z-Y-X" form (θ_Z, θ_Y, θ_X): $R = R_Z R_Y R_X$

1. Rotation around fixed X-axis by θ_X
2. Rotation around fixed Y-axis by θ_Y
3. Rotation around fixed Z-axis by θ_Z

Order matters!

Euler Angles

Note that...

1. Rotation around original X-axis by θ_X
2. Rotation around original Y-axis by θ_Y
3. Rotation around original Z-axis by θ_Z

$$R = R_Z R_Y R_X$$

Is different from...

1. Rotation around original Z-axis by θ_Z
2. Rotation around original Y-axis by θ_Y
3. Rotation around original X-axis by θ_X

Euler Angles

But...

1. Rotation around original X-axis by θ_X
2. Rotation around original Y-axis by θ_Y
3. Rotation around original Z-axis by θ_Z

$$R = R_Z R_Y R_X$$

Is equivalent to...

1. Rotation around original Z-axis by θ_Z
2. Rotation around **mobile** Y-axis by θ_Y
3. Rotation around **mobile** X-axis by θ_X

Slides adapted from Robert Peter Matthew