Lecture 19-20: Dynamics of Multibody Mechanisms

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Contents

Previously: kinematics of bodies and mechanisms

- Forward kinematics: from θ to g_{0n}
- Inverse kinematics: from g_{0n} to θ
- Forward differential kinematics: from $\dot{\theta}$ to V_{0n}^{0}
- Forward differential kinematics: from F_0 to τ
- Inverse differential kinematics: from V_{0n}^{0} to $\dot{\theta}$

This week: dynamics

- From geometry to physics
- Relation between torques τ and accelerations $\ddot{\theta}$

Introduction to Dynamics

Dynamics

Dynamics: relation between accelerations and forces

- For rigid bodies: accelerations $\dot{V}$ and wrenches F
- For mechanisms: accelerations $\ddot{\theta}$ and torques τ

Things we need to know to describe dynamics

- Mass and inertias of all the parts
- All relevant forces and torques (actuation, gravity, friction, interaction, ...)

Newton dynamics

Newton's Second Law for unconstrained point masses

$$f = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$$

with $\dot{x} \in \mathbb{R}^3$ the inertial velocity.

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Why?

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Why?

- Mechanism is not a point mass
- Coordinate space is not Euclidean or inertial

Generalized coordinates

Coordinates θ are called generalized coordinates

- Newton's law does not apply in configuration space
- However, system itself lives in Euclidean space

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Properties of generalized coordinates

- Not coordinates of the point masses, but other coordinates that uniquely define their positions
- Coordinates that are not constrained in any way
- For robotics: joint angles θ
- For general systems: usually denoted by q

Dynamics using generalized coordinates

Dynamics for systems with generalized coordinates

- Newton's Law does not apply
- Use the Euler-Lagrange equations instead

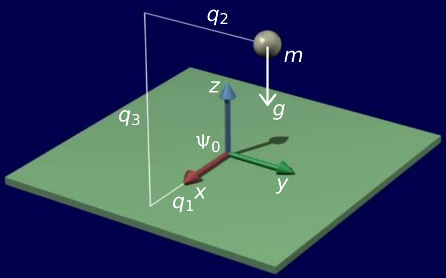
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \Gamma$$

with $L(q, \dot{q})$ the Lagrangian given by

$$L(q, \dot{q}) = \underbrace{T(q, \dot{q})}_{\text{kinetic}} - \underbrace{V(q)}_{\text{potential}}$$

and Γ the vector of input forces collocated with $\dot{q}$.

Example: point mass



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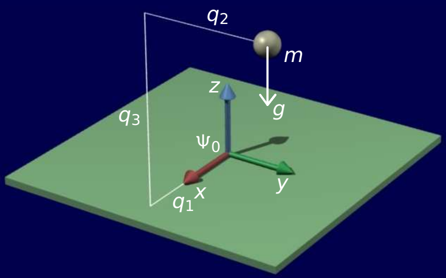
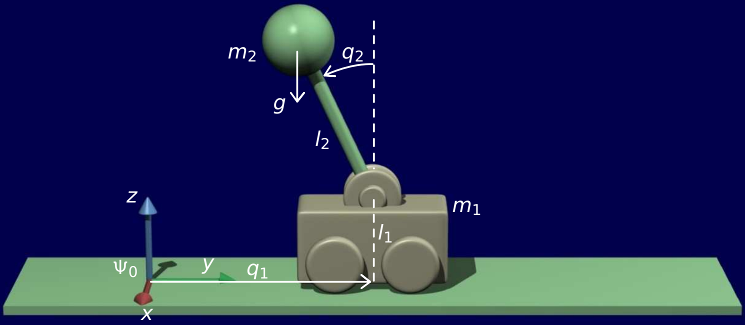


- Kinetic energy: $T(q, \dot{q}) = \frac{1}{2} m \dot{q}^T \dot{q}$
- Potential energy: $V(q) = mgq_3$
- Lagrangian: $L(q, \dot{q}) = T(q, \dot{q}) - V(q)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = m \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = 0$$

Example: cart with stick



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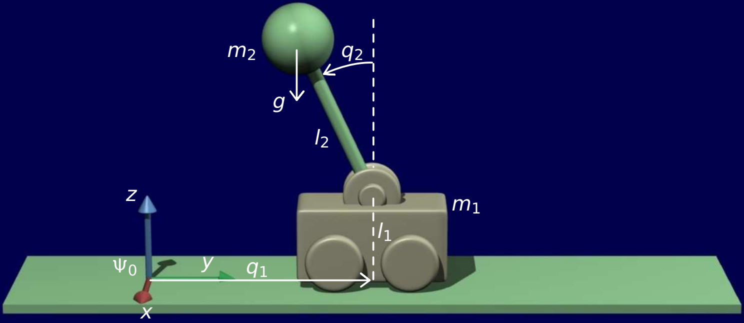


- Kinetic energy:

$$T(q,\dot{q}) = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\left(\dot{q}_1^2 + l_2^2\dot{q}_2^2 - 2l_2\cos(q_2)\dot{q}_1\dot{q}_2\right)$$

- Potential energy: $V(q) = m_2 g(l_1 + l_2\cos(q_2))$

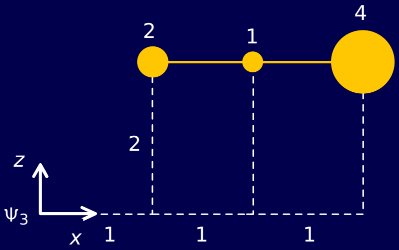
Application to mechanisms

In order to apply this to robotic mechanisms, we need

- Coordinates q: we choose joint angles θ
- Kinetic energy: energies of the links
- Potential energy: gravity on links

Energy of a Rigid Body

Center of mass



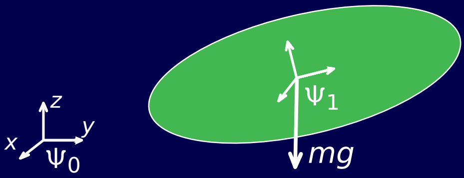
Center of mass (center of gravity)

- Balancing point on a rigid body
- If suspended from that point, body will not rotate

Compute CoM position as a weighted average

$$\begin{bmatrix} x_{\mathrm{CoM}} \\ y_{\mathrm{CoM}} \\ z_{\mathrm{CoM}} \end{bmatrix} = \frac{1}{m} \iiint_{x,y,z} \rho(x,y,z) \begin{bmatrix} x \\ y \\ z \end{bmatrix} dx\, dy\, dz$$

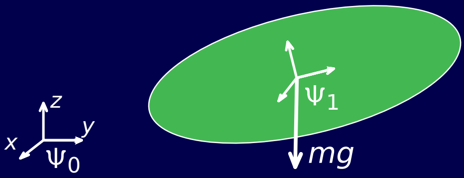
Potential energy



Gravity is a potential field with gravitational constant g

- Gravitational energy for a point $V = mgh$
- Gravitational energy for a rigid body $V = mgh$ (total mass m, height of the center of mass h)

Potential energy



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For a rigid body with frame Ψ_1 at the center of mass and inertial reference frame Ψ_0 with z-axis up

$$V = mgp_{01}[3] = mgg_{01}[3, 4]$$

where p_{01} is the position component of $g_{01} \in SE(3)$

Mass vs. inertia

Translation energy vs. Rotation energy

- Point mass m with $\dot{x} = v$: $T = \frac{1}{2}mv^2$
- Point mass m with $\dot{x} = \omega r$: $T = \frac{1}{2}m(\omega r)^2 = \frac{1}{2}(mr^2)\omega^2$

Both energies are the same kinetic energy, just expressed in different coordinates!

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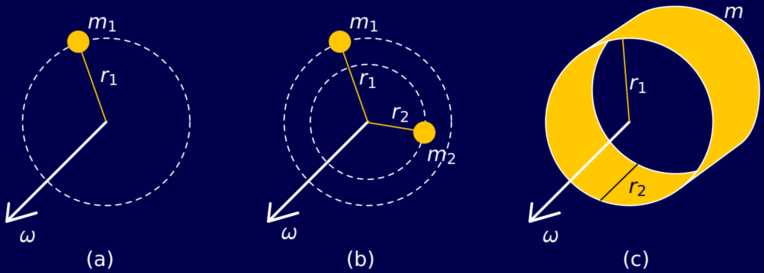
Inertia

- Inertia J of a mass m: 'angular weight' s.t. $T = \frac{1}{2}J\omega^2$
- $\omega \in \mathbb{R}$: angular velocity around a specific axis
- Inertia depends on axis direction and location!

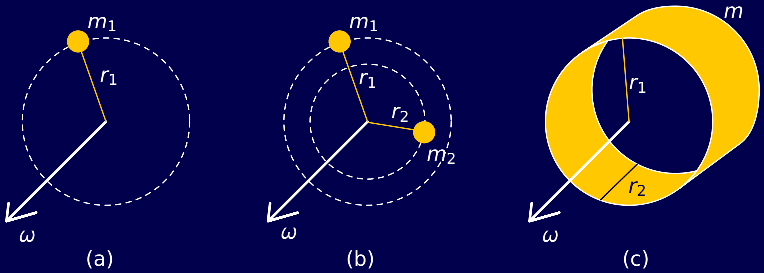
Inertias of some objects



Just like mass, inertia can be summed / integrated

$$J_a = m_1 r_1^2$$

Inertias of some objects



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$$J_a = m_1 r_1^2$$

$$J_b = m_1 r_1^2 + m_2 r_2^2$$

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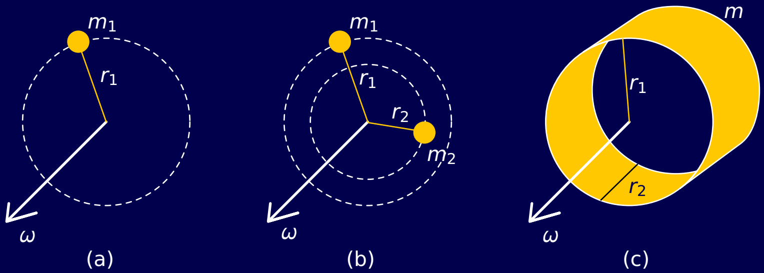

Just like mass, inertia can be summed / integrated

$$J_a = m_1 r_1^2$$

$$J_b = m_1 r_1^2 + m_2 r_2^2$$

$$J_c = \int_0^{r_2} \int_0^{2\pi} \rho(\theta, x) r^2(\theta, x) d\theta dx = m r_1^2$$

Kinetic energy

Kinetic energy T of a rigid body with density $\rho(r)$

- Integral of kinetic energy of all particles

$$T = \frac{1}{2}\int_V \rho(r)\|\dot{r}\|^2 dV$$

- If we express the velocity $\dot{r}$ using twists

$$T = \frac{1}{2}\int_V \rho(r)\|\hat{\omega}r + v\|^2 dV$$

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$$T = \frac{1}{2}\int_V \rho(r)\|\hat{\omega} r + v\|^2 dV = \frac{1}{2}\int_V \rho(r)\|v - \hat{r}\omega\|^2 dV$$

$$= \frac{1}{2}\begin{bmatrix} v^T & \omega^T \end{bmatrix} \underbrace{\left(\int_V \rho(r)\begin{bmatrix} I & \hat{r}^T \\ \hat{r} & \hat{r}^T\hat{r} \end{bmatrix} dV\right)}_{\text{inertia matrix } \mathcal{M}} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

Inertia matrix

Kinetic energy $T = \frac{1}{2} V_{01}^T \mathcal{M} V_{01}$ with

$$\mathcal{M} = \int_V \rho(r) \begin{bmatrix} I & \hat{r}^T \\ \hat{r} & \hat{r}^T \hat{r} \end{bmatrix} dV = \begin{bmatrix} \int_V \rho(r) I dV & \int_V \rho(r) \hat{r}^T dV \\ \int_V \rho(r) \hat{r} dV & \int_V \rho(r) \hat{r}^T \hat{r} dV \end{bmatrix}$$

The numerical values and properties depend on the choice of coordinate frame for V_{01} and $\mathcal{M}$

- For any choice: $\int_V \rho(r) dV = m$ with m the total mass
- Choose body frame Ψ_1: $\mathcal{M}$ constant!
- Choose Ψ_1 at the center of mass: $\int_V \rho(r) \hat{r} dV = 0$
- Choose Ψ_1 aligned with principle axes: $\mathcal{M}$ diagonal

Inertia matrix

Inertial properties of any rigid body are determined by

- Total mass
- Center of mass (balancing point)
- Inertias around three principle axes

Inertia matrix

Inertial properties of any rigid body are determined by

- Total mass
- Center of mass (balancing point)
- Inertias around three principle axes

Any rigid body of any shape behaves just like an ellipsoid with the same mass and inertia

=

Dynamics of a rigid body

Newton-Euler equations

$$\begin{bmatrix} mI & 0 \\ 0 & \mathcal{I} \end{bmatrix} \begin{bmatrix} \dot{v}_{01}^{1} \\ \dot{\omega}_{01}^{1} \end{bmatrix} + \begin{bmatrix} \omega_{01}^{1} \times m v_{01}^{1} \\ \omega_{01}^{1} \times \mathcal{I} \omega_{01}^{1} \end{bmatrix} = F_1$$

Derivation can be found in MLS Chapter 4.

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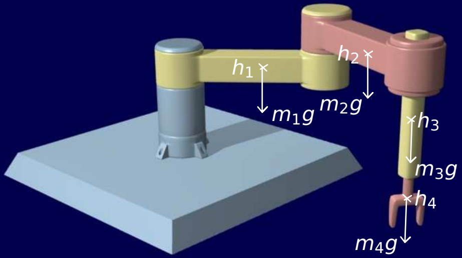
Interesting properties

- Energy is conserved: $\frac{d}{dt}(T+V) = F_1^T V_{01}^1$
- Momentum (in coordinates Ψ_0) is conserved
- Stability of rotations: the lunchbox experiment

Energy of a Robotic Mechanism

Potential energy



Sum potential energies of the links

- Coordinate frame Ψ_i attached to each link i
- Center of mass has constant coordinates in Ψ_i
- Compute height $h_i(\theta)$ of center of mass using FK
- Potential energy equals

$$V(\theta) = \sum_{i=1}^{n} m_i g h_i(\theta)$$

Kinetic energy

Express kinetic energy using θ and $\dot{\theta}$

- Kinetic energy of body i:

$$T_i = \frac{1}{2} \left(V^i_{0i}\right)^T \mathcal{M}_i \, V^i_{0i}$$

Kinetic energy

Express kinetic energy using θ and $\dot{\theta}$

- Kinetic energy of body i:

$$T_i = \frac{1}{2}\left(V_{0i}^{i}\right)^T \mathcal{M}_i \, V_{0i}^{i}$$

- Twist of body i can be written using body Jacobian

$$V_{0i}^{0} = J_i(\theta)\dot{\theta} = \begin{bmatrix} \xi_1' & \cdots & \xi_i' & 0 & \cdots & 0 \end{bmatrix} \dot{\theta}$$

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$$V_{0i}^0 = J_i(\theta)\dot{\theta} = \begin{bmatrix} \xi_1' & \cdots & \xi_i' & 0 & \cdots & 0 \end{bmatrix} \dot{\theta}$$

- Combining these two expressions

$$\begin{aligned} T_i &= \frac{1}{2}\left(\mathrm{Ad}_{g_{i0}} J_i \dot{\theta}\right)^T \mathcal{M}_i \left(\mathrm{Ad}_{g_{i0}} J_i \dot{\theta}\right) \\ &= \frac{1}{2}\dot{\theta}^T \left(J_i^T \mathrm{Ad}_{g_{i0}}^T \mathcal{M}_i \mathrm{Ad}_{g_{i0}} J_i\right) \dot{\theta} \end{aligned}$$

Kinetic energy

$$T_i(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T \left(J_i^T \, \mathrm{Ad}_{g_{i0}}^T \, \mathcal{M}_i \, \mathrm{Ad}_{g_{i0}} J_i\right) \dot{\theta}$$

Summing kinetic energy over all links

Kinetic energy

$$T_i(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T \left(J_i^T \operatorname{Ad}_{g_{i0}}^T \mathcal{M}_i \operatorname{Ad}_{g_{i0}} J_i \right) \dot{\theta}$$

Summing kinetic energy over all links

$$T(\theta, \dot{\theta}) = \sum_{i=1}^{n} \frac{1}{2}\dot{\theta}^T \left(J_i^T \operatorname{Ad}_{g_{i0}}^T \mathcal{M}_i \operatorname{Ad}_{g_{i0}} J_i \right) \dot{\theta}$$

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Kinetic energy

$$M(\theta) = \sum_{i=1}^{n} J_i^T(\theta)\,\mathrm{Ad}^T_{g_{i0}(\theta)}\,\mathcal{M}_i\,\mathrm{Ad}_{g_{i0}(\theta)} J_i(\theta)$$

$M(\theta)$: inertia matrix / mass matrix / inertia tensor

Kinetic energy

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- Symmetric: $M^T(\theta) = M(\theta)$

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- Positive (semi-)definite: $M(\theta) \geq 0$

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- Symmetric: $M^T(\theta) = M(\theta)$
- Positive (semi-)definite: $M(\theta) \geq 0$
- In practice strictly positive definite: $M(\theta) > 0$
- Physical reason: any motion $\dot{\theta}$ has kinetic energy (point masses and infinitely thin rods do not exist)

Kinetic energy

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- Physical reason: any motion $\dot{\theta}$ has kinetic energy (point masses and infinitely thin rods do not exist)
- Exception: badly chosen coordinates w/singularities
- Compare null-motions: $J(\theta)\dot{\theta} = 0$

Dynamics of a Robotic Mechanism

Euler-Lagrange equations

Remember the dynamics in generalized coordinates

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \Gamma$$

with $L(q, \dot{q})$ the Lagrangian given by

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and Γ the vector of input forces collocated with $\dot{q}$.

Lagrangian of a Mechanism

For a robotic mechanism with joint angles θ

$$T(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta}$$

$$V(\theta) = \sum_{i=1}^{n} m_i g h_i(\theta)$$

So the Lagrangian becomes

$$L(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} - V(\theta)$$

Dynamics of a mechanism

$$L(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} - V(\theta)$$

Substituting into the Euler-Lagrange equation gives

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + N(\theta) = \tau$$

with

$$N(\theta) = \frac{\partial V}{\partial \theta}$$

$$C_{ij}(\theta, \dot{\theta}) = \sum_{k=1}^{n} \Gamma_{ijk}\dot{\theta}_k = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ki}}{\partial \theta_j} - \frac{\partial M_{jk}}{\partial \theta_i}\right)\dot{\theta}_k$$

Dynamics of a mechanism

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + N(\theta) = \tau$$

- $C(\theta, \dot{\theta})$ represents Coriolis and centrifugal effects
- Efficient algorithms exist to find $M(\theta)$ and $C(\theta, \dot{\theta})$

Dynamics of a mechanism

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + N(\theta) = \tau$$

- $C(\theta, \dot{\theta})$ represents Coriolis and centrifugal effects
- Efficient algorithms exist to find $M(\theta)$ and $C(\theta, \dot{\theta})$
- Multiply by M^{-1} to find $\ddot{\theta}$ as function of τ
- Define states θ and $\dot{\theta}$ to get

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta} \\ -M^{-1}C\dot{\theta} - M^{-1}N\end{bmatrix} + \begin{bmatrix}0 \\ M^{-1}\end{bmatrix}\tau$$

- Result: first order ODE of the form $\dot{x} = f(x) + g(x)u$
- Use in simulation, analysis, nonlinear control design

* *

Limitations of the standard equation

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + N(\theta) = \tau$$

Only suitable for simple joints with $\theta \in \mathbb{R}$

- More general joints: ball joint, free-floating

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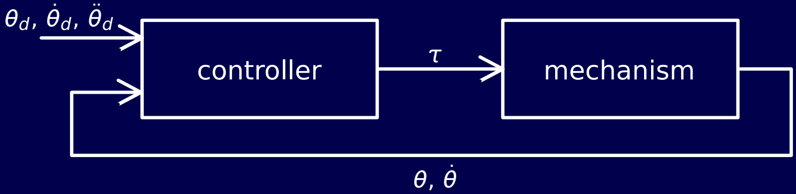
- More general joints: ball joint, free-floating

Only suitable for so-called holonomic joints

- Non-holonomic: dimension of velocity space is smaller than dimension of configuration space
- Examples of nonholonomic joints: unicycle, car...

Application in Computed Torque Control

Naive Computed Torque Control



The dynamics of the plant are given as

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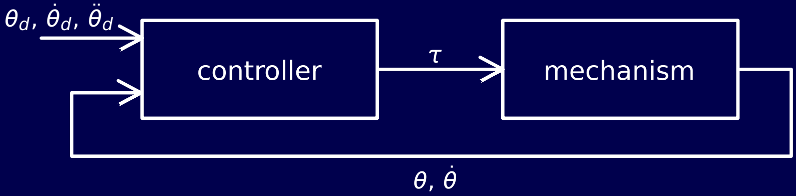
Naive Computed Torque Control



The dynamics of the plant are given as

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + N(\theta) = \tau$$

What if we choose the control inputs τ as follows?

$$\tau = M(\theta)\ddot{\theta}_d + C(\theta, \dot{\theta})\dot{\theta} + N(\theta)$$

For this choice, the interconnection of the two becomes

$$\ddot{\theta} = \ddot{\theta}_d$$

Extended Computed Torque Control

To deal with initial conditions and disturbances, choose

$$\tau = M(\theta)\ddot{\theta}_d + C(\theta,\dot{\theta})\dot{\theta} + N(\theta) - M(\theta)K_v\dot{e} - M(\theta)K_p e$$

with $e := \theta - \theta_d$.

Extended Computed Torque Control

To deal with initial conditions and disturbances, choose

$$\tau = M(\theta)\ddot{\theta}_d + C(\theta,\dot{\theta})\dot{\theta} + N(\theta) - M(\theta)K_v\dot{e} - M(\theta)K_p e$$

with $e := \theta - \theta_d$. The closed loop equation then becomes

$$\ddot{e} + K_v\dot{e} + K_p e = 0$$

which can be stabilized with suitable K_p, K_v.

- Matlab simulation example

Robustness of Computed Torque Control

What about an error in the model $M \rightarrow \tilde{M}$?

$$\tau = \tilde{M}(\theta)\ddot{\theta}_d + C(\theta, \dot{\theta})\dot{\theta} + N(\theta) - \tilde{M}(\theta)K_v\dot{e} - \tilde{M}(\theta)K_p e$$

The closed loop becomes

$$\ddot{\theta} = M^{-1}\tilde{M}\ddot{\theta}_d - M^{-1}\tilde{M}K_v\dot{e} - M^{-1}\tilde{M}K_p e$$

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This may be unstable, even for $\tilde{M}$ close to M!

- Matlab simulation example

Conclusions

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Dynamics of robotic mechanisms

- Lagrange eqns: dynamics in generalized coords
- Any rigid body acts like an ellipsoid
- We can write kinetic energy in terms of V or $\dot{\theta}$
- We can write potential energy in terms of g or θ
- Result: second-order explicit differential eqns

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + N(\theta) = \tau$$

- Use: simulation, analysis, controller design
- Computed torque: beware of model deviations