6. Dynamics

Introduction to Robotics

Slides by Robert Peter Matthew

Outline

- Equations of motion
  - Newtonian mechanics
    - Examples: particle, spring, pendulum
  - Lagrangian mechanics
    - Examples: particle, spring, pendulum

rpmatthew <at> berkeley.edu

Newton's 2ND Law

The applied **linear force f** equals the rate of change of linear momentum:

$$f = \frac{d}{dt}(mass \ \times \ velocity)$$

For a constant mass, we have:

$$f = mass \ \times \ acceleration$$

EULER'S ROTATION EQUATIONS

The applied **torque $\boldsymbol{\tau}$** equals the time rate of change of angular momentum:

$$\tau = \frac{d}{dt}(I'\omega^s)$$

where:

$I' = RIR^T$ is the inertia tensor w.r.t. to the inertial frame
ω^s is the angular velocity

Euler's Rotation Equations

Assume 2-D and no change in inertia w.r.t. time:

$$\tau = \frac{d}{dt}(I'\omega^s)$$

$$\tau = \frac{d}{dt}(RIR^T\ \omega^s)$$

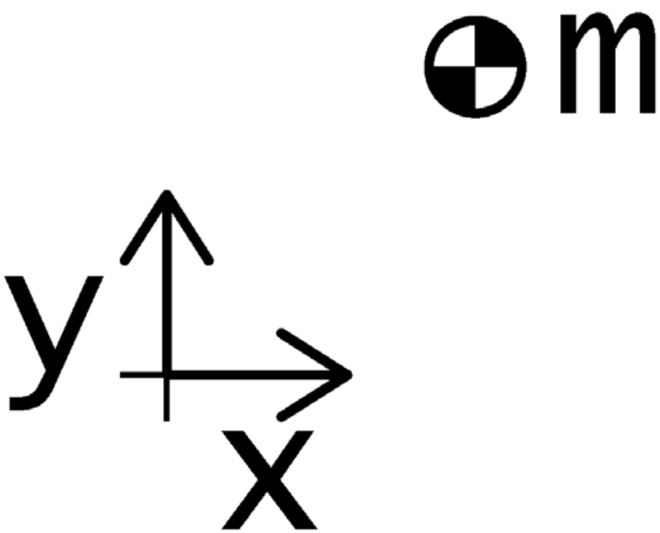
$$\tau = RIR^T\dot{\omega}^s + \dot{R}IR^T\omega^s + RI\dot{R}^T\omega^s$$

$$\tau = I'\dot{\omega}^s + \omega^s \times I'\omega^s - I'\omega^s \times \omega^s$$

$$\tau = I'\ddot{\theta}^s$$

Example I: Particle

$$f = ma$$
$$\tau = I'\ddot{\theta}$$



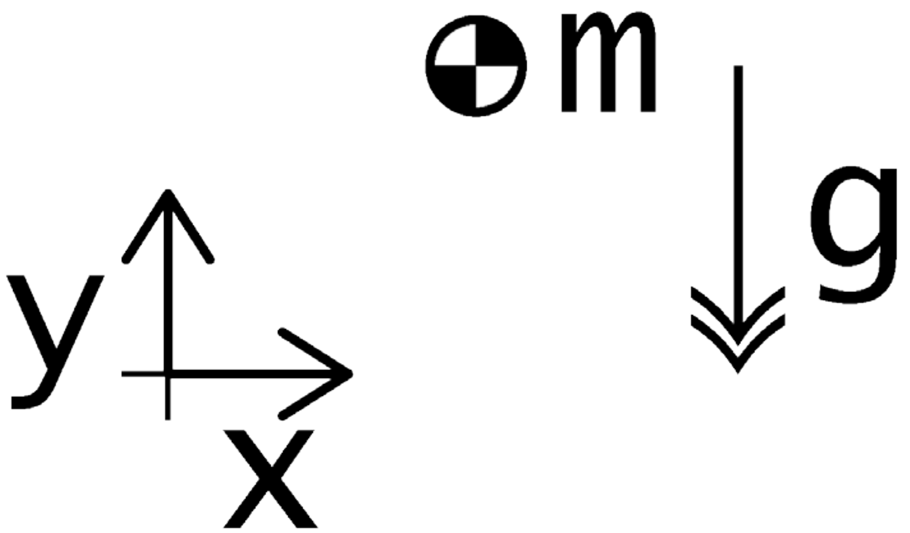
$$f_X = m\ddot{x} = 0$$
$$\rightarrow \ddot{x} = 0$$

$$f_Y = m\ddot{y} = 0$$
$$\rightarrow \ddot{y} = 0$$

rpmatthew <at> berkeley.edu

Example II: Particle & Gravity

$$f = ma$$
$$\tau = I'\ddot{\theta}$$



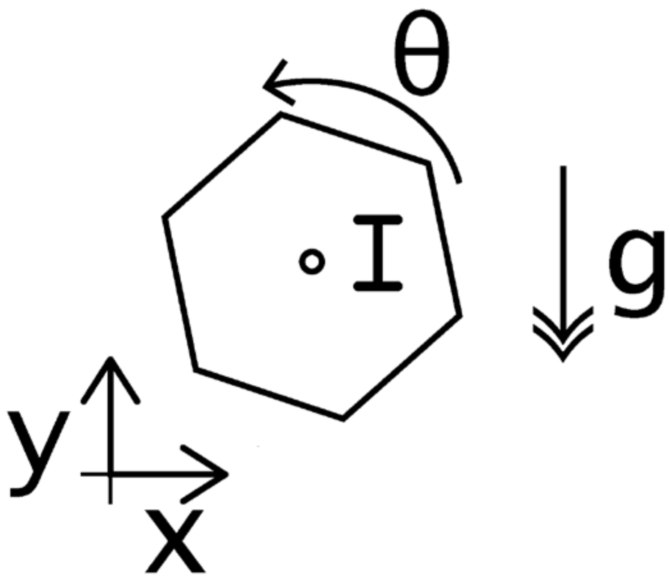
$$f_X = m\ddot{x} = 0$$
$$\rightarrow \ddot{x} = 0$$

$$f_Y = m\ddot{y} = \boldsymbol{-mg}$$
$$\rightarrow \ddot{y} = -g$$

Gravitational force $= \boldsymbol{-mg}$

EXAMPLE III: BODY & GRAVITY

$$f = ma$$
$$\tau = I'\ddot{\theta}$$



$$f_X = m\ddot{x} = 0$$
$$\rightarrow \ddot{x} = 0$$

$$f_Y = m\ddot{y} = -mg$$
$$\rightarrow \ddot{y} = -mg$$

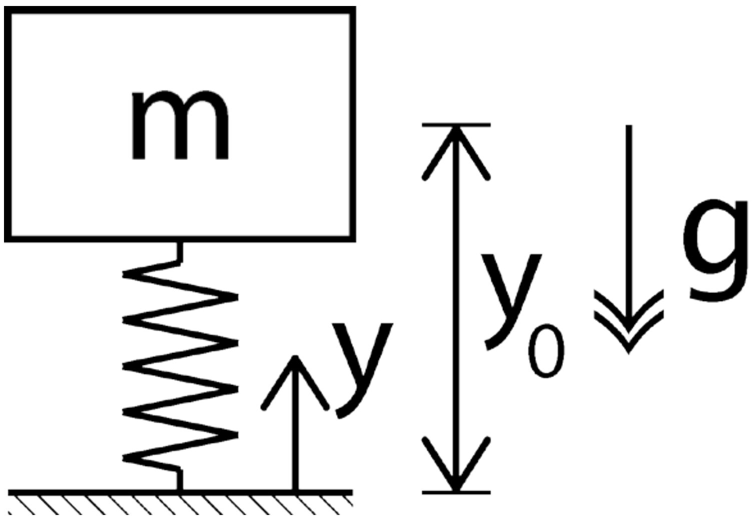
$$\tau = I'\ddot{\theta} = 0$$
$$\rightarrow \ddot{\theta} = 0$$

Example IV: Spring Mass

$$f = ma$$
$$\tau = I'\ddot{\theta}$$

A **linear spring** can be characterized by the equation:

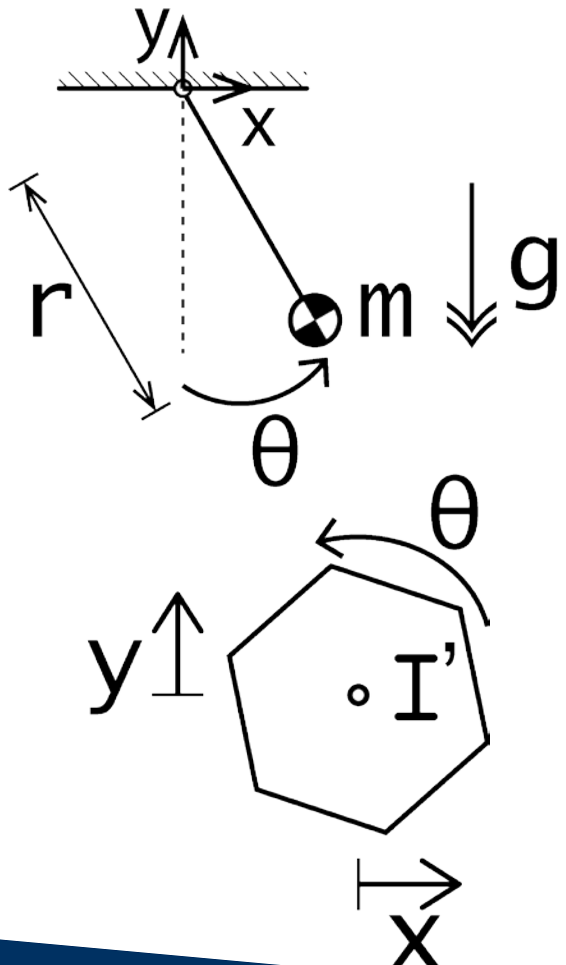
$$\boldsymbol{f_Y = -k(y - y_0)}$$



$$f_Y = m\ddot{y} = -k(y - y_0) - mg$$
$$\rightarrow \ddot{y} = -\frac{k}{m}(y - y_0) - g$$
$$\rightarrow y_0 = \frac{gm}{k} \qquad \rightarrow \ddot{y} = -\frac{k}{m}y$$

Example IV: Pendulum

$$f = ma \qquad (1)$$
$$\tau = I'\ddot{\theta} \qquad (2)$$



We can rewrite this system as a rotating object with the inertia: $\boldsymbol{I' = mr^2}$

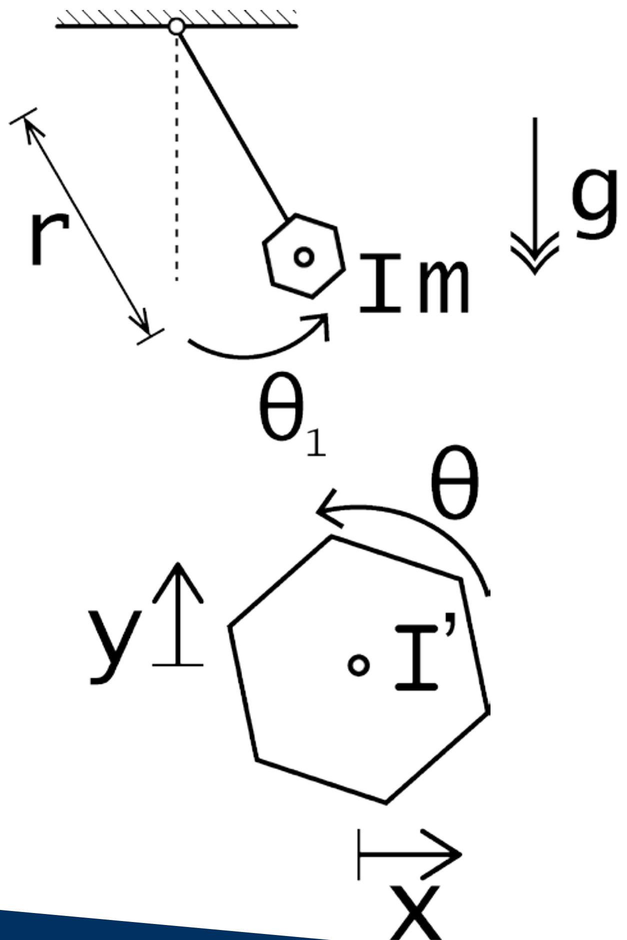
Torque generated by gravity:

$$\tau = -rmgsin(\theta) \qquad (3)$$

Combining (2) and (3):

$$mr^2\ddot{\theta} = -rmgsin(\theta)$$
$$\ddot{\theta} = -\frac{g}{r}sin(\theta)$$

Example V: Pendular Rod



$$f = ma$$
$$\tau = I'\ddot{\theta}$$

We can rewrite this system as a rotating object with the inertia:

$$\boldsymbol{I' = mr^2 + I}$$

This is known as the parallel axis theorem

$$\tau = I'\ddot{\theta} = -rmgsin(\theta)$$
$$(mr^2 + I)\ddot{\theta} = -rmgsin(\theta)$$
$$\ddot{\theta} = -\frac{rmg}{mr^2 + I} sin(\theta)$$

Berkeley
UNIVERSITY OF CALIFORNIA
rpmatthew <at> berkeley.edu

Lagrangian Dynamics

Lagrangian = Kinetic Energy - Potential Energy

$$L = T - V$$

The change in the Lagrange is equal to the non-conservative forces:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \gamma_i$$

where q_i is a generalized coordinate

where γ_i is the corresponding non-conservative force

KINETIC ENERGY

Kinetic Energy T: energy due to motion

Translational T: $E_t = \frac{1}{2} m v^2$

Rotational T: $E_r = \frac{1}{2} I \dot{\theta}^2$

Total KE: $E_k = E_t + E_r$

POTENTIAL ENERGY

Potential Energy V: energy due to position

Gravitational V: $V_g = mgh$

where
m is the mass
g is the accel. due to gravity
h is the distance along the gravitational axis

Elastic V: $V_e = \frac{1}{2}k\delta^2$

k is the spring constant
δ is the extension of the spring

Example I: Particle

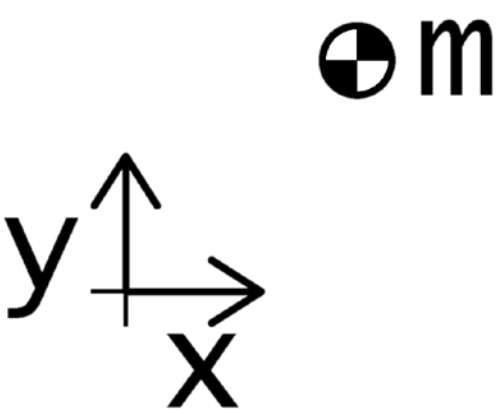
$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \gamma_i$$

Kinetic Energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

Potential Energy $V = 0$

Lagrangian $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}(m\dot{x}) - 0 = 0$$

$$m\ddot{x} = 0 \qquad \rightarrow \ddot{x} = 0$$

Example I: Particle

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \gamma_i$$

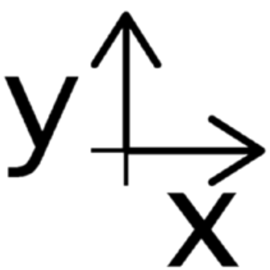
Kinetic Energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

Potential Energy $V = 0$

Lagrangian $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{d}{dt}(m\dot{y}) - 0 = 0$$



$$m\ddot{y} = 0 \qquad \rightarrow \ddot{y} = 0$$

Example II: Particle & Gravity

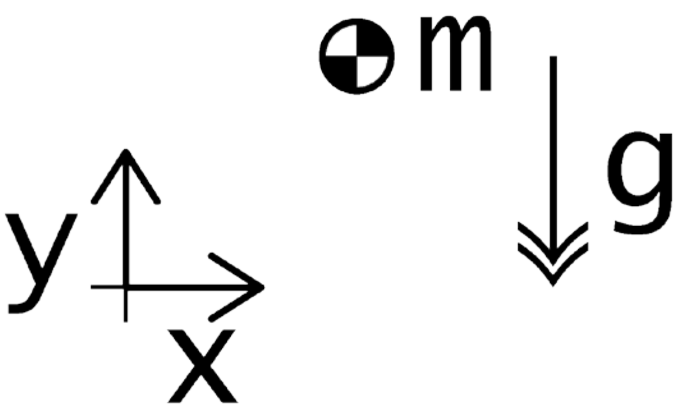
$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \gamma_i$$

Kinetic Energy $\quad T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

Potential Energy $\quad V = mgy$

Lagrangian $\quad L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} = 0 \qquad \rightarrow \ddot{x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = m\ddot{y} + mg = 0 \qquad \rightarrow \ddot{y} = -g$$

Example III: Body & Gravity

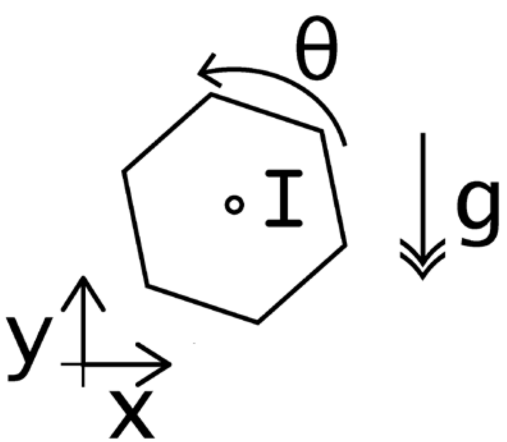
$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \gamma_i$$

Kinetic Energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2$

Potential Energy $V = mgy$

Lagrangian $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 - mgy$



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} = 0 \quad \rightarrow \ddot{x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = m\ddot{y} + mg = 0 \quad \rightarrow \ddot{y} = -g$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = I\ddot{\theta} = 0 \quad \rightarrow \ddot{\theta} = 0$$

Example IV: Spring Mass

$$L = T - V$$

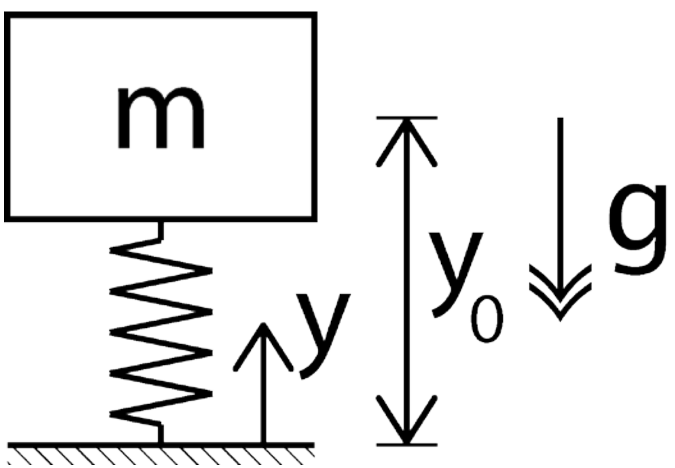
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \gamma_i$$

Kinetic Energy $T = \frac{1}{2}m(\dot{y}^2)$

Potential Energy $V = mgy + \boldsymbol{\frac{1}{2}k(y - y_0)^2}$

potential energy stored in a spring

Lagrangian $L = \frac{1}{2}m(\dot{y}^2) - \frac{1}{2}k(y - y_0)^2 - mgy$



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y} \qquad \frac{\partial L}{\partial y} = -k(y - y_0) - mg$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = m\ddot{y} + k(y - y_0) + mg = 0$$

$$\ddot{y} = -\frac{k}{m}(y - y_0) - g \qquad y_0 = \frac{gm}{k} \rightarrow \ddot{y} = -\frac{k}{m}y$$

Example V: Pendulum

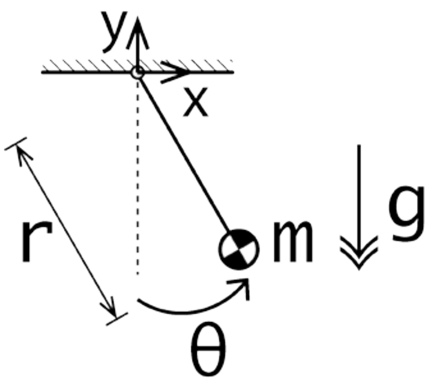
$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \gamma_i$$

Kinetic Energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left(r^2c^2\dot{\theta}^2 + r^2s^2\dot{\theta}^2\right) = \frac{1}{2}mr^2\dot{\theta}^2$

Potential Energy $V = mgy = -mgrc$

Lagrangian $L = \frac{1}{2}mr^2\dot{\theta}^2 + mgrc$



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} \qquad \frac{\partial L}{\partial \theta} = -mgrs$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = mr^2\ddot{\theta} + mgrs$$

$$\ddot{\theta} = -\frac{g}{r}s$$