Slides adapted from material by Robert Peter Matthew and
A Mathematical Introduction to Robot Manipulation (Murray, Li, Sastry)

EECS C106A/206A

Discussion #5: Jacobians and Wrenches

Agenda

- Logistics
- Lecture Review
    - Velocities
    - Manipulator Jacobians
    - Wrenches

Logistics

- Upcoming:
  - Midterm 1 regrades due on **10/13**
  - Midterm 2 on **11/8**
  - Homework 4 due **10/11**
- Office Hours
  - Still happening
  - Tuesdays & Thursdays @ 11:30 - 12:30, Locations on Piazza
  - By appointment: brentyi@berkeley.edu

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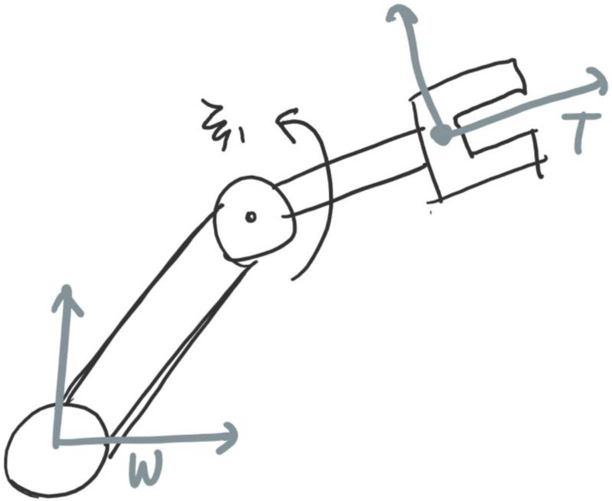
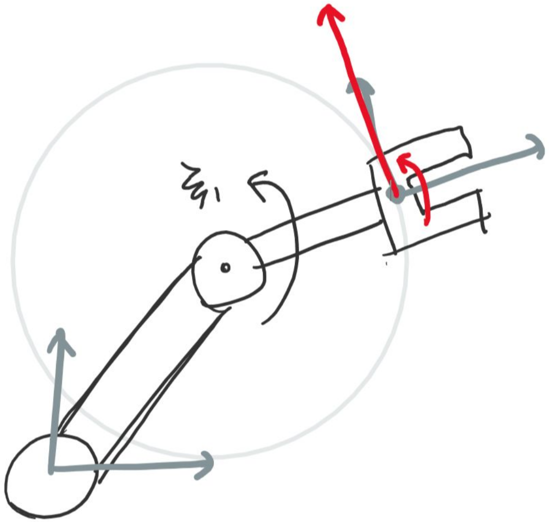
Velocities Review

1. How do we visualize body and spatial velocities?
2. How can we determine each by inspection?



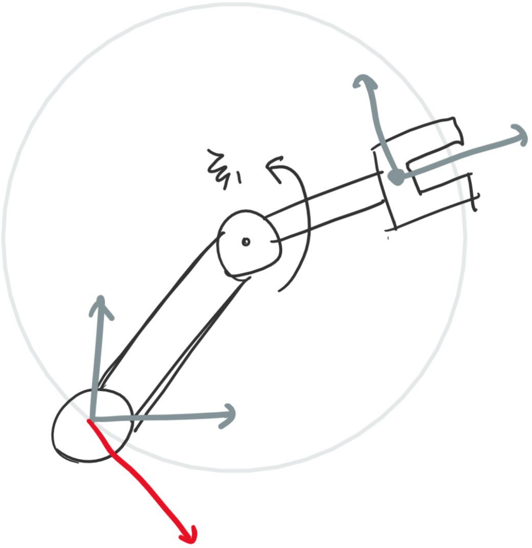
Body Velocities

- How is the tool frame moving wrt its own axes?



Spatial Velocities

- Imagine point at spatial origin, attached to the tool frame
- Express velocity wrt the spatial frame's axes:



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Manipulator Jacobians

Recall: Jacobians (Calculus)

- Matrix of partial derivatives
    - How does each element of $\boldsymbol{f}$ change wrt each element of $\boldsymbol{x}$?

$$x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$$
$$f(x) = \begin{bmatrix} f_1 & f_2 & \dots & f_m \end{bmatrix}^T$$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Recall: Manipulator Jacobians

- Manipulator Jacobians describe relationship between components of:
  - Joint velocity vector
  - End effector velocity

- End effector velocity can be expressed in either spatial or body frame

Recall: Spatial Jacobians

- Map joint velocities to an end effector spatial velocity
- For a robot with $\boldsymbol{n}$ joints:

$$J^s(\theta) = \begin{bmatrix} \xi_1 & \xi_2' & \cdots & \xi_n' \end{bmatrix}$$

- Twist are expressed in the current configuration:

$$\xi_i' = Ad_{e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_{i-1}\theta_{i-1}}}\xi_i$$

Recall: Spatial Jacobians

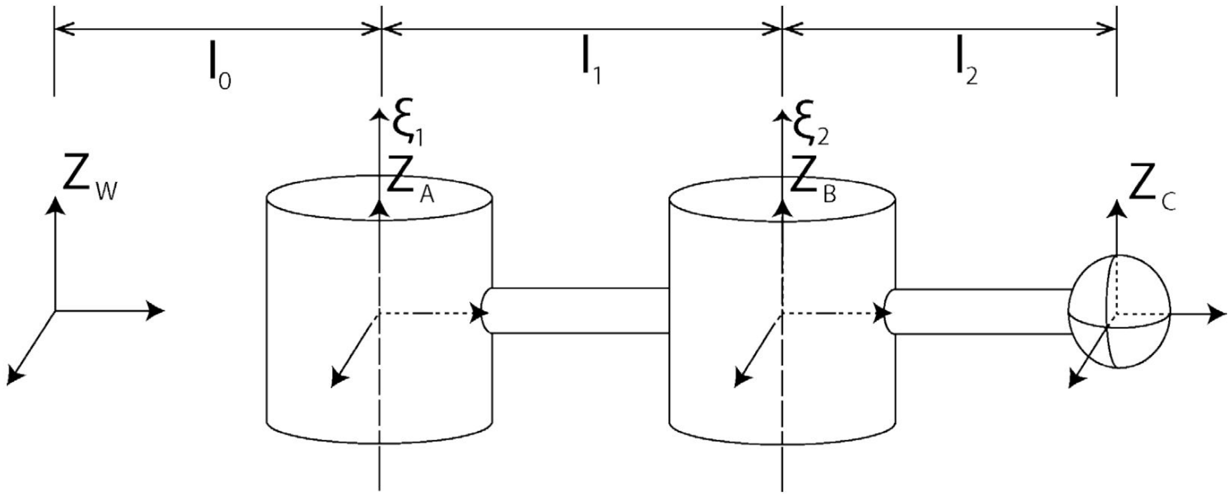
- Map joint velocities to an end effector spatial velocity

$$J^s(\theta) = \begin{bmatrix} \xi_1 & \xi_2' & \cdots & \xi_n' \end{bmatrix}$$

$$\begin{aligned} V_{WT}^S(\theta) &= J_{WT}^s(\theta) \begin{bmatrix} \dot\theta_1 \\ \vdots \\ \dot\theta_n \end{bmatrix} \\ &= J_{WT}^s(\theta)\dot\theta \end{aligned}$$

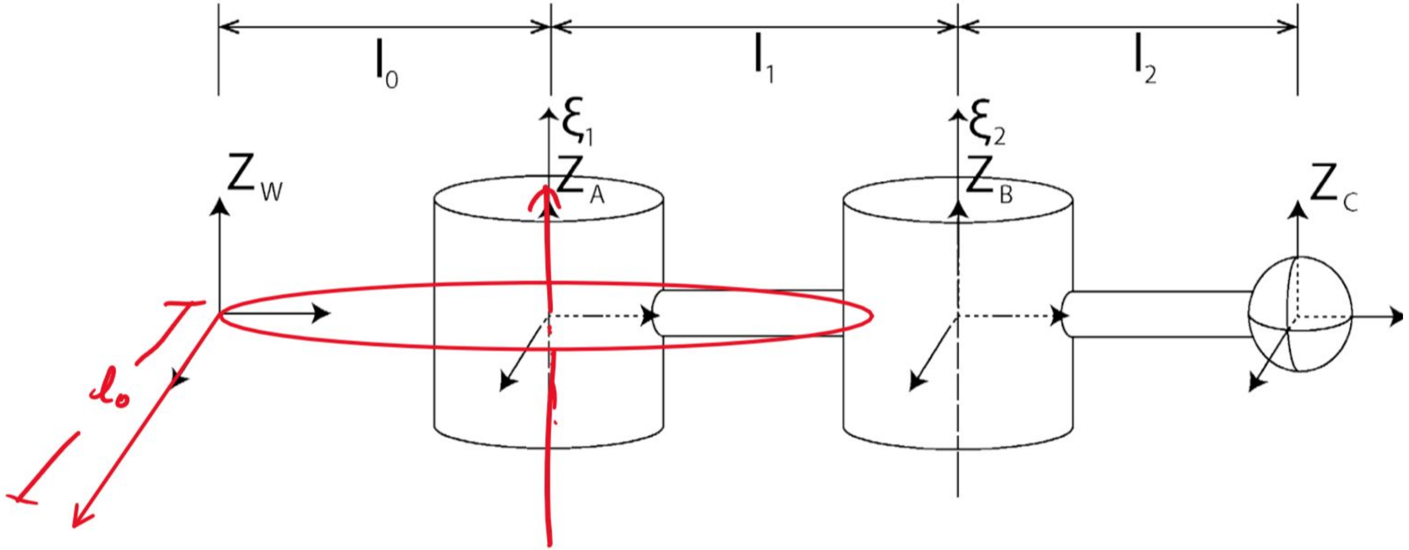
Exercise

- Find the spatial Jacobian for the manipulator in this configuration:



Visualizing Twists & Velocities

- Spatial twist: circle around joint, intersecting spatial frame origin



Recall: Body Jacobians

- Map joint velocities to an end effector body velocity
- For a robot with $\boldsymbol{n}$ joints:

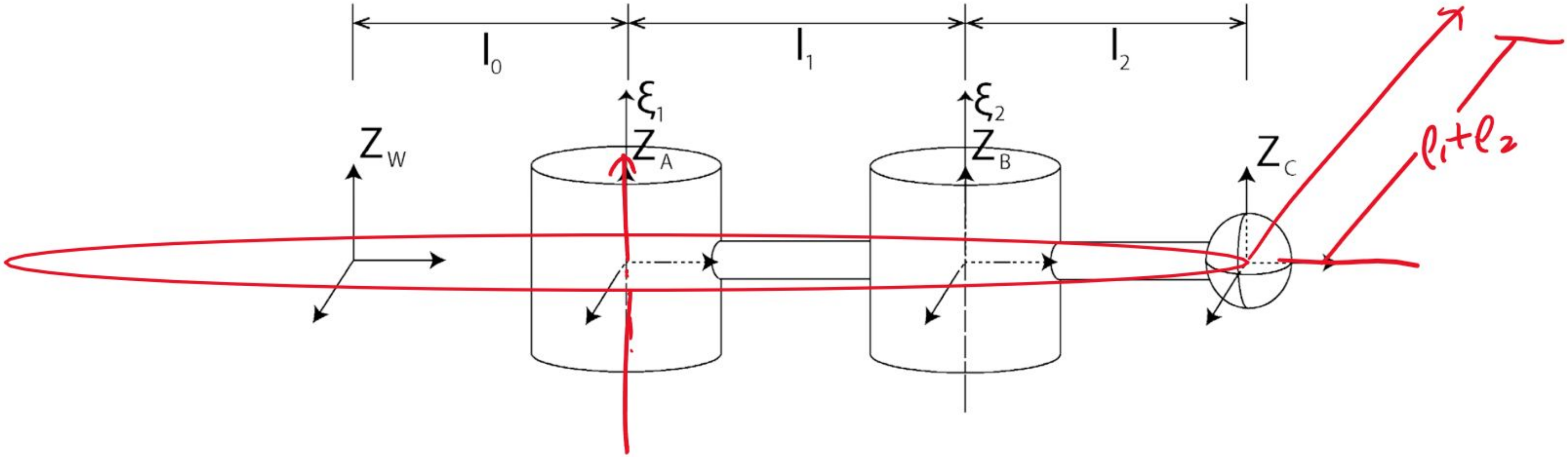
$$J^b(\theta) = \begin{bmatrix} \xi_1^\dagger & \xi_2^\dagger & \cdots & \xi_n^\dagger \end{bmatrix}$$

- Twists are re-expressed in the body frame, in the current configuration:

$$\xi_i^\dagger = Ad_{\left(e^{\hat{\xi}_i \theta_i} \ldots e^{\hat{\xi}_n \theta_n} g_{WT}(0)\right)^{-1}} \xi_i$$

Visualizing Twists & Velocities

- Body twist: circle around joint, intersecting body frame origin

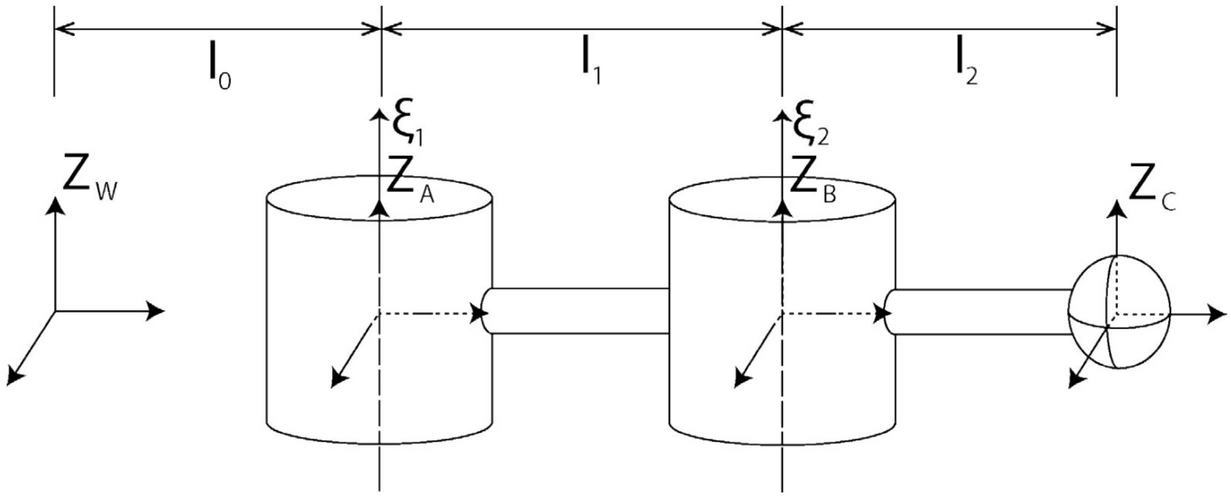


Jacobian Rank

- The rank (# of linearly independent columns) of the Jacobian describes its degrees of freedom
- Some configurations of robot can cause the Jacobian to drop rank: these are called **singularities**

Question

- Is this robot in a singular configuration?



Wrenches

- Recall generalized velocities with linear & angular components:

$$V = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^{6\times 1}$$

- Paralleled by **wrenches**, which generalize linear & angular forces:

$$F = \begin{bmatrix} f \\ \tau \end{bmatrix} \in \mathbb{R}^{6\times 1}$$

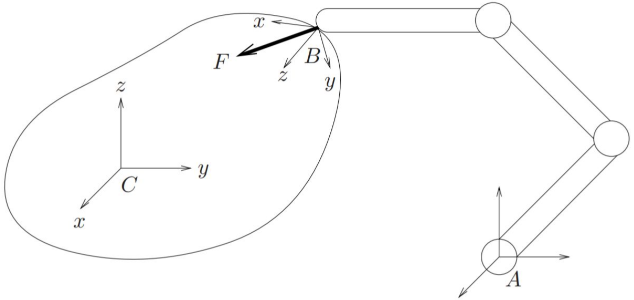
- Many properties of velocities & twists apply to wrenches as well!

Wrenches

- When they're expressed in the same frame, we can dot product *wrenches* and *velocities* to compute work:

$$W = \int_0^t \partial W \, dt$$

$$\partial W = V^s \cdot F^s = V^b \cdot F^b$$



- Two wrenches are equivalent if they produce the same work with every possible rigid body motion

Exercise

- Derive a mapping between spatial wrenches and body wrenches.
    - Hint:

$$\partial W = V^s \cdot F^s = V^b \cdot F^b$$

Wrenches

- Just as we use the Adjoint to express velocities in different frames:

$$V^s = \mathrm{Ad}_g\, V^b$$

- We can also use it to translate between different wrench representations:

$$F^b = \mathrm{Ad}_g^T\, F^s$$

$$F_a = \mathrm{Ad}_{g_{ba}}^T\, F_b$$

Joint Torques

- Let $\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \cdot \\ \cdot \\ \cdot \\ \tau_n \end{bmatrix} \in \mathbb{R}^{n\times 1}$ where τ_i is the torque of joint i

Joint Torques -> End Effector Wrench

- Recall the mapping from joint velocities to end effector velocities:

$$V^S(\theta) = J^s(\theta)\dot{\theta}$$

- Is there an analogous relationship between joint torques and end effector wrenches?

$$F_B, \ \tau$$

Exercise

- Derive a mapping between spatial wrenches and joint torques
    - Hint:

$$P = V^b \cdot F^b = \dot{\theta} \cdot \tau$$

Joint Torques -> End Effector Wrench

$$P = V^b \cdot F^b = \dot{\theta} \cdot \tau$$

$$V^b = J_b \dot{\theta} \implies (J_b \dot{\theta})^T F^b = \dot{\theta}^T \tau$$

$$\dot{\theta}^T J_b^T F^b = \dot{\theta}^T \tau$$

$$J_b^T F^b = \tau$$