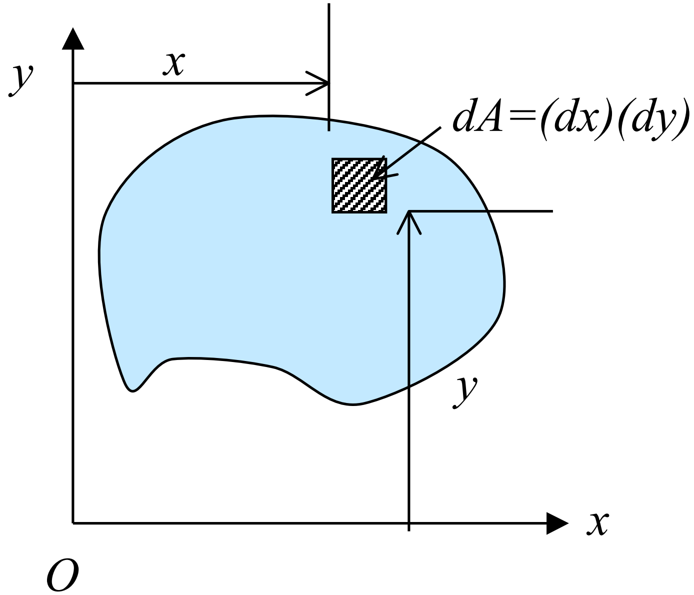
CHAPTER 9:
Moments of Inertia

- **Moment of Inertia of Areas**
  - **Second Moment, or Moment of Inertia, of an Area**
  - **Parallel-Axis Theorem**
  - **Radius of Gyration of an Area**
  - **Determination of the Moment of Inertia of an Area by Integration**
  - **Moments of Inertia of Composite Areas**
  - **Polar Moment of Inertia**

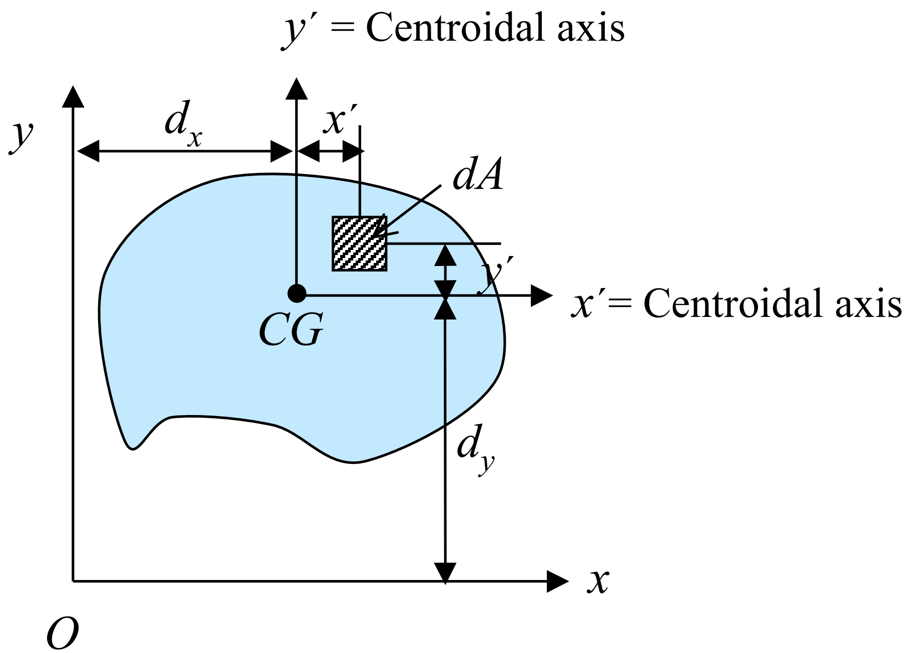
9.1 Moment of Inertia: Definition



$$I_x = \int_A (y)^2 \, dA$$

$$I_y = \int_A (x)^2 \, dA$$

9.2 Parallel-Axis Theorem of an Area



$$I_x = \int_A (y' + d_y)^2 \, dA$$

$$= \int_A [(y')^2 + 2(y')(d_y) + (d_y)^2] dA$$

$$= \int_A (y')^2 \, dA + \int_A 2(y')(d_y) dA + \int_A (d_y)^2 \, dA$$

$$0, \; \bar{y}' = 0$$

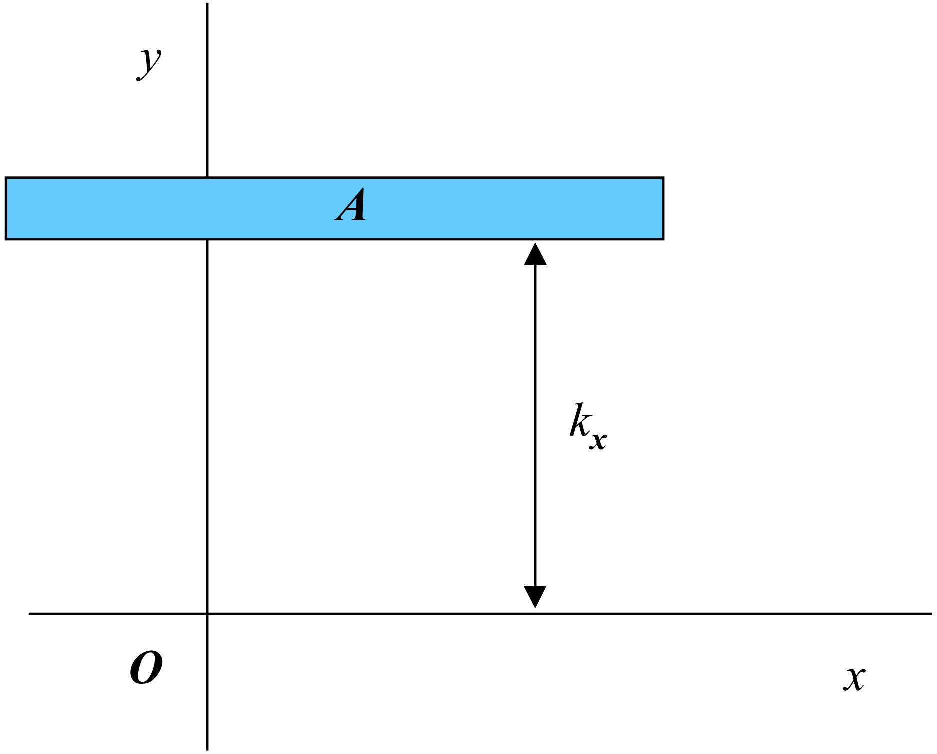
$$= \bar{I}_x + 2d_y \int_A y' \, dA + d_y{}^2 \int_A dA$$

$$\boxed{I_x = \bar{I}_x + 0 + d_y{}^2 A}$$

$$\boxed{I_y = \bar{I}_y + 0 + d_x{}^2 A}$$

$$\boxed{J_O = \bar{J}_C + Ad^2}$$

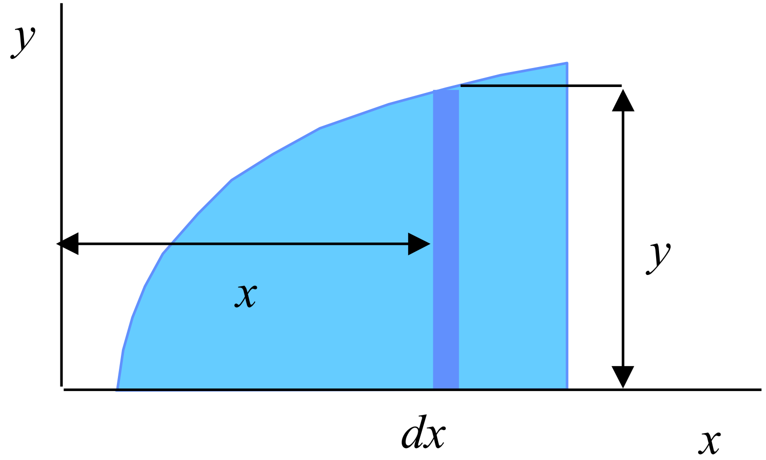
9.3 Radius of Gyration of an Area



The ***radius of gyration of an area*** A with respect to the x axis is defined as the distance k_x, where $I_x = k_x^2 A$. With similar definitions for the radii of gyration of A with respect to the y axis and with respect to O, we have

$$k_x = \sqrt{\frac{I_x}{A}} \qquad k_y = \sqrt{\frac{I_y}{A}} \qquad k_O = \sqrt{\frac{J_O}{A}}$$

9.4 Determination of the Moment of Inertia of an Area by Integration



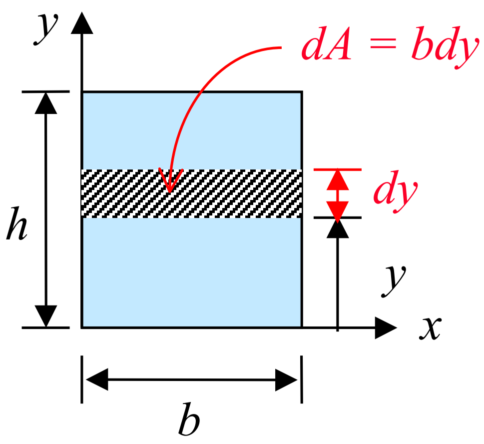
The ***rectangular moments of inertia*** $\boldsymbol{I_x}$ ***and*** $\boldsymbol{I_y}$ ***of an area*** are defined as

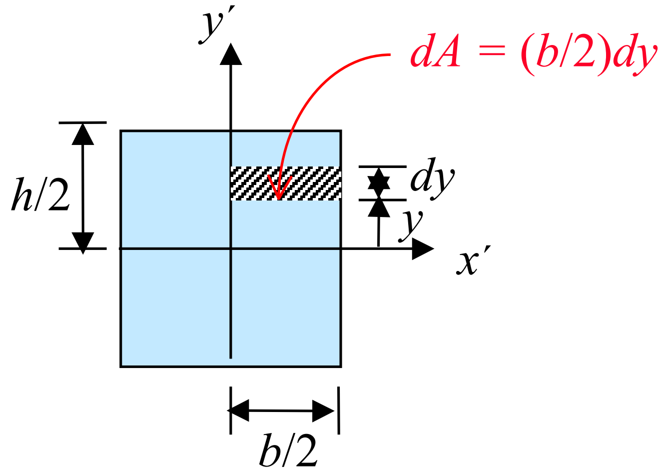
$$I_x = \int y^2 \, dA \qquad I_y = \int x^2 \, dA$$

These computations are reduced to single integrations by choosing dA to be a thin strip parallel to one of the coordinate axes. The result is

$$dI_x = \frac{1}{3} y^3 \, dx \qquad dI_y = x^2 y \, dx$$

• **Moment of Inertia of a Rectangular Area.**





$$I_x = \int_A y^2 dA$$

$$= \int_0^h y^2 (bdy)$$

$$= \left.\frac{(by^3)}{3}\right|_0^h$$

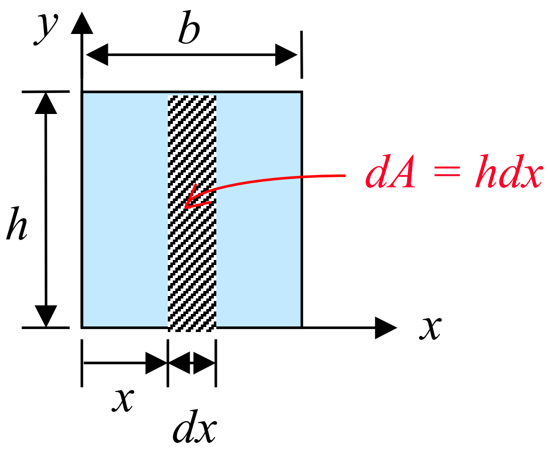
$$= \frac{bh^3}{3}$$

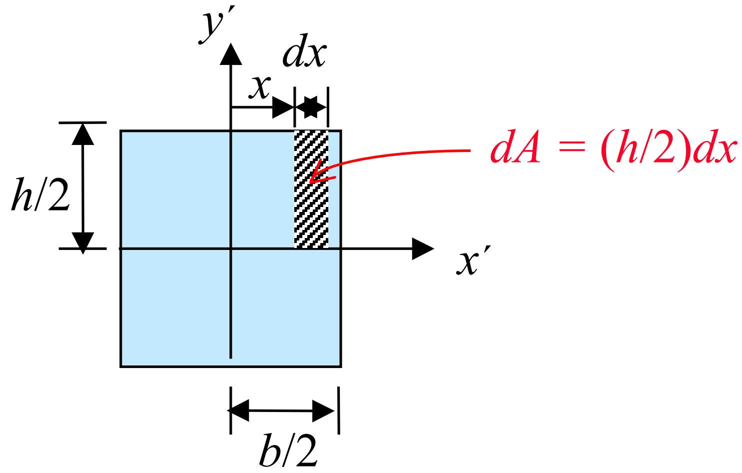
$$\bar{I}_x = I_{x'} = \int_A y^2 dA$$

$$= 4\int_0^h y^2 (\frac{b}{2} dy)$$

$$= 4(\frac{b}{2})\left.\frac{y^3}{3}\right|_0^{h/2}$$

$$= \frac{bh^3}{12}$$





$$I_y = \int_A x^2 dA$$

$$= \int_0^b x^2 (hdx)$$

$$= \frac{(hx^3)}{3}\Bigg|_0^b$$

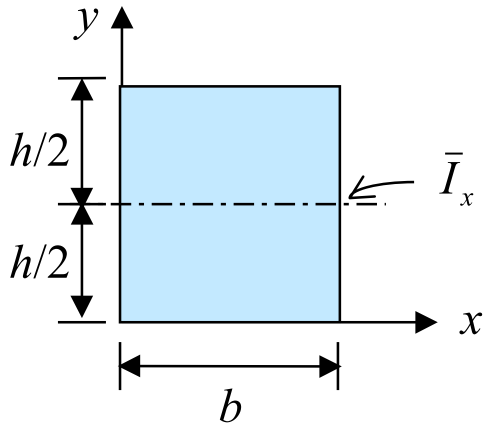
$$= \frac{hb^3}{3}$$

$$\bar{I}_y = I_{y'} = \int_A x^2 dA$$

$$= 4\int_0^h x^2 (\frac{h}{2} dx)$$

$$= 4(\frac{h}{2})\frac{x^3}{3}\Bigg|_0^{b/2}$$

$$= \frac{hb^3}{12}$$



$$\bar{I}_x = \frac{bh^3}{12}$$

$$I_x = \frac{bh^3}{3}$$

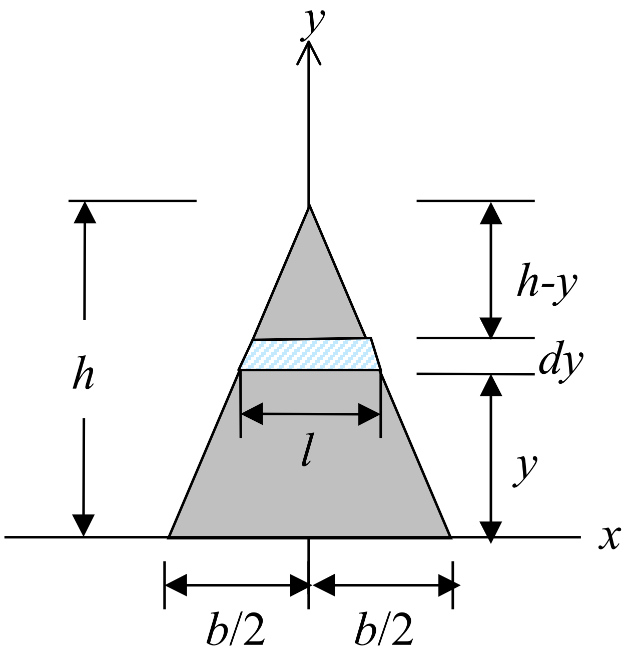
$$I_x = \bar{I}_x + Ad^2$$

$$= \frac{bh^3}{12} + (bh)(\frac{h}{2})^2$$

$$= \frac{bh^3}{12} + \frac{bh^3}{4}$$

$$I_x = \frac{bh^3}{3}$$

• Moment of Inertia of a Triangular Area.



$dI_x = y^2\, dA \qquad dA = l\, dy$

Using similar triangles, we have

$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

Integrating dI_x from $y = 0$ to $y = h$, we obtain

$$I_x = \int y^2\, dA$$

$$= \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h (hy^2 - y^3)dy$$

$$= \frac{b}{h}[h\frac{y^3}{3} - \frac{y^4}{4}]_0^h = \frac{bh^3}{12}$$

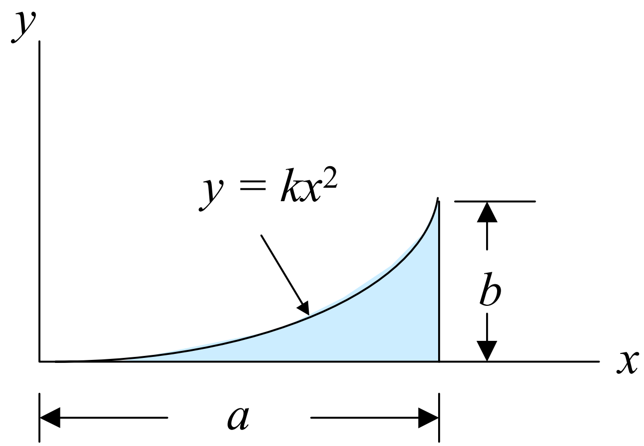
$$I_x = \bar{I}_x + Ad^2$$

$$\bar{I}_x = I_x - Ad^2$$

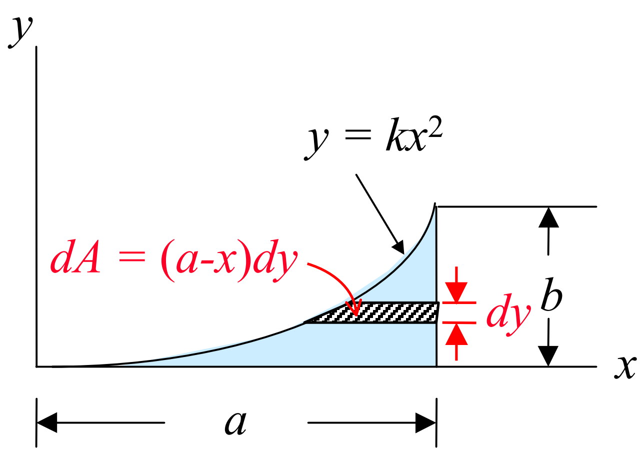
$$= \frac{bh^3}{12} - (\frac{bh}{2})(\frac{h}{3})^2 = \frac{bh^3}{36}$$

Example 9.1

Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes.



• **Moment of Inertia I_x.**



Substituting $x = a$ and $y=b$

$$y = kx^2$$

$$b = ka^2$$

$$k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2}x^2 \quad or \quad x = \frac{a}{b^{1/2}}y^{1/2}$$

$$I_x = \int_A y^2 dA$$

$$= \int_0^b y^2(a-x)dy$$

$$= \int_0^b y^2(a - \frac{a}{b^{1/2}}y^{1/2})dy$$

$$= a\int_0^b y^2 dy - \frac{a}{b^{1/2}}\int_0^b y^{5/2}dy$$

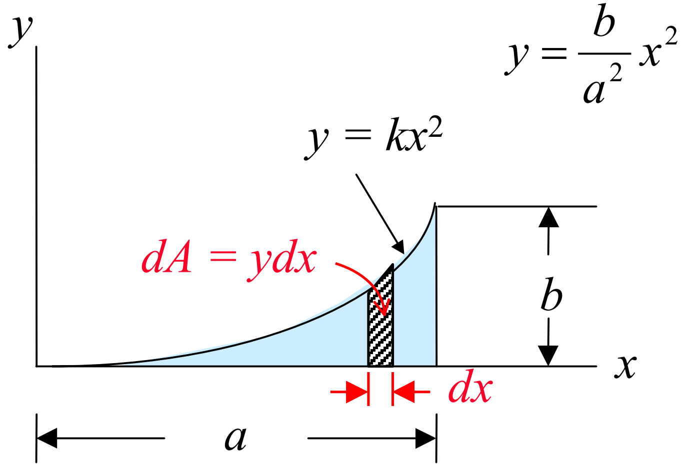
$$= \frac{ay^3}{3}\bigg|_0^b - \frac{a}{b^{1/2}}(\frac{2}{7}y^{7/2})\bigg|_0^b$$

$$= \frac{ab^3}{3} - \frac{a}{b^{1/2}}(\frac{2}{7}b^{7/2})$$

$$= \frac{ab^3}{3} - \frac{2ab^3}{7}$$

$$= \frac{ab^3}{21}$$

• **Moment of Inertia I_y.**



$$I_y = \int_A x^2 dA$$

$$= \int_0^a x^2 y dx$$

$$= \int_0^a x^2 (\frac{b}{a^2}x^2) dx$$

$$= \frac{b}{a^2}\int_0^a x^4 dx$$

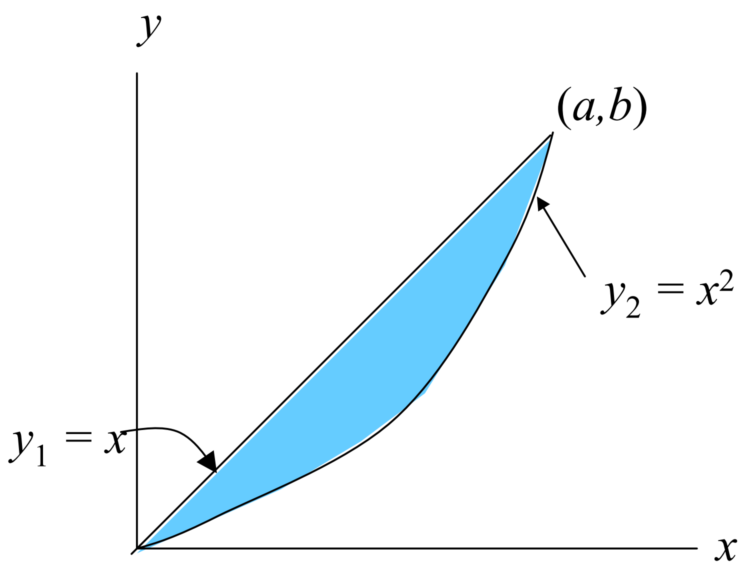
$$= (\frac{b}{a^2})(\frac{x^5}{5})\Big|_0^a$$

$$= (\frac{b}{a^2})(\frac{a^5}{5})$$

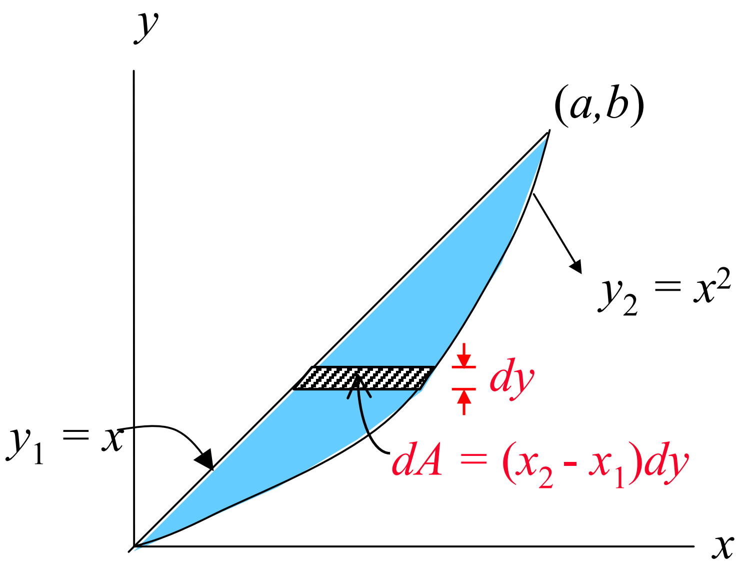
$$= \frac{a^3 b}{5}$$

Example 9.2

Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes.



• **Moment of Inertia I_x.**



$$I_x = \int_A y^2 dA$$

$$= \int_0^b y^2 (x_2 - x_1) dy$$

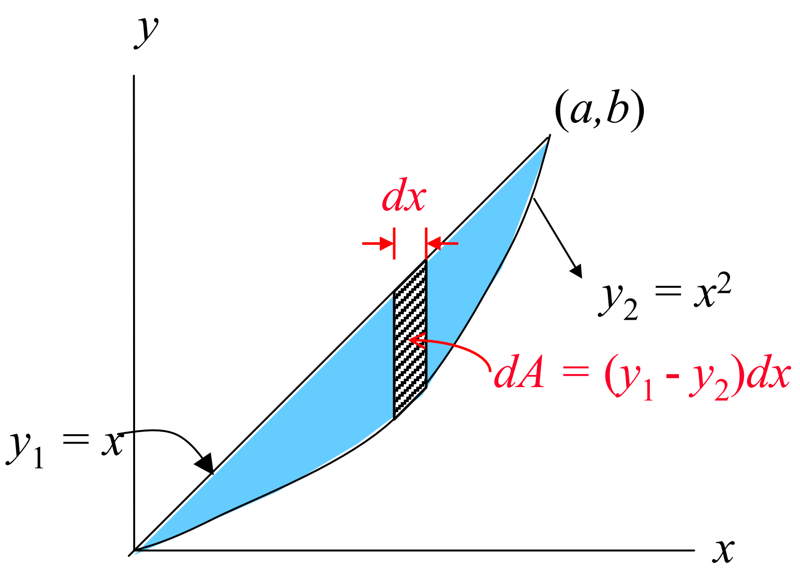
$$= \int_0^b y^2 (y^{1/2} - y) dy$$

$$= \int_0^b (y^{5/2}) dy - \int_0^b (y^3) dy$$

$$= \frac{2}{7} y^{7/2} \Big|_0^b - \frac{y^4}{4} \Big|_0^b$$

$$= \frac{2}{7} b^{7/2} - \frac{b^4}{4}$$

- **Moment of Inertia I_y.**



$$I_y = \int_A x^2 dA$$

$$= \int_0^a x^2 (y_1 - y_2) dx$$

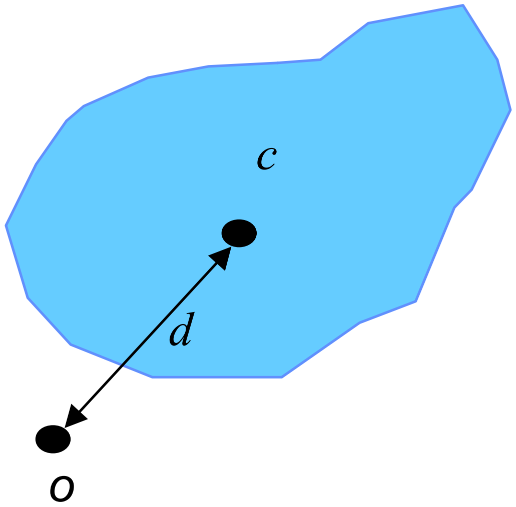
$$= \int_0^a x^2 (x - x^2) dx$$

$$= \int_0^a (x^3) dx - \int_0^a (x^4) dx$$

$$= \left.\frac{x^4}{4}\right|_0^a - \left.\frac{x^5}{5}\right|_0^a$$

$$= \frac{a^4}{4} - \frac{a^5}{5}$$

9.5 Moment of Inertia of Composite Areas



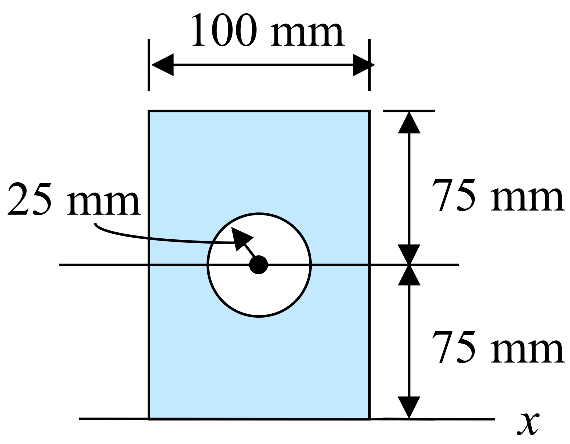
A similar theorem can be used with the polar moment of inertia. The polar moment of inertia $\bar{J}_O$ of an area about O and the polar moment of inertia J_C of the area about its centroid are related to the distance d between points C and O by the relationship

$$J_O = \bar{J}_C + Ad^2$$

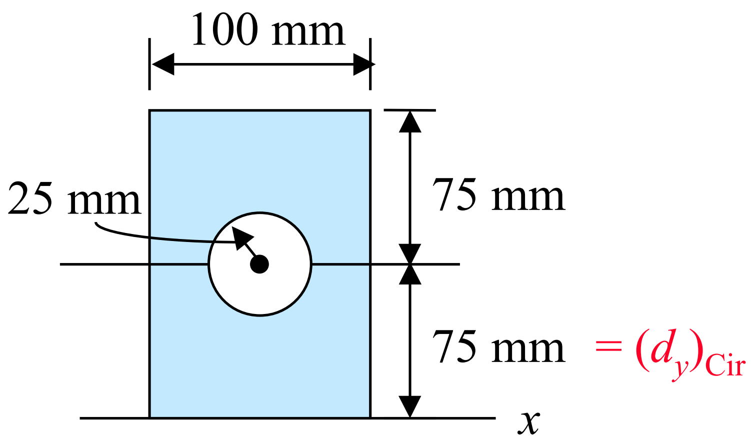
The parallel-axis theorem is used very effectively to compute the ***moment of inertia of a composite area*** with respect to a given axis.

Example 9.3

Compute the moment of inertia of the composite area shown.



SOLUTION



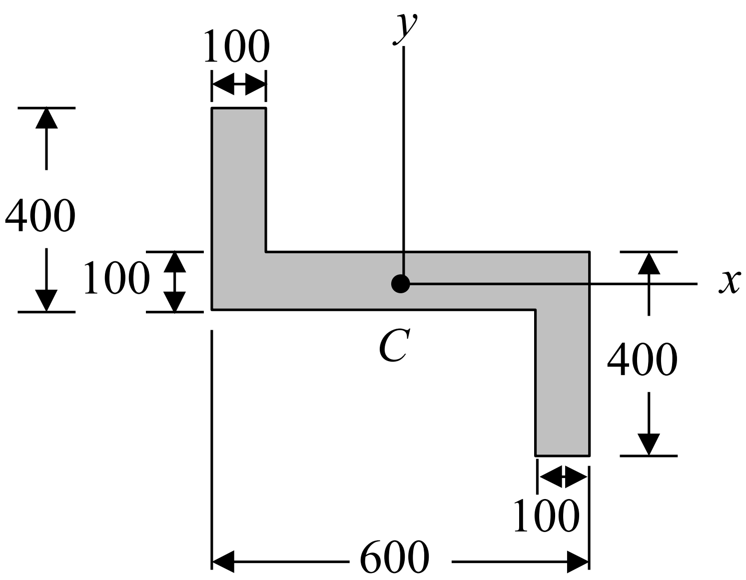
$$I_x = \left(\frac{bh^3}{3}\right)_{\text{Rect}} - (\bar{I}_x + Ad_y{}^2)_{Cir}$$

$$= \left[\frac{1}{3}(100)(150)^3\right]_{\text{Rect}} - \left[\frac{1}{4}\pi(25)^4 + (\pi \times 25^2)(75)^2\right]_{Cir}$$

$$= 101 \text{x} 10^6 \text{ mm}^4$$

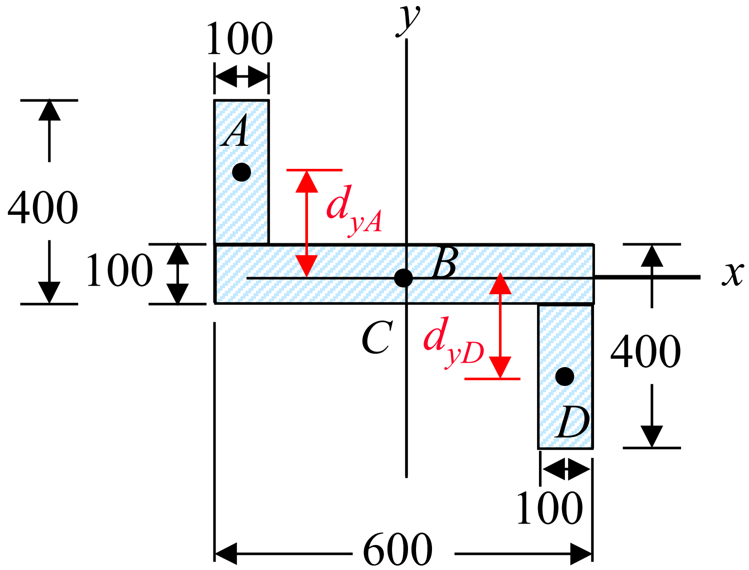
Example 9.4

Determine the moments of inertia of the beam's cross-sectional area shown about the *x* and *y* centroidal axes.



Dimension in mm

SOLUTION



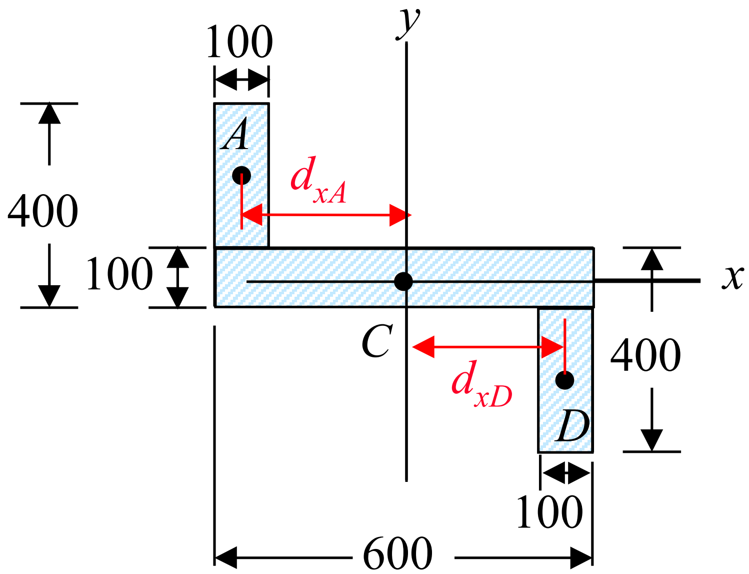
Dimension in mm

$$I_x = (\bar{I}_x + Ad_y^{\,2})_A + (\bar{I}_x + \cancel{Ad_y^{\,2}}^{\,0})_B + (\bar{I}_x + Ad_y^{\,2})_C$$

$$= [\frac{1}{12}(100)(300)^3 + (100\times300)(200)^2] + [\frac{1}{12}(600)(100)^3 + 0]$$

$$+ [\frac{1}{12}(100)(300)^3 + (100\times300)(200)^2]$$

$= 2.9\text{x}10^9\ \text{mm}^4$



Dimension in mm

$$I_y = (\bar{I}_y + Ad_x^{\ 2})_A + (\bar{I}_y + A\cancelto{0}{d_x^{\ 2}})_B + (\bar{I}_y + Ad_x^{\ 2})_C$$

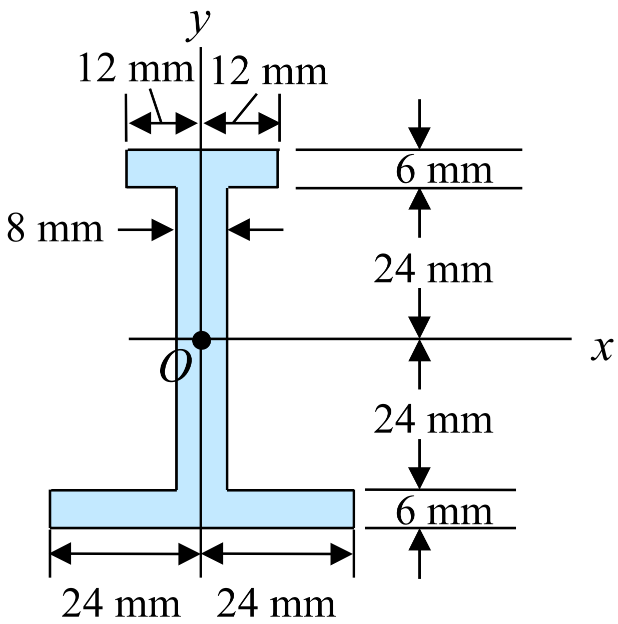
$$= [\frac{1}{12}(300)(100)^3 + (100 \times 300)(250)^2]_A + [\frac{1}{12}(100)(600)^3 + 0]_B$$

$$+ [\frac{1}{12}(300)(100)^3 + (100 \times 300)(250)^2]_C$$

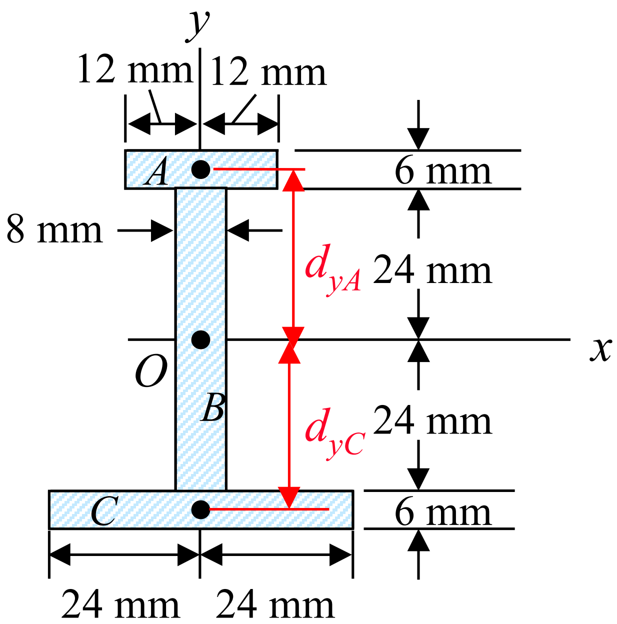
$= 5.6\text{x}10^9\ \text{mm}^4$

Example 9.5 (Problem 9.31,33)

Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes.



SOLUTION



$$I_x = (\bar{I}_x + Ad_y^{\,2})_A + (\bar{I}_x + A\overset{0}{\cancel{d_y^{\,2}}})_B + (\bar{I}_x + Ad_y^{\,2})_C$$

$$= [\frac{1}{12}(24)(6)^3 + (24\times 6)(27)^2]_A$$

$$+ [\frac{1}{12}(8)(48)^3 + 0]_B$$

$$+ [\frac{1}{12}(48)(6)^3 + (48\times 6)(27)^2]_C$$

$$I_x = 390\text{x}10^3 \text{ mm}^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{390\times 10^3}{[(24\times 6)+(8\times 48)+(48\times 6)]}} = 21.9\ mm$$

$$I_y = (\bar{I}_y + A\overset{0}{\cancel{d_x^{\,2}}})_A + (\bar{I}_y + A\overset{0}{\cancel{d_x^{\,2}}})_B + (\bar{I}_y + A\overset{0}{\cancel{d_x^{\,2}}})_C$$

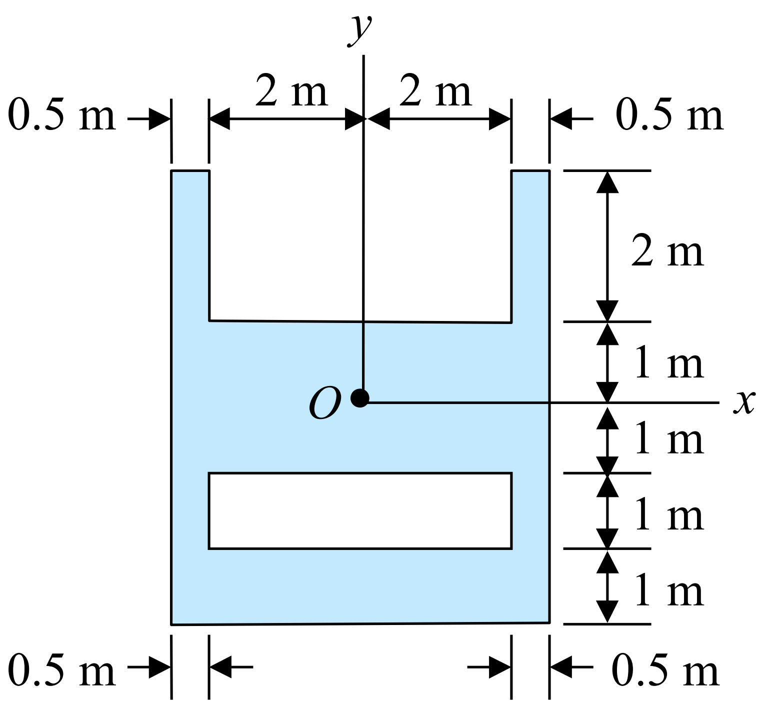
$$= [\frac{1}{12}(6)(24)^3]_A + [\frac{1}{12}(48)(8)^3]_B + [\frac{1}{12}(6)(48)^3]_C$$

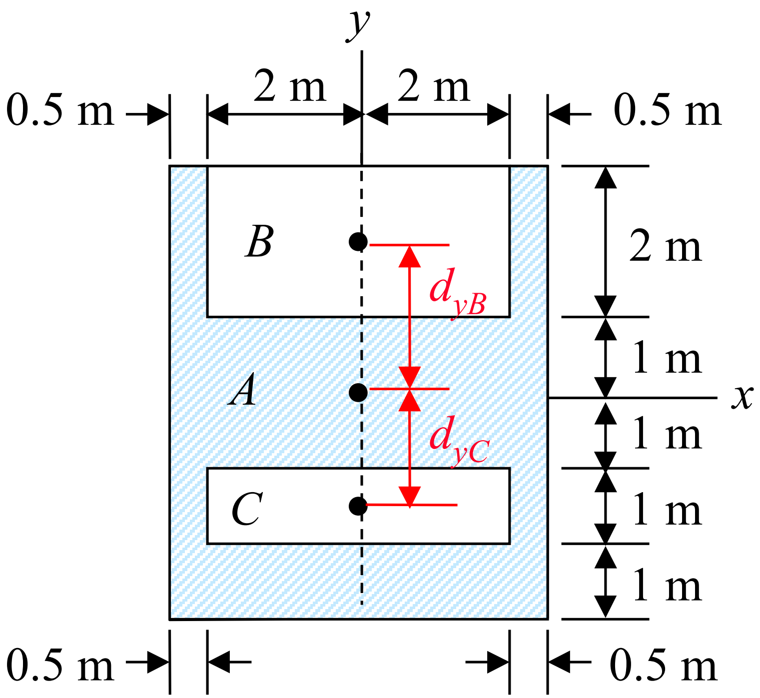
$$I_y = 64.3\text{x}10^3 \text{ mm}^4$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{64.3\times 10^3}{[(24\times 6)+(8\times 48)+(48\times 6)]}} = 8.87\ mm$$

Example 9.6 (Problem 9.32,34)

Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes.





$$I_x = (\bar{I}_x + \cancelto{0}{Ad_y^{\,2}})_{A5\times6} - (\bar{I}_x + Ad_y^{\,2})_{B4\times2} - (\bar{I}_x + Ad_y^{\,2})_{C4\times1}$$

$$= [\frac{1}{12}(5)(6)^3 + 0]_A - [\frac{1}{12}(4)(2)^3 + (2\times4)(2)^2]_B$$

$$-[\frac{1}{12}(4)(1)^3 + (4\times1)(1.5)^2]_C$$

$$I_x = 46 \text{ m}^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{46}{[(5\times6)-(4\times2)-(4\times1)]}} = 1.599\ m$$

$$I_y = (\bar{I}_y + \cancelto{0}{Ad_x^{\,2}})_A - (\bar{I}_y + \cancelto{0}{Ad_x^{\,2}})_B - (\bar{I}_y + \cancelto{0}{Ad_x^{\,2}})_C$$

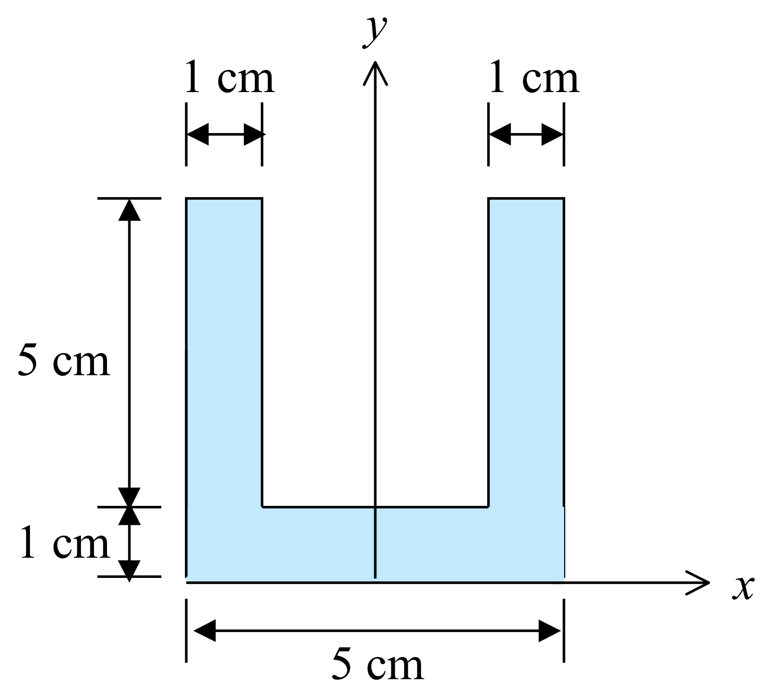
$$= [\frac{1}{12}(6)(5)^3]_A - [\frac{1}{12}(2)(4)^3]_B - [\frac{1}{12}(1)(4)^3]_C$$

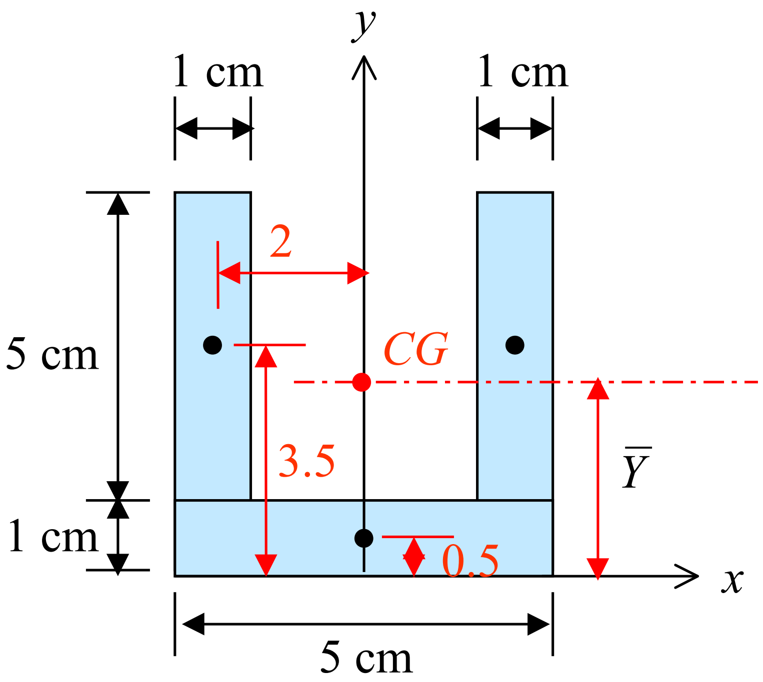
$$I_y = 46.5 \text{ m}^4$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{46.5}{[(5\times6)-(4\times2)-(4\times1)]}} = 1.607\ m$$

Example 9.7

Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes and at the centroidal axes.





$$\bar{Y}\sum A = \sum \bar{y}A$$

$$\bar{Y} = \frac{2[(3.5)(5\times 1)] + (0.5)(1\times 5)}{3(5\times 1)}$$

$$= 2.5\ cm$$

- **Moments of inertia about *x* axis**

$$I_x = 2[(\frac{1}{12}(1)(5)^3 + (5\times 1)(3.5)^2] + \frac{1}{3}(5)(1)^3$$

$$= 145\ cm^4$$

- **Moments of inertia about centroid**

$$\bar{I}_x = I_x - Ad_y{}^2$$

$$= 145 - (15)(2.5)^2$$

$$= 51.25\ cm^4$$

OR

$$\bar{I}_x = 2[(\frac{1}{12}(1)(5)^3 + (5\times 1)(1)^2]$$

$$+ [(\frac{1}{12}(5)(1)^3 + (5\times 1)(2)^2]$$

$$= 51.25\ cm^4$$

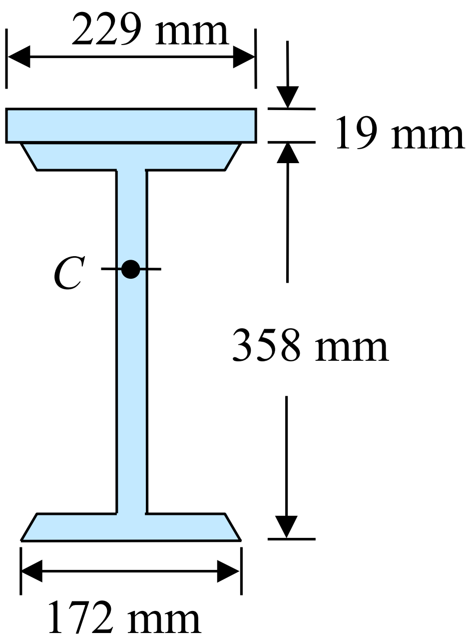
$$\bar{I}_y = I_y = 2[(\frac{1}{12}(5)(1)^3 + (5\times 1)(2)^2] + \frac{1}{12}(1)(5)^3$$

$$= 51.25\ cm^4$$

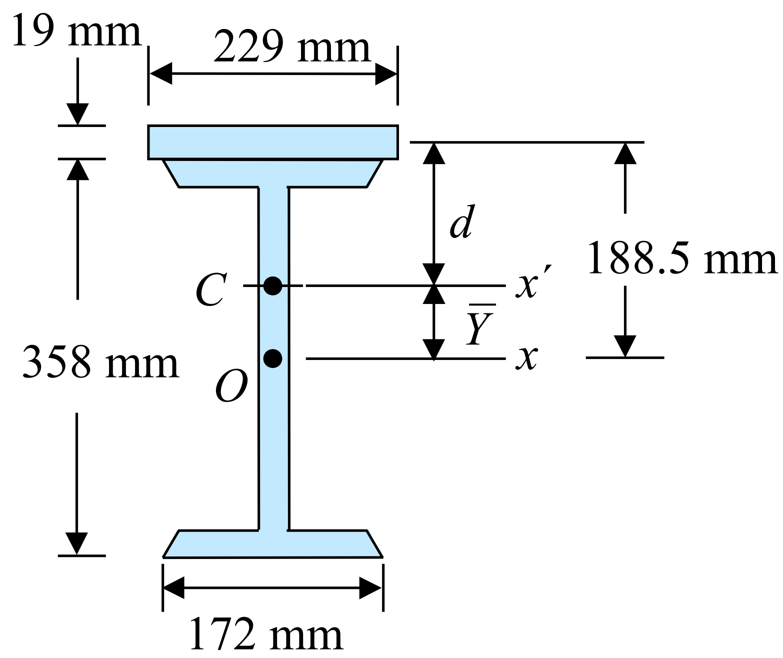
$$\bar{k}_x = \bar{k}_y = \sqrt{\frac{\bar{I}_x}{A}} = \sqrt{\frac{51.25}{15}} = 1.848\ cm$$

Example 9.8

The strength of a W360 x 57 rolled-steel beam is increased by attaching a 229 mm x 19 mm plate to its upper flange as shown. Determine the moment of inertia and the radius of gyration of the composite section with respect to an axis which is parallel to the plate and passes through the centroid *C* of the section.



SOLUTION



- **Centroid**

The wide-flange shape of $W360$ x 57 found by referring to Fig. 9.13

A = 7230 mm^2 $\quad \bar{I}_x = 160.2\ mm^4$

$$A_{\text{plate}} = (229)(19) = 4351 \text{ mm}^2$$

$$\bar{Y}\Sigma A = \Sigma \bar{y} A$$

$$\bar{Y}(4351 + 7230) = (188.5)(4351) + (0)(7230)$$

$$\bar{Y} = 70.8\ mm$$

- **Moment of Inertia**

$$I_{x'} = (I_{x'})_{plate} + (I_{x'})_{wide-flange}$$

$$= (\bar{I}_{x'} + Ad^2)_{plate} + (\bar{I}_{x'} + A\bar{Y}^2)_{wide-flange}$$

$$= \left[\frac{1}{12}(229)(19)^3 + (4351)(188.5 - 70.8)^2\right]$$

$$+ \left[160.2 \times 10^6 + (7230)(70.8)^2\right]$$

$$= 256.8 \times 10^6\ mm^4$$

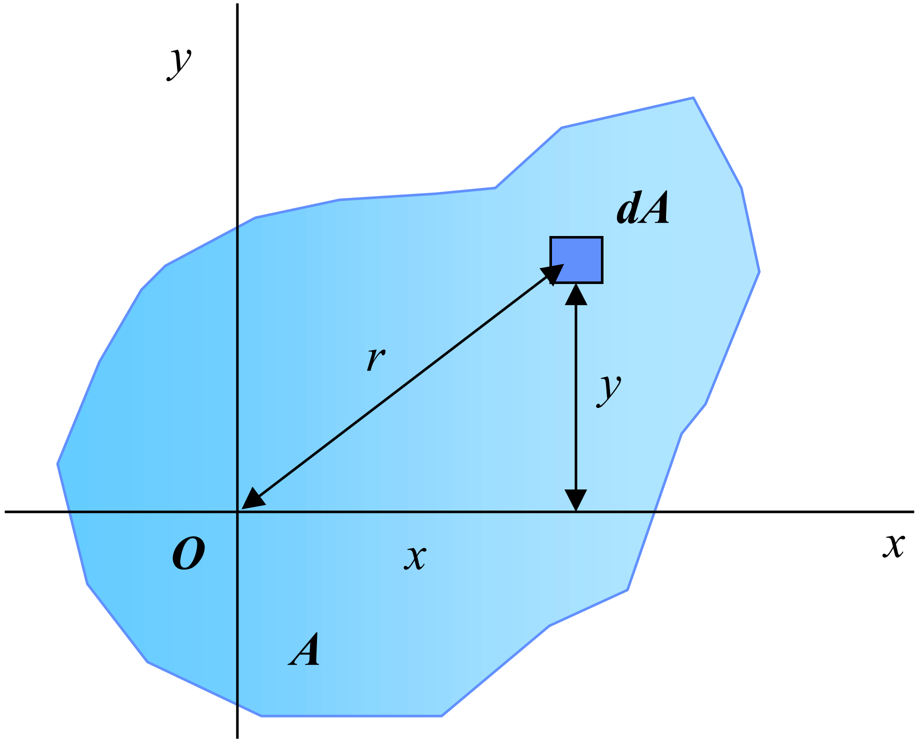
$$I_{x'} = 257 \times 10^6\ mm^4$$

- **Radius of Gyration**

$$k_{x'}^2 = \frac{I_{x'}}{A} = \frac{256.8 \times 10^6}{(4351 + 7230)}$$

$$k_{x'} = 149\ mm$$

9.6 Polar Moment of Inertia



The ***polar moment of inertia of an area A*** with respect to the pole *O* is defined as

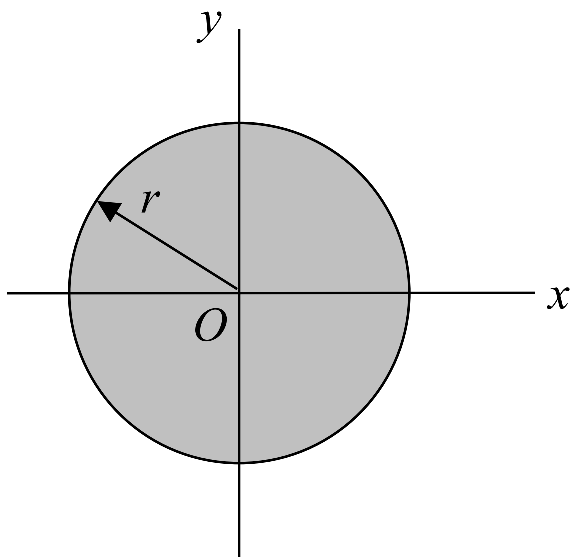
$$J_O = \int r^2 \, dA$$

The distance from O to the element of area dA is r. Observing that $r^2 = x^2 + y^2$, we established the relation

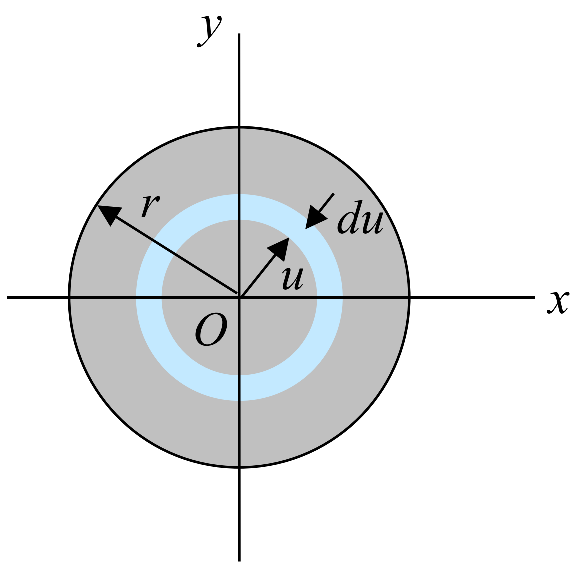
$$J_O = I_x + I_y$$

Example 9.9

(*a*) Determine the centroidal polar moment of inertia of a circular area by direct integration. (*b*) Using the result of part *a*, determine the moment of inertia of a circular area with respect to a diameter.



SOLUTION



a. Polar Moment of Inertia.

$$dJ_O = u^2\, dA \qquad dA = 2\pi u\, du$$

$$J_O = \int dJ_O = \int_0^r u^2 (2\pi u\, du) = 2\pi \int_0^r u^3 du$$

$$J_O = \frac{\pi}{2} r^4$$

b. Moment of Inertia with Respect to a Diameter.

$$J_O = I_x + I_y = 2I_x$$

$$\frac{\pi}{2} r^4 = 2I_x$$

$$I_{diameter} = I_x = \frac{\pi}{4} r^4$$