3. Inverse Kinematics

Introduction to Robotics

EE 106A/206A

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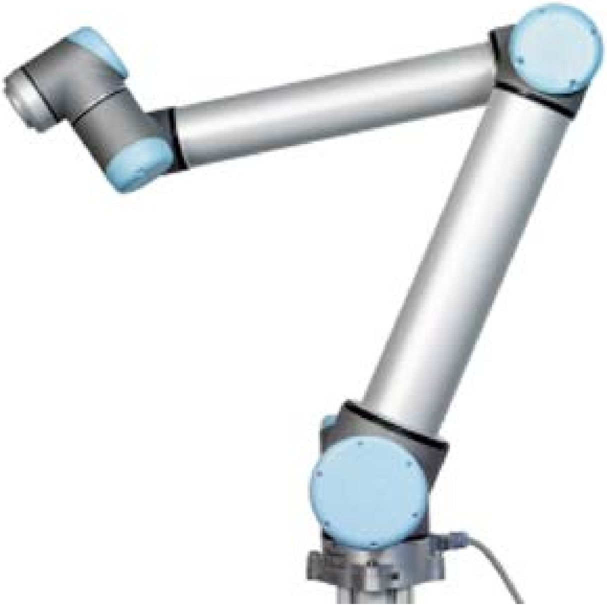
Outline

- Inverse Kinematics (IK)
- Example: Planar 2-Link Manipulator
- Paden-Kahan Subproblems (I, II, III)
- Example: 6-DOF Manipulator
- Discussion of IK Problems

Terminology

- Forward Kinematics:
  - Given valid joint positions, find end effector coordinates.
- Inverse Kinematics:
  - Given valid end effector coordinates, find associated joint positions.



Forward Kinematics Map

- For any reference frame at a zero configuration, the forward kinematics map is given by:

$$g(\theta) = e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_n \theta_n} g(0)$$

Inverse Kinematics Problem

- Given a desired configuration g_d for the tool frame, find the joint angles $\theta_1 \ldots \theta_n$ that achieve that configuration:

$$g_d = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \ldots e^{\hat{\xi}_n\theta_n} g(0)$$

- This is not a well-posed problem mathematically. There may be a **unique solution**, **multiple solutions**, or **no solution**.

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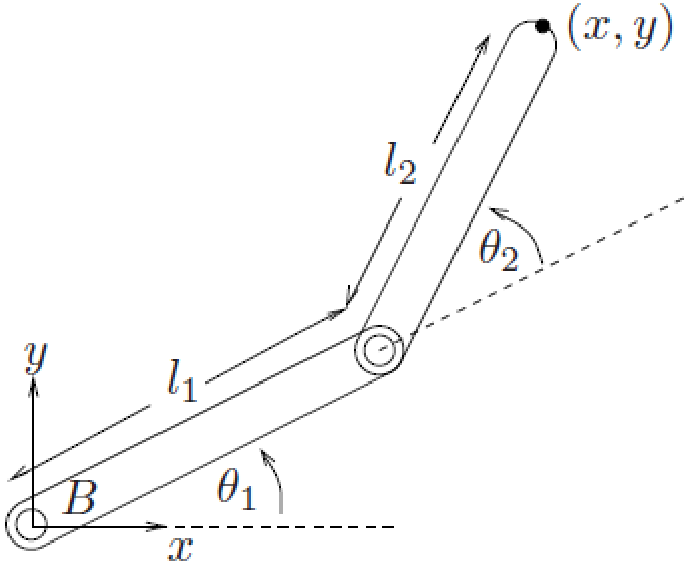
Example I – Planar 2-Link

- For reference: p. 97 in textbook.
- Given a desired point (x, y), find the corresponding joint angles (θ_1, θ_2).

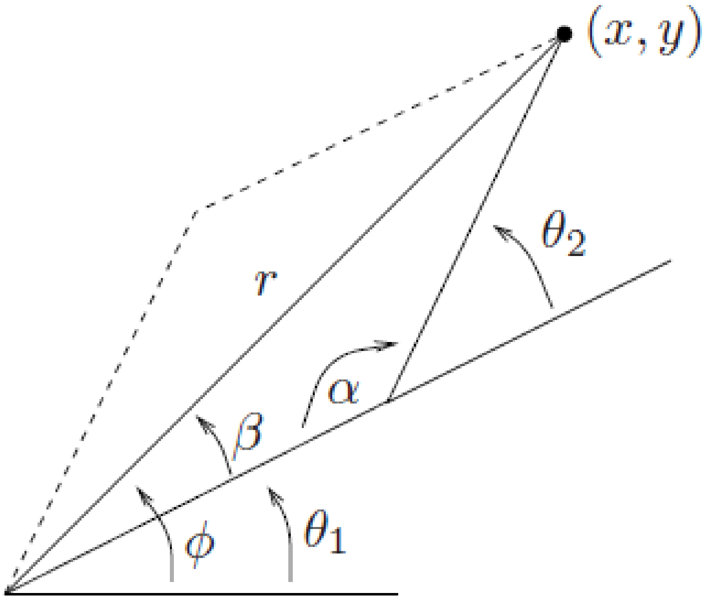


Example I – Planar 2-Link

- Notice that there will be **two solutions**!
- From a polar coordinate perspective, each target point (x, y) will have a corresponding (r, ϕ).

- For (x, y) to be in the workspace:

$$r \leq l_1 + l_2$$



Example I – Planar 2-Link

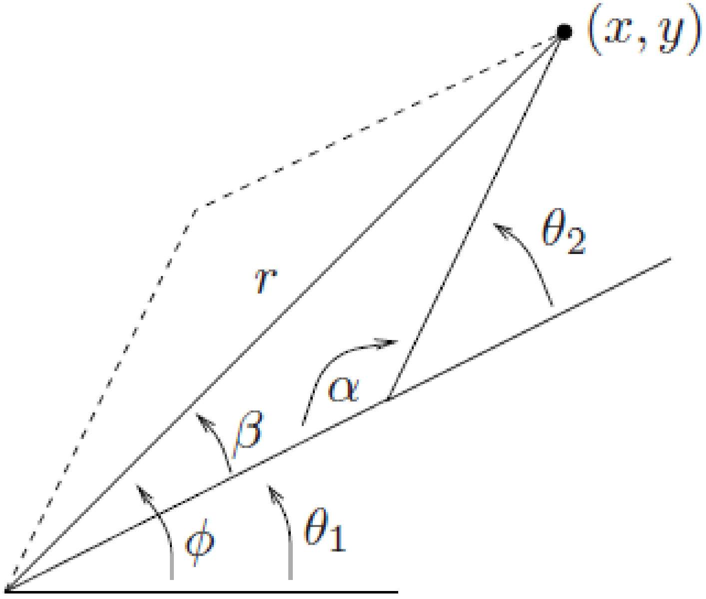
- Given a point (x, y); the lengths of r, l_1, and l_2 are known.

- Using the law of cosines:

$$\alpha = acos\left(\frac{l_1^2 + l_2^2 - r^2}{2l_1l_2}\right)$$

- Therefore:

$$\theta_2 = \pi \pm \alpha$$



Example I – Planar 2-Link

- Similarly:

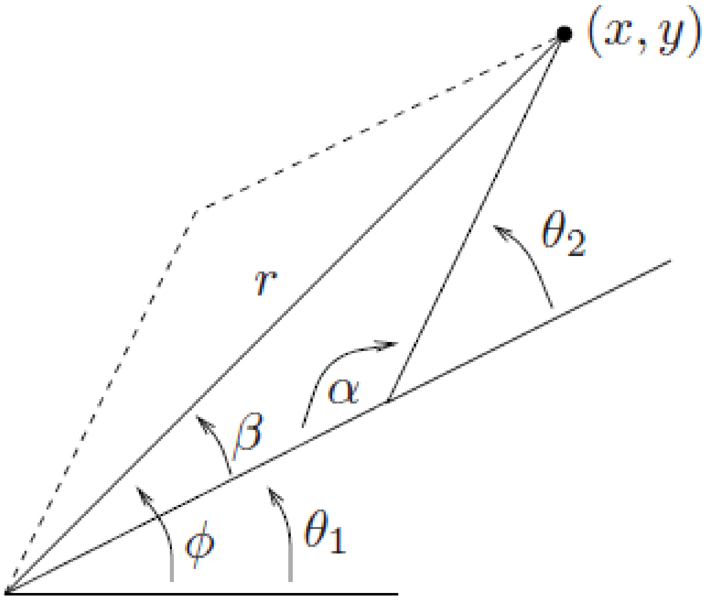
$$\phi = atan2(y, x)$$

- Using the law of cosines:

$$\beta = acos\left(\frac{r^2 + l_1^2 - l_2^2}{2l_1 r}\right)$$

- Therefore:

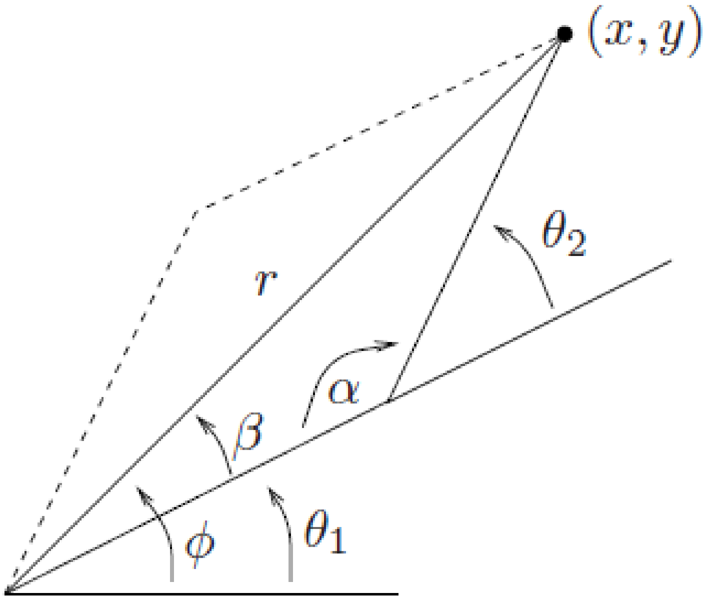
$$\theta_1 = \phi \pm \beta$$



Example I – Planar 2-Link

- Given a point (x, y), find the corresponding joint angles (θ_1, θ_2).

$$\theta_1 = \phi \pm \beta$$
$$\theta_2 = \pi \pm \alpha$$



- While (θ_1, θ_2) both determine the point (x, y), they independently control the radial distance and polar angle.

- **Separation is a useful IK strategy.**

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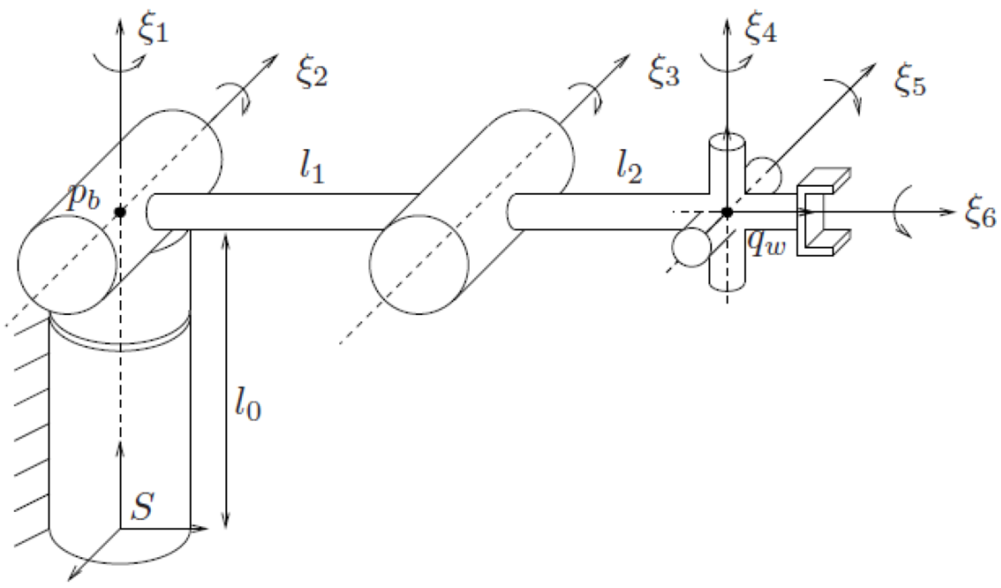
Example II – 6-DOF

- For reference: p. 104-106 in textbook.



- Given a valid end effector configuration, find the corresponding joint angles $(\theta_1 \ldots \theta_6)$

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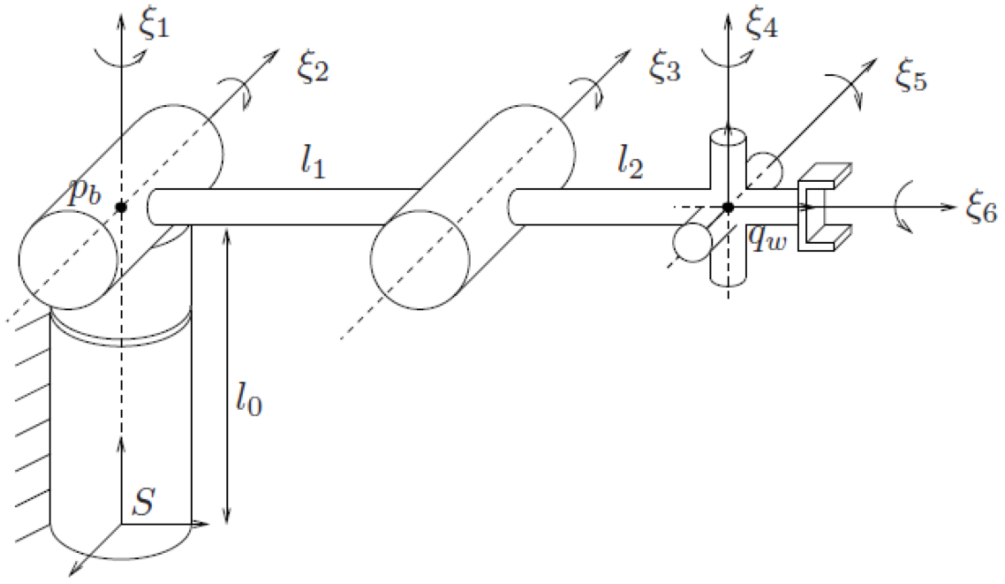
Example II – 6-DOF

- What do we need to know to specify the end effector configuration?
  - Distance $\|q_w - p_b\|$
  - Polar position of q_w
  - Orientation of the end effector



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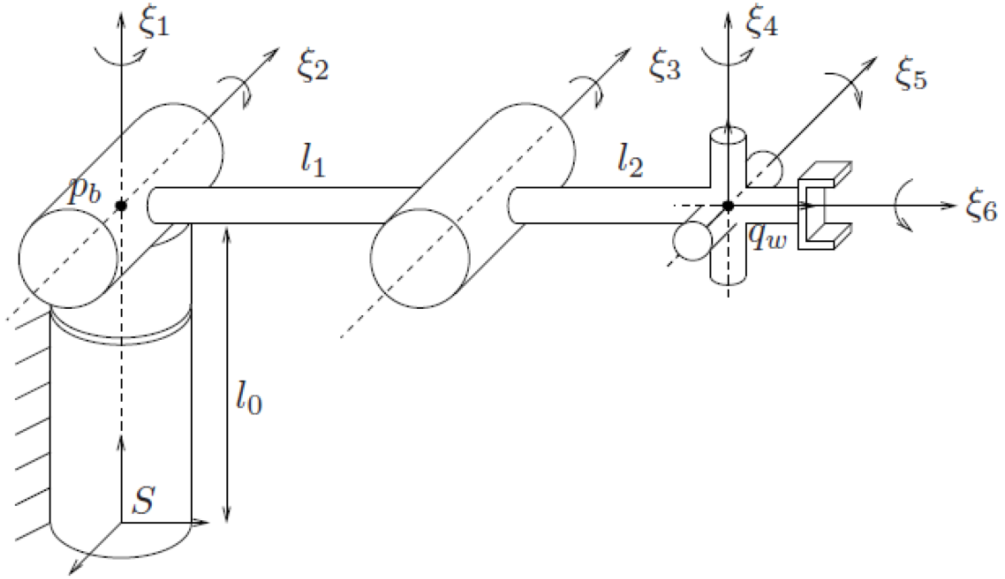
Example II – 6-DOF

- Which joints govern the variables of interest?
  - Distance $\|q_w - p_b\|$: θ_3
  - Polar position of q_w: $\theta_{1,2}$
  - Orientation of the end effector: $\theta_{4,5,6}$



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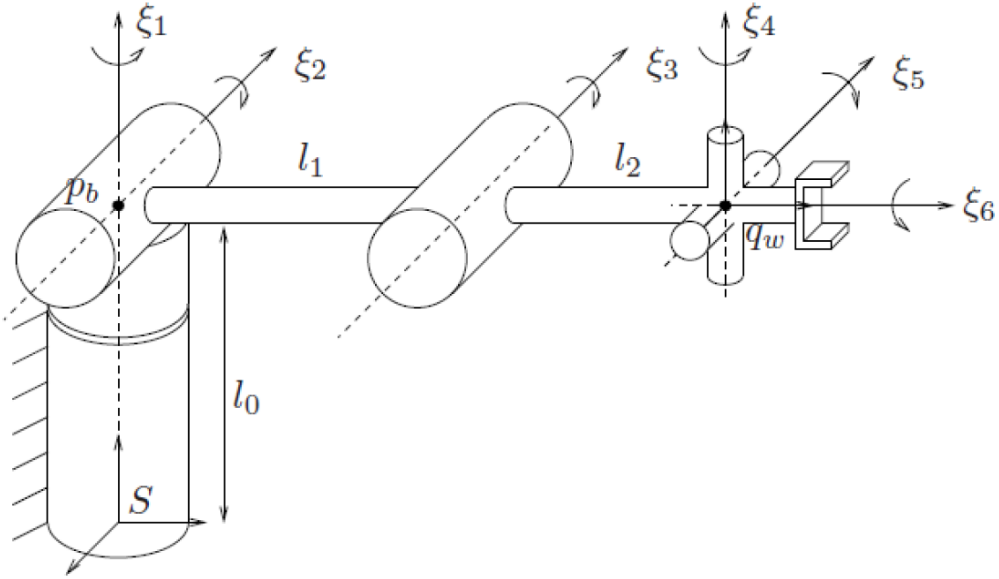
Invariant Points

- The points q_w and p_b are special:
  - p_b does not change with rotations from $\theta_{1,2}$.
  - q_w does not change with rotations from $\theta_{4,5,6}$.
  - These points are said to be **invariant**.



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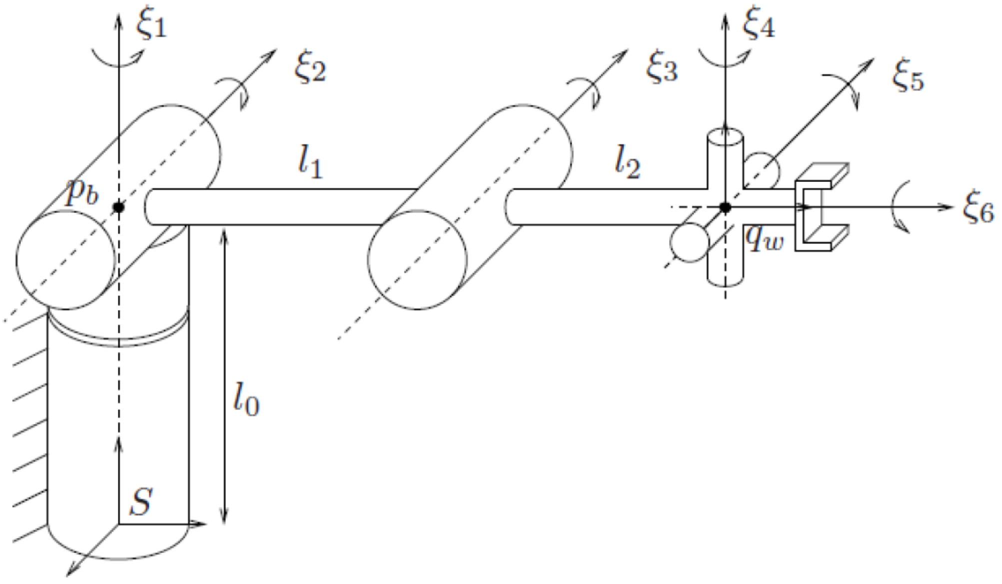
Invariant Points

- Invariant points lie on axes of rotation:

$$p = e^{\hat{\xi}_i \theta_i} p$$

- So we have:

$$p_b = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} p_b$$
$$q_w = e^{\hat{\xi}_4 \theta_4} e^{\hat{\xi}_5 \theta_5} e^{\hat{\xi}_6 \theta_6} q_w$$



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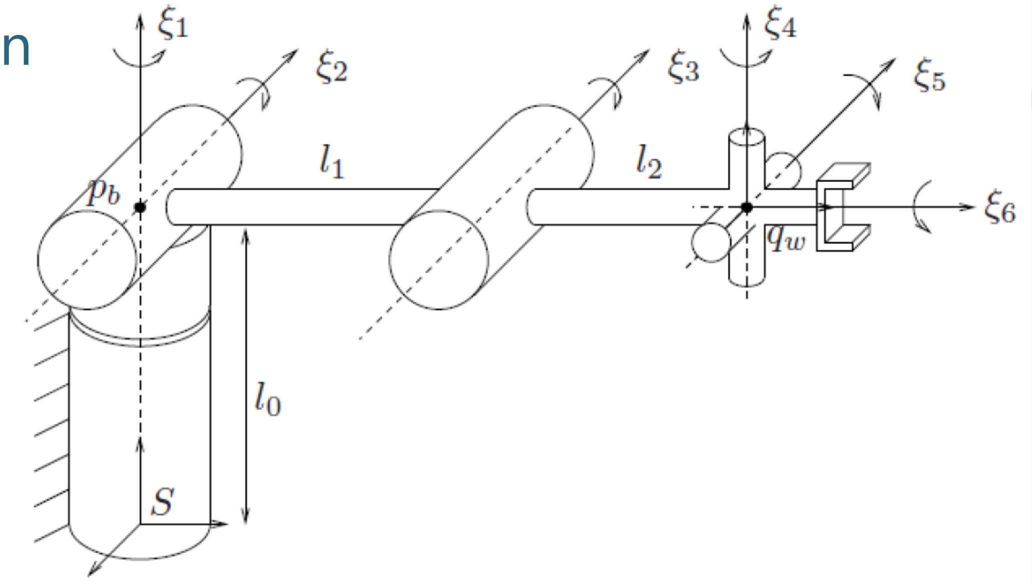
Invariant Points

- Given a desired end effector configuration g_d and an initial configuration g_0, find $\theta_{1\ldots6}$

$$g_d = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} g_0$$

$$\begin{aligned} g_1 &:= g_d g_0^{-1} \\ &= e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} \end{aligned}$$



Paden-Kahan Subproblems!

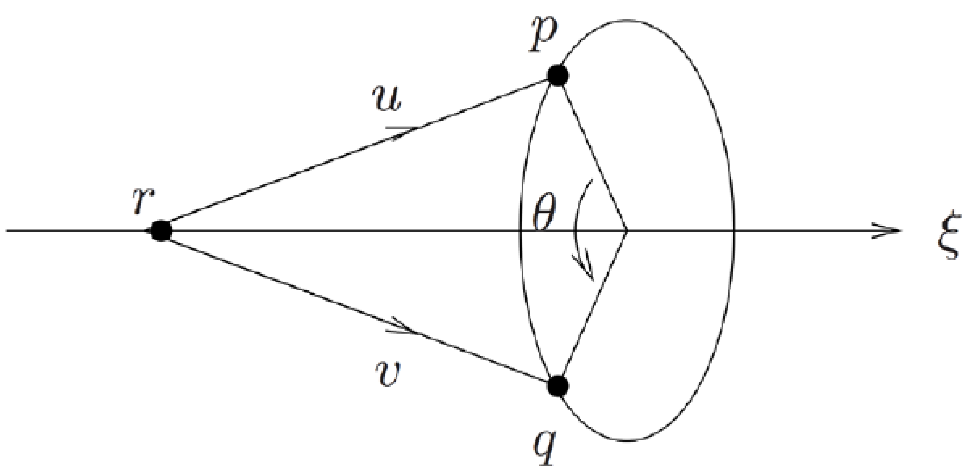
- Approach for analytically solving inverse kinematics problems!
  - Advantages: fast, efficient
  - Disadvantages: strict

- Methodology:
  - Divide the full manipulator into **subsections** and solve the inverse kinematics for those subsections individually.

- Other methods, both numerical and analytical, exist as well.

Paden-Kahan Subproblem I

- Rotation of a point p about a single axis ξ until it coincides with a second point q.

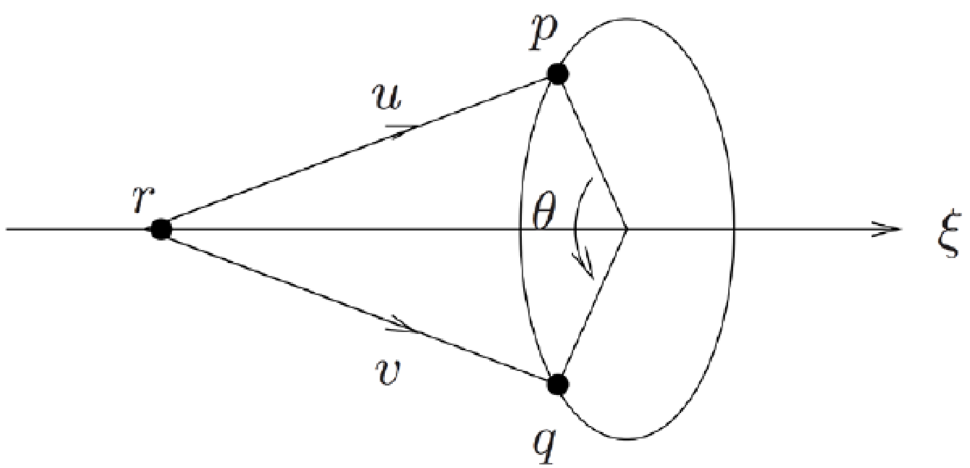


- $p, q \in \mathbb{R}^3$
- ξ is a **zero-pitch** twist.

- Find angle θ such that:

$$\boldsymbol{q} = e^{\hat{\xi}\theta}\boldsymbol{p}$$

Paden-Kahan Subproblem I



- Pick a point r on the axis of rotation ξ.
- Define relative coordinates:

$$\boldsymbol{u} := \boldsymbol{p} - \boldsymbol{r}$$
$$\boldsymbol{v} := \boldsymbol{q} - \boldsymbol{r}$$

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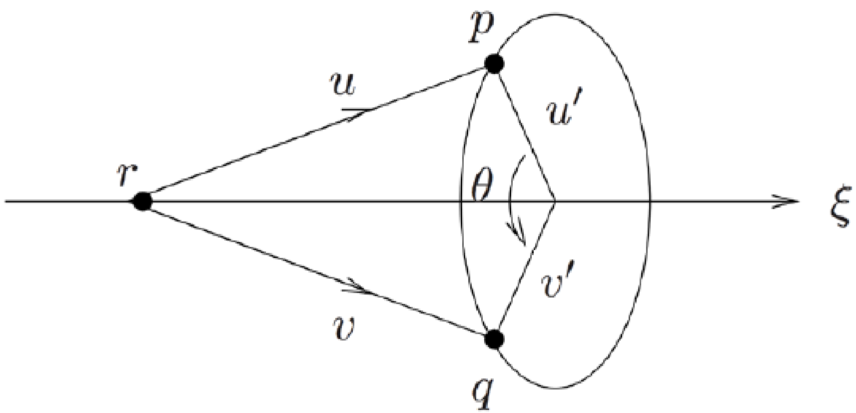
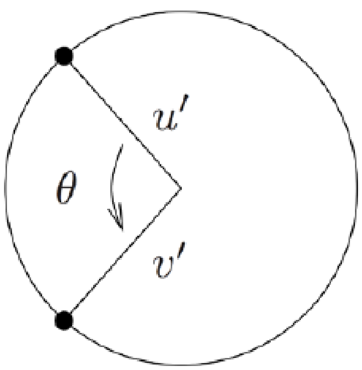
Paden-Kahan Subproblem I





- These relative points can then be projected onto the circle of revolution:

$$\boldsymbol{u'} = \boldsymbol{u} - \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{u}$$
$$\boldsymbol{v'} = \boldsymbol{v} - \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}$$

- Where ω is the unit vector in the direction of the axis of ξ.

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Paden-Kahan Subproblem I

- This problem only has a solution if the projections of u and v onto the ω–axis and onto the circle of revolution have equal lengths such that:

$$\boldsymbol{\omega}^T \boldsymbol{u} = \boldsymbol{\omega}^T \boldsymbol{v}$$
$$|\boldsymbol{v}'| = |u'|$$

- If a solution exists, we can use the following equations:

$$\boldsymbol{u}' \times \boldsymbol{v}' = \boldsymbol{\omega} \boldsymbol{sin}(\boldsymbol{\theta}_1) \|\boldsymbol{u}'\|_2 \|\boldsymbol{v}'\|_2$$
$$\boldsymbol{u}' \cdot \boldsymbol{v}' = \boldsymbol{cos}(\boldsymbol{\theta}_1) \|\boldsymbol{u}'\|_2 \|\boldsymbol{v}'\|_2$$

$$\boxed{\boldsymbol{\theta}_1 = \boldsymbol{atan2}\left(\boldsymbol{\omega}^T (\boldsymbol{u}' \times \boldsymbol{v}'), \boldsymbol{u}'^T \boldsymbol{v}'\right)}$$

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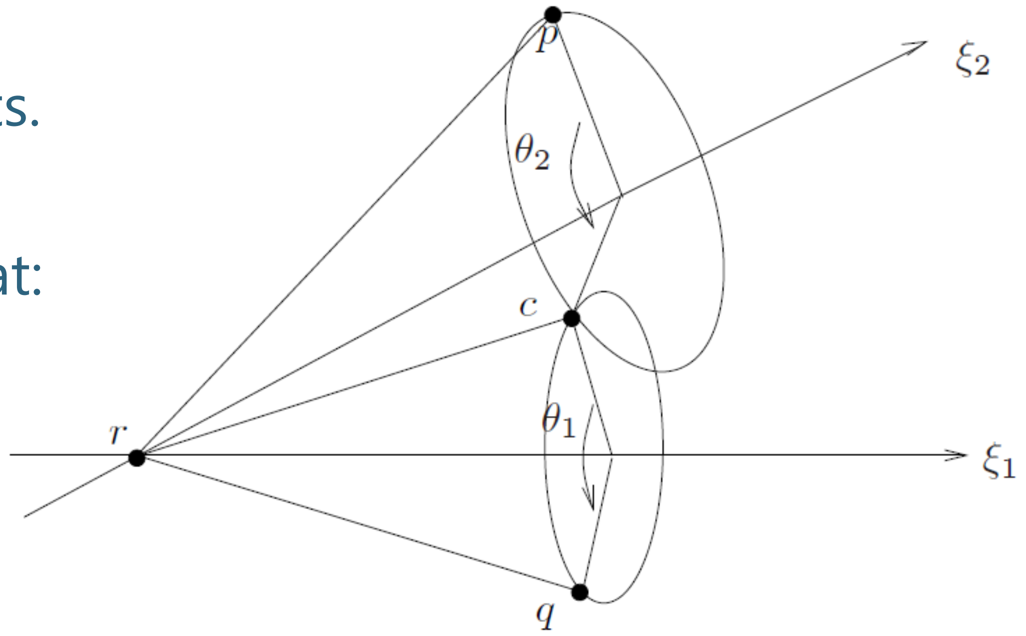
Paden-Kahan Subproblem II

- Rotation of a point p about two subsequent axes ξ_2, ξ_1 until it coincides with a point q.

- $p, q \in \mathbb{R}^3$
- ξ_2, ξ_1 are **zero-pitch** twists.

- Find angles θ_2, θ_1 such that:

$$q = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} p$$



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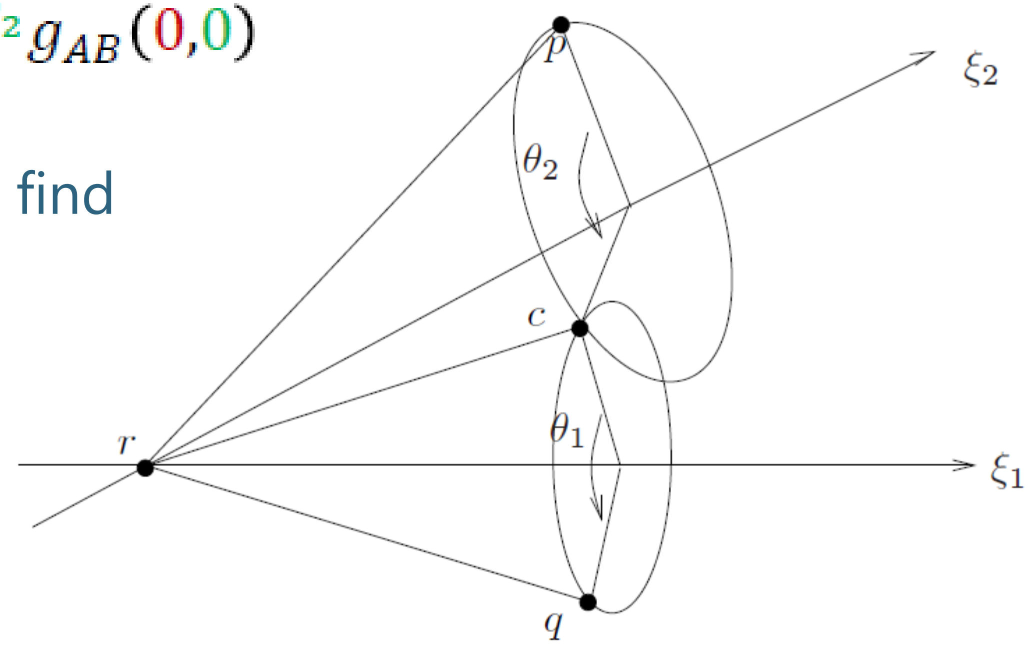
Paden-Kahan Subproblem II

- Rotation about subsequent axes ω_1, ω_2.

$$g_{AB}(\theta_1, \theta_2) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} g_{AB}(0,0)$$

- Given two points p and q, find the angles θ_1, θ_2.

- Find intersection point c.

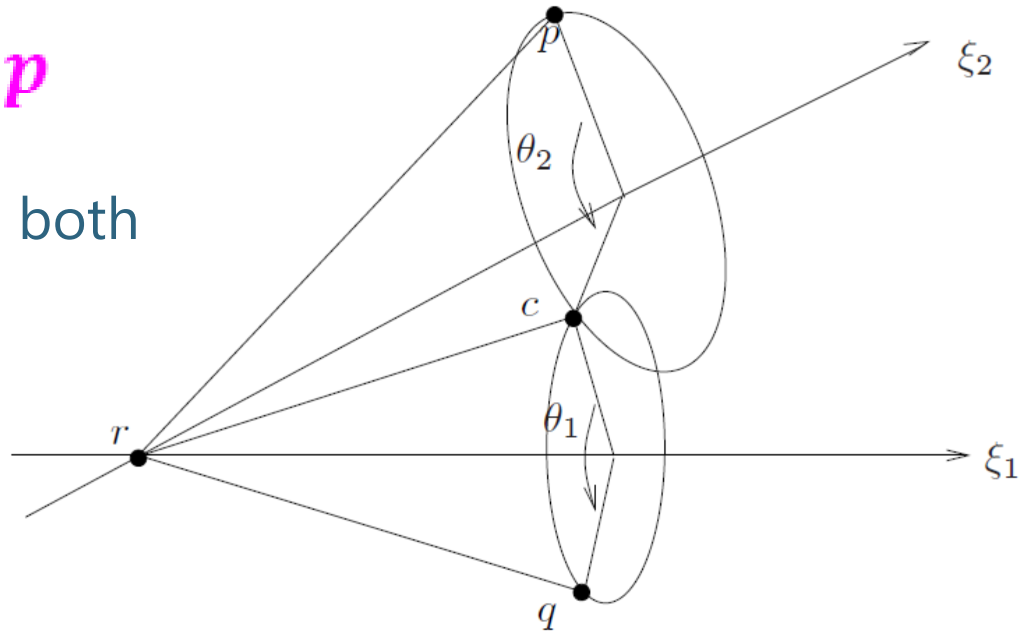


Paden-Kahan Subproblem II

$$q = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p$$
$$e^{-\hat{\xi}_1\theta_1}q = e^{\hat{\xi}_2\theta_2}p$$
$$e^{-\hat{\xi}_1\theta_1}q = c = e^{\hat{\xi}_2\theta_2}p$$

- Find a point r common to both axes.

$$u := p - r$$
$$z := c - r$$
$$v := q - r$$

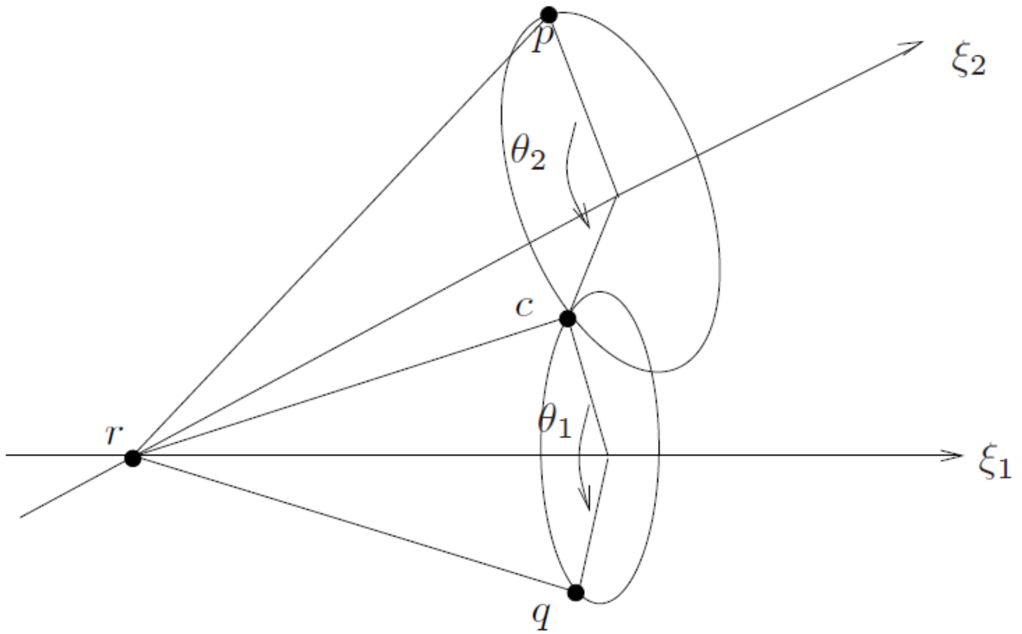


Paden-Kahan Subproblem II

$$u := p - r$$
$$z := c - r$$
$$v := q - r$$

$$e^{-\hat{\xi}_1\theta_1} q = c = e^{\hat{\xi}_2\theta_2} p$$

$$e^{-\hat{\xi}_1\theta_1} v = z = e^{\hat{\xi}_2\theta_2} u$$



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Paden-Kahan Subproblem II

$$e^{-\hat{\xi}_1 \theta_1} v = z = e^{\hat{\xi}_2 \theta_2} u$$

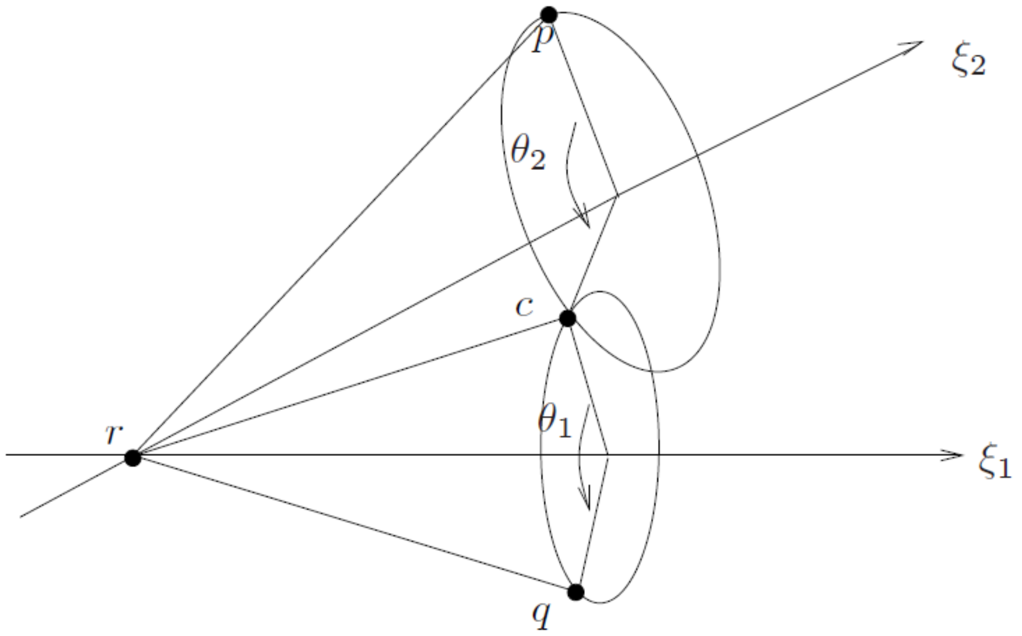
- NOTE: ω_1, ω_2 are linearly independent!

$$z = \alpha \omega_1 + \beta \omega_2 + \gamma(\omega_1 \times \omega_2)$$

$$\alpha = \frac{(\omega_1^T \omega_2)\omega_2^T u - \omega_1^T v}{(\omega_1^T \omega_2)^2 - 1}$$

$$\beta = \frac{(\omega_1^T \omega_2)\omega_1^T v - \omega_2^T u}{(\omega_1^T \omega_2)^2 - 1}$$

$$\boxed{\gamma^2 = \frac{\|u\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta\omega_1^T \omega_2}{\|\omega_1 \times \omega_2\|^2}}$$



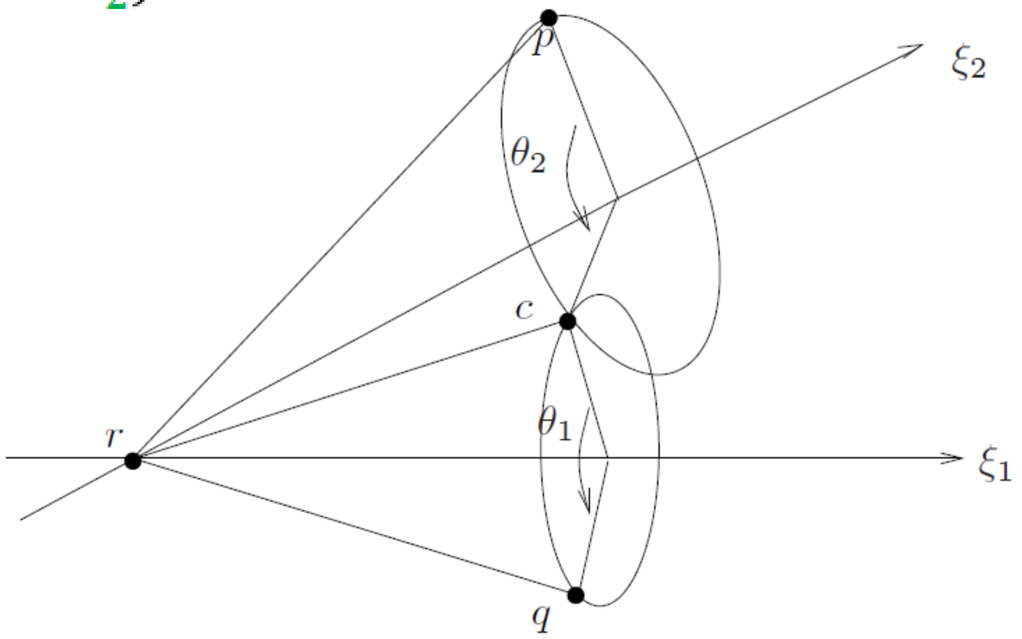
Paden-Kahan Subproblem II

$$e^{-\hat{\xi}_1\theta_1}v = z = e^{\hat{\xi}_2\theta_2}u$$

$$z_1 = \alpha\omega_1 + \beta\omega_2 + \gamma_1(\omega_1 \times \omega_2)$$
$$z_2 = \alpha\omega_1 + \beta\omega_2 + \gamma_2(\omega_1 \times \omega_2)$$

- Reduces to PK I:

$$e^{-\hat{\xi}_1\theta_1}v = z_1 = e^{\hat{\xi}_2\theta_2}u$$
$$e^{-\hat{\xi}_1\theta_1}v = z_2 = e^{\hat{\xi}_2\theta_2}u$$



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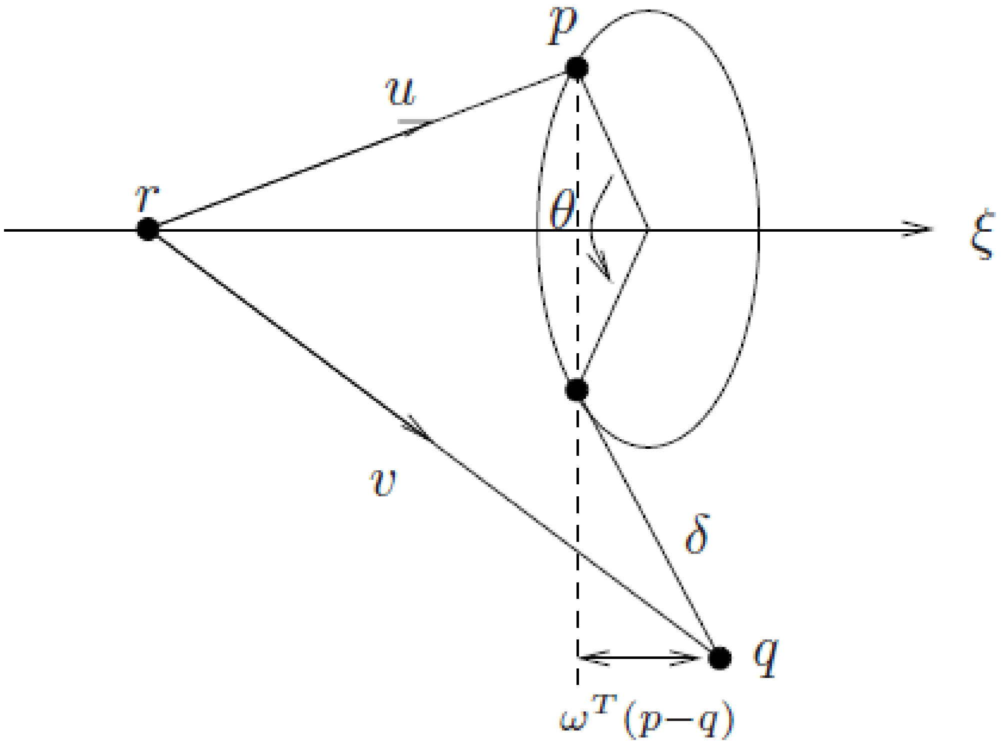
Paden-Kahan Subproblem III

- Rotation of a point p about an axis ξ until the point is a distance δ from a point q.

- $p, q \in \mathbb{R}^3$
- $\delta > 0 \in \mathbb{R}$
- ξ is a **zero-pitch** twist.

- Find angle θ such that:

$$\delta = \left\| q - e^{\hat{\xi}_1 \theta_1} p \right\|$$



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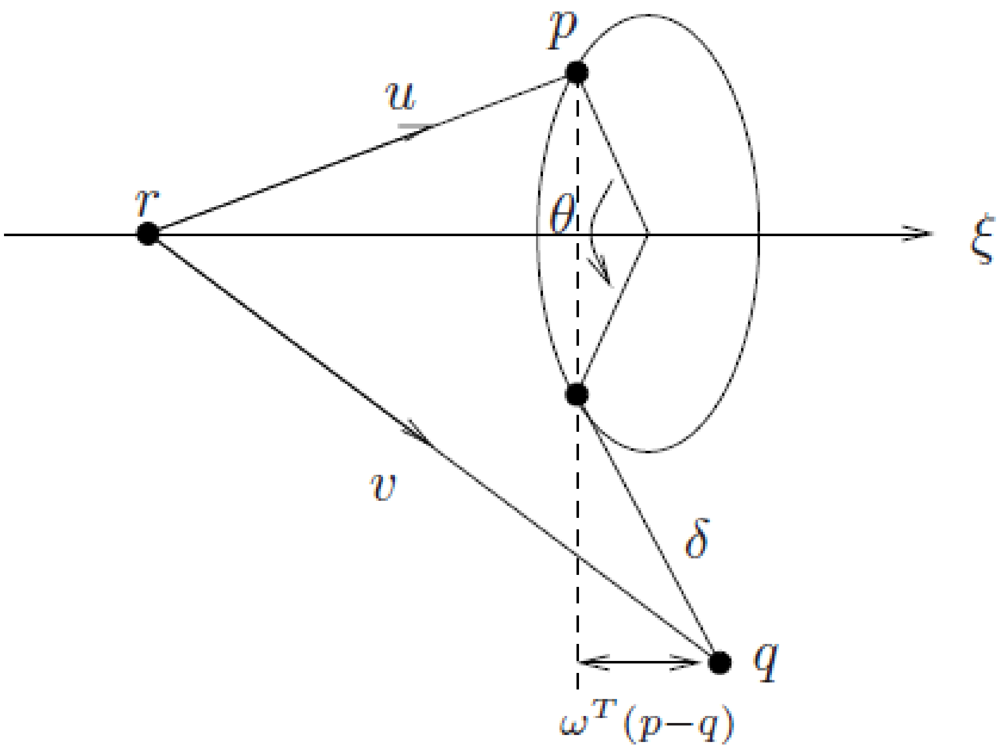
Paden-Kahan Subproblem III

- Rotation of a point p about an axis ξ until the point is a distance δ from a point q.

- Given two points p and q, find the angle θ_1.

$$\delta = \left\| q - e^{\hat{\xi}_1 \theta_1} p \right\|$$



- Think of δ as a sphere around the point q

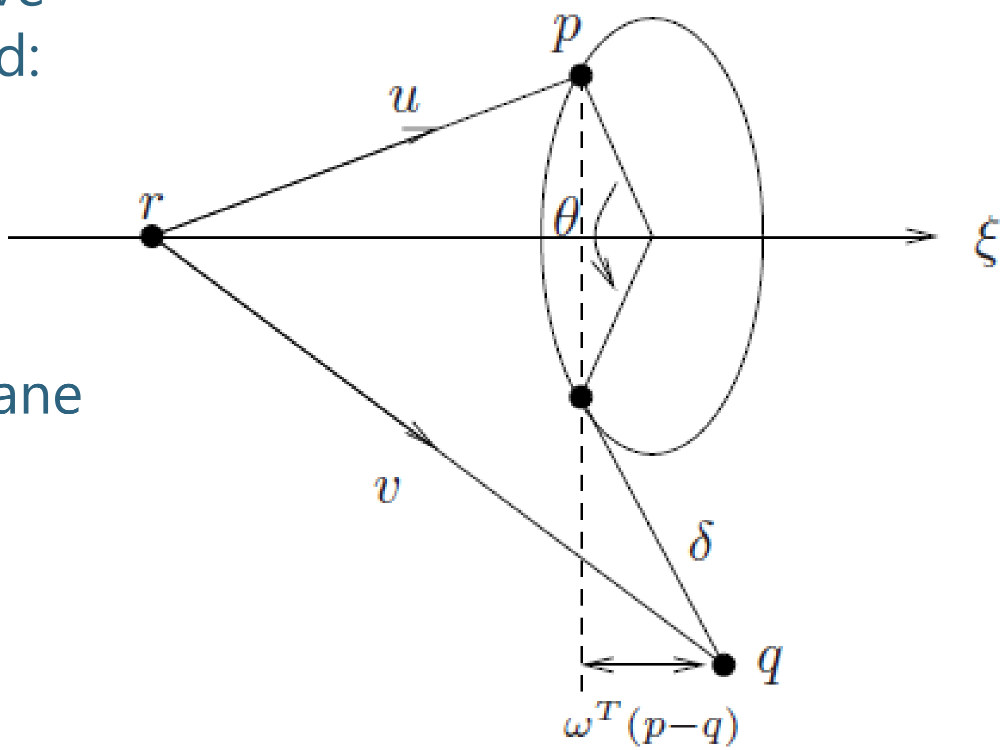
Paden-Kahan Subproblem III

- Given a point r on the rotational axis, the relative coordinates can be found:

$$u = p - r$$
$$v = q - r$$

- The projection on the plane of rotation is given by:

$$u' = u - \omega\omega^T u$$
$$v' = v - \omega\omega^T v$$



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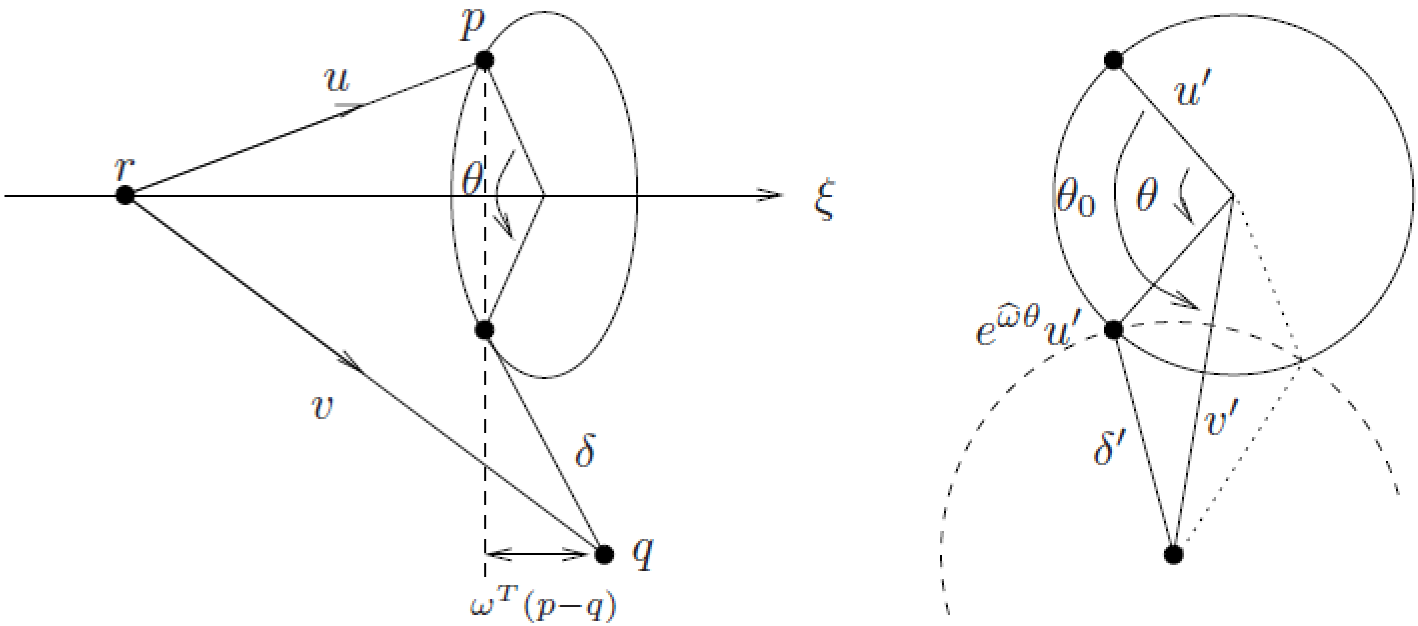
Paden-Kahan Subproblem III



- The projection of δ onto the rotational plane can be computed using:

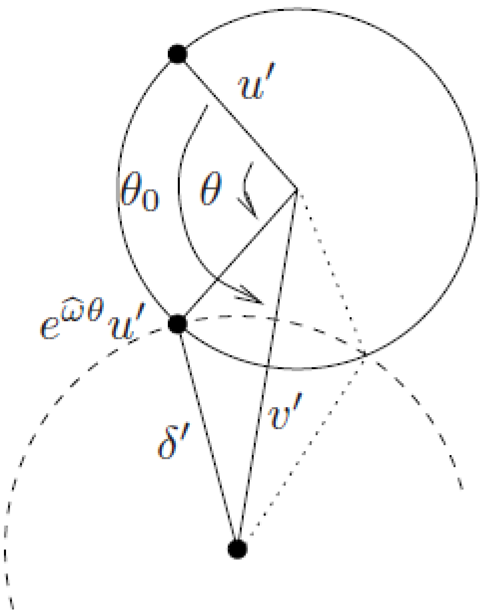
$$\delta'^2 = \delta^2 - |\omega^T(p-q)|^2$$

Paden-Kahan Subproblem III

- The two solutions will then take the following form:

$$\theta_0 = atan2(\omega^T(u' \times v'), u'^T v')$$

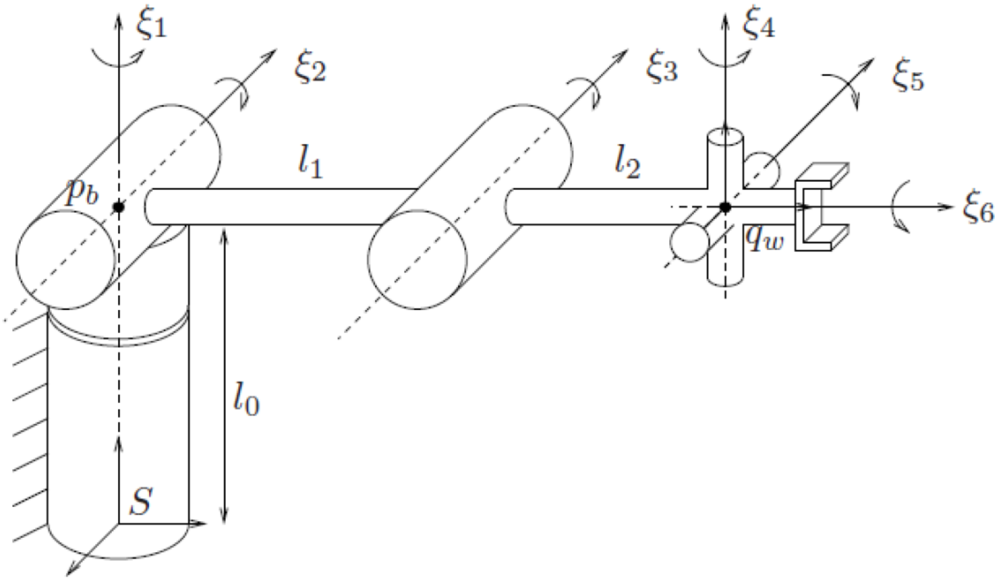
$$\boxed{\theta = \theta_0 \pm cos^{-1}\left(\frac{\|u'\|^2 + \|v'\|^2 - \delta'^2}{2\|u'\|\|v'\|}\right)}$$



Example II – 6-DOF

- Let's see if we can apply what we learned to this example!

- From earlier:
  - Distance $||q_w - p_b||$: θ_3
  - Polar position of q_w: $\theta_{1,2}$
  - Orientation of the end effector: $\theta_{4,5,6}$



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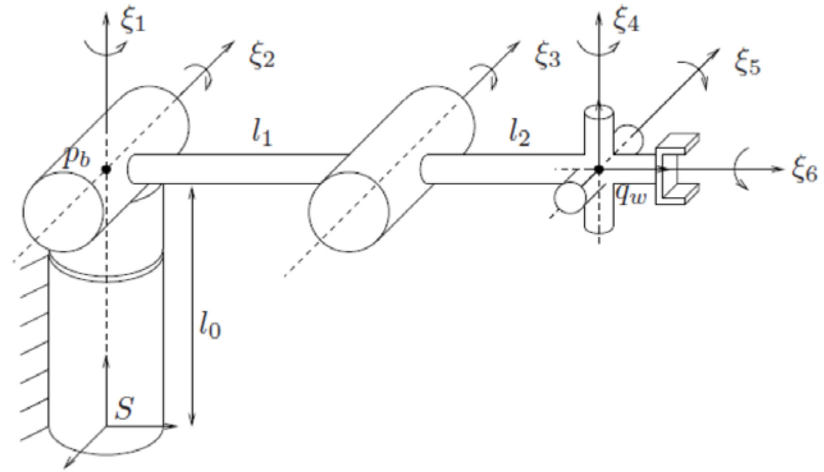
Example II – 6-DOF

- Let's see if we can apply what we learned to this example!

- Recall our **invariant points**:

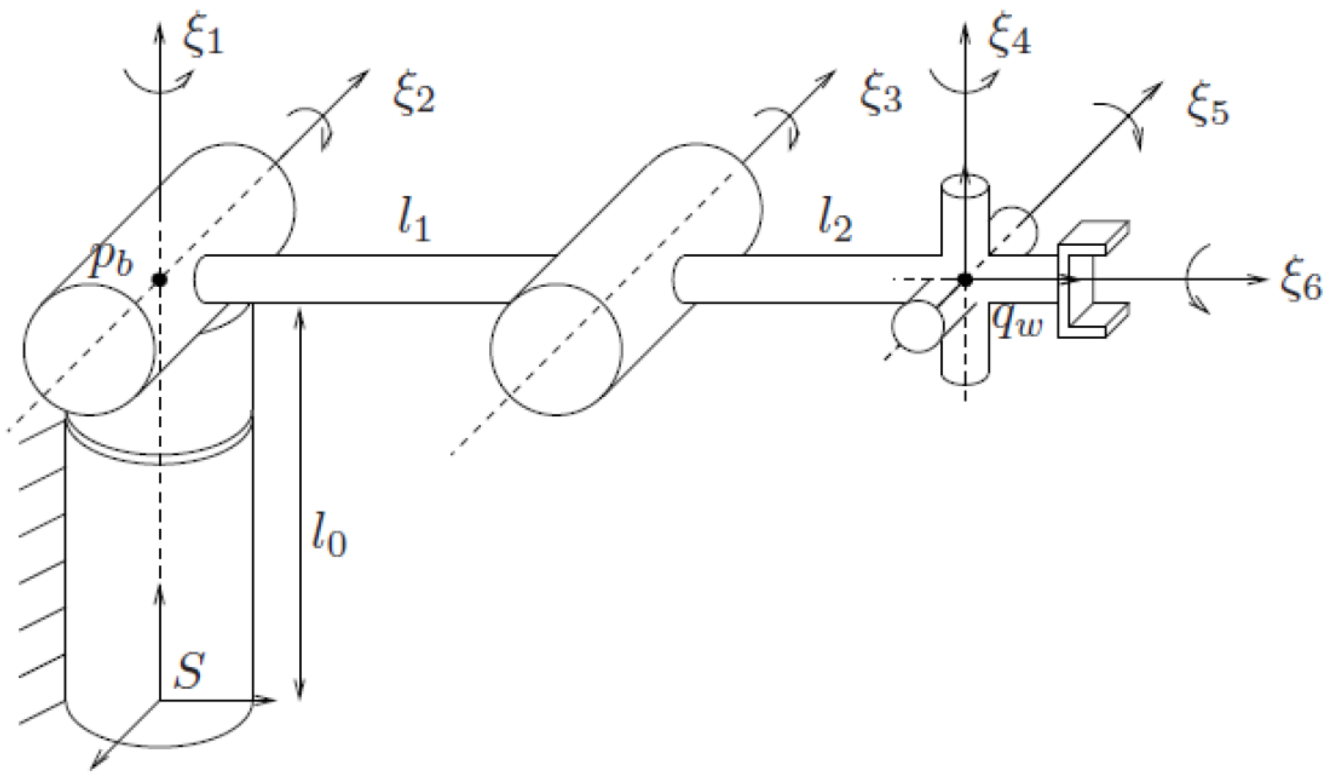
$$p_b = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p_b$$
$$q_w = e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}q_w$$

- Our goal is to determine the angles $\theta_{1\ldots6}$ such that:



$$g_1 = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}$$

Example II – 6-DOF



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