Slides adapted from material by Robert Peter Matthew and
A Mathematical Introduction to Robot Manipulation (Murray, Li, Sastry)

EECS C106A/206A

Discussion #2: Forward Kinematics

Agenda

- Logistics
- Lecture Review
  - Homogeneous Transforms
  - Forward Kinematics

Logistics

- Upcoming:
  - Homework 1 due **9/6**
  - Homework 2 due **9/13**
  - Midterm 1 on **9/27**
- Office Hours
  - Started this week!
  - Tuesdays & Thursdays @ 11:30 - 12:30, Locations on Piazza
  - By appointment: brentyi@berkeley.edu

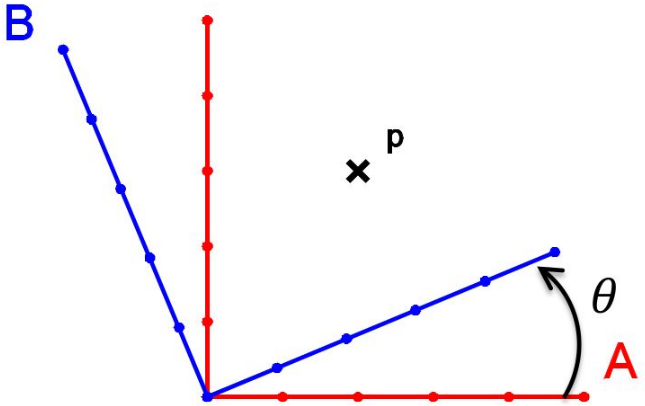
Logistics

- Discussions
    - Non-comprehensive review of lecture material
    - Attendance not required
- New GSI
    - Andrew Barkan
    - Mechanical Engineering PhD Student
    - Running discussions every other week, starting next week
    - He’s great and you’ll love him!

Homogeneous Transforms

Recall: Rotation Matrices

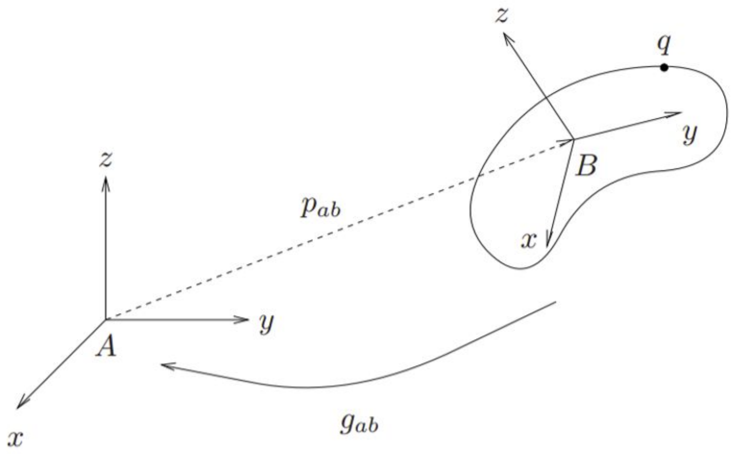
- Allow us to write a rotation as a matrix multiplication



$$q_A = R_{AB} q_B$$

Rigid Body Motion

- Translation of origin, relative rotation



$$q_A = R_{AB} q_B + p_{AB}$$

$$SE(3) = \{(p, R) : p \in \mathbb{R}^3, R \in SO(3)\}$$

Homogeneous Coordinates

For all points:

$$\bar{q}_A = \begin{bmatrix} q_A \\ 1 \end{bmatrix}$$

Homogeneous Transforms

Homogeneous coordinates allow us to represent both a rotation and a translation using a single matrix multiply.

$$q_A = R_{AB} q_B + p_{AB}$$

$$\begin{bmatrix} q_A \\ 1 \end{bmatrix} = \begin{bmatrix} R_{AB} & p_{AB} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} q_B \\ 1 \end{bmatrix}$$

$$\bar{q}_A = \begin{bmatrix} R_{AB} & p_{AB} \\ \vec{0}^T & 1 \end{bmatrix} \bar{q}_B$$

Homogeneous Transforms

This can be written alternatively as:

$$g_{AB} = \begin{bmatrix} R_{AB} & p_{AB} \\ \vec{0}^T & 1 \end{bmatrix}$$

$$\bar{q}_A = g_{AB}\bar{q}_B$$

Chaining Transforms

$$\bar{q}_A = \begin{bmatrix} R_{AB} & p_{AB} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_{BC} & p_{BC} \\ \vec{0}^T & 1 \end{bmatrix} \bar{q}_C$$

$$\bar{q}_A = g_{AB} g_{BC} \bar{q}_C$$

Inverting Transforms

$$\xcancel{g^{-1} = \begin{bmatrix} R & p \\ \vec{0}^T & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -p \\ \vec{0}^T & 1 \end{bmatrix}}$$

$$g^{-1} = \begin{bmatrix} R & p \\ \vec{0}^T & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T p \\ \vec{0}^T & 1 \end{bmatrix}$$

Questions so far?

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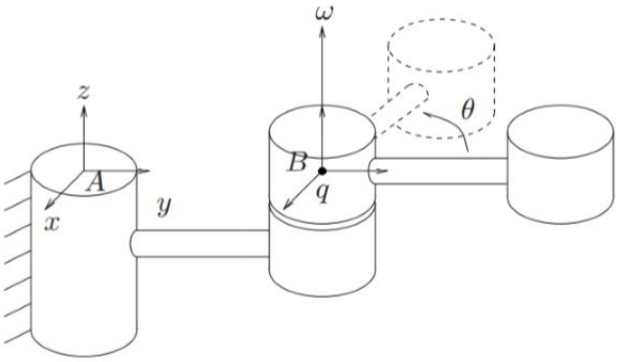
Forward Kinematics

Given a set of joints and their positions, how do we find the configuration of the end effector?

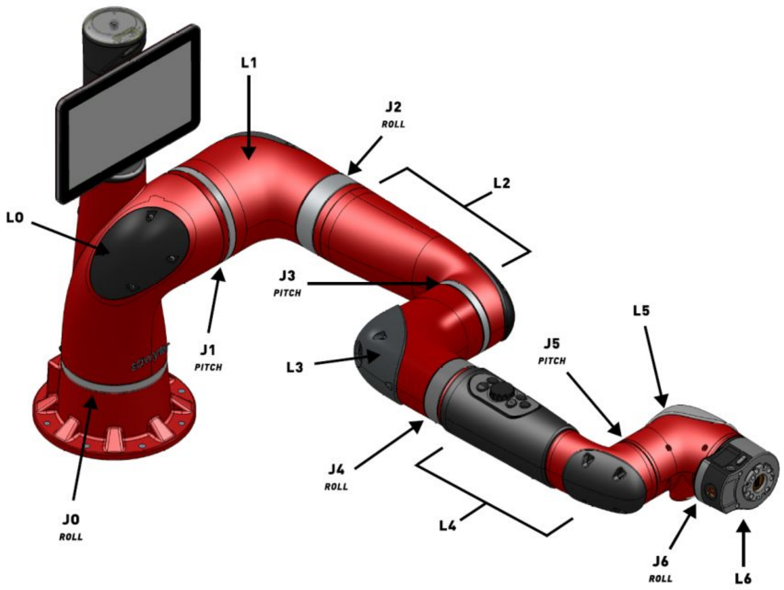
Joints: Rotational and Prismatic

- *Rotational joints*: second frame moves rotationally relative to first one
- *Prismatic joints*: second frame moves linearly relative to first one



Joints

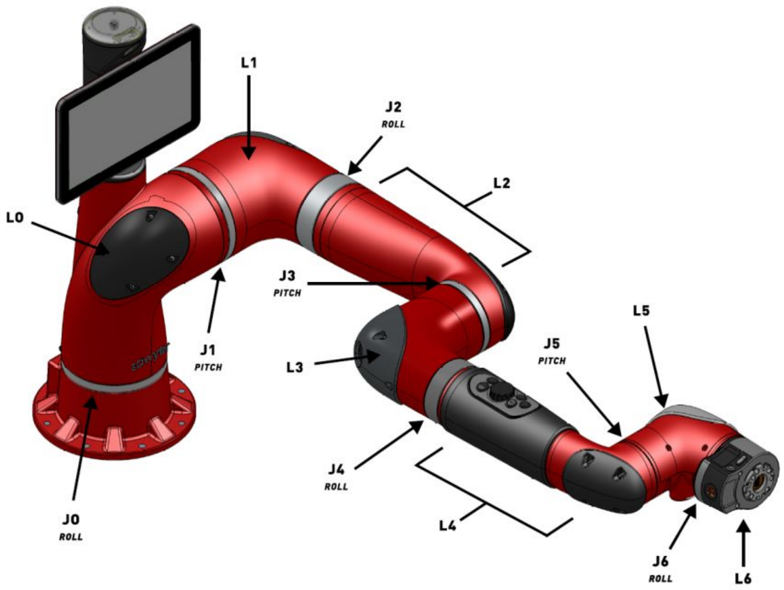
- What kind of joints does a Sawyer have?
- How many? Why?



Forward Kinematics

- We can describe the configuration of the end effector with a homogeneous transform **g**

- **FK:** given joint angles, we want to find a mapping from the initial transform to a final transform



Twists

We can characterize the kinematics of a joint with a twist:

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} \qquad \begin{array}{l} v \in \mathbb{R}^3 : \text{linear component} \\ \omega \in \mathbb{R}^3 : \text{angular component} \end{array}$$

Twists

In 106A, we mostly care about two kinds of twists:

- Purely translational:
  - Axis **v**

$$\xi = \begin{bmatrix} v \\ \vec{0} \end{bmatrix}$$

- Purely rotational:
  - Axis of rotation **ω**
  - Any point along **ω**, **q**

$$\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$$

- Normalized axes

Twist Operators

We define two operators for use in the exponential:

- "Wedge"

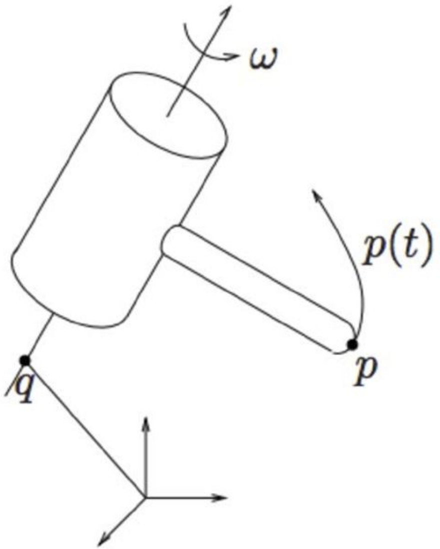
$$\hat{\xi} = \begin{bmatrix} v \\ \omega \end{bmatrix}^{\wedge} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$$

- "Vee"

$$(\hat{\xi})^{\vee} = \xi$$

Exponential Representation (Revolute)

Consider point **p**, rotating about axis **ω**.



Exponential Representation (Revolute)

We can describe the motion of **p** with:

$$\dot{p}(t) = \omega \times (p(t) - q)$$

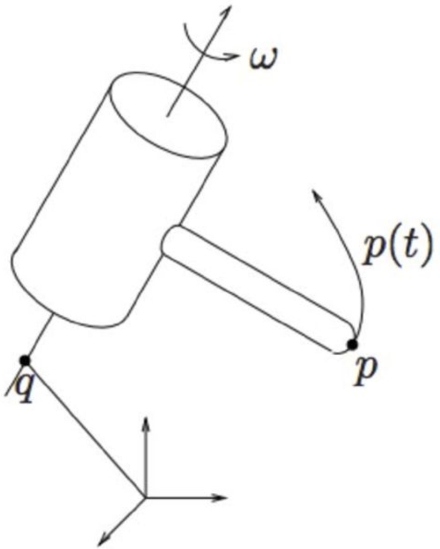
Which can equivalently be written as:

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \widehat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$= \hat{\xi} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

The solution for which is:

$$\bar{p}(\theta) = e^{\hat{\xi}\theta} \bar{p}(0)$$



Exponential Representation

Does this form look familiar?

$$p(\theta) = e^{\hat{\xi}\theta} p(0)$$

Similar to exponential computed by Rodrigues' Formula:

$$e^{\hat{\omega}\theta} = I_3 + \frac{\hat{\omega}}{||\omega||_2} \sin(||\omega||\theta) + \frac{\hat{\omega}^2}{||\omega||_2^2}(1 - \cos(||\omega||\theta))$$

...but generalized to all twists instead of just rotations.

Exponential Rigid Body Motion

- Given a homogeneous transform representing the initial configuration of the end effector (**World -> End Effector**):

$$g_{WE}(0) = \begin{bmatrix} R_{WE} & p_{WE} \\ \vec{0}^T & 1 \end{bmatrix}$$

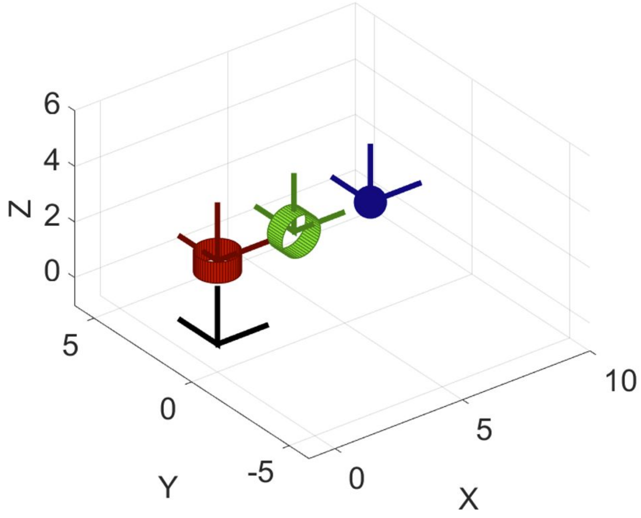
- We can compute a new transform parameterized by our joint angle as:

$$g_{WE}(\theta) = e^{\hat{\xi}\theta} g_{WE}(0)$$

Exponential Representations

- This also works for multiple joints:

$$g(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} g(0)$$



Exponential Solutions

- Screw/rotational motions:

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\widehat{\omega}\theta} & \left(I - e^{\widehat{\omega}\theta}\right)(\omega \times v) + \omega\omega^T v\theta \\ \mathbf{0} & 1 \end{bmatrix}$$

$$e^{\hat{\omega}\theta} = I_3 + \frac{\hat{\omega}}{||\omega||_2}\sin(||\omega||\theta) + \frac{\hat{\omega}^2}{||\omega||_2^2}(1 - \cos(||\omega||\theta))$$

- Translational motions:

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} \mathbb{I}_3 & \boldsymbol{v}\theta \\ \mathbf{0} & 1 \end{bmatrix}$$