

On the star forest polytope for 4-cactus graphs

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Abstract. This paper investigates the polyhedral structure of the Maximum Weight Star Forest Problem (MWSFP) in an undirected weighted graph $G = (V, E)$, where each edge has a non-negative weight. A star in G is either an isolated node or a connected subgraph in which all edges share a common endpoint, and a star forest is a collection of disjoint stars. The objective of the MWSFP is to find a star forest with the maximum total edge weight. This problem is NP-hard in general but can be solved in polynomial time when G is a cactus graph [19].

In this paper, we provide a complete polyhedral description of the star forest polytope $SFP(G)$ when G is a 4-cactus graph, a subclass of cactus graphs where each cycle has at most four edges. More precisely, we introduce a new class of facet-defining inequalities, called M -cactus inequalities, which hold for any graph. We then show that when G is a 4-cactus graph, the M -cactus inequalities, together with the non-negativity inequalities, completely describe $SFP(G)$.

Keywords: Graph theory · Spanning star forest · Polyhedral combinatorics · Cactus graph

1 Introduction

The Maximum Weight Star Forest Problem (MWSFP) arises in graph theory and optimization, where the objective is to find a collection of disjoint stars in an undirected graph $G = (V, E)$ that maximizes the total weight of the included edges. A *star* in G is defined as either an isolated node or a subgraph where all edges share a common endpoint known as the *center*. A node adjacent to the center is called a *leaf*. A *star forest* refers to a collection of disjoint stars in G . The weight of a star forest is the sum of the weights of its edges. Given a weight vector $c \in \mathbb{R}_+^{|E|}$, the MWSFP seeks to find a star forest in the graph G with the

maximum possible weight, i.e., $\max \sum_{e \in F} c_e$ subject to $F \in \mathcal{F}(G)$, where $\mathcal{F}(G)$

is the set of star forests in G . Introduced by Nguyen et al. [18], the problem is NP-hard since it can be reduced to the minimum dominating set problem in the unweighted case. The MWSFP has several applications. In computational biology, it is used for aligning genomic sequences with high duplication, improving methods such as the threaded blockset aligner program [16,18], and for comparing phylogenetic trees [9]. In the automobile industry, a directed version of the problem helps optimize configuration management in inclusion relationships, where the objective is to find a maximum-weight spanning subgraph in which each connected component forms an outward star [1].

Previous research on the MWSFP has primarily focused on two directions. The first involves developing approximation algorithms for the unweighted version and its generalizations. Nguyen et al. [18] introduced a 0.6-approximation algorithm, later improved to 0.803 by Athanassopoulos et al. [3] using its connection to the complementary set cover problem. Stronger inapproximability results and APX-hardness proofs have also been established for node-weighted and edge-weighted versions [18,11,14,10]. The second direction focuses on efficient algorithms for special graph classes in the weighted version. Nguyen et al. [18] developed a linear time algorithm for solving the MWSFP on trees, leading to a $\frac{1}{2}$ -approximation algorithm for general graphs. More recently, Nguyen extended this approach to cactus graphs by introducing a linear time algorithm for this class [19].

Despite significant progress in algorithmic studies of the MWSFP, its polyhedral structure remains largely unexplored. Given a star forest F in a graph G , let x_F be its incidence vector in $\mathbb{R}^{|E|}$. The star forest polytope of G , denoted $SFP(G)$, is the convex hull of the incidence vectors of all star forests in G . To the best of our knowledge, the only existing work on $SFP(G)$ is by Aider et al. [2], who provided a complete description of this polytope when G is a tree or a cycle. However, extending these results to more general graph classes remains an open problem. Motivated by this issue, this paper focuses on the polyhedral study of the MWSFP by introducing a new class of facet-defining inequalities for the star forest polytope in cactus graphs, which generalizes both trees and cycles. Formally, a cactus graph is a graph where each edge belongs to at most one cycle. These graphs are relevant in applications such as telecommunication network design in sparsely populated areas [17] and material handling systems using automated guided vehicles [15]. Using an extended formulation inspired by the work of Baiou and Barahona on the Uncapacitated Facility Location Problem [8], we provide a complete description of the star forest polytope for 4-cactus graphs, a subclass of cactus graphs in which each cycle has at most four edges. Furthermore, we demonstrate that the facet-defining inequalities for $SFP(G)$ when G is a 4-cactus can be extended to facet-defining inequalities for $SFP(G)$ in the case of an arbitrary graph.

The paper is structured as follows. In Section 2, we introduce a class of facet-defining inequalities, called the M -cactus inequalities for $SFP(G)$ in the general case of G . Section 3 presents how the M -cactus inequalities combined

with non-trivial inequalities, provide a complete polyhedral description when G is a 4-cactus graph. Finally, in Section 4, we summarize our work and discuss the directions for future research.

In the rest of this section, we introduce some useful definitions and notations used throughout this paper. Given an undirected graph $G = (V, E)$, a node u is a *pendant* node if it has degree 1, and a *non-pendant* node otherwise. A *2-matching* S in G is a subgraph where each component is either an edge or two edges sharing a common endpoint. For a subgraph M of G and an edge $e \in G \setminus M$, we denote $M + e$ as the graph obtained by adding the edge e to M , and $M - e$ as the graph obtained by removing the edge e from M . For $x \in \mathbb{R}^{|E|}$ and $F \subseteq E$, let $x(F) = \sum_{e \in F} x(e)$. A cycle $C = \{u_1, u_2, \dots, u_p\}$ in a graph of length $p \geq 3$ is a sequence of distinct vertices such that each pair u_i, u_{i+1} for $i = 1, \dots, p-1$ and (u_p, u_1) forms an edge in the graph. A 3-cycle (triangle) is a cycle of length 3, and a 4-cycle is a cycle of length 4. We denote by \mathcal{C}_3 and \mathcal{C}_4 the sets of all 3-cycles and 4-cycles in G , respectively.

2 Facet-defining inequalities for $SFP(G)$: M -cactus inequalities

This section discusses the general case where G is an arbitrary connected undirected graph. Aider et al. [2] introduced M -tree inequalities and proved that, together with non-negativity inequalities, they provide a complete description of $SFP(G)$ when G is a tree. Inspired by this result, we introduce a more general class of inequalities, called M -cactus inequalities, to extend this characterization beyond trees. Furthermore, we show that these inequalities define facets of $SFP(G)$ for general graphs. To begin, we define the concept of a M -cactus.

Definition 1. (M -cactus) A M -cactus is a cactus graph where

- Every node in a 3-cycle or 4-cycle is adjacent to at most one pendant node.
- Every remaining non-pendant node is adjacent to exactly one pendant node.

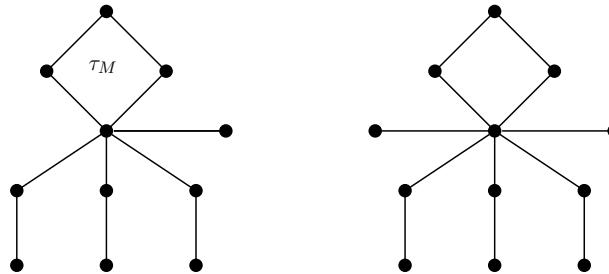


Fig. 1: A M -cactus τ_M (left) and a non M -cactus (right).

Observe that, a M -tree is a special case of a M -cactus that does not contain any cycles. For a M -cactus τ_M , we define an M -2-matching M as a cactus graph whose node set is exactly that of τ_M , including all its pendant nodes. Figure 1 illustrates a M -cactus τ_M that is not a M -tree, as well as an example of a non- M -cactus. Figure 2 illustrates the corresponding M -2-matching M of τ_M .

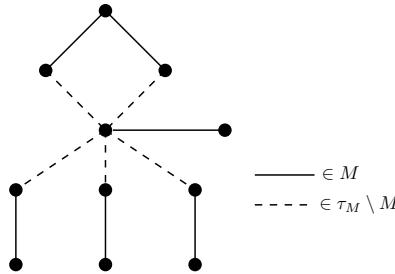


Fig. 2: The M -2-matching M corresponds to τ_M .

Definition 2. (M -cactus inequality) Given a M -cactus τ_M , let M be the corresponding M -2-matching. We define the M -cactus inequalities as follows:

$$x(\tau_M) \leq |M|. \quad (1)$$

Before proving the validity of the M -cactus inequality, we first recall an important property of $SFP(G)$ that is given by Aider et al. [2].

Lemma 1. [2] For any graph G , $SFP(G)$ is a full dimensional polytope.

Theorem 1. The M -cactus inequalities are valid for $SFP(G)$.

Proof. Assume that τ_M is a M -cactus and M is a M -2-matching of τ_M . To prove validity, we show that

$$|F \cap \tau_M| \leq |M|,$$

where F is any star forest of G . If F only contains the edges in M , i.e., $M \subseteq F$, then $|F \cap \tau_M| \leq |M|$. If $(\tau_M \setminus M) \subseteq F$, then, by the definition of a M -cactus graph we obtain

$$|F \cap \tau_M| \leq |\tau_M \setminus M| < |M|.$$

The remaining case is where F contains edges in both M and $\tau_M \setminus M$. If $F \cap \tau_M$ is a 2-matching, then $|F \cap \tau_M| \leq |M|$ because M is a maximum 2-matching in τ_M . Therefore, we now examine the case where $F \cap \tau_M$ contains a star S with center u , such that S includes at least two edges from F : one edge uu_1 in M and another edge uu_2 in $\tau_M \setminus M$. There exists an edge $u_2u_3 \in M$, where u_3 may coincide with u_1 , because M covers all vertices of τ_M . Since uu_1 and uu_2 already belong to F , it follows that $u_2u_3 \notin F$, that is, $u_2u_3 \in M \setminus F$. By the definition of a M -cactus, u_3 must either have degree 1 in τ_M or belong to the

cycle $\{u_1, u, u_2, u_3\} \in \mathcal{C}_4$, with $u_3u_1 \notin F$. This ensures that for each edge in $F \cap (\tau_M \setminus M)$, there is a corresponding edge in $M \setminus F$. Therefore,

$$|F \cap (\tau_M \setminus M)| \leq |M \setminus F|.$$

Combining these, we get

$$|(F \cap (\tau_M \setminus M)) \cup (F \cap M)| \leq |(M \setminus F) \cup (F \cap M)| = |M|.$$

Since $M \subseteq \tau_M$, we get that $|(F \cap (\tau_M \setminus M)) \cup (F \cap M)| = |F \cap \tau_M|$. As a result, we have $|F \cap \tau_M| \leq |M|$. Thus, the M -cactus inequalities are valid. \square

Theorem 2. *The M -cactus inequalities define facets of $SFP(G)$.*

Proof. Suppose there exists a facet-defining inequality

$$\alpha^t x \leq \beta \tag{2}$$

for $SFP(G)$ such that all star forests satisfying (1) at equality also satisfy (2) at equality. As stated in Lemma (1), since $SFP(G)$ is full dimensional, it suffices to demonstrate that Inequality (2) must be a positive multiple of the M -cactus inequality (1).

Let τ be any M -cactus and M be the M -2-matching associated with τ . Let us examine the corresponding M -cactus inequality

$$x(\tau) \leq |M|. \tag{3}$$

Observe that M is a star forest satisfying (1) at equality. Thus, M also satisfies (2) at equality. This implies that

$$\alpha(M) = \beta. \tag{4}$$

- We first show that $\alpha(e) = \alpha(e')$ for all $e, e' \in \tau$. Indeed, let uv be an edge in $\tau \setminus M$. We consider the following cases.

- *Case 1.* uv belongs to a 3-cycle $\{u, v, w\} \in \mathcal{C}_3$ with $uw, vw \in M$. We consider the star forests $M' = M - uw + uv$ and $M'' = M - vw + uv$ (see figure 3). Observe that M, M', M'' are star forests that satisfy (3) at equality. Thus, we obtain

$$\alpha(M) = \alpha(M - uw + uv) = \alpha(M - vw + uv).$$

It follows that

$$\alpha(uv) = \alpha(uw) = \alpha(vw).$$

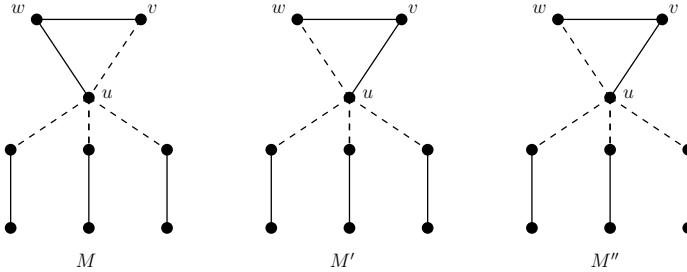


Fig. 3: An example of the star forests \$M, M', M''\$.

- *Case 2.* \$uv\$ belongs to a 4-cycle \$\{u, v, v', u'\} \in \mathcal{C}_4\$. We consider two subcases:
 - *Subcase 2.1.* If \$uu'\$ and \$vv'\$ are in \$M\$, then \$M\$, \$M - uu' - vv' + uv + v'u'\$, \$M - uu' + uv\$, and \$M - vv' + uv\$ are the star forests satisfying (3) at equality, which implies that

$$\alpha(uv) = \alpha(uu') = \alpha(vv') = \alpha(u'v').$$

- *Subcase 2.2.* If \$uu'\$ and \$u'v'\$ are in \$M\$, then using a similar argument, considering the star forests \$M\$, \$M - uu' - u'v' + uv + vv'\$, \$M - uu' + uv\$, and \$M - u'v' + uv\$, we conclude that

$$\alpha(uv) = \alpha(uu') = \alpha(u'v') = \alpha(vv').$$

- *Case 3.* \$uv\$ does not belong to any cycle \$C \in \{\mathcal{C}_3, \mathcal{C}_4\}\$. Let \$uu'\$ and \$vv'\$ be edges in \$M\$ that are incident to \$u\$ and \$v\$, respectively. Consider the star forests \$M' = M - uu' + uv\$ and \$M'' = M - vv' + uv\$. Since both \$M'\$ and \$M''\$ satisfy (3) at equality, it follows

$$\alpha(uv) = \alpha(uu') = \alpha(vv').$$

Thus, we have established that all edges in \$\tau\$ have the same coefficient, i.e., \$\alpha_{e'} = \alpha_e\$ for all \$e', e \in \tau\$.

- We now show that \$\alpha(e) = 0\$ for all \$e \notin \tau\$. Considering an edge \$uv\$ belongs to \$\tau\$, there are two possible cases.

- *Case 1.* Neither \$u\$ nor \$v\$ belongs to any cycle \$C \in \{\mathcal{C}_3, \mathcal{C}_4\}\$. Since \$M\$ and \$M + uv\$ are both star forests satisfying (3) at equality, it follows that

$$\alpha(M) = \alpha(M + uv) = \beta.$$

This implies that \$\alpha(uv) = 0\$.

- *Case 2.* Exactly one endpoint of the edge \$uv\$ belongs to a cycle \$C \in \{\mathcal{C}_3, \mathcal{C}_4\}\$. Suppose that this endpoint is \$u \in C\$. If \$u \notin M\$, then \$M + uv\$ remains a star forest satisfying (3), and we obtain \$\alpha(uv) = 0\$. Therefore, in the rest of this case, we only consider the scenario where \$u \in M\$. Since \$C \in \{\mathcal{C}_3, \mathcal{C}_4\}\$,

it follows that there are exactly two edges of C , denoted e_1 and e_2 belong to M . If three edges uv, e_1, e_2 share a common endpoint u , then $M + uv$ is also a star forest satisfying (3) at equality, and we also obtain $\alpha(uv) = 0$. Otherwise, without loss of generality, we assume that the edge sharing the common endpoint u with uv and e_1 is $e_3 \in C \setminus M$. In this case $M + uv + e_3 - e_2$ remains a star forest satisfying (3) at equality. Note that $\alpha(e_2) = \alpha(e_3)$ since e_2, e_3 are two edges in τ . Consequently, we obtain $\alpha(uv) = 0$.

From the above arguments, we conclude that $\alpha(e) = \alpha(e')$ for all $e, e' \in \tau$ and $\alpha(e) = 0$ for all $e \notin \tau$. Therefore, we obtain $\alpha^t x \leq \beta$ as a positive multiple of the M -cactus inequality, which completes the proof. \square

3 Complete description of $SFP(G)$ in 4-cactus

In this section, we focus on a special class of cactus graphs called 4-cactus graphs and provide a characterization of $SFP(G)$ for 4-cactus graphs. We begin by formally defining this concept.

Definition 3. A 4-cactus graph is a cactus graph in which every cycle has length at most 4, meaning that every cycle is either a 3-cycle or a 4-cycle.

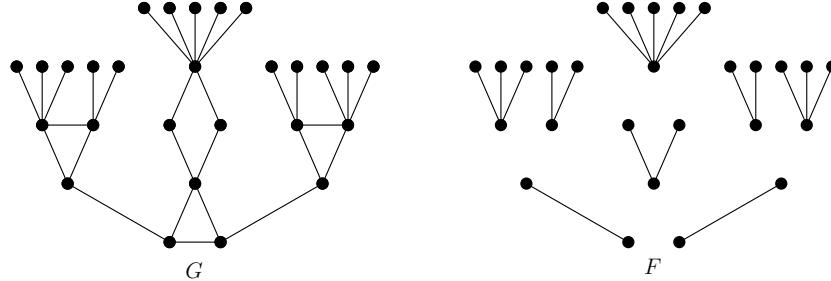


Fig. 4: A 4-cactus graph G (left) and a star forest F of G (right).

For the remainder of this section, we consider G as a 4-cactus graph. Consequently, any M -cactus graph in G is also a 4-cactus graph. With this setting, we now describe the size of maximal star forests in M -cactus in the following lemma.

Lemma 2. Every maximal star forest with respect to a M -cactus τ_M has cardinality $|M|$, where M is the M -2-matching in τ_M .

Proof. Let H be any maximal star forest with respect to τ_M . By the validity of M -cactus inequalities (1), we obtain $|H| \leq |M|$. It suffices to show that $|H| \geq |M|$. We demonstrate this by proving that each edge in M corresponds to one edge in H . Since M is a 2-matching, each of its components is either (1) two edges sharing a common endpoint or (2) an independent edge.

Type 1. The component in M that is an independent edge. Consider an independent edge $v_1v_2 \in M$ where v_1 is a pendant node in τ_M and v_2 is a non-pendant node. We examine two possibilities:

- If v_1 is a node in H , then v_1v_2 must also belong to H . Thus, it corresponds to itself.
- If v_1 is not a node in H , then by maximality of H , v_2 must be a leaf in some star of H . Otherwise, we could add v_1v_2 to H , contradicting its maximality. In this case, v_1v_2 corresponds to the edge incident to v_2 in H .

Type 2. The component in M that are two edges share a common endpoint. Suppose that two edges v_1v_2, v_1v_3 form a component in M . By definition of τ_M , the nodes v_1, v_2, v_3 must belong to the same cycle $C \in \{\mathcal{C}_3, \mathcal{C}_4\}$ of τ_M . We assume that C is a 3-cycle (similar arguments apply to the case of 4-cycle). We consider the following possible cases.

- If C has two edges in H , then these two edges correspond to v_1v_2 and v_1v_3 .
- If C has exactly one edge in H . Assume, without loss of generality, that the edge is $v_2v_3 \in H$ and that v_2 is a center of a star in H . This implies that the edges v_1v_2, v_2v_3 are not in H . Since H is a maximal star forest, v_1 should be a node in H (otherwise, adding the edge v_1v_2 to H would contradict its maximality). Moreover, v_1 must be a leaf (otherwise, we could also add the edge v_1v_2 to H and H would remain a star forest). In this case, the two edges v_1v_2 and v_1v_3 in M correspond to the edge v_2v_3 and the edge e' , that is incident to v_1 in H (see Figure 5).

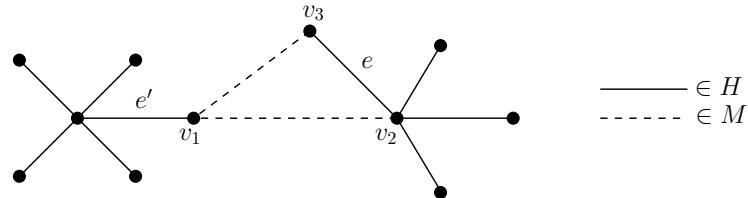


Fig. 5: The edges e, e' correspond to two edges $\{v_1v_2, v_1v_3\}$ in M when C has exactly one edge in H .

- If C has no edges in H , then by the maximality of H , at least two nodes of C must be the nodes in H since otherwise, we could add the edge connecting those two nodes to H and H remains a star forest. We now assume, without loss of generality, that v_2 and v_3 are the nodes in H . Then v_2 and v_3 should be two leaves belonging to two distinct stars in H . Indeed, if there is at least one such node—assume v_2 is a center or an isolated node in H , then we could add v_2v_1 to H and H would still be a star forest. This implies that the two edges v_1v_2 and v_1v_3 correspond to the edges e, e' , which are incident to v_2 and v_3 in H , respectively (see Figure 6).

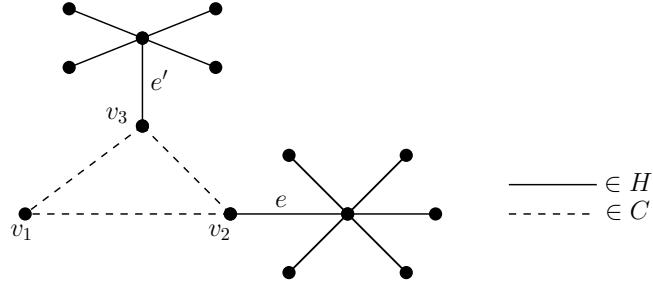


Fig. 6: The edges e, e' correspond to two edges $\{v_1v_2, v_1v_3\}$ in M when C has no edges in H .

Conclusion, $|H| \geq |M|$ and $|H| \leq |M|$, we then deduce that $|H| = |M|$. \square

To characterize the star forest polytope for 4-cactus graphs, we recall an important concept: *supported graph*. The support graph corresponding to the inequalities $a^t x \leq b$ is a subgraph of G containing all the edges with nonzero coefficients in $a^t x \leq b$. We now revisit the relationship between facet-defining inequalities and their associated support graphs, as shown in [2]. This relationship holds for any graph and directly applies to 4-cactus graphs.

Lemma 3. [2] Let $a^t x \leq b$ be a facet-defining inequality for $SFP(G)$, and let $G_a = (V_a, E_a)$ be its corresponding support graph. Then, every tight star forest associated with $a^T x \leq b$ is maximal with respect to E_a .

We now present the main result of this paper in the following theorem.

Theorem 3. Let G be a 4-cactus graph. Then, $SFP(G)$ is completely defined by the non-negativity inequalities and the M -cactus inequalities (1).

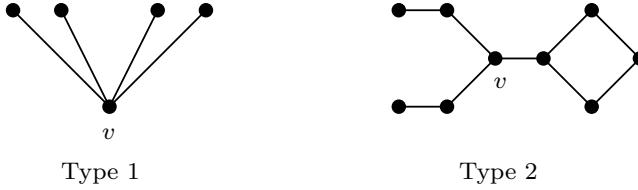
Proof. Assume that the inequality $a^t x \leq b$ defines a facet of $SFP(G)$ but is neither a non-negativity inequality nor a M -cactus inequality. Let G_a be the support graph associated with $a^t x \leq b$. We assume without loss of generality that G_a is a 4-cactus. G_a is thus a subgraph of G .

We now consider the case where G_a is a M -cactus graph. Consider any tight star forest S with respect to $a^t x \leq b$. By Lemma 3, S is a maximal subgraph of G_a satisfying the given inequality. Furthermore, Lemma 2 implies that S is also tight with respect to the M -cactus inequalities. This contradicts our assumption that $a^t x \leq b$ is a facet-defining inequality that is distinct from a M -cactus inequality. Hence, G_a cannot be a M -cactus graph.

Therefore, G_a must contain at least one non-pendant node v belonging to one of the following two categories (see Figure 7):

- Type 1. v is adjacent to at least two pendant nodes in G_a .
- Type 2. v does not belong to any cycle and is not connected to any pendant nodes in G_a .

We now analyze the two possible cases.

Fig. 7: An illustration of the two types of nodes v .

- *Case 1.* Every non-pendant node in G_a is of Type 1. This means that each such node is adjacent to at least two pendant nodes. In this situation, we modify G_a by keeping only one pendant node for each non-pendant node of Type 1, resulting in a M -cactus τ_M . Let M_τ be the M -2-matching associated with τ_M . Then, we have the corresponding M -cactus inequality $x(\tau_M) \leq |M_\tau|$.

Furthermore, observe that for any non-pendant node w of Type 1, any star forest S satisfying $a^t x \leq b$ at equality must be a maximal star forest in τ_M and contain either all or none of the pendant nodes connected to w . Let F be any tight star forest for $a^t x \leq b$. Then, $F \cap \tau_M$ is a maximal star forest in τ_M . By Lemma 2, every maximal star forest in τ_M satisfies the M -cactus inequality at equality. This implies that any star forest satisfying $a^t x \leq b$ at equality must also satisfy the corresponding M -cactus inequality at equality, leading to a contradiction.

- *Case 2.* G_a contains at least one non-pendant node of Type 2. We begin by proving the following claim.

Remark 1. There exists a non-pendant node p of Type 2 such that all the other non-pendant nodes of Type 2 belong to the same connected component obtained by removing p from G_a .

Proof. We prove this claim constructively. We begin by selecting any non-pendant node p_0 of Type 2 and setting $i = 0$. We define the iterative process as follows. At iteration i , let $G_1^i, \dots, G_{k_i}^i$ be the subgraphs obtained by removing p_i from G_a , and assume without loss of generality that G_1^i is the 4-cactus containing p_0 . Two cases arise:

- If all remaining non-pendant nodes of Type 2 are in a single 4-cactus G_{t_i} , then p_i satisfies the claim, and we stop.
- Otherwise, if Type 2 nodes exist in at least two 4-cactus graphs, assume $G_{k_i}^i$ is one such component. Choose p_{i+1} as an arbitrary non-pendant Type 2 node in $G_{k_i}^i$ and set $i \leftarrow i + 1$. This set of nodes forms the kernel.

Since each iteration adds a distinct node to the kernel, the process must terminate in at most $|V(G_a)|$ iterations, resulting in the required non-pendant node p of Type 2. \square

Let p be a non-pendant node of Type 2 that satisfies Remark 1 and let G_p be the connected component containing the other non-pendant Type 2 nodes,

obtained by removing p from G_a . Observe that p connects to G_p through at most two edges, and the subgraph S of G_a induced by the edges in $G_a \setminus G_p$ forms a 4-cactus (see Figure 8). It follows that S is either a M -cactus or a 4-cactus containing only Type 1 nodes. As demonstrated in Case 1, S includes a M -cactus τ_M , which includes all non-pendant nodes of S .

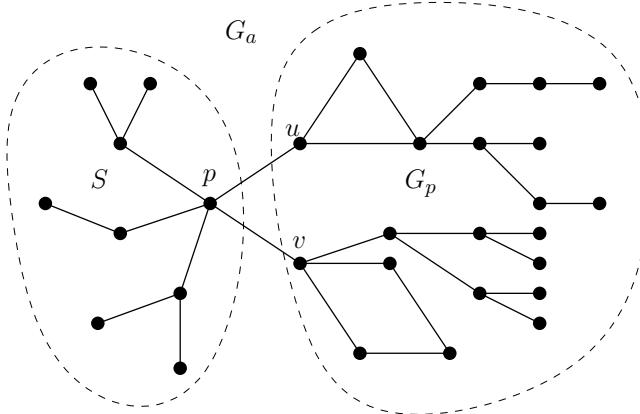


Fig. 8: The two subgraphs G_p and S of G_a .

Let u and v be neighbors of p in G_p . Note that u and v may coincide; however, the subsequent arguments remain unchanged. Thus, the edges pu and pv link S to G_p . Observe that any maximal star forest in G_a , whether it contains the edges pu and pv or not, must include a maximal star forest in S . By Case 1, this star forest contains a maximal star forest in τ_M . By Lemma 3, any star forest F satisfying $a^t x \leq b$ at equality is maximal in G_a , implying that $F \cap \tau_M$ is a maximal star forest in τ_M . By Lemma 2, $F \cap \tau_M$ satisfies the M -cactus inequality associated with τ_M at equality. This contradicts the assumption that $a^t x \leq b$ is facet-defining, which completes the proof. \square

4 Conclusions

In this paper, we provide a complete description of the star forest polytope for 4-cactus graphs. Our work extends previous polyhedral descriptions by introducing a new class of facet-defining inequalities. Furthermore, we demonstrate that the M -cactus inequalities define facets in the general case. Beyond 4-cactus graphs, our results provide useful insights that could aid in developing efficient approximation algorithms for the Maximum Weight Star Forest Problem in general graphs. Future research could leverage this polyhedral framework to improve combinatorial optimization techniques in broader graph structures.

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