Question 1

(a) Matrix form representation

As introduced, a state vector z_t that includes all relevant variables:

$$z_t = \begin{bmatrix} y_t \\ c_t \\ c_{t-1} \end{bmatrix}$$

The system of equations can be rewritten in matrix notation as:

$$\begin{bmatrix} y_t \\ c_t \\ c_{t-1} \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \beta_0 \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_3 & \beta_1 & \beta_2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ c_{t-1} \\ c_{t-2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_t \\ \nu_t \end{bmatrix}$$

Thus, in compact notation:

$$z_t = \mu + \mathrm{A} z_{t-1} + C\xi_t$$

Where:

State vector: $z_t = \begin{bmatrix} y_t \\ c_t \\ c_{t-1} \end{bmatrix}$

Constant vector: $\mu = \begin{bmatrix} \alpha_0 \\ \beta_0 \\ 0 \end{bmatrix}$

Transition matrix: $A = \begin{bmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_3 & \beta_1 & \beta_2 \\ 0 & 1 & 0 \end{bmatrix}$

Shock matrix: $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) Convert to an AR (1) representation

Now, we can express the system as an AR(1) process in the form:

$$z_t = Az_{t-1} + C\xi_t$$

Where:

The state vector remains: $z_t = \begin{bmatrix} y_t \\ c_t \\ c_{t-1} \end{bmatrix}$

The coefficient matrix A and shock matrix C remain the same as found in the part (a) above.

Thus, the AR(1) form of the system is:

$$\begin{bmatrix} y_t \\ c_t \\ c_{t-1} \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_3 & \beta_1 & \beta_2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ c_{t-1} \\ c_{t-2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_t \\ \nu_t \end{bmatrix}$$

This ensures that z_t follows a first-order vector autoregressive process.

Question 2

(a) Construct the transition matrix and validate

Given the information and as we have learned in the class, a Markov Chain is a stochastic process where the probability of moving to the next state depends only on the current state.

First, we identify the states, the Markov process categorizes borrowers into three states:

- Subprime (low credit quality)
- Average (medium credit quality)
- Prime (high credit quality)

Next, we extract probabilities from the diagram as provided.

The transition probabilities (probabilities of moving between states) are given in the diagram. Thus, the rows represent the current state and the columns represent the next state.

From ↓/ To →	**Subprime**	**Average**	**Prime**
Subprime	0.90	0.10	0.00
Average	0.20	0.65	0.15
Prime	0.10	0.05	0.85

Thus, the transition matrix is:

$$P = \begin{bmatrix} 0.90 & 0.10 & 0.00 \\ 0.20 & 0.65 & 0.15 \\ 0.10 & 0.05 & 0.85 \end{bmatrix}$$

To be continued, we validate the transition matrix

To be a valid Markov transition matrix, it must satisfy these conditions:

Each row sums to 1 (since probabilities must sum to 100%

- First row: $0.90 + 0.10 + 0.00 = 1.00$
- Second row: $0.20 + 0.65 + 0.15 = 1.00$
- Third row: $0.10 + 0.05 + 0.85 = 1.00$

And all probabilities are between 0 and 1, which means no negative values or values greater than 1. So, all values satisfy this condition.

In conclusion, this is a valid transition matrix because: (1) Every row sums to exactly 1 and (2) All probabilities lie within the range [0,1].

(b) Explaining the Markov diagram to a non-economist

Although this explanation is meant for someone without financial knowledge, we need to use simple, intuitive language.

First, we define the key idea

For example, we can talk about *"Imagine of thinking of this as a system that can tracks how borrowers move between different credit ratings over time".*

- Prime borrowers are people with excellent credit scores.
- Average borrowers have decent but not perfect credit.
- Subprime borrowers have poor credit histories.

Thus, lenders track how likely it is that someone will stay in their current category or move to another one based on their financial habits.

Next, we will explain the probabilities in simple terms.

If you have a subprime credit score (low quality), it can:

- There is a 90% chance you will stay subprime.
- There is a 10% chance you will improve to average credit.
- You cannot directly jump to prime status.

If you have an average credit score.

- You have a 65% chance of staying average.
- A 20% chance of falling to subprime (if you miss payments or increase debt).
- A 15% chance of improving to prime.

If you have a prime credit score (high quality)

- You have an 85% chance of staying prime.
- A 10% chance of dropping to subprime (if you default on loans or take too much debt).
- A 5% chance of dropping to average.

Next, we should provide a real-life example that helps someone understand easily.

Imagine you have a friend, Lina, who applies for a credit card.

- If Lina starts as a subprime borrower, she is very likely to stay subprime (90% chance).
- If Lina works on improving her credit (by paying bills on time), she might move up to average (10% chance).

- Once she becomes average, she has a 15% chance of becoming a prime borrower.
- If Lina becomes reckless with spending, she might drop back to subprime.

In conclusion, this Markov process shows how borrowers' credit scores evolve over time. Moreover, the longer someone maintains good financial behavior, the more likely they are to move up. Bad financial habits increase the chance of falling into subprime credit; it makes loans harder to get.

Question 3

Rouwenhorst's method is used to approximate a continuous AR(1) process as a discrete Markov Chain.

Step 1: We define inputs

- Set the number of states (N)
- Define persistence parameter (γ_1)
- Compute the standard deviation of the stationary distribution

Step 2: We create state space

- Construct a grid of N points centered around 0
- Compute the spacing between states

Step 3: We initialize transition matrix

- If N=2, define base transition matrix
- If N>2, use recursion to construct transition probabilities

Step 4: We normalize the transition matrix

- Ensure each row sums to 1

Step 5: Return state space and transition matrix

Thus, this pseudo-code provides the logical structure of the method.

Part (b), (c) and (d) please explore GitHub to see the result and explanation.

Question 4

(a) Writing the per-period utility function

First, define the cost function

As given information, we have the operational cost function is:

$$c(z_t) = a(z_t) + b(z_t)$$

Where:

$a(z_t)$ (Maintenance cost) which increases with age (z_t)

$b(z_t)$ (Breakdown cost) which means a random variable, but also increasing in (z_t) (For example: Older tanks have a higher breakdown risk).

Next, I am going to define the military's per-period utility function.

In particular, the military chooses between two options every year:

To keep the existing tank leads to paying maintenance and breakdown costs:

$$U_{keep}(z_t) = -c(z_t) = -(a(z_t) + b(z_t))$$

To replace the tank, leads to pay the full replacement cost but get a new tank:

$$U_{replace} = -(D_t + c(0))$$

So, the military's per-period utility function (cost-minimizing decision rule) is:

$$U(z_t) = \min\{-c(z_t), -\big(D_t + c(0)\big)\}$$

- In case $c(z_t)$ (keeping the tank) is lower than $D_t + c(0)$ (buying a new one), the military will keep the tank.
- In case $c(z_t)$ becomes too high, it is optimal to replace the tank.

Moreover, aging tanks have rising costs, which lead to eventual replacement.

Thus, per-period utility function is:

$$U(z_t) = \min\{-c(z_t), -\big(D_t + c(0)\big)\} = \min\{-(a(z_t) + b(z_t)), -\big(D_t + c(0)\big)\}$$

(b) Writing the Bellman equation

In this part, we will formulate the Bellman equation for the military's dynamic decision-making process in maintaining or replacing tanks accordingly.

First, define the state, control and uncertainty

To properly set up the Bellman equation, we need to define the relevant components to find the information:

With state variable z_t: The age of the tank (number of years in service)

With control variable d_t: The military's decision and we have two scenarios

- $d_t = 0, \mathit{which\ keep\ the\ existing\ tank}$
- $d_t = 1, \mathit{which\ replace\ the\ tank\ with\ a\ new\ one}$

With uncertainty ϵ_t: Random taste shocks affecting decisions.

Next, going to formulating the Bellman equation

Since the military minimizes costs by choosing whether to keep or replace the tank as mentioned. The Bellman equation captures this optimization:

$$V(z_t) = \min\{c(z_t) + BE[V(z_{t+1})], D_t + c(0) + \beta E[V(0)]\}$$

Where:

$V(z_t)$: The value function, which representing the minimum expected lifetime cost when the tank is of age z_t.

In case, if the military keeps the tank, we have:

- Pays operational cost $c(z_t)$
- Future state is $z_{t+1} = z_t + 1$ (if not replaced)
- Future cost is discounted by β, the discount factor.

In case, if the military replaces the tank, we have:

- Pays replacement cost $D_t + c(0)$
- Future state resets to $z_{t+1} = 0$

Next, incorporating taste shocks ϵ_t

The decision is subject to random shocks ϵ_t, which can affect preferences:

$$V(z_t) = \min\{c(z_t) + \beta E[V(z_{t+1})] + \epsilon_{keep}, D_t + c(0) + \beta E[V(0)] + \epsilon_{replace}\}$$

Where:

$\epsilon_{keep}, \epsilon_{replace} \sim i.i.d$, which captures random external factors (For example budget constraints and unexpected breakdowns).

Moreover, to interpret economic.

If $c(z_t) + \beta E[V(z_{t+1})] < D_t + c(0) + \beta E[V(0)]$, which the military keeps the tank

If the replacement costs are lower than in expectation, which leading to the military replaces the tank.

The probability of replacement increases with age, as $c(z_t)$ rises over time.

And taste shocks ϵ_t introduce randomness, it makes the replacement decision probabilistic.

Thus, Bellman equation:

$$V(z_t) = \min\{c(z_t) + \beta E[V(z_{t+1})] + \epsilon_{keep}, D_t + c(0) + \beta E[V(0)] + \epsilon_{replace}\}$$

(c) Describing the transition probabilities for z_t

First, we need to understand the state transition

In particular, the state variable z_t, which represent the age of a tank. The military decides whether to replace or keep the tank, which affects its transition probability.

We can consider two possible transitions:

If the tank is not replaced ($d_t = 0$):

- The tank ages by one year, thus: $z_{t+1} = z_t + 1$
- This happens with probability $1 - p_r(z_t)$, where $p_r(z_t)$ is the probability of replacement.

If the tank is replaced ($d_t = 1$):

- The military purchases a new tank, resetting its age: $z_{t+1} = 0$
- This happens with probability $p_r(z_t)$, which means the probability of replacement.

Next, defining the transition probabilities

The transition probabilities are:

$$P(z_{t+1}|z_t, d_t = \begin{cases} 1 - p_r(z_t), if\ z_{t+1} = z_t + 1 (tank\ is\ kept) \\ p_r(z_t), if\ z_{t+1} = 0\ (tank\ is\ replaced) \end{cases}$$

Where:

$p_r(z_t)$ increases with age, which means older tanks are more likely to be replaced.

$p_r(z_t)$ could be modeled as a function of age, such as:

$$p_r(z_t) = \frac{c(z_t)}{c(z_t) + D_t}$$

Where higher operational costs lead to higher replacement probability.

Next, when z_t reaches maximum age:

Tanks cannot last forever and after reaching a critical age Z_{max}, the tank must be replaced.

Thus, for $z_t = Z_{max}$:

$$P(z_{t+1} = 0|z_t = Z_{max}) = 1$$

Which means the tank must be replaced with certainty

In conclusion, interpret economic:

If z_t is small, the military is less likely to replace the tank as maintenance costs are still low.

As z_t increases, maintenance and breakdown costs increase, it makes replacement more likely.

At Z_{max} replacement is inevitable, which ensures the fleet is modernized over time.

This transition structure ensures a continuous decision process, where old tanks are gradually phased out.

Thus, transition probabilities is:

$$P(z_{t+1}|z_t, d_t) = \begin{cases} 1 - p_r(z_t), if\ z_{t+1} = z_t + 1 (tank\ is\ kept) \\ p_r(z_t), if\ z_{t+1} = 0\ (tank\ is\ replaced) \end{cases}$$

Where:

$p_r(z_t)$ increases with age

$P(z_{t+1} = 0|Z_{max}) = 1$ ensures full replacement at maximum age.

Question 5

(a) Ho Chi Minh City (HCMC) is Vietnam's largest economic hub, contributing approximately 16% of the country's GDP and accounting for 19-20% of national economic growth. As a globalized financial center, HCMC is a key driver of Vietnam's foreign investment, trade, and services sector. The city is home to over 320,000 registered businesses, making it the country's most dynamic commercial center (Trinh, 2024).

Key economic growth indicators (Time series data)

Indicator	**2019**	**2020**	**2021**	**2022**	**2023**	**Source**
GRDP Growth (%)	7.81	1.39	-6.78	9.03	5.81	GSO Vietnam
GRDP per capita (USD)	6,430	6,110	5,600	6,800	7,500	GSO Vietnam
FDI Inflows (Billion USD)	8.3	4.4	3.6	5.9	4.43	MPI Vietnam
Unemployment Rate (%)	3.0	3.6	6.29	3.9	3.3	CEIC Data
Inflation Rate (CPI, %)	3.2	2.8	2.3	3.8	4.2	GSO Vietnam
Export Value (Billion USD)	44.4	38.7	39.5	46.2	47.9	GSO Vietnam

For economic structure of HCMC

- With services sector: HCMC dominates the economy, which contribute 64% of GRDP and specialize in finance, real estate and retail.

- With industry and manufacturing: The city represents 30.5% of GRDP, which focus on electronics, textiles and high-tech industries.
- With agriculture: This city has a minor sector at 0.5%, which reflecting the city's urbanization.

For economic trends and challenges

- With recovery post-pandemic: HCMC has rebounded with 9.03% growth in 2022 after a sharp -6.78% contraction in 2021 and slowed to 5.81% in 2023.
- With rising inflation and costs: The city has a CPI increased to 4.2% in 2023 and this driven by rising housing and food prices.
- With FDI and investment growth: HCMC has remain Vietnam's top FDP destination, which securing $4.43 billion in 2023.

Ho Chi Minh City continues to be Vietnam's economic engine, leading in GRDP, FDI, and exports. In contrast, inflation, rising costs, and post-pandemic recovery had challenged impact its long-term growth trajectory. HCMC's continued focus on infrastructure, innovation, and foreign investment will shape its future development.

(b) Several macroeconomic and structural factors contribute to HCMC's growth, it includes investment, the labor market, technological advancement, and infrastructure.

FDI and capital inflows

- FDI accounts for approximately 40% of Vietnam's total inflows, with HCMC also attracting $4.43 billion in 2023.
- Major sectors receiving FDI with high-tech, real estate, export-oriented manufacturing, and real estate.
- However, it makes more challenges, such as high costs of land and infrastructure constraints.

This policy should be used (1) Investment incentives with tax breaks and streamlined business registration for foreign investors. (2) Infrastructure expansion with upgrading industrial zones and transport networks.

Industrialization and manufacturing growth

HCMC is a top manufacturing hub in Vietnam, especially electronics, textiles and machinery leading exports. Moreover, the manufacturing sector which contributes approximately 30.5% of GRDP and supporting jobs and supply chain networks. However, it makes reliance on imported raw material affects productivity and export competitiveness.

This policy should be used (1) Develop local supply chains by promoting domestic raw material production to help reduce dependency on imports. (2) Boost R&D investment by increasing funding for automation and high-tech manufacturing.

Labor market and workforce development

With a labor force of 5 million, HCMC has a skill gaps exist in high-tech and digital sectors. Unemployment peaked at 6.29% in 2021 but declined to 3.3% in 2023 which showing recovery (Tuyet, 2023).

This policy should be used (1) Expand vocational training by increasing funding for skill development programs (2) Tech-driven workforce by strengthening partnerships between universities and industries for innovation.

Infrastructure and smart city development

Recently, HCMC has Metro Line aims to reduce congestion and boost urban mobility. Moreover, smart city projects focus on AI, 5G makes better governance and efficiency. However, traffic congestion and housing shortages reduce productivity.

This policy should be used (1) Expand public transport by boosting fast-track Metro expansion and electric bus networks (For example using electric bus in D2). (2) Digital governance with implement AI for urban planning and traffic control.

Inflation and cost of living

HCMC has inflation rose to 4.2% in 2023 which driven by housing, food and energy prices. Rising costs impact consumer spending and business operations.

This policy should be used (1) Subsidies on essential goods with control price surges for food and transport. (2) Monetary policy with regulate interest rates to stabilize inflation.

References

Trinh, K. (2024, December 31). Ho Chi Minh City confidently rises in the new era. *Nhan Dan Online*. https://en.nhandan.vn/ho-chi-minh-city-confidently-rises-in-the-new-era-post142835.html

General Statistics Office of Vietnam. (2023). *Key economic growth indicators: Time series data for Ho Chi Minh City (2019-2023)* [Unpublished data set]. General Statistics Office of Vietnam.

Ministry of Planning and Investment of Vietnam. (2023). *Foreign direct investment in Vietnam, 2019-2023* [Unpublished data set]. Ministry of Planning and Investment of Vietnam.

CEIC Data. (2023). *Vietnam key economic indicators, 2019-2023* [Data set]. CEIC. https://www.ceicdata.com/en/country/vietnam

Vinh H. V.-. H. (2024, September 25). HCM City's hi-tech industries expected to account for 40% of GRDP. *Nhịp Sống Kinh Tế Việt Nam & Thế Giới*. https://vneconomy.vn/hcm-citys-hi-tech-industries-expected-to-account-for-40-of-grdp.htm

Tuyet, L. (2023, November 6). Number of unemployment benefit applicants rises 11% in HCMC. *VnExpress International – Latest News, Business, Travel and Analysis From Vietnam*. https://e.vnexpress.net/news/business/economy/number-of-unemployment-benefit-applicants-rises-11-in-hcmc-4673234.html