Proof of Theorem 1

Channel Characterization of UAV-RIS-aided Systems with Adaptive phase shift Configuration

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The CDF of the G2A SINR in the paper can be derived as

$$F_{\Gamma^{\star}_{\mathsf{g2a}}}(x) = \Pr\left[\frac{\bar{\gamma}_{\mathsf{g2a}}\gamma_{\mathsf{g2a}}}{\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1} < x\right] = \Pr\left[\gamma_{\mathsf{g2a}} < \frac{x}{\bar{\gamma}_{\mathsf{g2a}}}(\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1)\right] \tag{1}$$

$$= \mathbb{E}_{\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1}\left\{F_{\gamma_{\mathsf{g2a}}}\left(\frac{x}{\bar{\gamma}_{\mathsf{g2a}}}(\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1)\right)\right\}, \tag{2}$$

where $F_{\gamma_{\mathsf{g2a}}}(x)$ can be written using the proposed Lemma as

$$F_{\gamma_{\mathsf{g2a}}}(x) = 1 - \sum_{k=0}^{M\psi_{\mathrm{SU}}-M}\sum_{n=0}^{M\psi_{\mathrm{SU}}-k-1}\frac{\chi_{M,k}}{n!}\frac{x^n}{\alpha_{\mathsf{g2a}}^n}e^{-\frac{x}{\alpha_{\mathsf{g2a}}}}, x > 0. \tag{3}$$

Here, $\alpha \triangleq \alpha_{\mathsf{g2a}}$ is determined in the proposed Lemma.

Plugging the above CDF into (2) and using the linearity property of the expectation operators, i.e., $\mathbb{E}\{aX+bY\} = a\mathbb{E}\{X\}+b\mathbb{E}\{Y\}$, we obtain

$$F_{\Gamma^{\star}_{\mathsf{g2a}}}(x) = 1 - \sum_{k=0}^{M\psi_{\mathrm{SU}}-M}\sum_{n=0}^{M\psi_{\mathrm{SU}}-k-1}\frac{\chi_{M,k}}{n!}\left(\frac{x}{\alpha_{\mathsf{g2a}}\bar{\gamma}_{\mathsf{g2a}}}\right)^n \times \mathbb{E}_{\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1}\left\{(\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1)^n e^{-\frac{x}{\alpha_{\mathsf{g2a}}\bar{\gamma}_{\mathsf{g2a}}}(\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1)}\right\}. \tag{4}$$

Using the definition of expectation operator, where $\mathbb{E}\{g(X)\} = \int_{-\infty}^{\infty} g(x)f_X(x)\mathrm{d}x$ with $f_X(x)$ being the PDF of X, we obtain

$$F_{\Gamma^{\star}_{\mathsf{g2a}}}(x) = 1 - \sum_{k=0}^{M\psi_{\mathrm{SU}}-M}\sum_{n=0}^{M\psi_{\mathrm{SU}}-k-1}\frac{\chi_{M,k}}{n!}\left(\frac{x}{\alpha_{\mathsf{g2a}}\bar{\gamma}_{\mathsf{g2a}}}\right)^n\int_0^{\infty}\gamma^n e^{-\frac{x}{\alpha_{\mathsf{g2a}}\bar{\gamma}_{\mathsf{g2a}}}\gamma} f_{\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1}(\gamma)\mathrm{d}\gamma. \tag{5}$$

It is noted that

$$\int_0^{\infty}\gamma^n e^{-\frac{x}{\alpha_{\mathsf{g2a}}\bar{\gamma}_{\mathsf{g2a}}}\gamma} f_{\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1}(\gamma)\mathrm{d}\gamma = (-1)^n\left[\frac{\mathrm{d}^n}{\mathrm{d}s^n}\int_0^{\infty}e^{-s\gamma}f_{\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1}(\gamma)\mathrm{d}\gamma\right]\Bigg|_{s\to s_0}, \tag{6}$$

where $s_0 \triangleq \frac{x}{\alpha_{\mathrm{g2a}}\bar{\gamma}_{\mathrm{g2a}}}$. In addition, the integral $\int_0^\infty e^{-s\gamma} f_{\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1}(\gamma)\mathrm{d}\gamma$ specifies the Laplace transform of $\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1$, denoted as $\mathfrak{L}_{\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1}(s)$. Since γ_{IU} and γ_{JU} are statistically independent, we have

$$\mathfrak{L}_{\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1}(s) = e^{-s}\mathfrak{L}_{\gamma_{\mathrm{IU}}}(s)\mathfrak{L}_{\gamma_{\mathrm{JU}}}(s), s>0. \tag{7}$$

Furthermore, $\gamma_{\mathrm{IU}} = \sum_{l=1}^{L} \gamma_{\mathrm{I}_l\mathrm{U}}$ and $\gamma_{\mathrm{JU}} = \sum_{k=1}^{K} \gamma_{\mathrm{J}_k\mathrm{U}}$, where $\gamma_{\mathrm{I}_l\mathrm{U}}$ and $\gamma_{\mathrm{J}_k\mathrm{U}}$ are statistically mutually independent, we obtain that

$$\begin{aligned}\mathfrak{L}_{\gamma_{\mathrm{IU}}+\gamma_{\mathrm{JU}}+1}(s) &= e^{-s}\prod_{l=1}^{L}\mathfrak{L}_{\gamma_{\mathrm{I}_l\mathrm{U}}}(s)\prod_{k=1}^{K}\mathfrak{L}_{\gamma_{\mathrm{J}_k\mathrm{U}}}(s) &(8)\\ &= e^{-s}\prod_{l=1}^{L}\frac{e^{-K_{\mathrm{I}_l\mathrm{U}}+\frac{K_{\mathrm{I}_l\mathrm{U}}}{1+\bar{\gamma}_{\mathrm{I}_l\mathrm{U}}s}}}{1+\bar{\gamma}_{\mathrm{I}_l\mathrm{U}}s}\prod_{k=1}^{K}\frac{e^{-K_{\mathrm{J}_k\mathrm{U}}+\frac{K_{\mathrm{J}_k\mathrm{U}}}{1+\bar{\gamma}_{\mathrm{J}_k\mathrm{U}}s}}}{1+\bar{\gamma}_{\mathrm{J}_k\mathrm{U}}s}, s>0. &(9)\end{aligned}$$

With (9), (6), and (4), we rewrite the CDF of G2A SINR in the paper as follows:

$$\begin{aligned}F_{\Gamma^\star_{\mathrm{g2a}}}(x) = 1 - &\sum_{k=0}^{M\psi_{\mathrm{SU}}-M}\sum_{n=0}^{M\psi_{\mathrm{SU}}-k-1}\chi_{M,k}(-s_0)^n\\ &\times\frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}s^n}e^{-s}\prod_{l=1}^{L}\mathfrak{L}_{\gamma_{\mathrm{I}_l\mathrm{U}}}(s)\prod_{k=1}^{K}\mathfrak{L}_{\gamma_{\mathrm{J}_k\mathrm{U}}}(s)\right]\Bigg|_{s\to s_0}.\end{aligned} \tag{10}$$

In (10) , the n-th order derivatives, denoted as $\Delta_{\mathrm{U}}^{(n)}(s)$, is required to derive the closed-form expression of the above CDF. The derivation of $\Delta_{\mathrm{U}}^{(n)}(s)$ is quite lengthy and not straightforward; thus, it is omitted in the letter. In recursive form, we have

$$\begin{aligned}\Delta_{\mathrm{U}}^{(n)}(s) &= \frac{\mathrm{d}^n}{\mathrm{d}s^n}e^{-s}\mathfrak{L}_{\gamma_{\mathrm{IU}}}(s)\mathfrak{L}_{\gamma_{\mathrm{JU}}}(s) &(11)\\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}\Delta_{\mathrm{U}}^{(n-k-1)}(s)S_{k+1}(s), &(12)\end{aligned}$$

where $\Delta_{\mathrm{U}}^{(0)}(s) = e^{-s}\mathfrak{L}_{\gamma_{\mathrm{IU}}}(s)\mathfrak{L}_{\gamma_{\mathrm{JU}}}(s)$. In closed-form expression, we have that

$$\begin{aligned}\Delta_{\mathrm{U}}^{(n)}(s) &= \Delta_{\mathrm{U}}^{(0)}(s)\sum_{r=0}^{n}\widetilde{\sum_{r,n}}\frac{n!}{\prod_{i=1}^{r}p_i!}\prod_{i=1}^{\varrho(\boldsymbol{\Sigma})}\frac{(-S_{p_{\langle i\rangle}}(s))^{\nu_i}}{\nu_i!} &(13)\\ &= \Delta_{\mathrm{U}}^{(0)}(s)\left(1+\sum_{r=1}^{n}\widetilde{\sum_{r,n}}\frac{n!}{\prod_{i=1}^{r}p_i!}\prod_{i=1}^{\varrho(\boldsymbol{\Sigma})}\frac{(-S_{p_{\langle i\rangle}}(s))^{\nu_i}}{\nu_i!}\right). &(14)\end{aligned}$$

Proof: First, we define the following functions

$$\delta_n(a,c,s) = (n-1)!\frac{(-c)^n}{(1+cs)^n}\left(1+\frac{na}{1+cs}\right), \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\delta_n(a,c,s) = \delta_{n+1}(a,c,s), \tag{16}$$

$$S_n(s) \triangleq \sum_{l=1}^{L} \delta_n(K_{\mathrm{I}_l\mathrm{U}}, \lambda_{\mathrm{I}_l\mathrm{U}}, s) + \sum_{k=1}^{K} \delta_n(K_{\mathrm{J}_k\mathrm{U}}, \lambda_{\mathrm{J}_k\mathrm{U}}, s) - C_n, \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}s} S_n(s) = S_{n+1}(s), \tag{18}$$

where $C_1 = 1$ and $C_{n\geq 2} = 0$.

By using the Leibniz rule of derivative, we derive $\Delta_{\mathrm{U}}^{(1)}(s)$ as

$$\begin{aligned}
\Delta_{\mathrm{U}}^{(1)}(s) = & -e^{-s}\mathfrak{L}_{\gamma_{\mathrm{IU}}}(s)\mathfrak{L}_{\gamma_{\mathrm{JU}}}(s) \\
& + e^{-s}\mathfrak{L}_{\gamma_{\mathrm{JU}}}(s)\sum_{l=1}^{L}\mathfrak{L}_{\gamma_{\mathrm{I}_l\mathrm{U}}}(s)\frac{-\lambda_{\mathrm{I}_l\mathrm{U}}}{1+\lambda_{\mathrm{I}_l\mathrm{U}}s}\left(1+\frac{K_{\mathrm{I}_l\mathrm{U}}}{1+\lambda_{\mathrm{I}_l\mathrm{U}}s}\right)\prod_{t=1,t\neq l}^{L}\mathfrak{L}_{\gamma_{\mathrm{I}_t\mathrm{U}}}(s) \\
& + e^{-s}\mathfrak{L}_{\gamma_{\mathrm{IU}}}(s)\sum_{k=1}^{K}\mathfrak{L}_{\gamma_{\mathrm{J}_k\mathrm{U}}}(s)\frac{-\lambda_{\mathrm{J}_k\mathrm{U}}}{1+\lambda_{\mathrm{J}_k\mathrm{U}}s}\left(1+\frac{K_{\mathrm{J}_k\mathrm{U}}}{1+\lambda_{\mathrm{J}_k\mathrm{U}}s}\right)\prod_{t=1,t\neq k}^{K}\mathfrak{L}_{\gamma_{\mathrm{J}_t\mathrm{U}}}(s)
\end{aligned} \tag{19}$$

$$= \Delta_{\mathrm{U}}^{(0)}(s)\left\{\sum_{l=1}^{L}\delta_1(K_{\mathrm{I}_l\mathrm{U}}, \lambda_{\mathrm{I}_l\mathrm{U}}, s) + \sum_{k=1}^{K}\delta_1(K_{\mathrm{J}_k\mathrm{U}}, \lambda_{\mathrm{J}_k\mathrm{U}}, s) - 1\right\} \tag{20}$$

$$= \Delta_{\mathrm{U}}^{(0)}(s)S_1(s), \tag{21}$$

Then, $\Delta_{\mathrm{U}}^{(2)}(s)$, $\Delta_{\mathrm{U}}^{(3)}(s)$, and $\Delta_{\mathrm{U}}^{(4)}(s)$ can be derived as

$$\Delta_{\mathrm{U}}^{(2)}(s) = \Delta_{\mathrm{U}}^{(1)}(s)S_1(s) + \Delta_{\mathrm{U}}^{(0)}(s)S_2(s), \tag{22}$$

$$\Delta_{\mathrm{U}}^{(3)}(s) = \Delta_{\mathrm{U}}^{(2)}(s)S_1(s) + 2\Delta_{\mathrm{U}}^{(1)}(s)S_2(s) + \Delta_{\mathrm{U}}^{(0)}(s)S_3(s), \tag{23}$$

$$\Delta_{\mathrm{U}}^{(4)}(s) = \Delta_{\mathrm{U}}^{(3)}(s)S_1(s) + 3\Delta_{\mathrm{U}}^{(2)}(s)S_2(s) + 3\Delta_{\mathrm{U}}^{(1)}(s)S_3(s) + \Delta_{\mathrm{U}}^{(0)}(s)S_4(s). \tag{24}$$

By induction, we can prove that

$$\Delta_{\mathrm{U}}^{(n)}(s) = \sum_{k=0}^{n-1}\binom{n-1}{k}\Delta_{\mathrm{U}}^{(n-k-1)}(s)S_{k+1}(s). \tag{25}$$

Assuming that (25) is correct, we need to prove that

$$\Delta_{\mathrm{U}}^{(n+1)}(s) = \sum_{k=0}^{n}\binom{n}{k}\Delta_{\mathrm{U}}^{(n-k)}(s)S_{k+1}(s). \tag{26}$$

We have that

$$\Delta_{\mathrm{U}}^{(n+1)}(s) = \frac{\mathrm{d}}{\mathrm{d}s}\Delta_{\mathrm{U}}^{(n)}(s) = \sum_{k=0}^{n-1}[\Delta_{\mathrm{U}}^{(n-k)}S_{k+1}(s) + \Delta_{\mathrm{U}}^{(n-k-1)}S_{k+2}(s)], \tag{27}$$

$$\begin{aligned}
& = \binom{n-1}{0}\Delta_{\mathrm{U}}^{(n)}S_1(s) + \binom{n-1}{n-1}\Delta_{\mathrm{U}}^{(0)}S_{n+1}(s) \\
& \quad + \sum_{k=1}^{n-1}\left[\binom{n-1}{k} + \binom{n-1}{k-1}\right]\Delta_{\mathrm{U}}^{(n-k)}(s)S_{k+1}(s)
\end{aligned} \tag{28}$$

$$\stackrel{(a)}{=} \Delta_{\mathrm{U}}^{(n)} S_1(s) + \Delta_{\mathrm{U}}^{(0)} S_{n+1}(s) + \sum_{k=1}^{n-1} \binom{n}{k} \Delta_{\mathrm{U}}^{(n-k)}(s) S_{k+1}(s), \tag{29}$$

where (a) is due to the fact that $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$. It is noted that (29) equals (26). By induction, we conclude that (25) is correct. In addition, with induction, we also achieve the closed-form expression of $\Delta_{\mathrm{U}}^{(n)}$ in (14). This completes the proof. ■