HA NOI UNIVERSITY OF SCIENCE AND TECHNOLOGY
SCHOOL OF INFORMATION AND COMMUNICATION TECHNOLOGY

Computer Vision

Chapter 3.2: Image transform

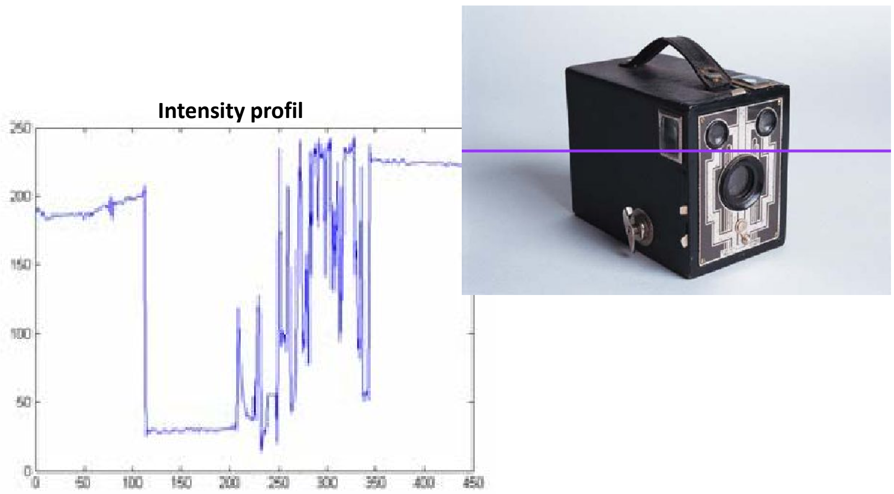
Computer Vision

Chapter 3.2 Image transform

Content

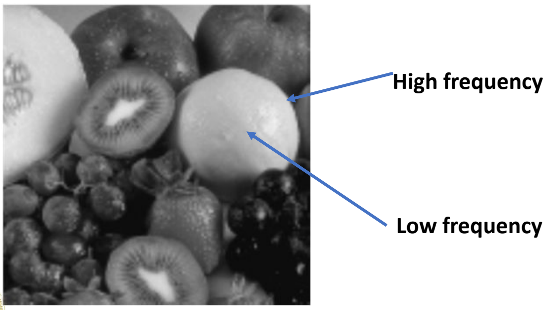
- Rappel: digital image representation
- Point Processing
- Convolution and Linear filtering
- More neighborhood operators
- Image transforms
  - Frequency domain
    - Frequencies in images
    - Fourier transform
    - Frequential Processing (frequential filters)
  - PCA (additional reading)

Frequencies in images



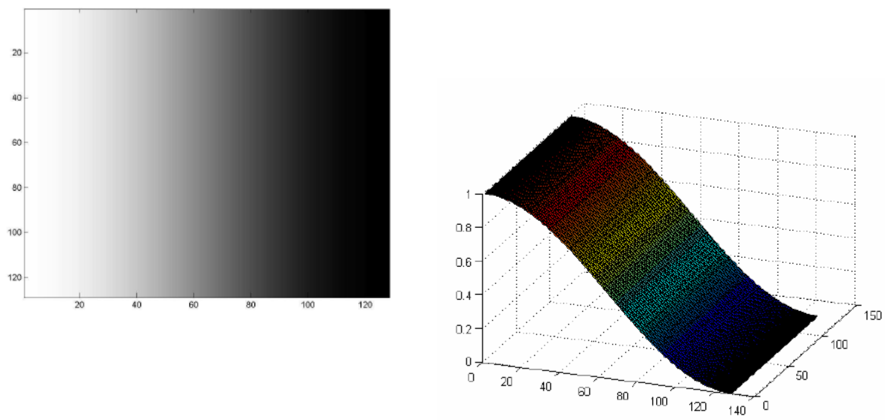
Frequencies in images

- What are the (low/high) frequencies in an image?
  - Frequency **= intensity change**
  - Slow changes (homogeneous /blur regions): low frequency
  - fast/abrupt changes (egde, contour, noise): high frequency

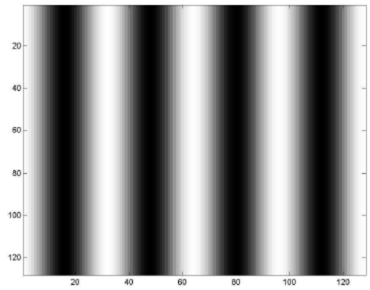


Most of energy concentrated in low frequencies

Low frequencies



High frequencies



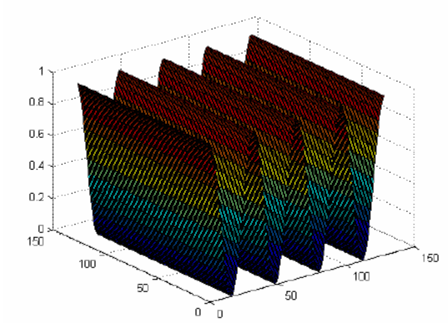


Image spectral analysis

- An image is a visual signal
  - We can analyse the frequencies of the signal
- How?
  - we will create a new « image » which will contains all frequencies of the image
    - Like a 2D frequency graphic
  - The basic tool for it is the **Fourier Transform**

- We talk about the **frequency domain**, opposing to the **spatial domain** (image)

Frequencies in a signal

High frequency signal →

...

Low frequency signal →

Sum of all the above signals →

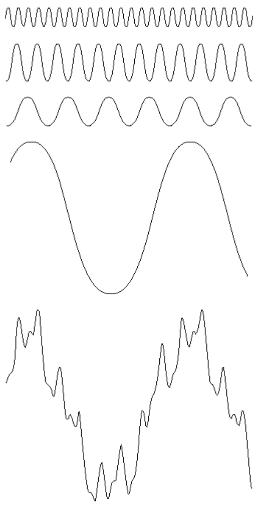


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Source : *Gonzalez and Woods. Digital Image Processing. Prentice-Hall, 2002.*

Fourier series

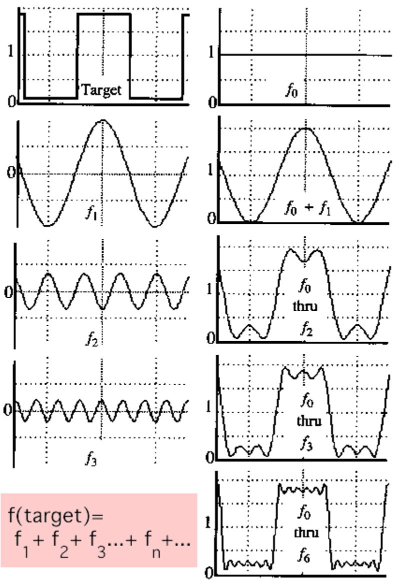
A bold idea (1807) - Jean Baptiste Joseph Fourier (1768-1830):

***Any** univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.*

Our building block:

$$A\sin(\omega t) + B\cos(\omega t)$$

Add enough of them to get any signal *g(t)* you want!



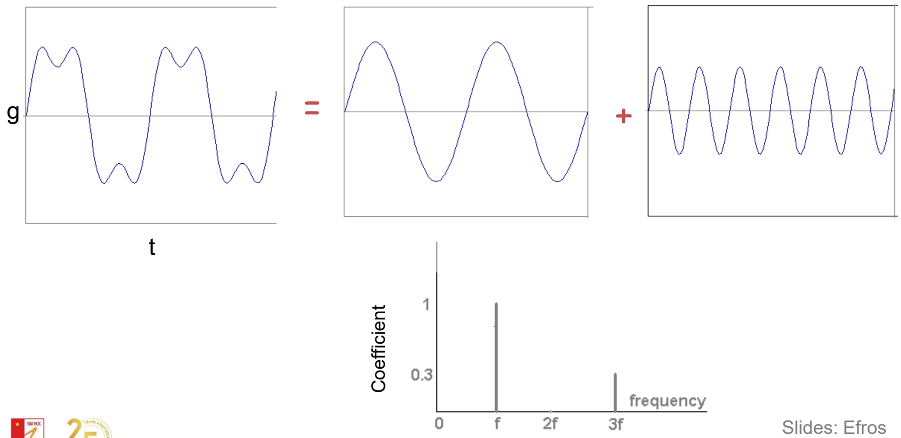
f(target)= $f_1 + f_2 + f_3 ... + f_n + ...$

Hays

Example

$t = [0,2], f = 1$

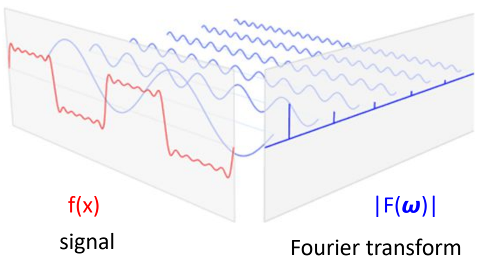
$g(t) \quad = \quad \sin(2\pi f\, t) \quad + (1/3)\sin(2\pi(3f)\, t)$



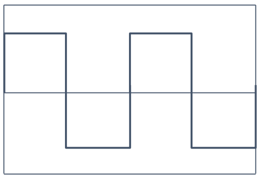
Slides: Efros

Fourier Transform

- Fourier transform is a mathematical transform that
  - Decomposes functions depending on space or time into functions depending on spatial or temporal frequency



Fourier Series



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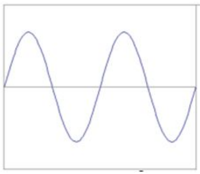
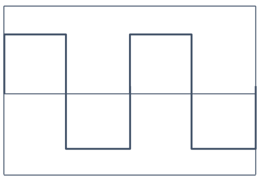


We want to get this function

Slide by Alexei A. Efros

Fourier Series



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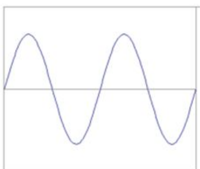
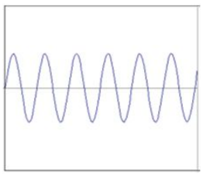


+



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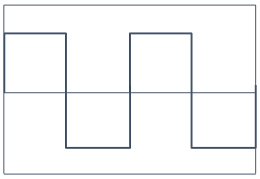
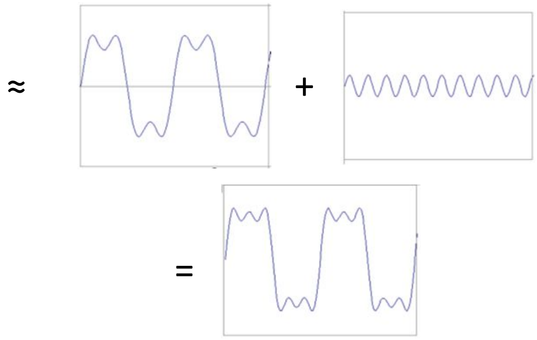
We want to get this function

Slide by Alexei A. Efros

Fourier Series



We want to get this function



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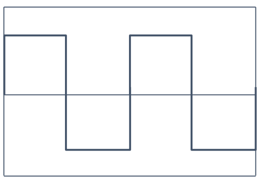
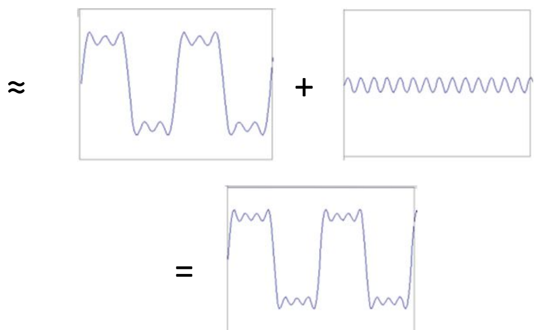
Fourier Series

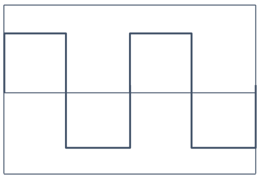


We want to get this function

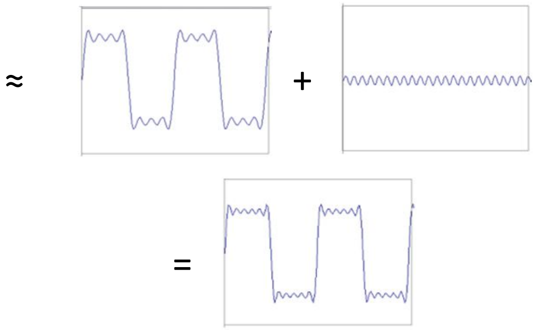


Slide by Alexei A. Efros

Fourier Series

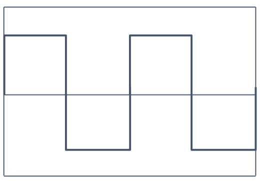


We want to get this function

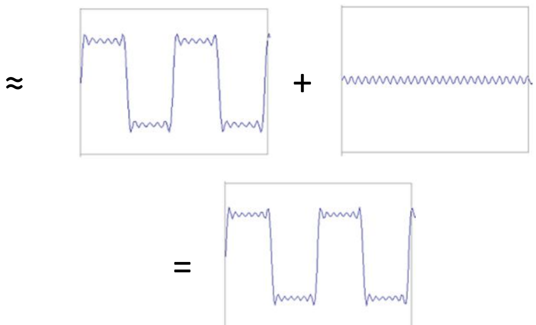


Slide by Alexei A. Efros

Fourier Series

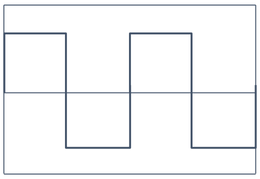


We want to get this function



Slide by Alexei A. Efros

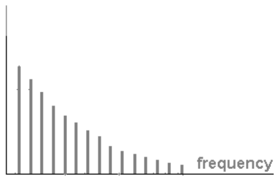
Fourier Series



We want to get this function

$$= A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)$$

We'll get there in the limit



Slide by Alexei A. Efros

The math

$$\text{Fourier Transform}: \quad F(\omega) = \int_{-\infty}^{+\infty} f(x)e^{-i\omega x}dx$$

$$\text{Inverse Fourier Transform}: \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega x}d\omega$$

- Where are the sines and cosines? $e^{i\omega x} = \cos(\omega x) + i\sin(\omega x)$
- The result is a complex function $F(\omega) = R(\omega) + iI(\omega)$
- We've been showing only the **amplitude A (spectre)** so far:
- Phase is also encoded: $A = \pm\sqrt{R(\omega)^2 + I(\omega)^2}$

$$\phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

Slide by Steve Seitz

Magnitude and phase

- Fourier transform stores the **magnitude** and **phase** at each frequency
  - Magnitude encodes how much signal there is at a particular frequency
  - Phase encodes spatial information (indirectly)
  - For mathematical convenience, this is often notated in terms of real and complex numbers

Amplitude:

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2}$$

Phase:

$$\phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

Slide by Rober Pless

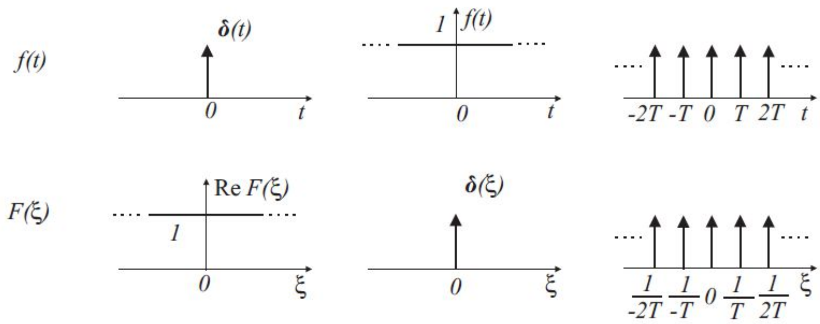
Discrete Fourier transform

$$H_{f_j} = \frac{1}{N}\sum_k h_{t_k} e^{2\pi i f_j t_k}$$

$$h_{t_j} = \frac{1}{N}\sum_k H_{f_k} e^{-2\pi i f_k t_j}$$

where the t_k are the time corresponding to my signal in the time domain h_{t_k}, f_k are the corresponding frequency to my signal in the frequency domain, and N is the number of points of the signal data.

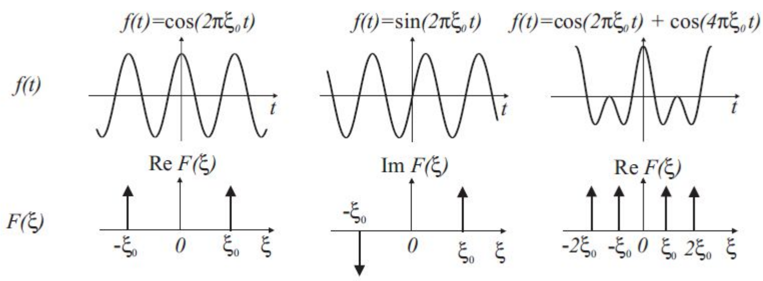
Basic Fourier Transform pairs



Dirac | constant | ∞ sequence of Diracs

Source: Václav Hlavác - **Fourier transform, in 1D and in 2D**

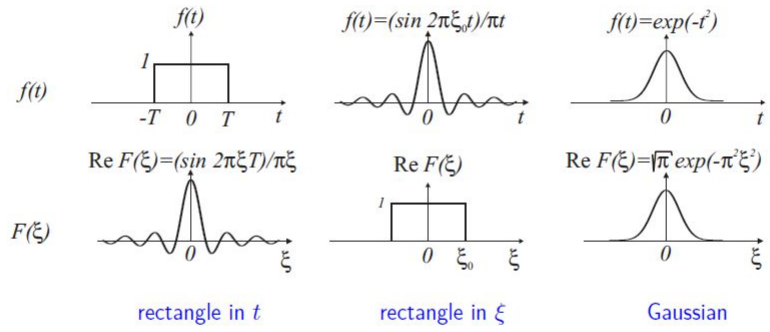
Basic Fourier Transform pairs



cosine | sine | two cosines mixture

Source: Václav Hlavác - **Fourier transform, in 1D and in 2D**

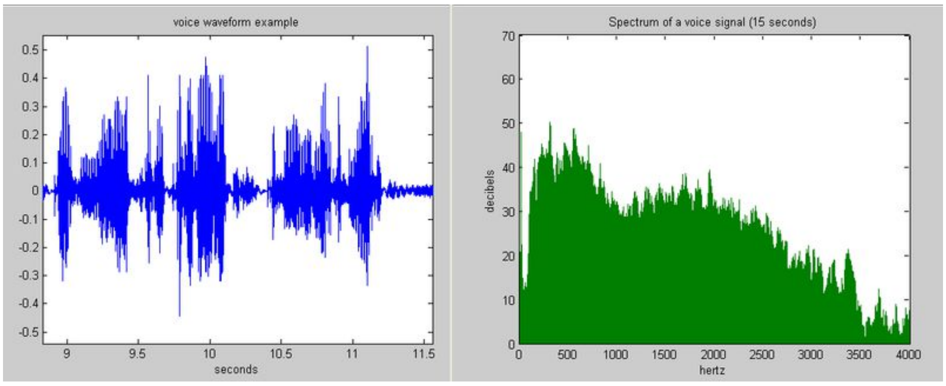
Basic Fourier Transform pairs



Source: *Václav Hlavác* - ***Fourier transform, in 1D and in 2D***

Example: Music

- We think of music in terms of frequencies at different magnitudes



Slide: Hoiem

2D FFT - discrete

Direct transform

$$F(u,v) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)\,\exp\left[-2\pi i\left(\frac{mu}{M}+\frac{nv}{N}\right)\right],$$

$$u = 0, 1, \ldots, M-1\,, \qquad v = 0, 1, \ldots, N-1\,,$$

Inverse transform

$$f(m,n) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\,\exp\left[2\pi i\left(\frac{mu}{M}+\frac{nv}{N}\right)\right],$$

$$m = 0, 1, \ldots, M-1\,, \qquad n = 0, 1, \ldots, N-1\,.$$

Image Fourrier transform

Original image | **Spectra | F(u,v) |** | **Enhanced Spectra log (1+ | F(u,v) |)**

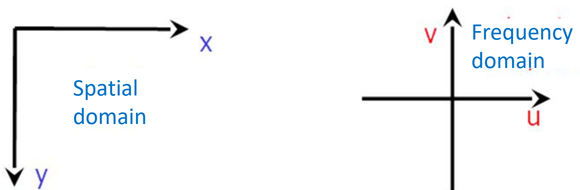


Image Fourrier transform

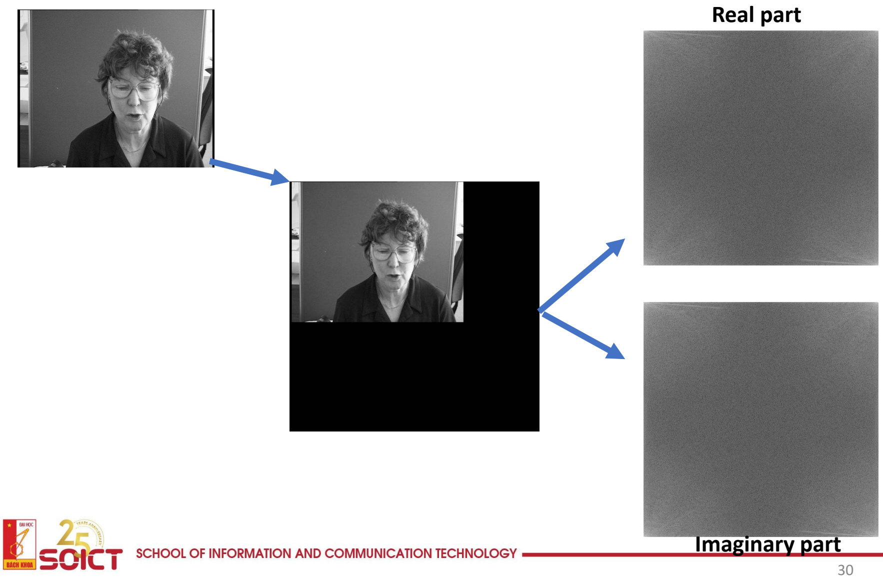
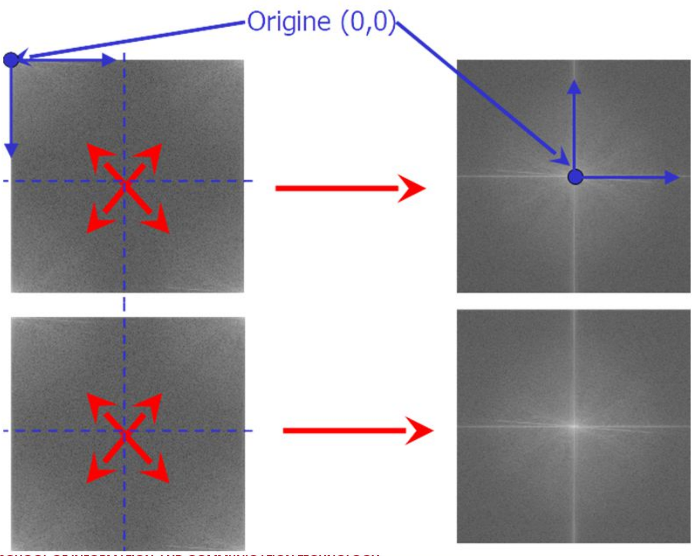
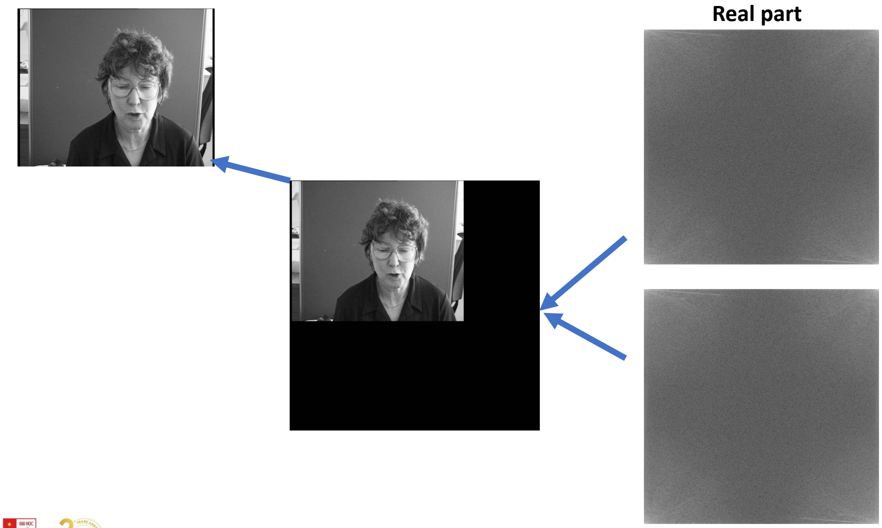


Image Fourrier transform

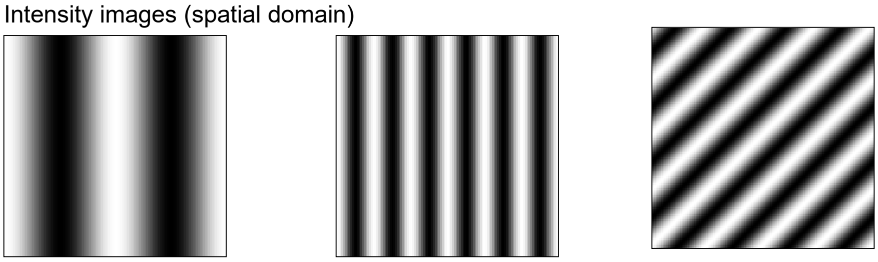


Inverse Fourrier transform

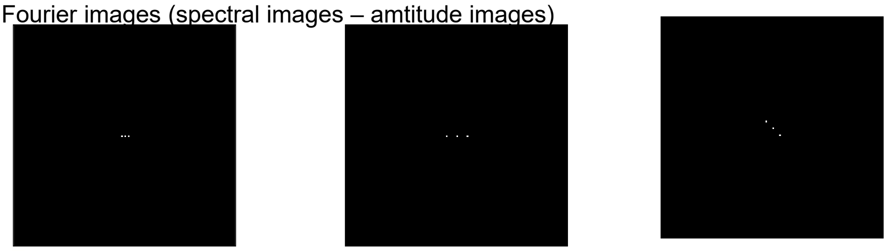


Fourier analysis in images

Intensity images (spatial domain)



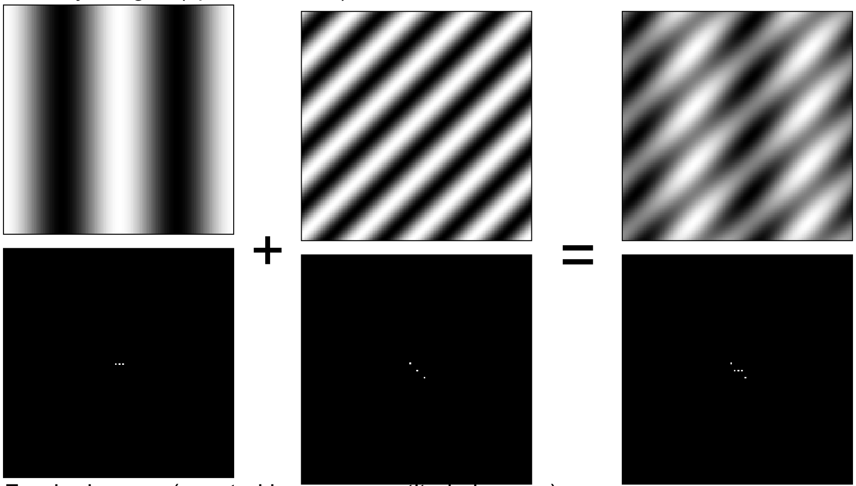
Fourier images (spectral images – amtitude images)



http://sharp.bu.edu/~slehar/fourier/fourier.html#filtering More: http://www.cs.unm.edu/~brayer/vision/fourier.html

Signals can be composed

Intensity images (spatial domain)



Fourier images (spectral images – amtitude images)

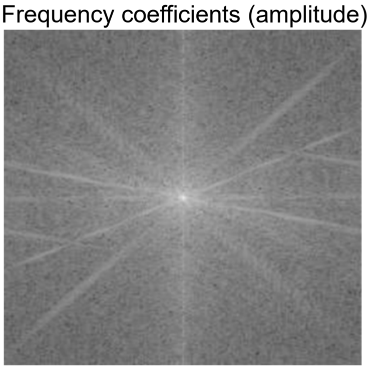
http://sharp.bu.edu/~slehar/fourier/fourier.html#filtering

Fourier Transform of an image

Natural image

f(x,y)

Fourier decomposition
Frequency coefficients (amplitude)



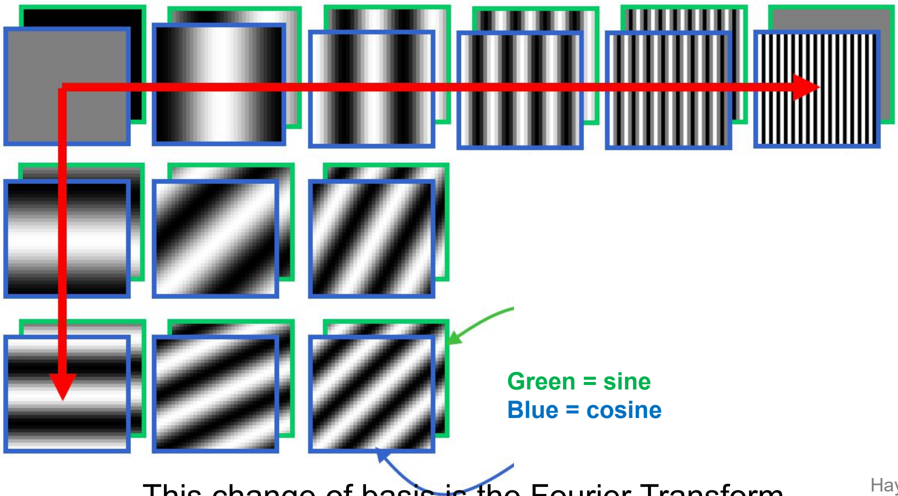
|F(ω)|

What does it mean to be at pixel x,y?
What does it mean to be more or less bright in the Fourier decomposition image?

Slide by Steve Seitz

Fourier Bases

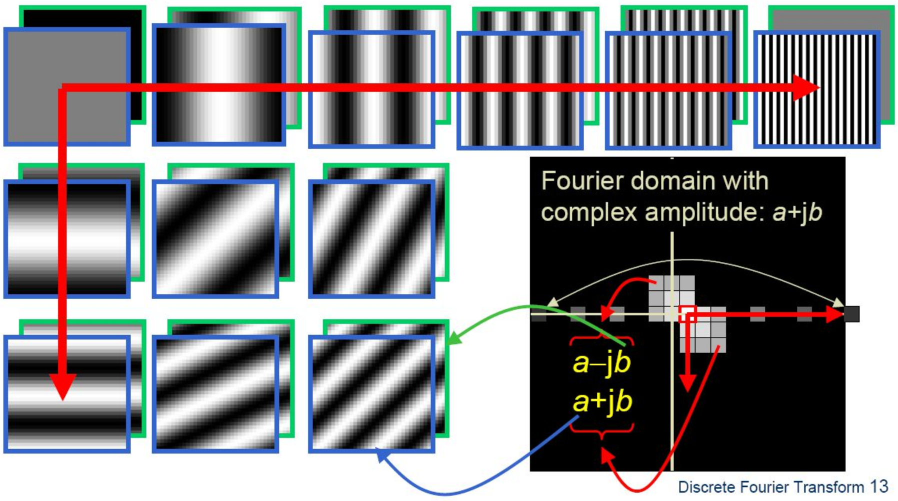
Teases away ‘fast vs. slow’ changes in the image.



This change of basis is the Fourier Transform

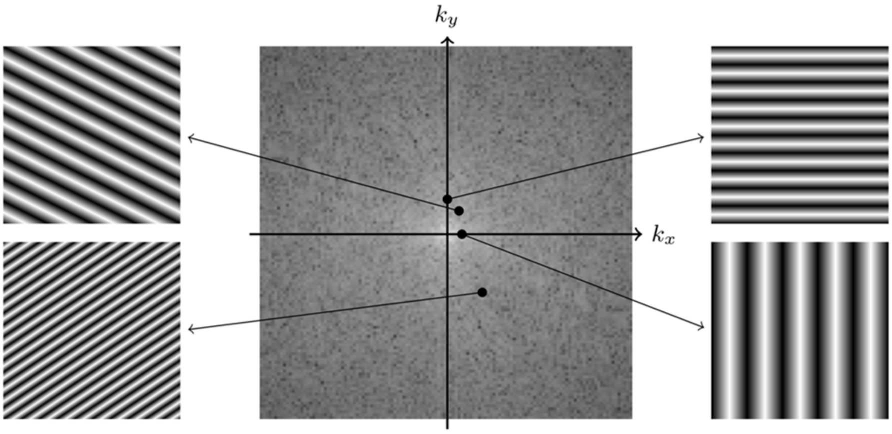
Hays

Fourier Bases



Hays

2D Fourier Transform

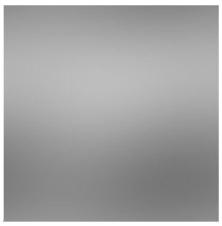


Slide by Steve Seitz

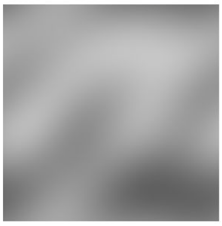
Basis reconstruction

Full image

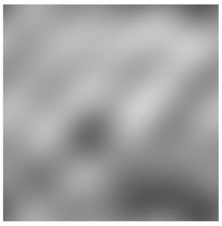
First 1 basis fn



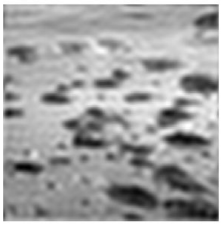
First 4 basis fns



First 9 basis fns



First 16 basis fns



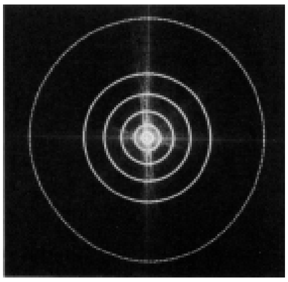
First 400 basis fns

Danny Alexander

2D Fourier transform

Image

Fourier Spectrum



Percentage of image power enclosed in circles (small to large) :
90, 95, 98, 99, 99.5, 99.9

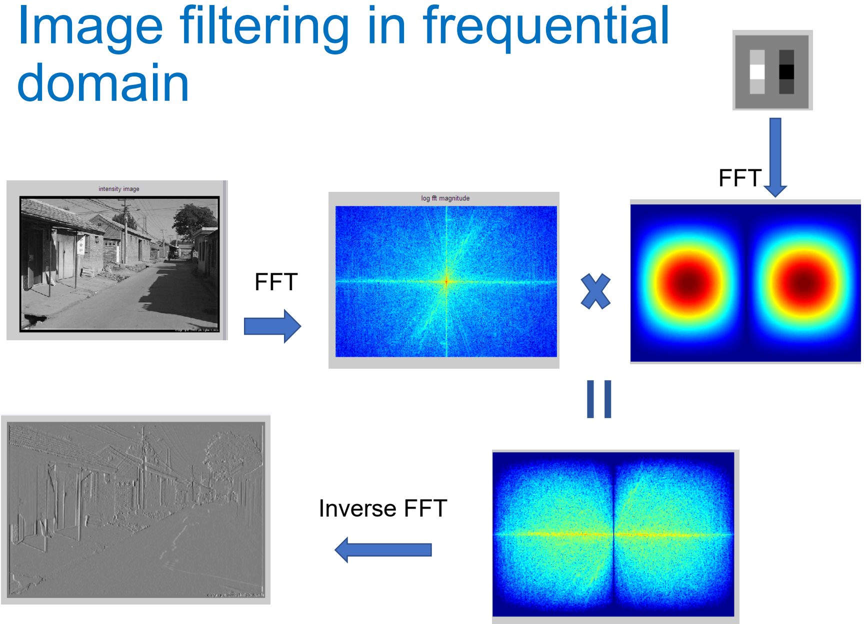
Most of energy concentrated in low frequencies

Image filtering in the frequential domain

- We will be dealing only with functions (images) of finite duration so we will be interested only in Fourier Transform

Original Function in Spatial Domain $\xrightarrow{\text{Fourier Transform}}$ Image in Frequency Domain $\longrightarrow$ Apply Filter in Frequency Domain $\xrightarrow{\text{Inverse Fourier Transform}}$ Enhanced Image in Spatial Domain

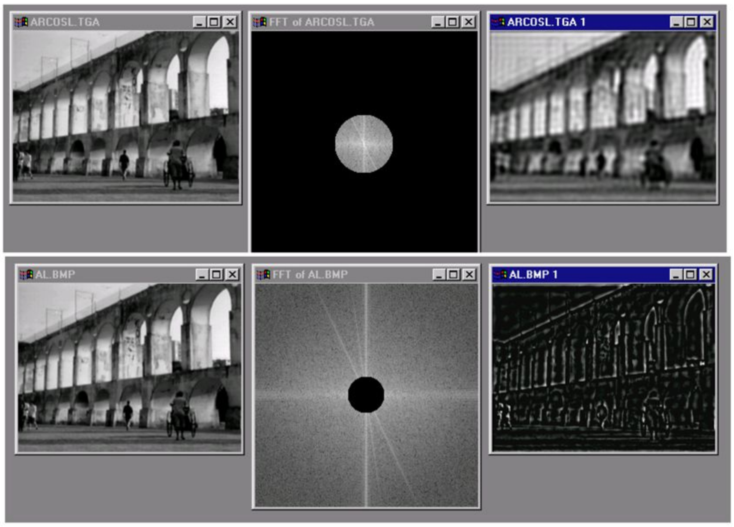
Image filtering in frequential domain



Slide by Derek Hoiem

Now we can edit frequencies!

Low-pass and high-pass filtering



Slide by Alexei Efros

Low-pass filter

spatial domain | frequency domain

f(x,y) → F(u,v)

G(u,v) = F(u,v) • H(u,v)

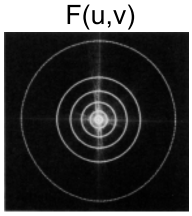
filter

g(x,y) ← G(u,v)

Purpose: smoothing noise

f(x,y)

F(u,v)



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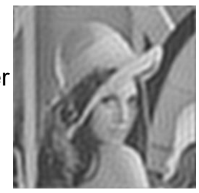
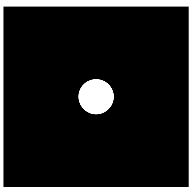


g(x,y)



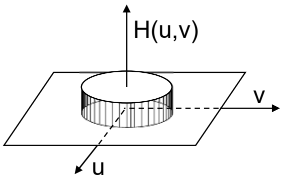
H(u,v)

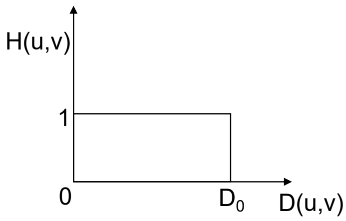
H(u,v) - Ideal low-pass filter

$$H(u,v) = \begin{cases} 1 & D(u,v) \le D_0 \\ 0 & D(u,v) > D_0 \end{cases}$$

$$D(u,v) = \sqrt{u^2 + v^2}$$

D_0 = cut off frequency





Blurring - Ideal low-pass filters

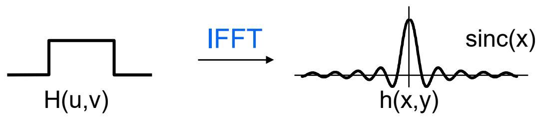
99.7% 99.37% 98.65%

The ringing problem

$$G(u,v) = F(u,v) \bullet H(u,v)$$

$$\big\downarrow \text{ Convolution Theorem}$$

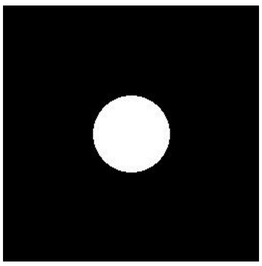
$$g(x,y) = f(x,y) * h(x,y)$$



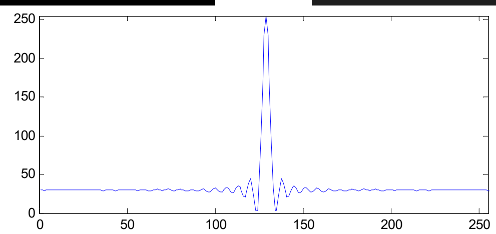
$\uparrow D_0 \longrightarrow \downarrow$ Ringing radius + $\downarrow$ blur

The ringing problem

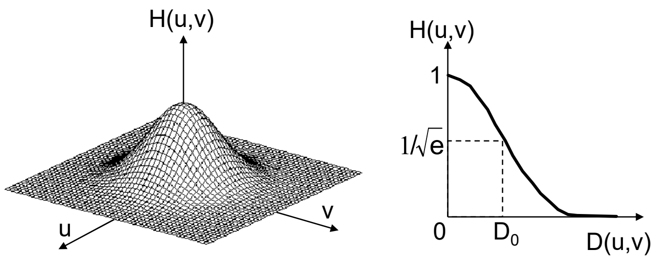
Freq. domain



Spatial domain



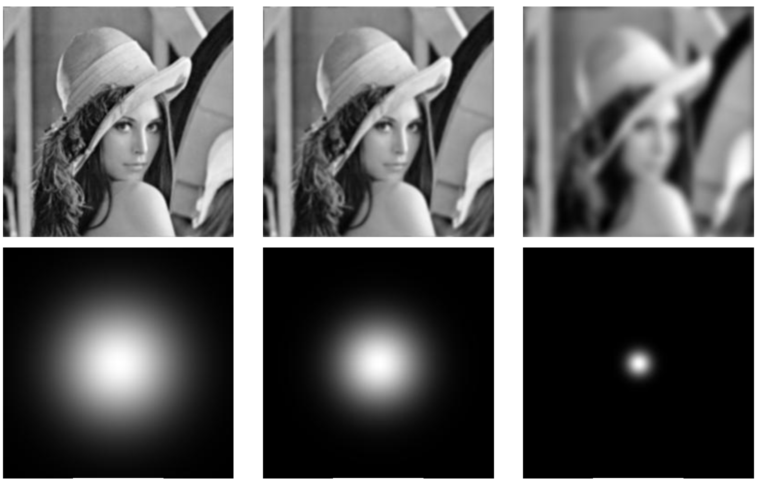
H(u,v) - Gaussian filter



$$H(u,v) = e^{-D^2(u,v)/(2D^2_0)} \qquad D(u,v) = \sqrt{u_2 + v_2}$$

Softer Blurring + no Ringing

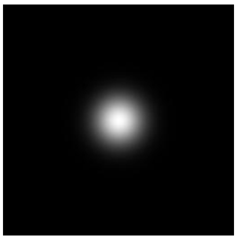
Blurring - Gaussain lowpass filter



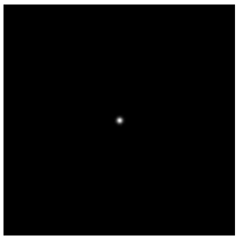
99.11% 98.74% 96.44%

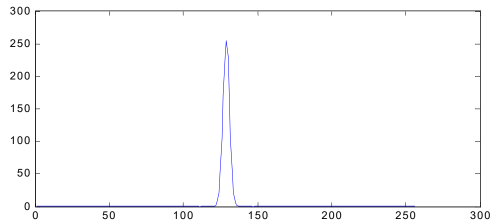
The Gaussian lowpass filter

Freq. domain



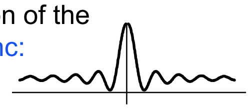
Spatial domain





Blurring in the Spatial Domain

Averaging = convolution with $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ = point multiplication of the transform with sinc:



Gaussian Averaging = convolution with $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

= point multiplication of the transform with a gaussian.

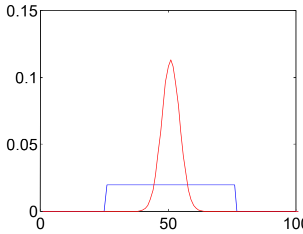
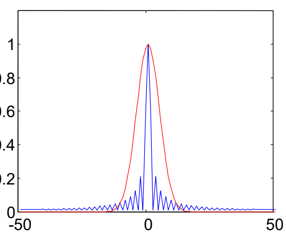


Image Domain



Frequency Domain

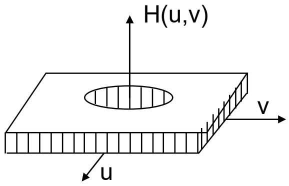
High-pass filter

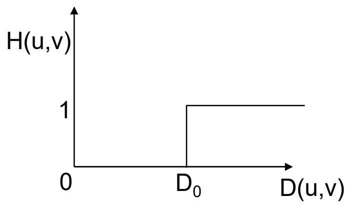
H(u,v) - Ideal Filter

$$H(u,v) = \begin{cases} 0 & D(u,v) \le D_0 \\ 1 & D(u,v) > D_0 \end{cases}$$

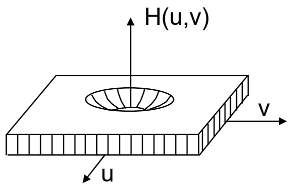
$D(u,v) = \sqrt{u^2 + v^2}$

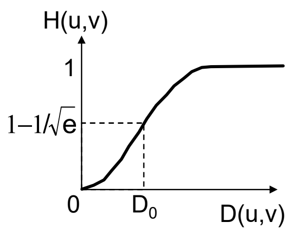
D_0 = cut off frequency





High-pass gaussian filter



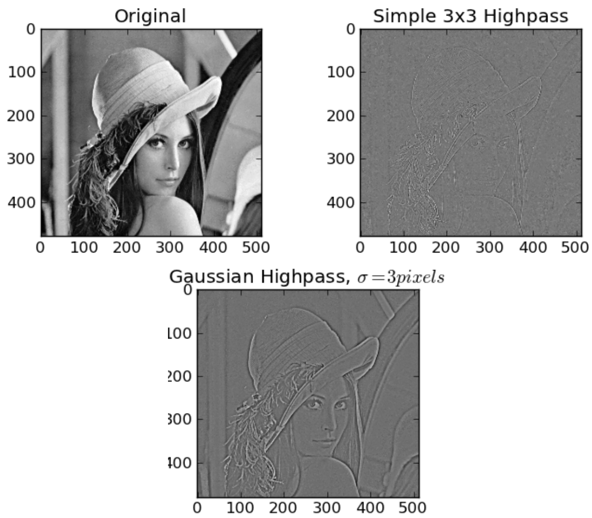


$$H(u,v) = 1 - e^{-D^2(u,v)/(2D^2_0)}$$

$$D(u,v) = \sqrt{u_2 + v_2}$$

High-pass filtering

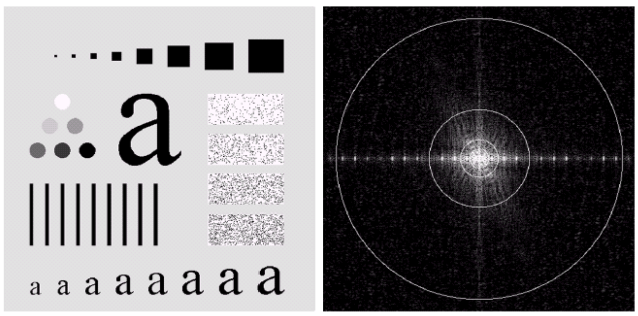
$$h_1(3x3) = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$



High pass filtering

Original image

Spectre with filters

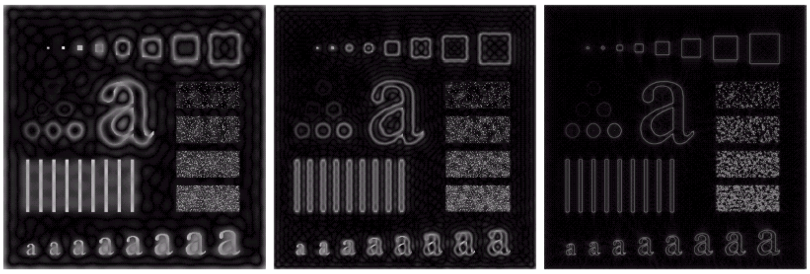


a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

Source : Gonzalez and Woods. Digital Image Processing. Prentice-Hall, 2002.

High pass filtering



a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

Source : Gonzalez and Woods. Digital Image Processing. Prentice-Hall, 2002.

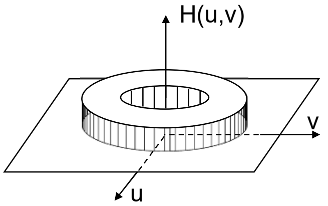
Band-pass filtering

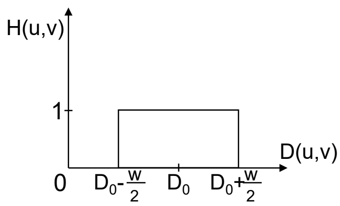
$$H(u,v) = \begin{cases} 0 & D(u,v) \le D_0 - \frac{w}{2} \\ 1 & D_0 - \frac{w}{2} \le D(u,v) \le D_0 + \frac{w}{2} \\ 0 & D(u,v) > D_0 + \frac{w}{2} \end{cases}$$

$D(u,v) = \sqrt{u^2 + v^2}$

D_0 = cut off frequency

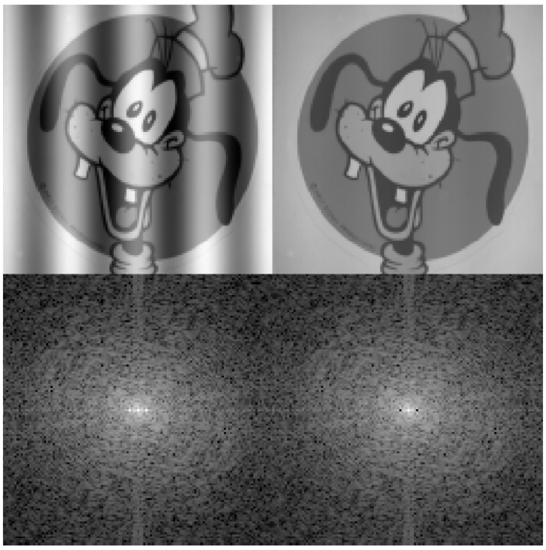
w = band-width





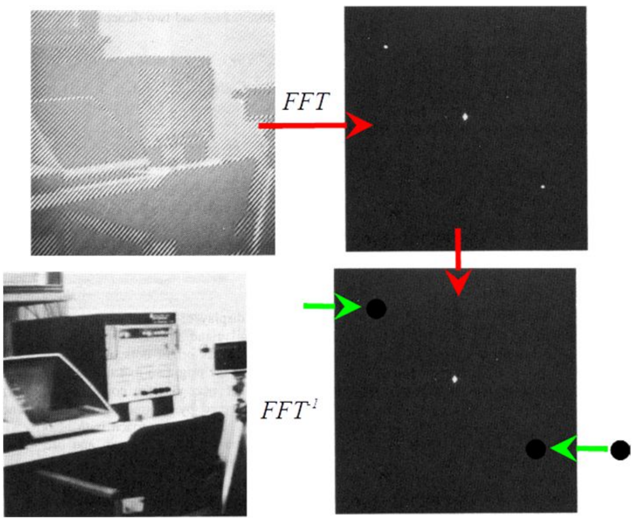
Can be obtained by multiplying the filter functions of **a low-pass and of a high-pass** in the frequency domain

Removing sinus noise

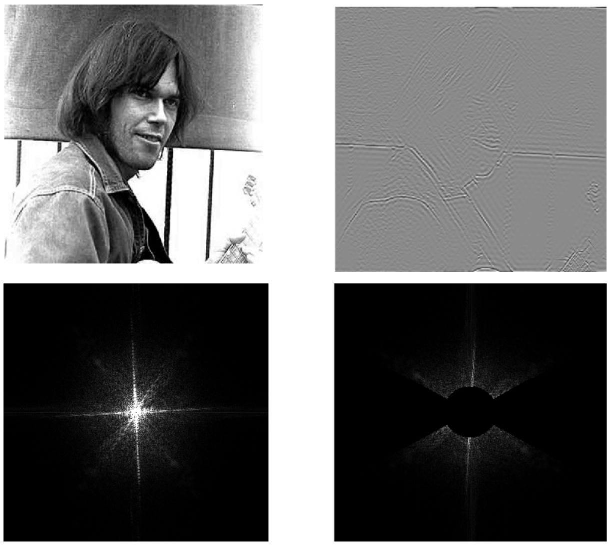


Brayer

Removing sinus noise



High-pass filtering + orientation



Hybrid Images

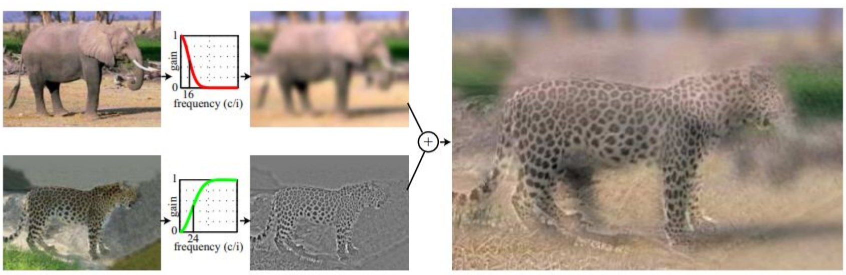
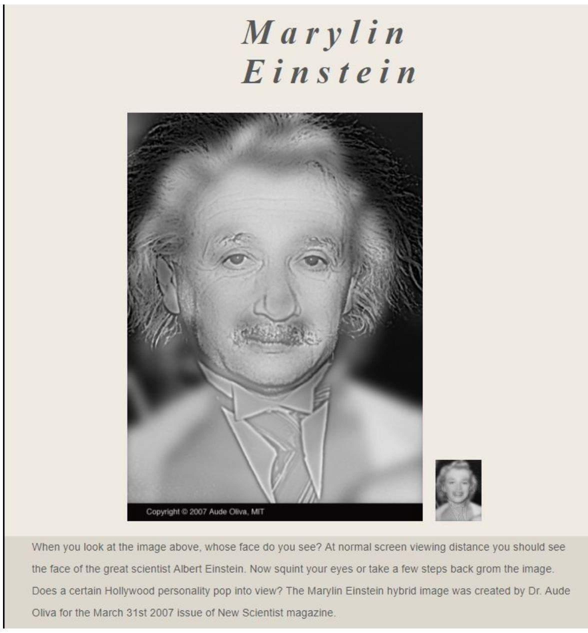


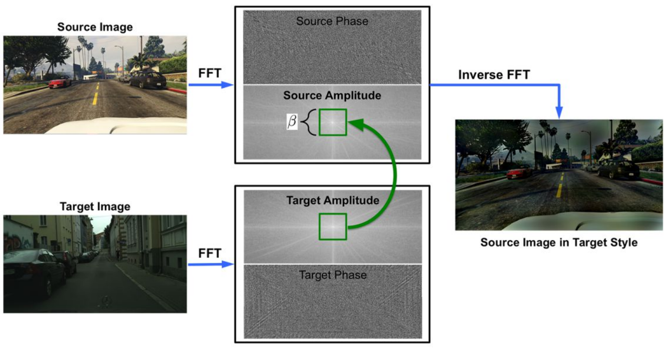
Figure 2: hybrid images are generated by superimposing two images at two different spatial scales: the low-spatial scale is obtained by filtering one image with a low-pass filter, and the high spatial scale is obtained by filtering a second image with a high-pass filter. The final hybrid image is composed by adding these two filtered images.

A. Oliva, A. Torralba, P.G. Schyns, SIGGRAPH 2006



http://cvcl.mit.edu/hybrid_gallery/monroe_einstein.html

Style transfer



Content

- Rappel: digital image representation
- Point Processing
- Convolution and Linear filtering
- More neighborhood operators
- Image transforms
  - Frequency domain
  - PCA (additional reading)
    - PCA
    - Example of using PCA for face recognition

Principle Component Analysis - PCA (Karhunen-Loeve transformation)

- **PCA** transforms the original input space into a lower dimensional space
  - By constructing dimensions that are linear combinations of the given features
- The objective: consider independent dimensions along which data have largest variance (i.e., greatest variability)

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Principal Component Analysis (cont.)

– **PCA** enables transform a number of possibly correlated variables into a smaller number of uncorrelated variables called principal components

– The first principal component accounts for as much of the variability in the data as possible

– Each succeeding component (orthogonal to the previous ones) accounts for as much of the remaining variability as possible

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Principal Component Analysis (cont.)

- PCA is the most commonly used dimension reduction technique.
- Data samples $x_1, \cdots, x_N$
- Compute the mean $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$
- Computer the covariance matrix:

$$\Sigma_x = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(x_i - \bar{x})^T$$

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Principal Component Analysis (cont.)

- Compute the eigenvalues λ and eigenvectors e of the matrix Σ_x
- Solve $\Sigma_x x = \lambda x$
- Order them by magnitude:

$$\lambda_1 \geq \lambda_2 \geq .\lambda_N.$$

- PCA reduces the dimension by keeping direction e such that $\lambda < T$.

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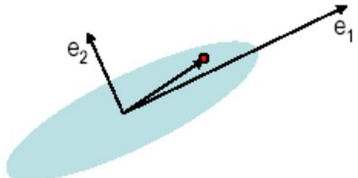
Principal Component Analysis (cont.)

- For many datasets, most of the eigenvalues are negligible and can be discarded.

The eigenvalue λ measures the variation
In the direction of corresponding eigenvector

Example:
$\lambda_1 \neq 0, \lambda_2 = 0.$



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Principal Component Analysis (cont.)

- How to get uncorrelated components which Capture most of the variance
- Project the data onto the selected eigenvectors:

$$y_i = e_i^T (x_i - \bar{x})$$

- If we consider first M eigenvectors we get new lower dimensional representation

$$[y_1, \cdots, y_M]$$

- Proportion covered by first M eigenvalues

$$\frac{\sum_{i=1}^{M} \lambda_i}{\sum_{i=1}^{N} \lambda_i}$$

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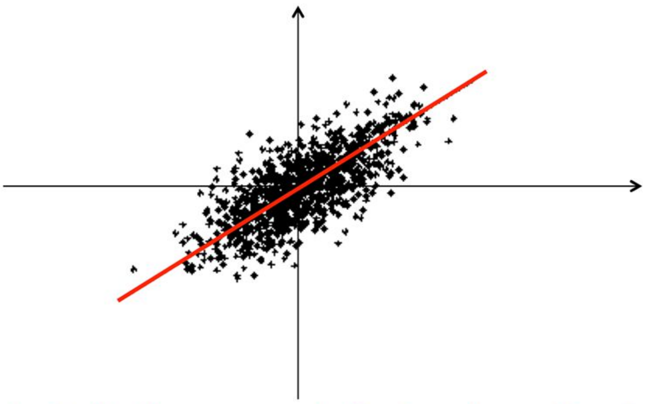
Illustration of PCA



First principal component of a two-dimensional data set.

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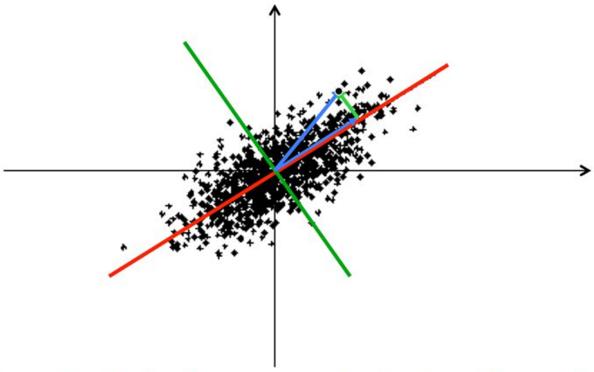
Illustration of PCA



Second principal component of a two-dimensional data set.

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Determining the number of components

- Plot the eigenvalues
  - each eigenvalue is related to the amount of variation explained by the corresponding axis (eigenvector)
  - If the points on the graph tend to level out (show an "elbow" shape), these eigenvalues are **usually close enough to zero that they can be ignored**

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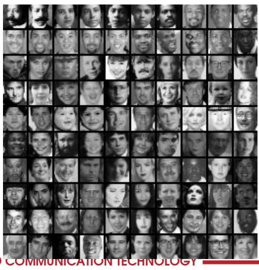
The space of all face images

- When viewed as vectors of pixel values, face images are extremely high-dimensional
  - 100x100 image = 10,000 dimensions
- However, relatively few 10,000-dimensional vectors correspond to valid face images
- We want to effectively model the subspace of face images



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Eigenfaces: Key idea

- Assume that most face images lie on a low-dimensional subspace determined by the first k ($k<d$) directions of maximum variance
- Use PCA to determine the vectors or "eigenfaces" **u**1,...,**u**k that span that subspace
- Represent all face images in the dataset as linear combinations of eigenfaces

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Eigenfaces example- Traning images

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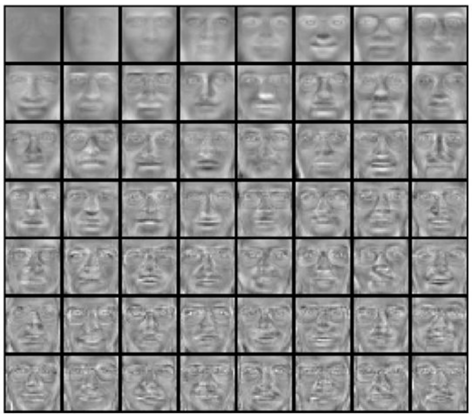
Eigenfaces example

Top eigenvectors: $u_1, \ldots u_k$

Mean: μ



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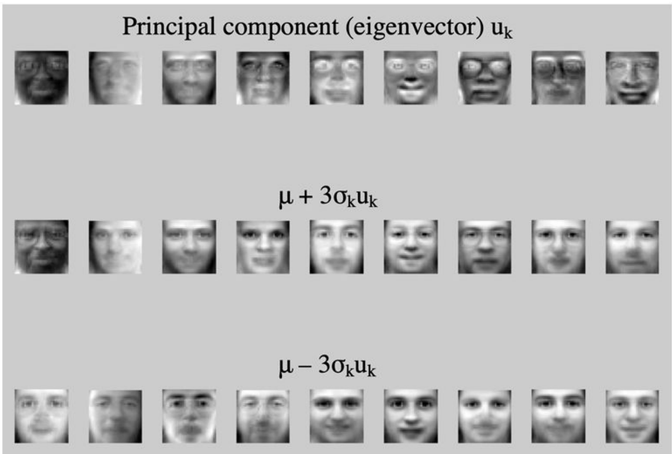
Eigenfaces example



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Eigenfaces examples

- Representation

$(w_{i1},\ldots,w_{ik}) = (\mathbf{u}_1^T(\mathbf{x}_i - \boldsymbol{\mu}), \ldots, \mathbf{u}_k^T(\mathbf{x}_i - \boldsymbol{\mu}))$

- Reconstruction

 =  + 

$$\hat{x} = \mu + w_1 u_1 + w_2 u_2 + w_3 u_3 + w_4 u_4 + \ldots$$

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Recognition with eigenfaces

- Process labeled training images:
- Find mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$
- Find k principal components (eigenvectors of $\boldsymbol{\Sigma}$) $\mathbf{u}_1,\ldots\mathbf{u}_k$
- Project each training image $\mathbf{x}_i$ onto subspace spanned by principal components:
  $(w_{i1},\ldots,w_{ik}) = (\mathbf{u}_1^T(\mathbf{x}_i - \boldsymbol{\mu}), \ldots, \mathbf{u}_k^T(\mathbf{x}_i - \boldsymbol{\mu}))$

- Given novel image $\mathbf{x}$:
- Project onto subspace:
  $(w_1,\ldots,w_k) = (\mathbf{u}_1^T(\mathbf{x} - \boldsymbol{\mu}), \ldots, \mathbf{u}_k^T(\mathbf{x} - \boldsymbol{\mu}))$
- Optional: check reconstruction error $\mathbf{x} - \hat{\mathbf{x}}$ to determine whether image is really a face
- Classify as closest training face in k-dimensional subspace

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