Contents

Exercise 1

Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function.

i) Show that
   - $xf(x)$ is an even function
   - $x^2 f(x)$ is an odd function

ii) Show that
   - The function $g_1 : \mathbb{R} \to \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function
   - The function $g_2 : \mathbb{R} \to \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function

iii) Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h(x) = x^i f(x^j)$, where i and j are positive integers. When is $h(x)$ an odd function?

Answer

$f(x)$ is an odd function, which means:

$$\begin{aligned} f(x) &= -f(-x) \\ f(-x) &= -f(x) \end{aligned}$$

i) Let $f_1 : \mathbb{R} \to \mathbb{R}$ be defined as $f_1(x) = xf(x)$.

$$\begin{aligned} f_1(-x) &= -xf(-x) \\ &= xf(x) \\ &= f_1(x) \end{aligned}$$

Which means $f_1(x)$ is an even function.

Let $f_2 : \mathbb{R} \to \mathbb{R}$ be defined as $f_2(x) = x^2 f(x)$.

$$\begin{aligned} f_2(-x) &= (-x)^2 f(-x) \\ &= x^2 f(-x) \\ &= -x^2 f(x) \\ &= -f_2(x) \end{aligned}$$

Which means $f_2(x)$ is an odd function.

ii) We have

$$\begin{aligned} g_1(-x) &= f((-x)^2) \\ &= f(x^2) \\ &= g_1(x) \end{aligned}$$

Which means $g_1(x)$ is an even function.

$$\begin{aligned} g_2(-x) &= f((-x)^3) \\ &= f(-x^3) \\ &= -f(x^3) \\ &= -g_2(x) \end{aligned}$$

Which means $g_2(x)$ is an odd function.

iii) Doing some transformation

$$
\begin{aligned}
h(x) &= x^i f(x^j) \\
h(-x) &= (-x)^i f((-x)^j) \\
&= (-1)^i x^i f((-x)^j) \\
&= (-1)^i x^i f((-1)^j x^j) \\
&= (-1)^i (-1)^j x^i f(x^j) \\
&= (-1)^{i+j} x^i f(x^j) \\
&= (-1)^{i+j} h(x)
\end{aligned}
$$

Because

- $(-1)^{i+j} = -1$ when $i + j$ is odd, and
- $(-1)^{i+j} = 1$ when $i + j$ is even

Then

- $h(x) = -h(x)$ or $h(x)$ is an odd function, when $i + j$ is odd, and
- $h(x) = h(x)$ or $h(x)$ is an even function, when $i + j$ is even

Exercise 2

Let

- $S(n, 2) = \sum_{k=1}^{n} k^2$ and
- $S(n, 3) = \sum_{k=1}^{n} k^3$

i) Let $T(n, 2, x) = \sum_{k=1}^{n} k^2 x^k$.

   Use formula

$$T(n, j + 1, x) = x \frac{d}{dx}(T(n, j, x)) \quad \forall j \geq 0$$

   for $j = 1$, i.e.,

$$T(n, 2, x) = x \frac{d}{dx}(T(n, 1, x))$$

   And formula

$$T(n, 1, x) = \frac{x - (n + 1)x^{n+1} + nx^{n+2}}{(1 - x)^2}$$

   for $T(n, 1, x)$, to show that

$$T(n, 2, x) = \frac{x + x^2 - (n + 1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1 - x)^3}$$

ii) Note that $S(n, 2) = T(n, 2, 1)$. Use l'Hopital's rule to evaluate $T(n, 2, 1)$, and conclude that $S(n, 2) = \frac{n(n+1)(2n+1)}{6}$

iii) Compute $T(n, 3, x) = \sum_{k=1}^{n} k^3 x^k$ using formula

$$T(n, j + 1, x) = x \frac{d}{dx}(T(n, j, x)) \quad \forall j \geq 0$$

   for $j = 2$, i.e,

$$T(n,3,x) = x\frac{d}{dx}(T(n,2,x))$$

iv) Note that $S(n,3) = T(n,3,1)$. Use l'Hopital's rule to evaluate $T(n,3,1)$ and conclude that $S(n,3) = \left(\frac{n(n+1)}{2}\right)^2$.

Answer

i)

$$T(n,2,x) = x\frac{d}{dx}\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$$

Using the quotient rule

$$\left[\frac{u(x)}{v(x)}\right]\frac{d}{dx} = \frac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2}$$

With

$$\begin{aligned}
u(x) &= x-(n+1)x^{n+1}+nx^{n+2} \\
u'(x) &= 1-(n+1)^2x^n+n(n+2)x^{n+1} \\
v(x) &= (1-x)^2 \\
v'(x) &= -2(1-x)
\end{aligned}$$

$$\begin{aligned}
u'(x)v(x) &= \left(1-(n+1)^2x^n+n(n+2)x^{n+1}\right)(1-x)^2 \\
&= (1-x)^2\left(1-(n+1)^2x^n+n(n+2)x^{n+1}\right) \\
u(x)v'(x) &= \left(x-(n+1)x^{n+1}+nx^{n+2}\right)-2(1-x) \\
&= -2(1-x)\left(x-(n+1)x^{n+1}+nx^{n+2}\right)
\end{aligned}$$

$$\begin{aligned}
u'(x)v(x)-u(x)v'(x) &= (1-x)^2\left(1-(n+1)^2x^n+n(n+2)x^{n+1}\right) \\
&\quad +2(1-x)\left(x-(n+1)x^{n+1}+nx^{n+2}\right) \\
&= (1-x)\big[(1-x)\left(1-(n+1)^2x^n+n(n+2)x^{n+1}\right) \\
&\quad +2\left(x-(n+1)x^{n+1}+nx^{n+2}\right)\big] \\
&= (1-x)\left(1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}\right) \\
\frac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2} &= \frac{(1-x)\left(1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}\right)}{(1-x)^4} \\
&= \frac{1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}}{(1-x)^3}
\end{aligned}$$

Which means

$$\begin{aligned}
x\frac{d}{dx}(T(n,1,x)) &= x\frac{1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}}{(1-x)^3} \\
&= \frac{x+x^2-(n+1)^2x^{n+1}+(2n^2+2n-1)x^{n+2}-n^2x^{n+3}}{(1-x)^3} \qquad \square
\end{aligned}$$

ii)

$$
\begin{aligned}
T(n,2,x) &= \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \\
T(n,2,1) &= \frac{1 + 1 - (n+1)^2 + (2n^2+2n-1) - n^2}{0} \\
&= \frac{2 - (n^2+2n+1) + (2n^2+2n-1) - n^2}{0} \\
&= \frac{0}{0}
\end{aligned}
$$

Which is indeterminate. Apply l'Hopital's rule:

$$
\begin{aligned}
\lim_{x\to 1} T(n,2,x) &= \lim_{x\to 1} \frac{\frac{d}{dx}\left[x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}\right]}{\frac{d}{dx}\left[(1-x)^3\right]} \\
&= \lim_{x\to 1} \frac{1 + 2x - (n+1)^3 x^n + (n+2)(2n^2+2n-1)x^{n+1} - (n+3)n^2 x^{n+2}}{-3(1-x)^2} \\
&= \lim_{x\to 1} \frac{2 - n(n+1)^3 x^{n-1} + (n+1)(n+2)(2n^2+2n-1)x^n - (n+2)(n+3)n^2 x^{n+1}}{6(1-x)} \\
&= \lim_{x\to 1} \frac{-(n-1)n(n+1)^3 x^{n-2} + n(n+1)(n+2)(2n^2+2n-1)x^{n-1} - (n+1)(n+2)(n+3)n^2 x^n}{-6} \\
&= \frac{-(n-1)n(n+1)^3 + n(n+1)(n+2)(2n^2+2n-1) - (n+1)(n+2)(n+3)n^2}{-6} \\
&= \frac{n(n+1)\,-(n-1)(n+1)^2 + n(n+1)\,(n+2)(2n^2+2n-1) + n(n+1)\,-n(n+2)(n+3)}{-6} \\
&= \frac{n(n+1)\left[-(n-1)(n+1)^2 + (n+2)(2n^2+2n-1) - n(n+2)(n+3)\right]}{-6} \\
&= \frac{n(n+1)[-2n-1]}{-6} = \frac{n(n+1)(2n+1)}{6}
\end{aligned}
$$

iii)

$$
\begin{aligned}
T(n,3,x) &= x\frac{d}{dx}\left(T(n,2,x)\right) \\
&= x\frac{d}{dx}\left[\frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}\right]
\end{aligned}
$$

Using the quotient rule

$$
\left[\frac{u(x)}{v(x)}\right]\frac{d}{dx} = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}
$$

With

$$
\begin{aligned}
u(x) &= x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3} \\
u'(x) &= 1 + 2x - (n+1)^3 x^n + (2n^2 + 2n - 1)(n+2)x^{n+1} - (n+3)n^2 x^{n+2} \\
v(x) &= (1-x)^3 \\
v'(x) &= -3(1-x)^2 \\
[v(x)]^2 &= (1-x)^6
\end{aligned}
$$

$$
\begin{aligned}
u'(x)v(x) &= \left(1 + 2x - (n+1)^3 x^n + (2n^2 + 2n - 1)(n+2)x^{n+1} - (n+3)n^2 x^{n+2}\right)(1-x)^3 \\
&= (1-x)\left(1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}\right)(1-x)^2 \\
u(x)v'(x) &= \left(x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}\right) \times -3(1-x)^2 \\
&= \left(-3x - 3x^2 + 3(n+1)^2 x^{n+1} - 3(2n^2 + 2n - 1)x^{n+2} + 3n^2 x^{n+3}\right)(1-x)^2
\end{aligned}
$$

Let $g(x) = (1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2})$, which means $u'(x)v(x) = (1-x)g(x)\,(1-x)^2$. Here is the calculation of $(1-x)g(x)$ putting onto a grid:

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$g(x)$	1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)$	$-(n+3)n^2$	
$xg(x)$		1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)$	$-(n+3)n^2$
$(1-x)\,g(x)$	1	1	-2	$-(n+1)^3$	...	...	$n^3 + 3n^2$

Because

$$
\begin{aligned}
(n+2)(2n^2 + 2n - 1) - [-(n+1)^3] &= (2n^3 + 4n^2 + 2n^2 + 4n - n - 2) \\
&\quad + (n^3 + 3n^2 + 3n + 1) \\
&= (2n^3 + 6n^2 + 3n - 2) \\
&\quad + (n^3 + 3n^2 + 3n + 1) \\
&= 3n^3 + 9n^2 + 6n - 1
\end{aligned}
$$

and

$$
\begin{aligned}
-(n+3)n^2 - (n+2)(2n^2 + 2n - 1) &= -(n^3 + 3n^2) \\
&\quad - (2n^3 + 6n^2 + 3n - 2) \\
&= -3n^3 - 9n^2 - 3n + 2
\end{aligned}
$$

We have

$$
\begin{aligned}
(1-x)\,g(x) &= 1 + x - 2x^2 - (n+1)^3 x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1} \\
&\quad + (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3} \\
u'(x)v(x) &= (1-x)\,g(x)\,(1-x)^2 \\
&= [1 + x - 2x^2 - (n+1)^3 x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1} \\
&\quad + (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3}]\,(1-x)^2
\end{aligned}
$$

Let

$$
\begin{aligned}
h_1(x) &= [1 + x - 2x^2 - (n+1)^3 x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1} \\
&\quad + (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3}] \\
u'(x)v(x) &= h_1(x)\,(1-x)^2 \\
h_2(x) &= -3x - 3x^2 + 3(n+1)^2 x^{n+1} \\
&\quad - 3(2n^2 + 2n - 1)x^{n+2} + 3n^2 x^{n+3} \\
u(x)v'(x) &= h_2(x)\,(1-x)^2
\end{aligned}
$$

$$
u'(x)v(x) - u(x)v'(x) = (h_1(x) - h_2(x))\,(1-x)^2
$$

The calculation can be put into a grid like this:

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$h_1(x)$	1	1	-2	$-(n+1)^3$	$3n^3 + 9n^2 + 6n - 1$	$-3n^3 - 9n^2 - 3n + 2$	$n^3 + 3n^2$
$h_2(x)$		-3	-3		$3(n+1)^2$	$-3(2n^2 + 2n - 1)$	$3n^2$
$h_1(x) - h_2(x)$	1	4	1	$-(n+1)^3$	…	…	n^3

Because

$$
\begin{aligned}
(3n^3 + 9n^2 + 6n - 1) - 3(n+1)^2 &= (3n^3 + 9n^2 + 6n - 1) - (3n^2 + 6n + 3) \\
&= 3n^3 + 6n^2 - 4
\end{aligned}
$$

$$
\begin{aligned}
(-3n^3 - 9n^2 - 3n + 2) - (-3)(2n^2 + 2n - 1) &= (-3n^3 - 9n^2 - 3n + 2) + (6n^2 + 6n - 3) \\
&= -3n^3 - 3n^2 + 3n - 1
\end{aligned}
$$

Then

$$
\begin{aligned}
u'(x)v(x) - u(x)v'(x) &= [1 + 4x + x^2 - (n+1)^3 x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3 x^{n+3}]\,(1-x)^2 \\
\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} &= \frac{\begin{gathered}[1 + 4x + x^2 - (n+1)^3 x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\ (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3 x^{n+3}]\end{gathered}}{(1-x)^4} \\
T(n,3,x) &= x\frac{\begin{gathered}[1 + 4x + x^2 - (n+1)^3 x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\ (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3 x^{n+3}]\end{gathered}}{(1-x)^4} \\
&= \frac{\begin{gathered}[x + 4x^2 + x^3 - (n+1)^3 x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} + \\ (-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3 x^{n+4}]\end{gathered}}{(1-x)^4}
\end{aligned}
$$

—

The second way to solve this

$$
\begin{aligned}
F(x) &= \frac{P(x)}{(1-x)^m} \\
&= \frac{u(x)}{v(x)} \\
F'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} \\
&= \frac{P'(x)(1-x)^m - P(x)(-m)(1-x)^{m-1}}{(1-x)^{2m}} \\
&= \frac{P'(x) + \frac{mP(x)}{1-x}}{(1-x)^m} \\
&= \frac{(1-x)P'(x) + mP(x)}{(1-x)^{m+1}}
\end{aligned}
$$

Let

$$
\begin{aligned}
P(x) &= x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3} \\
P'(x) &= 1 + 2x - (n+1)^3 x^n + (n+2)(2n^2+2n-1)x^{n+1} - (n+3)n^2 x^{n+2}
\end{aligned}
$$

Using a grid to calculate $(1-x)\,P'(x)$

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$P(x)$		1	1		$-(n+1)^2$	$2n^2+2n-1$	$-n^2$
$P'(x)$	1	2		$-(n+1)^3$	$(n+2)(2n^2+2n-1)2$	$-(n+3)(2n^2+2n-1)$	
$xP'(x)$		1	2		$-(n+1)^3$	$(n+2)(2n^2+2n-1)2$	$-(n+3)(2n^2+2n-1)$
$(1-x)P'(x)$	1	1	-2	$-(n+1)^3$	$\dots$	$\dots$	$(n+3)(2n^2+2n-1)$

iv)

$$
T(n,3,x) = \frac{[x + 4x^2 + x^3 - (n+1)^3 x^{n+1} + (3n^3+6n^2-4)x^{n+2} + (-3n^3-3n^2+3n-1)x^{n+3} + n^3 x^{n+4}]}{(1-x)^4}
$$

$$
\begin{aligned}
T(n,3,1) &= \frac{[1+4+1-(n+1)^3 + (3n^3+6n^2-4) + (-3n^3-3n^2+3n-1) + n^3]}{0} \\
&= \frac{[6 - (1+3n+3n^2+n^3) + (-4+6n^2+3n^3) + (-1+3n-3n^2-3n^3) + n^3]}{0}
\end{aligned}
$$

Simplifying the numerator using a grid we have:

	1	n	n^2	n^3
	6			
	-1	-3	-3	-1
	-4		6	3
	-1	3	-3	-3

				1
Total	0	0	0	0

Which means the numerator is 0 and $T(n, 3, 1) = \frac{0}{0}$.

To apply l'Hopital's rule, let $f(x)$ be the numerator and $g(x)$ be the denominator and

$$\begin{aligned} f(x) &= [x + 4x^2 + x^3 - (n+1)^3 x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} + \\ &\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3 x^{n+4}] \\ &= [x + 4x^2 + x^3 - (n+1)^3 x^{n+1} + l(n)x^{n+2} + m(n)x^{n+3} + n^3 x^{n+4}] \\ &= f_1(x) + f_{21(x)} + f_{22(x)} + f_{23(x)} + f_{24(x)} \\ g(x) &= (1-x)^4 \end{aligned}$$

where

$$\begin{aligned} f_1(x) &= x + 4x^2 + x^3 \\ f_{21}(x) &= -(n+1)^3 x^{n+1} \\ f_{22}(x) &= (3n^3 + 6n^2 - 4)x^{n+2} \\ f_{23}(x) &= (-3n^3 - 3n^2 + 3n - 1)x^{n+3} \\ f_{24}(x) &= n^3 x^{n+4} \end{aligned}$$

Derivatives of $g(x)$ and $f_1(x)$ and other functions:

$$\begin{aligned} f_1(x) &= x + 4x^2 + x^3 \\ f_1'(x) &= 1 + 8x + 3x^2 \\ f_1^{(2)}(x) &= 8 + 6x \\ f_1^{(3)}(x) &= 6 \\ f_1^{(4)}(x) &= 0 \end{aligned}$$

$$\begin{aligned} f_{21}(x) &= -(n+1)^3 x^{n+1} \\ f_{21}'(x) &= -(n+1)^4 x^n \\ f_{21}^{(2)}(x) &= -n(n+1)^4 x^{n-1} \\ f_{21}^{(3)}(x) &= -(n-1)n(n+1)^4 x^{n-2} \\ f_{21}^{(4)}(x) &= -(n-2)(n-1)n(n+1)^4 x^{n-3} \end{aligned}$$

$$
\begin{aligned}
f_{22}(x) &= (3n^3+6n^2-4)x^{n+2} \\
f'_{22}(x) &= (n+2)(3n^3+6n^2-4)x^{n+1} \\
f_{22}^{(2)}(x) &= (n+1)(n+2)(3n^3+6n^2-4)x^{n} \\
f_{22}^{(3)}(x) &= n(n+1)(n+2)(3n^3+6n^2-4)x^{n-1} \\
f_{22}^{(4)}(x) &= (n-1)n(n+1)(n+2)(3n^3+6n^2-4)x^{n-2}
\end{aligned}
$$

$$
\begin{aligned}
f_{23}(x) &= (-3n^3-3n^2+3n-1)x^{n+3} \\
f'_{23}(x) &= (n+3)(-3n^3-3n^2+3n-1)x^{n+2} \\
f_{23}^{(2)}(x) &= (n+2)(n+3)(-3n^3-3n^2+3n-1)x^{n+1} \\
f_{23}^{(3)}(x) &= (n+1)(n+2)(n+3)(-3n^3-3n^2+3n-1)x^{n} \\
f_{23}^{(4)}(x) &= n(n+1)(n+2)(n+3)(-3n^3-3n^2+3n-1)x^{n-1}
\end{aligned}
$$

$$
\begin{aligned}
f_{24}(x) &= n^3x^{n+4} \\
f'_{24}(x) &= (n+4)n^3x^{n+3} \\
f_{24}^{(2)}(x) &= (n+3)(n+4)n^3x^{n+2} \\
f_{24}^{(3)}(x) &= (n+2)(n+3)(n+4)n^3x^{n+1} \\
f_{24}^{(4)}(x) &= (n+1)(n+2)(n+3)(n+4)n^3x^{n}
\end{aligned}
$$

$$
\begin{aligned}
g(x) &= (1-x)^4 \\
g'(x) &= -4(1-x)^3 \\
g^{(2)}(x) &= 12(1-x)^2 \\
g^{(3)}(x) &= -24(1-x) \\
g^{(4)}(x) &= 24
\end{aligned}
$$

$$
\begin{aligned}
f_{21}^{(4)}(x)+f_{22}^{(4)}(x) &= -(n-2)(n-1)n(n+1)^4x^{n-3} \\
&\quad + (n-1)n(n+1)(n+2)(3n^3+6n^2-4)x^{n-2} \\
f_{21}^{(4)}(1)+f_{22}^{(4)}(1) &= -(n-2)(n-1)n(n+1)^4 \\
&\quad + (n-1)n(n+1)(n+2)(3n^3+6n^2-4) \\
&= (n-1)n(n+1)\left[-(n-2)(n+1)^3\right] \\
&\quad + (n-1)n(n+1)\left[(n+2)(3n^3+6n^2-4)\right] \\
&= (n-1)n(n+1)\left[-(n-2)(n+1)^3+(n+2)(3n^3+6n^2-4)\right] \\
&= (n-1)n(n+1)\left[(-n+2)(n^3+3n^2+3n+1)\right. \\
&\quad \left.+ (n+2)(3n^3+6n^2-4)\right] \\
&= (n-1)n(n+1)\left[\left((-n^4+2n^3)+(-3n^3+6n^2)+(-3n^2+6n)+(-n+2)\right)\right. \\
&\quad \left.+ \left((3n^4+6n^3)+(6n^3+12n^2)+(-4n-8)\right)\right] \\
&= (n-1)n(n+1)\left[(-1+3)n^4+(2-3+6+6)n^3\right. \\
&\quad \left.+ (6-3+12)n^2+(6-1-4)n+(2-8)\right] \\
&= (n-1)n(n+1)\left[2n^4+11n^3+15n^2+n-6\right]
\end{aligned}
$$

—

$$
\begin{aligned}
f_{23}^{(4)}(x)+f_{24}^{(4)}(x) &= n(n+1)(n+2)(n+3)(-3n^3-3n^2+3n-1)x^{n-1} \\
&\quad + (n+1)(n+2)(n+3)(n+4)n^3x(n) \\
f_{23}^{(4)}(1)+f_{24}^{(4)}(1) &= n(n+1)(n+2)(n+3)(-3n^3-3n^2+3n-1) \\
&\quad + (n+1)(n+2)(n+3)(n+4)n^3 \\
&= n(n+1)(n+2)(n+3)\left[-3n^3-3n^2+3n-1\right] \\
&\quad + n(n+1)(n+2)(n+3)\left[(n+4)n^2\right] \\
&= n(n+1)(n+2)(n+3)\left[(-3n^3-3n^2+3n-1)+(n+4)n^2\right] \\
&= n(n+1)(n+2)(n+3)\left[(-3n^3-3n^2+3n-1)+(n^3+4n^2)\right] \\
&= n(n+1)(n+2)(n+3)\left[(-3+1)n^3+(-3+4)n^2+3n-1\right] \\
&= n(n+1)(n+2)(n+3)\left[-2n^3+n^2+3n-1\right]
\end{aligned}
$$

Because $f_1^{(4)}(x)=0$

$$
\begin{aligned}
f^{(4)}(1) &= f_{21}^{(4)}(1)+f_{22}^{(4)}(1)+f_{23}^{(4)}(1)+f_{24}^{(4)}(1) \\
&= (n-1)n(n+1)\left[2n^4+11n^3+15n^2+n-6\right] \\
&\quad + n(n+1)(n+2)(n+3)\left[2n^3+n^2+3n-1\right] \\
&= n(n+1)\left[(n-1)(2n^4+11n^3+15n^2+n-6)\right] \\
&\quad + n(n+1)\left[(n+2)(n+3)(-2n^3+n^2+3n-1)\right] \\
&= n(n+1)\left[(n-1)(2n^4+11n^3+15n^2+n-6)+\right. \\
&\quad \left.+ (n+2)(n+3)(-2n^3+n^2+3n-1)\right]
\end{aligned}
$$

Let

$$\begin{aligned}
h_1(n) &= (n-1)(2n^4 + 11n^3 + 15n^2 + n - 6) \\
&= (2n^5 - 2n^4) + (11n^4 - 11n^3) + (15n^3 - 15n^2) + (n^2 - n) + (-6n + 6) \\
&= 2n^5 + (-2 + 11)n^4 + (-11 + 15)n^3 + (-15 + 1)n^2 + (-1 - 6)n + 6 \\
&= 2n^5 + 9n^4 + 4n^3 - 14n^2 - 7n + 6 \\
h_2(n) &= (n+2)(n+3)(-2n^3 + n^2 + 3n - 1) \\
&= (n^2 + 5n + 6)(-2n^3 + n^2 + 3n - 1) \\
&= (-2n^5 - 10n^4 - 12n^3) + (n^4 + 5n^3 + 6n^2) \\
&\quad + (3n^3 + 15n^2 + 18n) + (-n^2 - 5n - 6) \\
&= -2n^5 + (-10 + 1)n^4 + (-12 + 5 + 3)n^3 + (6 + 15 - 1)n^2 + (18 - 5)n - 6 \\
&= -2n^5 - 9n^4 - 4n^3 + 20n^2 + 13n - 6
\end{aligned}$$

$$\begin{aligned}
h_1(n) + h_2(n) &= (2n^5 + 9n^4 + 4n^3 - 14n^2 - 7n + 6) \\
&\quad + (-2n^5 - 9n^4 - 4n^3 + 22n^2 + 13n - 6) \\
&= 6n^2 + 6n \\
&= 6n(n+1)
\end{aligned}$$

Which means

$$\begin{aligned}
f^4(1) &= n(n+1)\,6n(n+1) \\
&= 6(n(n+1))^2 \\
\frac{f^4(1)}{g^4(1)} &= \frac{6(n(n+1))^2}{24} \\
&= \frac{n(n+1)^2}{4} \\
&= \left(\frac{n(n+1)}{2}\right)^2 \\
S(n,3) &= T(n,3,1) \\
&= \lim_{x\to 1} \frac{f(1)}{g(1)} \\
&= \frac{f^4(1)}{g^4(1)} \\
&= \left(\frac{n(n+1)}{2}\right)^2 \quad \square
\end{aligned}$$

Exercise 3

Compute $S(n,4) = \sum_{k=1}^{n} k^4$ using the recursion formula

$$S(n,i) = \frac{1}{i+1}\left((n+i)^{i+1} - 1 - \sum_{j=0}^{i-1} \binom{i+1}{j} S(n,j)\right) \forall i \geq 1$$

The term $\binom{i+1}{j}$ is the binomial coefficient defined as follows:

$$\binom{i+1}{j} = \frac{(i+1)!}{j!(i+1-j)!}$$

Answer

$$S(n,i) = \frac{1}{i+1}\left((n+1)^{i+1} - 1 - \sum_{j=0}^{i-1}\binom{i+1}{j}S(n,j)\right) \forall i \geq 1$$

$$\begin{aligned}
S(n,4) &= \frac{1}{5}\left((n+1)^5 - 1 - \sum_{j=0}^{3}\binom{5}{j}S(n,j)\right) \\
&= \frac{1}{5}\left((n+1)^5 - 1 - \left[\binom{5}{0}S(n,0) + \binom{5}{1}S(n,1) + \binom{5}{2}S(n,2) + \binom{5}{3}S(n,3)\right]\right) \\
&= \frac{1}{5}\left((n+1)^5 - 1 - \left[\binom{5}{0}n + \binom{5}{1}\frac{n(n+1)}{2} + \binom{5}{2}\frac{n(n+1)(2n+1)}{6} + \binom{5}{3}\left(\frac{n(n+1)}{2}\right)^2\right]\right) \\
&= \frac{1}{5}\left((n+1)^5 - 1 - \left[n + 5\frac{n(n+1)}{2} + 10\frac{n(n+1)(2n+1)}{6} + 10\left(\frac{n(n+1)}{2}\right)^2\right]\right) \\
&= \frac{1}{5}\left((n+1)^5 - \left[(n+1) + \frac{5n(n+1)}{2} + \frac{10n(n+1)(2n+1)}{6} + \left(10\frac{n(n+1)}{2}\right)^2\right]\right) \\
&= \frac{n+1}{5}\left((n+1)^4 - \left[1 + \frac{5n}{2} + \frac{10n(2n+1)}{6} + \frac{10n^2(n+1)}{4}\right]\right) \\
&= \frac{n+1}{5}\left((n+1)^4 - \left[1 + \frac{5n}{2} + \frac{5n(2n+1)}{3} + \frac{5n^2(n+1)}{2}\right]\right) \\
&= \frac{n+1}{30}(6(n+1)^4 - [6 + 15n + 10n(2n+1) + 15n^2(n+1)]) \\
&= \frac{n+1}{30}(6(n^4 + 4n^3 + 6n^2 + 4n + 1) \\
&\quad - [6 + 15n + (20n^2 + 10n) + 15n^3 + 15n^2]) \\
&= \frac{n+1}{30}([6n^4 + 24n^3 + 36n^2 + 24n + 6] \\
&\quad - [15n^3 + 35n^2 + 25n + 6]) \\
&= \frac{n+1}{30}(6n^4 + 9n^3 + n^2 - n) \\
&= \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30} \quad \square
\end{aligned}$$

Exercise 4

It's easy to see that the sequence $(x_n)_{n\geq 1}$ given by $x_n = \sum_{k=1}^{n} k^2$ satisfies the recursion

$$x_{n+1} = x_n + (n+1)^2, \forall n \geq 1$$

with $x_1 = 1$

i) By substituting $n+1$ for n in the above formula, obtain that

$$x_{n+2} = x_{n+1} + (n+2)^2$$

Subtract x_{n+1} from x_{n+2} to find that

$$x_{n+2} = 2x_{n+1} - x_n + 2n + 3 \quad \forall n \geq 1$$

with $x_1 = 1$ and $x_2 = 5$.

ii) Similarly, substitute $n+1$ for n in the above finding and obtain that

$$x_{n+3} = 2x_{n+2} - x_{n+1} + 2(n+1) + 3$$

Subtract x_{n+2} from x_{n+3} to find that

$$x_{n+3} = 3x_{n+2} - 3x_{n+1} + x_n + 2 \quad \forall n \geq 1$$

with $x_1 = 1$, $x_2 = 5$, and $x_3 = 14$.

iii) Use a similar method to prove that the sequence $(x_n)_{n \geq 0}$ satisfies the linear recursion

$$x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} + x_n = 0$$

The characteristic polynomial associated to the above recursion function is:

$$P(z) = z^4 - 4z^3 + 6z^2 - 4z + 1 = (z-1)^4$$

Use the fact that $x_1 = 1$, $x_2 = 5$, $x_3 = 14$, and $x_4 = 30$ to show that

$$x_n = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \geq 1$$

and conclude that

$$S(n,2) = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \geq 1$$

Answer

i)

$$\begin{aligned}
x_{n+1} &= x_n + (n+1)^2 \\
x_{n+2} &= x_{n+1} + (n+2)^2 \\
x_{n+2} - x_{n+1} &= \left[x_{n+1} + (n+2)^2\right] - \left[x_n + (n+1)^2\right] \\
x_{n+2} - x_{n+1} &= x_{n+1} - x_n + (n+2)^2 - (n+1)^2 \\
x_{n+2} - x_{n+1} &= x_{n+1} - x_n + \left(n^2 + 4n + 4\right) - \left(n^2 + 2n + 1\right) \\
x_{n+2} - x_{n+1} &= x_{n+1} - x_n + 2n + 3 \\
x_{n+2} &= 2x_{n+1} - x_n + 2n + 3
\end{aligned}$$

ii)

$$\begin{aligned}
x_{n+2} &= 2x_{n+1} - x_n + 2n + 3 \\
x_{n+3} &= 2x_{n+2} - x_{n+1} + 2(n+1) + 3 \\
x_{n+3} - x_{n+2} &= \left[2x_{n+2} - x_{n+1} + 2(n+1) + 3\right] - \left[2x_{n+1} - x_n + 2n + 3\right] \\
&= 2x_{n+2} + (-1-2)x_{n+1} + x_n + (2-2)n + (2+3-3) \\
&= 2x_{n+2} - 3x_{n+1} + x_n + 2 \\
x_{n+3} &= 3x_{n+2} - 3x_{n+1} + x_n + 2
\end{aligned}$$

iii)
$$\begin{aligned}
x_{n+3} &= 3x_{n+2} - 3x_{n+1} + x_n + 2 \\
x_{n+4} &= 3x_{n+3} - 3x_{n+2} + x_{n+1} + 2 \\
x_{n+4} - x_{n+3} &= [3x_{n+3} - 3x_{n+2} + x_{n+1} + 2] - [3x_{n+2} - 3x_{n+1} + x_n + 2] \\
&= 3x_{n+3} + (-3-3)x_{n+2} + (1+3)x_{n+1} - x_n + (2-2) \\
&= 3x_{n+3} - 6x_{n+2} + 4x_{n+1} - x_n \\
x_{n+4} &= 4x_{n+3} - 6x_{n+2} + 4x_{n+1} - x_n \\
x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} + x_n &= 0
\end{aligned}$$

The characteristic polynomial is:

$$P(z) = z^4 - 4z^3 + 6z^2 - 4z + 1 = (z-1)^4$$

Which means the only root of $P(z)$ is $\lambda = 1$. The general form of x_n:

$$x_n = C_1 + C_2 n + C_3 n^2 + C_4 n^3$$

As

$$\begin{aligned}
x_1 &= 1 \\
x_2 &= 5 \\
x_3 &= 14 \\
x_4 &= 30
\end{aligned}$$

We have this linear system:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 14 \\ 30 \end{pmatrix}$$

Solving the system yields

$$\begin{aligned}
C_1 &= 0 \\
C_2 &= \frac{1}{6} \\
C_3 &= \frac{1}{2} \\
C_4 &= \frac{1}{3}
\end{aligned}$$

Which means

$$\begin{aligned}
x_n &= C_1 + C_2 n + C_3 n^2 + C_4 n^3 \\
&= 0 + \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} \\
&= \frac{n(n+1)(2n+1)}{6} \quad \square
\end{aligned}$$

Exercise 5

Find the general form of the sequence $x(n)_{n \geq 0}$ satisfiying the linear recursion

$$x_{n+3} = 2x_{n+1} + x_n, \quad \forall n \geq 0$$

With $x_0 = 1$, $x_1 = -1$ and $x_2 = 1$

Answer

$$x_{n+3} = 2x_{n+1} + x_n x_{n+3} - 2x_{n+1} - x_n = 0$$

The characteristic polynomial is:

$$\begin{aligned} P(z) &= z^3 - 2z - 1 \\ &= (z+1)(z^2 - z - 1) \\ &= (z+1)\left[z - \frac{1+\sqrt{5}}{2}\right]\left[z - \frac{1-\sqrt{5}}{2}\right] \end{aligned}$$

Which means the roots of $P(z)$ are:

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= \frac{1+\sqrt{5}}{2} \\ \lambda_3 &= \frac{1-\sqrt{5}}{2} \end{aligned}$$

The general form of x_n is:

$$x_n = C_1 \lambda_1^n + C_2 \lambda_2^n + C_3 \lambda_3^n$$

With $x_0 = 1$, $x_1 = -1$ and $x_2 = 1$:

$$\begin{aligned} 1 &= C_1 + C_2 + C_3 \\ -1 &= C_1 + \frac{1+\sqrt{5}}{2} C_2 + \frac{1-\sqrt{5}}{2} C_3 \\ 1 &= C_1 + \frac{\left(1+\sqrt{5}\right)^2}{4} C_2 + \frac{\left(1-\sqrt{5}\right)^2}{4} C_3 \\ &= C_1 + \frac{3+\sqrt{5}}{2} C_2 + \frac{3-\sqrt{5}}{2} C_3 \end{aligned}$$

Solving the linear equation system yields:

$$\begin{aligned} C_1 &= -1 \\ C_2 &= \frac{1+\sqrt{5}}{\sqrt{5}} \\ C_3 &= \frac{-1+\sqrt{5}}{\sqrt{5}} \end{aligned}$$

Which means the general form of x_n is:

$$x_n = -1 + \left[\frac{1+\sqrt{5}}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n\right] + \left[\frac{-1+\sqrt{5}}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \quad \square$$

Exercise 6

The sequence $(x_n)_{n\geq 0}$ satisfies the recursion

$$x_{n+1} = 3x_n + 2, \quad \forall n \geq 0$$

with $x_0 = 1$.

i) Show that the sequence $(x_n)_{n\geq 0}$ satisfies the linear recursion

$$x_{n+2} = 4x_{n+1} - 3x_n, \quad \forall n \geq 0$$

with $x_0 = 1$ and $x_1 = 5$.

ii) Find the general formula for $x_n, n \geq 0$.

Answer

i) Replacing $n + 2$ into x_{n+1}, we have

$$\begin{aligned} x_{n+1} &= 3x_n + 2 \\ x_{n+2} &= 3x_{n+1} + 2 \end{aligned}$$

$$\begin{aligned} x_{n+2} - x_{n+1} &= 3x_{n+1} - 3x_n \\ x_{n+2} &= 4x_{n+1} - 3x_n \end{aligned}$$

ii) From the above transformation, we have:

$$x_{n+2} - 4x_{n+1} + 3x_n = 0$$

The characteristic polynomial is:

$$P(z) = z^2 - 4z + 3 = (z - 3)(z - 1)$$

Which means the roots of $P(z)$ are:

$$\begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= 1 \end{aligned}$$

The general form of x_n is:

$$x_n = C_1 \lambda_1^n + C_2 \lambda_2^n$$

With $x_0 = 1$ and $x_1 = 5$, we have this system of equations:

$$\begin{aligned} 1 &= C_1 + C_2 \\ 5 &= 3C_1 + C_2 \end{aligned}$$

Solving it yields:

$$\begin{aligned} C_1 &= 2 \\ C_2 &= -1 \end{aligned}$$

Then the general form of x_n is:

$$x_n = 2 \times 3^n - 1$$

Exercise 7

The sequence $(x_n)_{n\geq 0}$ satisfies the recursion

$$x_{n+1} = 3x_n + n + 2, \quad \forall n \geq 0$$

with $x_0 = 1$.

i) Show that the sequence $(x_n)_{n \geq 0}$ satisfies the linear recursion

$$x_{n+3} = 5x_{n+2} - 7x_{n+1} + 3x_n, \quad \forall n \geq 0$$

With $x_0 = 1$, $x_1 = 5$, and $x_2 = 18$.

ii) Find the general formula for x_n, $n \geq 0$.

Answer

i)

$$\begin{aligned}
x_{n+1} &= 3x_n + n + 2 \\
x_{n+2} &= 3x_{n+1} + (n+1) + 2 \\
x_{n+2} - x_{n+1} &= [3x_{n+1} + (n+1) + 2] - [3x_n + n + 2] \\
&= 3x_{n+1} - 3x_n + (1-1)n + (1+2-2) \\
&= 3x_{n+1} - 3x_n + 1 \\
x_{n+2} &= 4x_{n+1} - 3x_n + 1 \\
x_{n+3} &= 4x_{n+2} - 3x_{n+1} + 1 \\
x_{n+3} - x_{n+2} &= [4x_{n+2} - 3x_{n+1} + 1] - [4x_{n+1} - 3x_n + 1] \\
&= 4x_{n+2} + (-3-4)x_{n+1} + 3x_n + (1-1) \\
&= 4x_{n+2} - 7x_{n+1} + 3x_n \\
x_{n+3} &= 5x_{n+2} - 7x_{n+1} + 3x_n \quad \blacksquare
\end{aligned}$$

ii) From the above transformation, we have:

$$x_{n+3} - 5x_{n+2} + 7x_{n+1} - 3x_n = 0$$

The characteristic polynomial is:

$$P(z) = z^3 - 5z^2 + 7z - 3 = (z-3)(z-1)^2$$

Which means the roots of $P(z)$ are:

$$\begin{aligned}
\lambda_1 &= 3 \\
\lambda_2 &= 1
\end{aligned}$$

The general form of x_n is:

$$x_n = C_1 \lambda_1^n + C_2 + C_3 n$$

With $x_0 = 1$, $x_1 = 5$, and $x_2 = 18$, we have this system of equations:

$$\begin{aligned}
1 &= C_1 + C_2 \\
5 &= 3C_1 + C_2 + C_3 \\
18 &= 9C_1 + C_2 + 2C_3
\end{aligned}$$

Solving it yields:

$$C_1 = \frac{9}{4}$$
$$C_2 = -\frac{5}{4}$$
$$C_3 = -\frac{1}{2}$$

Then the general form of x_n is:

$$x_n = \frac{9}{4} \times 3^n - \frac{5}{4} - \frac{n}{2}$$

Exercise 8

Let $P(z) = \sum_{i=0}^{k} a_i z^i$ be the characteristic polynomial corresponding to the linear recursion

$$\sum_{i=0}^{k} a_i x_{n+i} = 0, \quad \forall n \geq 0$$

Assume that λ is a root of multiplicity of 2 of $P(z)$. Show that the sequence $(y_n)_{n\geq 0}$ is given by

$$y_n = Cn\lambda^n, \quad n \geq 0$$

Where C is an arbitrary constant, satistifes the recursion above.

Hint: Show that

$$\sum_{i=1}^{k} a_i y_{n+i} = Cn\lambda^n P(\lambda) + C\lambda^{n+1} P'(\lambda), \quad \forall n \geq 0$$

and recall that λ is a root of multiplicity 2 of the polynomial $P(z)$ if and only if $P(\lambda) = 0$ and $P'(\lambda) = 0$.

Answer

Because λ is a root of multiplicity 2 of $P(z)$

$$P(z) = (z - \lambda)^2 Q(z)$$

Where $Q(z)$ is some polynomial with $Q(\lambda) \neq 0$. It's easily see that:

$$P(\lambda) = (\lambda - \lambda)^2 Q(z) = 0$$

Using product rule:

$$\begin{aligned}
P'(z) &= \frac{d}{dz}[(z - \lambda)^2 Q(z)] \\
&= 2(z - \lambda)Q(z) + (z - \lambda)^2 Q'(z) \\
&= (z - \lambda)[2Q(z) + (z - \lambda)Q'(z)] \\
P'(\lambda) &= (\lambda - \lambda)[2Q(\lambda) + (\lambda - \lambda)Q'(z)] \\
&= 0[2Q(\lambda) + 0(\lambda - \lambda)Q'(z)] \\
&= 0
\end{aligned}$$

We then have

$$
\begin{aligned}
P(z) &= \sum_{i=0}^{k} a_i z^i \\
P(\lambda) &= \sum_{i=0}^{k} a_i \lambda^i = 0 \\
P'(z) &= \left(\sum_{i=0}^{k} a_i z^i\right)' = \sum_{i=1}^{k} i a_i z^{i-1} \\
P'(\lambda) &= \sum_{i=1}^{k} i a_i \lambda^{i-1} = 0
\end{aligned}
$$

Because $y_n = Cn\lambda^n$ then $y_{n+i} = C(n+i)\lambda^{n+1}$ and

$$
\begin{aligned}
\sum_{i=0}^{k} a_i y_{n+i} &= \sum_{i=0}^{k} a_i C(n+i)\lambda^{n+i} \\
&= Cn \sum_{i=0}^{k} a_i \lambda^{n+i} + C \sum_{i=0}^{k} i a_i \lambda^{n+i} \\
&= Cn\lambda^n \sum_{i=0}^{k} a_i \lambda^i + C\lambda^{n+1} \sum_{i=0}^{k} i a_i \lambda^{i-1} \\
&= Cn\lambda^n P(\lambda) + C\lambda^{n+1} P'(\lambda) \\
&= Cn\lambda^n \times 0 + C\lambda^{n+1} \times 0 \\
&= 0
\end{aligned}
$$

Or $(y_n)_{n\geq 0}$ satisfies the linear recursion.

Exercise 9

Let $n > 0$. Show that

$$
\begin{aligned}
O(x^n) + O(x^n) &= O(x^n), \quad \text{as } x \to 0 \\
o(x^n) + o(x^n) &= o(x^n), \quad \text{as } x \to 0
\end{aligned}
$$

For example, to prove the first equation, let $f(x) = O(x^n)$, and $g(x) = O(x^n)$ as $x \to 0$, and show that $f(x) + g(x) = O(x^n)$ as $x \to 0$, i.e., that

$$
\limsup_{x\to 0} \left| \frac{f(x) + g(x)}{x^n} \right| < \infty
$$

Answer

Let

- $O(x^n) = f(x)$ as $x \to 0$ iif $\limsup_{x\to 0} \left|\frac{f(x)}{x^n}\right| < \infty$
- $O(x^n) = g(x)$ as $x \to 0$ iif $\limsup_{x\to 0} \left|\frac{g(x)}{x^n}\right| < \infty$

We have

$$\left|\frac{f(x)+g(x)}{x^n}\right| \leq \left|\frac{f(x)}{x^n}\right| + \left|\frac{g(x)}{x^n}\right|$$

$$\limsup_{x\to 0}\left|\frac{f(x)+g(x)}{x^n}\right| \leq \limsup_{x\to 0}\left|\frac{f(x)}{x^n}\right| + \limsup_{x\to 0}\left|\frac{g(x)}{x^n}\right| < \infty$$

$$\limsup_{x\to 0}\left|\frac{f(x)+g(x)}{x^n}\right| < \infty$$

Because

$$\limsup_{x\to 0}\left|\frac{f(x)+g(x)}{x^n}\right| < \infty \Leftrightarrow f(x)+g(x) = O(x^n)$$

$$f(x) = O(x^n)$$

$$g(x) = O(x^n)$$

Therefore, $O(x^n) + O(x^n) = O(x^n)$.

—

Let

- $o(x^n) = h(x)$ as $x \to 0$ iif $\limsup_{x\to 0}\left|\frac{h(x)}{x^n}\right| = 0$
- $o(x^n) = i(x)$ as $x \to 0$ iif $\limsup_{x\to 0}\left|\frac{i(x)}{x^n}\right| = 0$

We have

$$\left|\frac{h(x)+i(x)}{x^n}\right| \leq \left|\frac{h(x)}{x^n}\right| + \left|\frac{i(x)}{x^n}\right|$$

$$\lim_{x\to 0}\left|\frac{h(x)+i(x)}{x^n}\right| \leq 0$$

Because

$$\left|\frac{h(x)+i(x)}{x^n}\right| \geq 0$$

Therefore,

$$\lim_{x\to 0}\left|\frac{h(x)+i(x)}{x^n}\right| = 0 \Leftrightarrow o(x^n) = h(x)+i(x)$$

Exercise 10

Prove that

$$\sum_{k=1}^{n} k^2 = O(n^3), \text{as } n \to \infty;$$

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + O(n^2), \text{as } n \to \infty;$$

i.e., show that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} k^2}{n^3} < \infty$$

and that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} k^2 - \frac{n^3}{3}}{n^2} < \infty$$

Similarly, prove that

$$\sum_{k=1}^{n} k^3 = O(n^4), \text{as } n \to \infty;$$

$$\sum_{k=1}^{n} k^3 = \frac{n^4}{4} + O(n^3), \text{as } n \to \infty;$$

Answer

Because

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} \\
&= \frac{(n^2+n)(2n+1)}{6} \\
&= \frac{(2n^3+2n^2)+(n^2+n)}{6} \\
&= \frac{2n^3+3n^2+n}{6} \\
&= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}
\end{aligned}$$

Then

$$\begin{aligned}
\limsup_{n\to\infty} \left| \frac{\sum_{k=1}^{n} k^2}{n^3} \right| &= \limsup_{n\to\infty} \left| \frac{\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}}{n^3} \right| \\
&= \limsup_{n\to\infty} \left| \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right| \\
&= \frac{1}{3} < \infty
\end{aligned}$$

$$\limsup_{n\to\infty} \left| \frac{\sum_{k=1}^{n} k^2}{n^3} \right| < \infty \Leftrightarrow \sum_{k=1}^{n} k^2 = O(n^3)$$

Also, we have

$$\begin{aligned}\limsup_{n\to\infty}\left|\frac{\sum_{k=1}^{n}k^2-\frac{n^3}{3}}{n^2}\right| &= \limsup_{n\to\infty}\left|\frac{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}-\frac{n^3}{3}}{n^2}\right| \\ &= \limsup_{n\to\infty}\left|\frac{\frac{n^2}{2}+\frac{n}{6}}{n^2}\right| \\ &= \limsup_{n\to\infty}\left|\frac{1}{2}+\frac{1}{6n}\right| \\ &= \frac{1}{3}<\infty\end{aligned}$$

$$\begin{aligned}\limsup_{n\to\infty}\left|\frac{\sum_{k=1}^{n}k^2-\frac{n^3}{3}}{n^2}\right|<\infty \Leftrightarrow \sum_{k=1}^{n}k^2-\frac{n^3}{3} &= O(n^2) \\ \sum_{k=1}^{n}k^2 &= O(n^2)+\frac{n^3}{3}\end{aligned}$$

For $\sum_{k=1}^{n}k^3$, because

$$\begin{aligned}\sum_{k=1}^{n}k^3 &= \left(\frac{n(n+1)}{2}\right)^2 \\ &= \frac{n^2(n+1)^2}{4} \\ &= \frac{n^2(n^2+2n+1)}{4} \\ &= \frac{n^4+2n^3+n^2}{4} \\ &= \frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}\end{aligned}$$

Then

$$\begin{aligned}\limsup_{n\to\infty}\left|\frac{\sum_{k=1}^{n}k^3}{n^4}\right| &= \limsup_{n\to\infty}\left|\frac{\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}}{n^4}\right| \\ &= \limsup_{n\to\infty}\left|\frac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}\right| \\ &= \frac{1}{4}<\infty\end{aligned}$$

$$\limsup_{n\to\infty}\left|\frac{\sum_{k=1}^{n}k^3}{n^4}\right|<\infty \Leftrightarrow \sum_{k=1}^{n}k^3=O(n^4)$$

$$
\begin{aligned}
\limsup_{n\to\infty}\left|\frac{\sum_{k=1}^{n} k^3 - \frac{n^4}{4}}{n^4}\right| &= \limsup_{n\to\infty}\left|\frac{\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}-\frac{n^4}{4}}{n^4}\right| \\
&= \limsup_{n\to\infty}\left|\frac{\frac{n^3}{2}+\frac{n^2}{4}}{n^4}\right| \\
&= \limsup_{n\to\infty}\left|\frac{1}{2n}+\frac{1}{4n^2}\right| \\
&= \frac{1}{2} < \infty
\end{aligned}
$$

$$
\limsup_{n\to\infty}\left|\frac{\sum_{k=1}^{n} k^3 - \frac{n^4}{4}}{n^4}\right| < \infty \Leftrightarrow \sum_{k=1}^{n} k^2 - \frac{n^4}{4} = O(n^3)
$$

$$
\sum_{k=1}^{n} k^2 = \frac{n^4}{4} + O(n^3) \quad \blacksquare
$$

Supplemental Exercise 1

Let $a > 0$ be a positive number. Compute:

$$
\sqrt{a+\sqrt{a+\sqrt{a+\ldots}}}
$$

Answer

Let $y = \sqrt{a+\sqrt{a+\sqrt{a+\ldots}}}$, $y > 0$. It means

$$
\begin{aligned}
y^2 &= a+\sqrt{a+\sqrt{a+\ldots}} \\
&= a + y \\
y^2 - y - a &= 0
\end{aligned}
$$

Use quadratic formula:

$$
y = \frac{-b \pm \sqrt{b^2 - 4a'c}}{2a'}
$$

With:

$$
\begin{aligned}
a' &= 1 \\
b &= -1 \\
c &= -a
\end{aligned}
$$

Since $b^2 - 4a'c = 1 + 4a > 0$, we have two roots:

$$
\begin{aligned}
y_1 &= \frac{1+\sqrt{1+4a}}{2} \\
y_2 &= \frac{1-\sqrt{1+4a}}{2}
\end{aligned}
$$

We easily see that $y_2 < 0$, so the only valid answer is y_1.