

05 Exercises

Exercise 1

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an odd function.

- i) Show that
 - $xf(x)$ is an even function
 - $x^2f(x)$ is an odd function
- ii) Show that
 - The function $g_1 : \mathbb{R} \rightarrow \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function
 - The function $g_2 : \mathbb{R} \rightarrow \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function
- iii) Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $h(x) = x^i f(x^j)$, where i and j are positive integers. When is $h(x)$ an odd function?

Answer

$f(x)$ is an odd function, which means:

$$\begin{aligned}f(x) &= -f(-x) \\f(-x) &= -f(x)\end{aligned}$$

- i) Let $f_1 : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f_1(x) = xf(x)$.

$$\begin{aligned}f_1(-x) &= -xf(-x) \\&= xf(x) \\&= f_1(x)\end{aligned}$$

Which means $f_1(x)$ is an even function.

Let $f_2 : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f_2(x) = x^2f(x)$.

$$\begin{aligned}f_2(-x) &= (-x)^2f(-x) \\&= x^2f(-x) \\&= -x^2f(x) \\&= -f_2(x)\end{aligned}$$

Which means $f_2(x)$ is an odd function.

- ii) We have

$$\begin{aligned}g_1(-x) &= f((-x)^2) \\&= f(x^2) \\&= g_1(x)\end{aligned}$$

Which means $g_1(x)$ is an even function.

$$\begin{aligned}g_2(-x) &= f((-x)^3) \\&= f(-x^3) \\&= -f(x^3) \\&= -g_2(x)\end{aligned}$$

Which means $g_2(x)$ is an odd function.

iii) Doing some transformation

$$\begin{aligned}
 h(x) &= x^i f(x^j) \\
 h(-x) &= (-x)^i f((-x)^j) \\
 &= (-1)^i x^i f((-x)^j) \\
 &= (-1)^i x^i f((-1)^j x^j) \\
 &= (-1)^i (-1)^j x^i f(x^j) \\
 &= (-1)^{i+j} x^i f(x^j) \\
 &= (-1)^{i+j} h(x)
 \end{aligned}$$

Because

- $(-1)^{i+j} = -1$ when $i + j$ is odd, and
- $(-1)^{i+j} = 1$ when $i + j$ is even

Then

- $h(x) = -h(x)$ or $h(x)$ is an odd function, when $i + j$ is odd, and
- $h(x) = h(x)$ or $h(x)$ is an even function, when $i + j$ is even

Exercise 2

Let

- $S(n, 2) = \sum_{k=1}^n k^2$ and
 - $S(n, 3) = \sum_{k=1}^n k^3$
- i) Let $T(n, 2, x) = \sum_{k=1}^n k^2 x^k$.

Use formula

$$T(n, j+1, x) = x \frac{d}{dx} (T(n, j, x)) \quad \forall j \geq 0$$

for $j = 1$, i.e.,

$$T(n, 2, x) = x \frac{d}{dx} (T(n, 1, x))$$

And formula

$$T(n, 1, x) = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

for $T(n, 1, x)$, to show that

$$T(n, 2, x) = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}$$

- ii) Note that $S(n, 2) = T(n, 2, 1)$. Use l'Hopital's rule to evaluate $T(n, 2, 1)$, and conclude that $S(n, 2) = \frac{n(n+1)(2n+1)}{6}$
- iii) Compute $T(n, 3, x) = \sum_{k=1}^n k^3 x^k$ using formula

$$T(n, j+1, x) = x \frac{d}{dx} (T(n, j, x)) \quad \forall j \geq 0$$

for $j = 2$, i.e.,

$$T(n, 3, x) = x \frac{d}{dx} (T(n, 2, x))$$

iv) Note that $S(n, 3) = T(n, 3, 1)$. Use l'Hopital's rule to evaluate $T(n, 3, 1)$ and conclude that $S(n, 3) = \left(\frac{n(n+1)}{2}\right)^2$.

Answer

i)

$$T(n, 2, x) = x \frac{d}{dx} \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

Using the quotient rule

$$\left[\frac{u(x)}{v(x)} \right] \frac{d}{dx} = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

With

$$u(x) = x - (n+1)x^{n+1} + nx^{n+2}$$

$$u'(x) = 1 - (n+1)^2x^n + n(n+2)x^{n+1}$$

$$v(x) = (1-x)^2$$

$$v'(x) = -2(1-x)$$

$$u'(x)v(x) = (1 - (n+1)^2x^n + n(n+2)x^{n+1})(1-x)^2$$

$$= (1-x)^2(1 - (n+1)^2x^n + n(n+2)x^{n+1})$$

$$u(x)v'(x) = (x - (n+1)x^{n+1} + nx^{n+2}) - 2(1-x)$$

$$= -2(1-x)(x - (n+1)x^{n+1} + nx^{n+2})$$

$$u'(x)v(x) - u(x)v'(x) = (1-x)^2(1 - (n+1)^2x^n + n(n+2)x^{n+1})$$

$$+ 2(1-x)(x - (n+1)x^{n+1} + nx^{n+2})$$

$$= (1-x)[(1-x)(1 - (n+1)^2x^n + n(n+2)x^{n+1})$$

$$+ 2(x - (n+1)x^{n+1} + nx^{n+2})]$$

$$= (1-x)(1+x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - n^2x^{n+2})$$

$$\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} = \frac{(1-x)(1+x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - n^2x^{n+2})}{(1-x)^4}$$

$$= \frac{1+x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - n^2x^{n+2}}{(1-x)^3}$$

Which means

$$x \frac{d}{dx} (T(n, 1, x)) = x \frac{1+x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - n^2x^{n+2}}{(1-x)^3}$$

$$= \frac{x + x^2 - (n+1)^2x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2x^{n+3}}{(1-x)^3} \quad \square$$

ii)

$$\begin{aligned}
 T(n, 2, x) &= \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \\
 T(n, 2, 1) &= \frac{1 + 1 - (n+1)^2 + (2n^2 + 2n - 1) - n^2}{0} \\
 &= \frac{2 - (n^2 + 2n + 1) + (2n^2 + 2n - 1) - n^2}{0} \\
 &= \frac{0}{0}
 \end{aligned}$$

Which is indeterminate. Apply l'Hopital's rule:

$$\begin{aligned}
 \lim_{x \rightarrow 1} T(n, 2, x) &= \lim_{x \rightarrow 1} \frac{\frac{d}{dx} [x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}]}{\frac{d}{dx} [(1-x)^3]} \\
 &= \lim_{x \rightarrow 1} \frac{1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}}{-3(1-x)^2} \\
 &= \lim_{x \rightarrow 1} \frac{2 - n(n+1)^3 x^{n-1} + (n+1)(n+2)(2n^2 + 2n - 1)x^n - (n+2)(n+3)n^2 x^{n+1}}{6(1-x)} \\
 &= \lim_{x \rightarrow 1} \frac{-(n-1)n(n+1)^3 x^{n-2} + n(n+1)(n+2)(2n^2 + 2n - 1)x^{n-1} - (n+1)(n+2)(n+3)n^2 x^n}{-6} \\
 &= \frac{-(n-1)n(n+1)^3 + n(n+1)(n+2)(2n^2 + 2n - 1) - (n+1)(n+2)(n+3)n^2}{-6} \\
 &= \frac{n(n+1) - (n-1)(n+1)^2 + n(n+1)(n+2)(2n^2 + 2n - 1) + n(n+1) - n(n+2)(n+3)}{-6} \\
 &= \frac{n(n+1)[-(n-1)(n+1)^2 + (n+2)(2n^2 + 2n - 1) - n(n+2)(n+3)]}{-6} \\
 &= \frac{n(n+1)[-2n-1]}{-6} = \frac{n(n+1)(2n+1)}{6}
 \end{aligned}$$

iii)

$$\begin{aligned}
 T(n, 3, x) &= x \frac{d}{dx} (T(n, 2, x)) \\
 &= x \frac{d}{dx} \left[\frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \right]
 \end{aligned}$$

Using the quotient rule

$$\left[\frac{u(x)}{v(x)} \right] \frac{d}{dx} = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

With

$$\begin{aligned}
u(x) &= x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3} \\
u'(x) &= 1 + 2x - (n+1)^3 x^n + (2n^2 + 2n - 1)(n+2)x^{n+1} - (n+3)n^2 x^{n+2} \\
v(x) &= (1-x)^3 \\
v'(x) &= -3(1-x)^2 \\
[v(x)]^2 &= (1-x)^6
\end{aligned}$$

$$\begin{aligned}
u'(x)v(x) &= (1 + 2x - (n+1)^3 x^n + (2n^2 + 2n - 1)(n+2)x^{n+1} - (n+3)n^2 x^{n+2}) (1-x)^3 \\
&= (1-x)(1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}) (1-x)^2 \\
u(x)v'(x) &= (x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}) \times -3(1-x)^2 \\
&= (-3x - 3x^2 + 3(n+1)^2 x^{n+1} - 3(2n^2 + 2n - 1)x^{n+2} + 3n^2 x^{n+3}) (1-x)^2
\end{aligned}$$

Let $g(x) = (1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2})$, which means $u'(x)v(x) = (1-x)g(x)(1-x)^2$. Here is the calculation of $(1-x)g(x)$ putting onto a grid:

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$g(x)$	1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)$	$-(n+3)n^2$	
$xg(x)$		1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)$	$-(n+3)n^2$
$(1-x)g(x)$	1	1	-2	$-(n+1)^3$	$n^3 + 3n^2$

Because

$$\begin{aligned}
(n+2)(2n^2 + 2n - 1) - [-(n+1)^3] &= (2n^3 + 4n^2 + 2n^2 + 4n - n - 2) \\
&\quad + (n^3 + 3n^2 + 3n + 1) \\
&= (2n^3 + 6n^2 + 3n - 2) \\
&\quad + (n^3 + 3n^2 + 3n + 1) \\
&= 3n^3 + 9n^2 + 6n - 1
\end{aligned}$$

and

$$\begin{aligned}
-(n+3)n^2 - (n+2)(2n^2 + 2n - 1) &= -(n^3 + 3n^2) \\
&\quad - (2n^3 + 6n^2 + 3n - 2) \\
&= -3n^3 - 9n^2 - 3n + 2
\end{aligned}$$

We have

$$\begin{aligned}
(1-x)g(x) &= 1 + x - 2x^2 - (n+1)^3 x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1} \\
&\quad + (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3} \\
u'(x)v(x) &= (1-x)g(x)(1-x)^2 \\
&= [1 + x - 2x^2 - (n+1)^3 x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1} \\
&\quad + (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3}] (1-x)^2
\end{aligned}$$

Let

$$\begin{aligned}
h_1(x) &= [1 + x - 2x^2 - (n+1)^3x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1} \\
&\quad + (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3}] \\
u'(x)v(x) &= h_1(x)(1-x)^2 \\
h_2(x) &= -3x - 3x^2 + 3(n+1)^2x^{n+1} \\
&\quad - 3(2n^2 + 2n - 1)x^{n+2} + 3n^2x^{n+3} \\
u(x)v'(x) &= h_2(x)(1-x)^2 \\
u'(x)v(x) - u(x)v'(x) &= (h_1(x) - h_2(x))(1-x)^2
\end{aligned}$$

The calculation can be put into a grid like this:

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$h_1(x)$	1	1	-2	$-(n+1)^3$	$3n^3 + 9n^2 + 6n - 1$	$-3n^3 - 9n^2 - 3n + 2$	$n^3 + 3n^2$
$h_2(x)$		-3	-3		$3(n+1)^2$	$-3(2n^2 + 2n - 1)$	$3n^2$
$h_1(x) - h_2(x)$	1	4	1	$-(n+1)^3$	n^3

Because

$$\begin{aligned}
(3n^3 + 9n^2 + 6n - 1) - 3(n+1)^2 &= (3n^3 + 9n^2 + 6n - 1) - (3n^2 + 6n + 3) \\
&= 3n^3 + 6n^2 - 4 \\
(-3n^3 - 9n^2 - 3n + 2) - (-3)(2n^2 + 2n - 1) &= (-3n^3 - 9n^2 - 3n + 2) + (6n^2 + 6n - 3) \\
&= -3n^3 - 3n^2 + 3n - 1
\end{aligned}$$

Then

$$\begin{aligned}
u'(x)v(x) - u(x)v'(x) &= [1 + 4x + x^2 - (n+1)^3x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3x^{n+3}](1-x)^2 \\
&\quad [1 + 4x + x^2 - (n+1)^3x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3x^{n+3}] \\
\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} &= \frac{(1-x)^4}{[1 + 4x + x^2 - (n+1)^3x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3x^{n+3}] \\
T(n, 3, x) &= x \frac{(1-x)^4}{[1 + 4x + x^2 - (n+1)^3x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3x^{n+4}]} \\
&= \frac{(1-x)^4}{(1-x)^4}
\end{aligned}$$

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The second way to solve this

$$\begin{aligned}
F(x) &= \frac{P(x)}{(1-x)^m} \\
&= \frac{u(x)}{v(x)} \\
F'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} \\
&= \frac{P'(x)(1-x)^m - P(x)(-m)(1-x)^{m-1}}{(1-x)^{2m}} \\
&= \frac{P'(x) + \frac{mP(x)}{1-x}}{(1-x)^m} \\
&= \frac{(1-x)P'(x) + mP(x)}{(1-x)^{m+1}}
\end{aligned}$$

Let

$$\begin{aligned}
P(x) &= x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3} \\
P'(x) &= 1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}
\end{aligned}$$

Using a grid to calculate $(1-x)P'(x)$

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$P(x)$		1	1		$-(n+1)^2$	$2n^2 + 2n - 1$	$-n^2$
$P'(x)$	1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)2$	$-(n+3)(2n^2 + 2n - 1)$	
$xP'(x)$		1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)2$	$-(n+3)(2n^2 + 2n - 1)$
$(1-x)P'(x)$	1	1	-2	$-(n+1)^3$	$(n+3)(2n^2 + 2n - 1)$

$$\begin{aligned}
\text{iv)} \quad & [x + 4x^2 + x^3 - (n+1)^3 x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} + \\
& (-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3 x^{n+4}] \\
T(n, 3, x) &= \frac{(1-x)^4}{[1 + 4 + 1 - (n+1)^3 + (3n^3 + 6n^2 - 4) + (-3n^3 - 3n^2 + 3n - 1) + n^3]} \\
T(n, 3, 1) &= \frac{[6 - (1 + 3n + 3n^2 + n^3) + (-4 + 6n^2 + 3n^3) + (-1 + 3n - 3n^2 - 3n^3) + n^3]}{0} \\
&= \frac{[6 - (1 + 3n + 3n^2 + n^3) + (-4 + 6n^2 + 3n^3) + (-1 + 3n - 3n^2 - 3n^3) + n^3]}{0}
\end{aligned}$$

Simplifying the numerator using a grid we have:

	1	n	n^2	n^3
	6			
	-1	-3	-3	-1
	-4		6	3
	-1	3	-3	-3

				1
Total	0	0	0	0

Which means the numerator is 0 and $T(n, 3, 1) = \frac{0}{0}$.

To apply l'Hopital's rule, let $f(x)$ be the numerator and $g(x)$ be the denominator and

$$\begin{aligned}
f(x) &= [x + 4x^2 + x^3 - (n+1)^3x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3x^{n+4}] \\
&= [x + 4x^2 + x^3 - (n+1)^3x^{n+1} + l(n)x^{n+2} + m(n)x^{n+3} + n^3x^{n+4}] \\
&= f_1(x) + f_{21(x)} + f_{22(x)} + f_{23(x)} + f_{24(x)}
\end{aligned}$$

$$g(x) = (1 - x)^4$$

where

$$f_1(x) = x + 4x^2 + x^3$$

$$f_{21}(x) = -(n+1)^3x^{n+1}$$

$$f_{22}(x) = (3n^3 + 6n^2 - 4)x^{n+2}$$

$$f_{23}(x) = (-3n^3 - 3n^2 + 3n - 1)x^{n+3}$$

$$f_{24}(x) = n^3x^{n+4}$$

Derivatives of $g(x)$ and $f_1(x)$ and other functions:

$$f_1(x) = x + 4x^2 + x^3$$

$$f_1'(x) = 1 + 8x + 3x^2$$

$$f_1^{(2)}(x) = 8 + 6x$$

$$f_1^{(3)}(x) = 6$$

$$f_1^{(4)}(x) = 0$$

$$f_{21}(x) = -(n+1)^3x^{n+1}$$

$$f_{21}'(x) = -(n+1)^4x^n$$

$$f_{21}^{(2)}(x) = -n(n+1)^4x^{n-1}$$

$$f_{21}^{(3)}(x) = -(n-1)n(n+1)^4x^{n-2}$$

$$f_{21}^{(4)}(x) = -(n-2)(n-1)n(n+1)^4x^{n-3}$$

$$f_{22}(x) = (3n^3 + 6n^2 - 4)x^{n+2}$$

$$f'_{22}(x) = (n+2)(3n^3 + 6n^2 - 4)x^{n+1}$$

$$f_{22}^{(2)}(x) = (n+1)(n+2)(3n^3 + 6n^2 - 4)x^n$$

$$f_{22}^{(3)}(x) = n(n+1)(n+2)(3n^3 + 6n^2 - 4)x^{n-1}$$

$$f_{22}^{(4)}(x) = (n-1)n(n+1)(n+2)(3n^3 + 6n^2 - 4)x^{n-2}$$

$$f_{23}(x) = (-3n^3 - 3n^2 + 3n - 1)x^{n+3}$$

$$f'_{23}(x) = (n+3)(-3n^3 - 3n^2 + 3n - 1)x^{n+2}$$

$$f_{23}^{(2)}(x) = (n+2)(n+3)(-3n^3 - 3n^2 + 3n - 1)x^{n+1}$$

$$f_{23}^{(3)}(x) = (n+1)(n+2)(n+3)(-3n^3 - 3n^2 + 3n - 1)x^n$$

$$f_{23}^{(4)}(x) = n(n+1)(n+2)(n+3)(-3n^3 - 3n^2 + 3n - 1)x^{n-1}$$

$$f_{24}(x) = n^3x^{n+4}$$

$$f'_{24}(x) = (n+4)n^3x^{n+3}$$

$$f_{24}^{(2)}(x) = (n+3)(n+4)n^3x^{n+2}$$

$$f_{24}^{(3)}(x) = (n+2)(n+3)(n+4)n^3x^{n+1}$$

$$f_{24}^{(4)}(x) = (n+1)(n+2)(n+3)(n+4)n^3x^n$$

$$g(x) = (1-x)^4$$

$$g'(x) = -4(1-x)^3$$

$$g^{(2)}(x) = 12(1-x)^2$$

$$g^{(3)}(x) = -24(1-x)$$

$$g^{(4)}(x) = 24$$

$$\begin{aligned}
f_{21}^{(4)}(x) + f_{22}^{(4)}(x) &= -(n-2)(n-1)n(n+1)^4x^{n-3} \\
&\quad + (n-1)n(n+1)(n+2)(3n^3+6n^2-4)x^{n-2} \\
f_{21}^{(4)}(1) + f_{22}^{(4)}(1) &= -(n-2)(n-1)n(n+1)^4 \\
&\quad + (n-1)n(n+1)(n+2)(3n^3+6n^2-4) \\
&= (n-1)n(n+1)[-(n-2)(n+1)^3] \\
&\quad + (n-1)n(n+1)[(n+2)(3n^3+6n^2-4)] \\
&= (n-1)n(n+1)[-(n-2)(n+1)^3 + (n+2)(3n^3+6n^2-4)] \\
&= (n-1)n(n+1)[(-n+2)(n^3+3n^2+3n+1) \\
&\quad + (n+2)(3n^3+6n^2-4)] \\
&= (n-1)n(n+1)[((-n^4+2n^3) + (-3n^3+6n^2) + (-3n^2+6n) + (-n+2)) \\
&\quad + ((3n^4+6n^3) + (6n^3+12n^2) + (-4n-8))] \\
&= (n-1)n(n+1)[(-1+3)n^4 + (2-3+6+6)n^3 \\
&\quad + (6-3+12)n^2 + (6-1-4)n + (2-8)] \\
&= (n-1)n(n+1)[2n^4 + 11n^3 + 15n^2 + n - 6]
\end{aligned}$$

—

$$\begin{aligned}
f_{23}^{(4)}(x) + f_{24}^{(4)}(x) &= n(n+1)(n+2)(n+3)(-3n^3-3n^2+3n-1)x^{n-1} \\
&\quad + (n+1)(n+2)(n+3)(n+4)n^3x(n) \\
f_{23}^{(4)}(1) + f_{24}^{(4)}(1) &= n(n+1)(n+2)(n+3)(-3n^3-3n^2+3n-1) \\
&\quad + (n+1)(n+2)(n+3)(n+4)n^3 \\
&= n(n+1)(n+2)(n+3)[-3n^3-3n^2+3n-1] \\
&\quad + n(n+1)(n+2)(n+3)[(n+4)n^2] \\
&= n(n+1)(n+2)(n+3)[(-3n^3-3n^2+3n-1) + (n+4)n^2] \\
&= n(n+1)(n+2)(n+3)[(-3n^3-3n^2+3n-1) + (n^3+4n^2)] \\
&= n(n+1)(n+2)(n+3)[(-3+1)n^3 + (-3+4)n^2 + 3n-1] \\
&= n(n+1)(n+2)(n+3)[-2n^3+n^2+3n-1]
\end{aligned}$$

Because $f_1^{(4)}(x) = 0$

$$\begin{aligned}
f^{(4)}(1) &= f_{21}^{(4)}(1) + f_{22}^{(4)}(1) + f_{23}^{(4)}(1) + f_{24}^{(4)}(1) \\
&= (n-1)n(n+1)[2n^4 + 11n^3 + 15n^2 + n - 6] \\
&\quad + n(n+1)(n+2)(n+3)[2n^3 + n^2 + 3n - 1] \\
&= n(n+1)[(n-1)(2n^4 + 11n^3 + 15n^2 + n - 6) \\
&\quad + n(n+1)[(n+2)(n+3)(-2n^3 + n^2 + 3n - 1)]] \\
&= n(n+1)[(n-1)(2n^4 + 11n^3 + 15n^2 + n - 6) + \\
&\quad + (n+2)(n+3)(-2n^3 + n^2 + 3n - 1)]
\end{aligned}$$

Let

$$\begin{aligned}
h_1(n) &= (n-1)(2n^4 + 11n^3 + 15n^2 + n - 6) \\
&= (2n^5 - 2n^4) + (11n^4 - 11n^3) + (15n^3 - 15n^2) + (n^2 - n) + (-6n + 6) \\
&= 2n^5 + (-2 + 11)n^4 + (-11 + 15)n^3 + (-15 + 1)n^2 + (-1 - 6)n + 6 \\
&= 2n^5 + 9n^4 + 4n^3 - 14n^2 - 7n + 6 \\
h_2(n) &= (n+2)(n+3)(-2n^3 + n^2 + 3n - 1) \\
&= (n^2 + 5n + 6)(-2n^3 + n^2 + 3n - 1) \\
&= (-2n^5 - 10n^4 - 12n^3) + (n^4 + 5n^3 + 6n^2) \\
&\quad + (3n^3 + 15n^2 + 18n) + (-n^2 - 5n - 6) \\
&= -2n^5 + (-10 + 1)n^4 + (-12 + 5 + 3)n^3 + (6 + 15 - 1)n^2 + (18 - 5)n - 6 \\
&= -2n^5 - 9n^4 - 4n^3 + 20n^2 + 13n - 6 \\
h_1(n) + h_2(n) &= (2n^5 + 9n^4 + 4n^3 - 14n^2 - 7n + 6) \\
&\quad + (-2n^5 - 9n^4 - 4n^3 + 22n^2 + 13n - 6) \\
&= 6n^2 + 6n \\
&= 6n(n+1)
\end{aligned}$$

Which means

$$\begin{aligned}
f^4(1) &= n(n+1)6n(n+1) \\
&= 6(n(n+1))^2 \\
\frac{f^4(1)}{g^4(1)} &= \frac{6(n(n+1))^2}{24} \\
&= \frac{n(n+1)^2}{4} \\
&= \left(\frac{n(n+1)}{2}\right)^2 \\
S(n, 3) &= T(n, 3, 1) \\
&= \lim_{x \rightarrow 1} \frac{f(1)}{g(1)} \\
&= \frac{f^4(1)}{g^4(1)} \\
&= \left(\frac{n(n+1)}{2}\right)^2 \quad \square
\end{aligned}$$

Exercise 3

Compute $S(n, 4) = \sum_{k=1}^n k^4$ using the recursion formula

$$S(n, i) = \frac{1}{i+1} \left((n+i)^{i+1} - 1 - \sum_{j=0}^{i-1} \binom{i+1}{j} S(n, j) \right) \forall i \geq 1$$

The term $\binom{i+1}{j}$ is the binomial coefficient defined as follows:

$$\binom{i+1}{j} = \frac{(i+1)!}{j!(i+1-j)!}$$

Answer

$$\begin{aligned} S(n, i) &= \frac{1}{i+1} \left((n+i)^{i+1} - 1 - \sum_{j=0}^{i-1} \binom{i+1}{j} S(n, j) \right) \forall i \geq 1 \\ S(n, 4) &= \frac{1}{5} \left((n+4)^5 - 1 - \sum_{j=0}^3 \binom{5}{j} S(n, j) \right) \\ &= \frac{1}{5} \left((n+4)^5 - 1 - \left[\binom{5}{0} S(n, 0) + \binom{5}{1} S(n, 1) \right. \right. \\ &\quad \left. \left. + \binom{5}{2} S(n, 2) + \binom{5}{3} S(n, 3) \right] \right) \\ &= \frac{1}{5} \left((n+4)^5 - 1 - \left[\binom{5}{0} n + \binom{5}{1} \frac{n(n+1)}{2} \right. \right. \\ &\quad \left. \left. + \binom{5}{2} \frac{n(n+1)(2n+1)}{6} + \binom{5}{3} \left(\frac{n(n+1)}{2} \right)^2 \right] \right) \\ &= \frac{1}{5} \left((n+4)^5 - 1 - \left[n + 5 \frac{n(n+1)}{2} + 10 \frac{n(n+1)(2n+1)}{6} + 10 \left(\frac{n(n+1)}{2} \right)^2 \right] \right) \\ &= \frac{1}{5} \left((n+4)^5 - 1 - \left[n + 5 \frac{n(n+1)}{2} + 10 \frac{n(n+1)(2n+1)}{6} + 10 \left(\frac{n(n+1)}{2} \right)^2 \right] \right) \\ &= \frac{1}{5} \left((n+4)^5 - (1+n) - 5 \frac{n(n+1)}{2} - 10 \frac{n(n+1)(2n+1)}{6} - 10 \left(\frac{n(n+1)}{2} \right)^2 \right) \\ &= \frac{(n+4)^5 - (n+1)}{5} - \frac{n(n+1)}{2} - n(n+1) \frac{2n+1}{3} - \frac{(n(n+1))^2}{2} \\ &= \frac{(n+4)^5 - (n+1)}{5} - 3 \frac{n(n+1)}{6} - 2n(n+1) \frac{2n+1}{6} - 3 \frac{(n(n+1))^2}{6} \\ &= \frac{(n+4)^5 - (n+1)}{5} - \frac{n(n+1)}{6} (3 + 2(2n+1) + 3n(n+1)) \\ &= \frac{(n+4)^5 - (n+1)}{5} - \frac{n(n+1)}{6} (3 + (4n+2) + (3n^2 + 3n)) \\ &= \frac{(n+4)^5 - (n+1)}{5} - \frac{n(n+1)(3n^2 + 7n + 5)}{6} \\ &= \frac{6((n+4)^5 - (n+1)) - 5(n(n+1)(3n^2 + 7n + 5))}{30} \\ &= \frac{6(n+4)^5 - (6(n+1) + 5(n(n+1)(3n^2 + 7n + 5)))}{30} \end{aligned}$$

Let

$$\begin{aligned}
f(x) &= 6(n+4)^5 \\
&= 6(n^5 + 5n^4 \cdot 4 + 10n^3 \cdot 4^2 + 10n^2 \cdot 4^3 + 5n \cdot 4^4 + 4^5) \\
&= 6(n^5 + 20n^4 + 160n^3 + 640n^2 + 1280n + 1024) \\
g(x) &= 6(n+1) + 5(n(n+1)(3n^2 + 7n + 5)) \\
&= 6n + 6 + 5(n^2 + n)(3n^2 + 7n + 5) \\
&= 6n + 6 + 5((3n^4 + 3n^3) + (7n^3 + 7n^2) + (5n^2 + 5n)) \\
&= 6n + 6 + 5(3n^4 + 10n^3 + 12n^2 + 5n) \\
&= 6n + 6 + (15n^4 + 50n^3 + 60n^2 + 25n) \\
&= 15n^4 + 50n^3 + 60n^2 + 31n + 6
\end{aligned}$$