05 Exercises

Exercise 1

Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function.

i) Show that
   - $xf(x)$ is an even function
   - $x^2 f(x)$ is an odd function

ii) Show that
   - The function $g_1 : \mathbb{R} \to \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function
   - The function $g_2 : \mathbb{R} \to \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function

iii) Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h(x) = x^i f(x^j)$, where i and j are positive integers. When is $h(x)$ an odd function?

Answer

$f(x)$ is an odd function, which means:

$$\begin{aligned} f(x) &= -f(-x) \\ f(-x) &= -f(x) \end{aligned} \tag{1.}$$

i) Let $f_1 : \mathbb{R} \to \mathbb{R}$ be defined as $f_1(x) = xf(x)$.

$$\begin{aligned} f_1(-x) &= -xf(-x) \\ &= xf(x) \\ &= f_1(x) \end{aligned} \tag{2.}$$

Which means $f_1(x)$ is an even function.

Let $f_2 : \mathbb{R} \to \mathbb{R}$ be defined as $f_2(x) = x^2 f(x)$.

$$\begin{aligned} f_2(-x) &= (-x)^2 f(-x) \\ &= x^2 f(-x) \\ &= -x^2 f(x) \\ &= -f_2(x) \end{aligned} \tag{3.}$$

Which means $f_2(x)$ is an odd function.

ii) We have

$$\begin{aligned} g_1(-x) &= f((-x)^2) \\ &= f(x^2) \\ &= g_1(x) \end{aligned} \tag{4.}$$

Which means $g_1(x)$ is an even function.

$$\begin{aligned} g_2(-x) &= f((-x)^3) \\ &= f(-x^3) \\ &= -f(x^3) \\ &= -g_2(x) \end{aligned} \tag{5.}$$

Which means $g_2(x)$ is an odd function.

iii) Doing some transformation

$$\begin{aligned} h(x) &= x^i f(x^j) \\ h(-x) &= (-x)^i f((-x)^j) \\ &= (-1)^i x^i f((-x)^j) \\ &= (-1)^i x^i f((-1)^j x^j) \\ &= (-1)^i (-1)^j x^i f(x^j) \\ &= (-1)^{i+j} x^i f(x^j) \\ &= (-1)^{i+j} h(x) \end{aligned} \tag{6.}$$

Because

- $(-1)^{i+j} = -1$ when $i + j$ is odd, and
- $(-1)^{i+j} = 1$ when $i + j$ is even

Then

- $h(x) = -h(x)$ or $h(x)$ is an odd function, when $i + j$ is odd, and
- $h(x) = h(x)$ or $h(x)$ is an even function, when $i + j$ is even

Exercise 2

Let

- $S(n, 2) = \sum_{k=1}^{n} k^2$ and
- $S(n, 3) = \sum_{k=1}^{n} k^3$

i) Let $T(n, 2, x) = \sum_{k=1}^{n} k^2 x^k$.

Use formula

$$T(n, j + 1, x) = x\frac{d}{dx}(T(n, j, x)) \quad \forall j \geq 0 \tag{7.}$$

for $j = 1$, i.e.,

$$T(n, 2, x) = x\frac{d}{dx}(T(n, 1, x)) \tag{8.}$$

And formula

$$T(n, 1, x) = \frac{x - (n + 1)x^{n+1} + nx^{n+2}}{(1 - x)^2} \tag{9.}$$

for $T(n, 1, x)$, to show that

$$T(n, 2, x) = \frac{x + x^2 - (n + 1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1 - x)^3} \tag{10.}$$

ii) TBA

iii) TBA

Answer

i) Replacing Equation 9 into Equation 8, we have:

$$
\begin{aligned}
T(n,2,x) &= x\frac{d}{dx}\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2} \\
&= \frac{x^2-(n+1)x^{n+2}+nx^{n+3}}{(1-x)^2}\frac{d}{dx}
\end{aligned}
\qquad 11.
$$

Using the quotient rule

$$
\left[\frac{u(x)}{v(x)}\right]\frac{d}{dx} = \frac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2} \qquad 12.
$$

With

$$
\begin{aligned}
u(x) &= \left(x^2-(n+1)x^{n+2} \qquad\quad + nx^{n+3}\right) \\
u'(x) &= \left(2x-(n+1)(n+2)x^{n+1}+n(n+3)x^{n+2}\right)
\end{aligned}
\qquad 13.
$$

and

$$
\begin{aligned}
v(x) &= (1-x)^2 \\
v'(x) &= 2(-1)(1-x) = -2(1-x) \\
v^2(x) &= (1-x)^4
\end{aligned}
\qquad 14.
$$

and

$$
\begin{aligned}
u'(x)v(x) &= \left(\left(2x-(n+1)(n+2)x^{n+1}+n(n+3)x^{n+2}\right)(1-x)^2\right) \\
u(x)v'(x) &= \left(x^2-(n+1)x^{n+2}+nx^{n+3}\right)(-2(1-x)) \qquad = \left(x^2-(n+1)x^{n+2}+nx^{n+3}\right)(-2(1-x))
\end{aligned}
\qquad 15.
$$