1.1 Brief review of differentiation

The function $f : R \to R$ is differentiable at the point $x \in \mathbb{R}$ if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, in which case the derivative $f'(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The function $f(x)$ is called differentiable if it is differentiable at all points x.

Theorem 1.1. (Product Rule.)

The product $f(x)g(x)$ of two differentiable functions $f(x)$ and $g(x)$ is differentiable, and

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

Theorem 1.2. (Quotient Rule.)

The quotient $\frac{f(x)}{g(x)}$ of two differentiable functions $f(x)$ and $g(x)$ is differentiable at every point x where the function $\frac{f(x)}{g(x)}$ is well defined, and

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Theorem 1.3. (Chain Rule.)

The composite function $(g \circ f) = g(f(x))$ of two differentiable functions $f(x)$ and $g(x)$ is differentiable at every point x where $g(f(x))$ is well defined and

$$(g(f(x)))' = g'(f(x))f'(x)$$

Example: Chain Rule is often used for power functions, exponential functions, and logarithmic functions.

$$\frac{d}{dx}((f(x))^n) = n(f(x))^{n-1} f'(x)$$

$$\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} f'(x)$$

$$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{\ln(f(x))}$$

Lemma 1.1.

Let $f : [a, b] \to [c, d]$ be a differentiable function, and assume that $f(x)$ has an inverse function denoted by $f^{-1}(x)$ with $f^{-1} : [c, d] \to [a, b]$. The function $f^{-1}(x)$ is differentiable at every point $x \in [c, d]$ where $f'(f^{-1}(x)) \neq 0$ and

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

$$\left(\frac{1}{f(x)}\right)' = \frac{1}{f'(f^{-1}(x))}$$

Reuse the Chain Rule

$$(g(f(z)))' = g'(f(z))f'(z)$$

With $g = f^{-1}$, we have

$$\begin{aligned}
\left(f^{-1}(f(z))\right)' &= \left(f^{-1}\right)'(f(z))f'(z) \\
z' &= \left(f^{-1}\right)'(f(z))f'(z) \\
1 &= \left(f^{-1}\right)'(f(z))f'(z)
\end{aligned}$$

Let $z = f^{-1}(x)$, then $f(z) = f(f^{-1}(x)) = x$ and

$$\begin{aligned}
1 &= \left(f^{-1}\right)'(x)f'\left(f^{-1}(x)\right) \\
\frac{1}{f'(f^{-1}(x))} &= \left(f^{-1}\right)'(x) \\
&= \left(f^{-1}(x)\right)'
\end{aligned}$$

Examples:

$$\begin{aligned}
\frac{d}{dx}\left(xe^{3x^2-1}\right) &= x'e^{3x^2-1} + x\left(e^{3x^2-1}\right)' \\
&= e^{3x^2-1} + x\left(3x^2-1\right)'\left(e^{3x^2-1}\right) \\
&= e^{3x^2-1} + 6x^2\left(e^{3x^2-1}\right) \\
&= \left(1+6x^2\right)\left(e^{3x^2-1}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\sqrt{3x^2-1}}{\sqrt{3x^2-1}+4}\right) &= \frac{d}{dx}\left(\frac{\sqrt{3x^2-1}+4-4}{\sqrt{3x^2-1}+4}\right) \\
&= \frac{d}{dx}\left(1-\frac{4}{\sqrt{3x^2-1}+4}\right) \\
&= \left(\frac{4}{\sqrt{3x^2-1}+4}\right)'
\end{aligned}$$

Let $f(x) = \sqrt{3x^2-1}+4$, then

$$\begin{aligned}
f^{-1}(x) &= \frac{1}{\sqrt{3x^2-1}+4} \\
\left(f^{-1}(x)\right)^2 &= \left(\frac{1}{\sqrt{3x^2-1}+4}\right)^2 \\
&= \frac{1}{(3x^2-1)+2\sqrt{3x^2-1}\cdot 4+16} \\
&= \frac{1}{3x^2+8\sqrt{3x^2-1}+15}
\end{aligned}$$

$$
\begin{aligned}
f'(x) &= \left(\sqrt{3x^2-1}+4\right)' \\
&= \sqrt{3x^2-1}' \\
&= \left((3x^2-1)^{\frac{1}{2}}\right)' \\
&= \frac{1}{2}(3x^2-1)'(3x^2-1)^{-\frac{1}{2}} \\
&= \frac{1}{2}\,6x \div \sqrt{3x^2-1} \\
&= \frac{3x}{\sqrt{3x^2-1}}
\end{aligned}
$$

and

$$
\begin{aligned}
\left(f^{-1}(x)\right)' &= \frac{1}{f'(f^{-1}(x))} \\
&= \frac{1}{3(f^{-1}(x)) \div \sqrt{3\left(f^{-1}(x)\right)^2-1}} \\
&= \frac{\sqrt{3\left(f^{-1}(x)\right)^2-1}}{3(f^{-1}(x))} \\
&= \frac{\sqrt{3\left(f^{-1}(x)\right)^2-1}}{3(f^{-1}(x))}
\end{aligned}
$$