1.3 Differentiating definite integrals

A definite integral of the form $\int_a^b f(x)$ is a real number. However, if a definite integral as functions as limits of integration, e.g.,

$$\int_{a(t)}^{b(t)} f(x)dx$$

Or if the function to be integrated is a function of the integrating variable and another variables, e.g.,

$$\int_a^b f(x,t)dx$$

Then the result of the integration is a function (of the variable t in both cases above). If certain conditions are met, this function is differentiable.

Lemma 1.2.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then,

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x)dx\right) = f(b(t))b'(t) - f(a(t))a'(t)$$

Where $a(t)$ and $b(t)$ are differentiable functions.

Lemma 1.3.

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that the partial derivative $\frac{\partial f}{\partial t}(x,t)$ exists and is continuous in both variables x and t. Then,

$$\frac{d}{dt}\left(\int_a^b f(x,t)dx\right) = \int_a^b$$

Lemma 1.4.

Let $f(x,t)$ to be a continuous function such that the partial derivative $\frac{\partial f}{\partial t}(x,t)$ exists and is continuous. Then,

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x,t)dx\right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}dx + f(b(t),t)b'(t) - f(a(t),t)a'(t)$$

Note that Lemma 1.2 and Lemma 1.3 are special cases of Lemma 1.4.