05 Exercises

Exercise 1

Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function.

i) Show that
   - $xf(x)$ is an even function
   - $x^2 f(x)$ is an odd function

ii) Show that
   - The function $g_1 : \mathbb{R} \to \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function
   - The function $g_2 : \mathbb{R} \to \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function

iii) Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h(x) = x^i f(x^j)$, where i and j are positive integers. When is $h(x)$ an odd function?

Answer

$f(x)$ is an odd function, which means:

$$\begin{aligned} f(x) &= -f(-x) \\ f(-x) &= -f(x) \end{aligned}$$

i) Let $f_1 : \mathbb{R} \to \mathbb{R}$ be defined as $f_1(x) = xf(x)$.

$$\begin{aligned} f_1(-x) &= -xf(-x) \\ &= xf(x) \\ &= f_1(x) \end{aligned}$$

Which means $f_1(x)$ is an even function.

Let $f_2 : \mathbb{R} \to \mathbb{R}$ be defined as $f_2(x) = x^2 f(x)$.

$$\begin{aligned} f_2(-x) &= (-x)^2 f(-x) \\ &= x^2 f(-x) \\ &= -x^2 f(x) \\ &= -f_2(x) \end{aligned}$$

Which means $f_2(x)$ is an odd function.

ii) We have

$$\begin{aligned} g_1(-x) &= f((-x)^2) \\ &= f(x^2) \\ &= g_1(x) \end{aligned}$$

Which means $g_1(x)$ is an even function.

$$\begin{aligned} g_2(-x) &= f((-x)^3) \\ &= f(-x^3) \\ &= -f(x^3) \\ &= -g_2(x) \end{aligned}$$

Which means $g_2(x)$ is an odd function.

iii) Doing some transformation

$$
\begin{aligned}
h(x) &= x^i f(x^j) \\
h(-x) &= (-x)^i f((-x)^j) \\
&= (-1)^i x^i f((-x)^j) \\
&= (-1)^i x^i f((-1)^j x^j) \\
&= (-1)^i (-1)^j x^i f(x^j) \\
&= (-1)^{i+j} x^i f(x^j) \\
&= (-1)^{i+j} h(x)
\end{aligned}
$$

Because

- $(-1)^{i+j} = -1$ when $i + j$ is odd, and
- $(-1)^{i+j} = 1$ when $i + j$ is even

Then

- $h(x) = -h(x)$ or $h(x)$ is an odd function, when $i + j$ is odd, and
- $h(x) = h(x)$ or $h(x)$ is an even function, when $i + j$ is even

Exercise 2

Let

- $S(n, 2) = \sum_{k=1}^{n} k^2$ and
- $S(n, 3) = \sum_{k=1}^{n} k^3$

i) Let $T(n, 2, x) = \sum_{k=1}^{n} k^2 x^k$.

Use formula

$$T(n, j + 1, x) = x \frac{d}{dx}(T(n, j, x)) \quad \forall j \geq 0$$

for $j = 1$, i.e.,

$$T(n, 2, x) = x \frac{d}{dx}(T(n, 1, x))$$

And formula

$$T(n, 1, x) = \frac{x - (n + 1)x^{n+1} + nx^{n+2}}{(1 - x)^2}$$

for $T(n, 1, x)$, to show that

$$T(n, 2, x) = \frac{x + x^2 - (n + 1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1 - x)^3}$$

ii) Note that $S(n, 2) = T(n, 2, 1)$. Use l'Hopital's rule to evaluate $T(n, 2, 1)$, and conclude that $S(n, 2) = \frac{n(n+1)(2n+1)}{6}$

iii) Compute $T(n, 3, x) = \sum_{k=1}^{n} k^3 x^k$ using formula

$$T(n, j + 1, x) = x \frac{d}{dx}(T(n, j, x)) \quad \forall j \geq 0$$

for $j = 2$, i.e,

$$T(n,3,x) = x\frac{d}{dx}(T(n,2,x))$$

iv) Note that $S(n,3) = T(n,3,1)$. Use l'Hopital's rule to evaluate $T(n,3,1)$ and conclude that $S(n,3) = \left(\frac{n(n+1)}{2}\right)^2$.

<u>Answer</u>

i)

$$T(n,2,x) = x\frac{d}{dx}\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$$

Using the quotient rule

$$\left[\frac{u(x)}{v(x)}\right]\frac{d}{dx} = \frac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2}$$

With

$$\begin{aligned} u(x) &= x-(n+1)x^{n+1}+nx^{n+2} \\ u'(x) &= 1-(n+1)^2x^n+n(n+2)x^{n+1} \\ v(x) &= (1-x)^2 \\ v'(x) &= -2(1-x) \end{aligned}$$

$$\begin{aligned} u'(x)v(x) &= \left(1-(n+1)^2x^n+n(n+2)x^{n+1}\right)(1-x)^2 \\ &= (1-x)^2\left(1-(n+1)^2x^n+n(n+2)x^{n+1}\right) \\ u(x)v'(x) &= \left(x-(n+1)x^{n+1}+nx^{n+2}\right)-2(1-x) \\ &= -2(1-x)\left(x-(n+1)x^{n+1}+nx^{n+2}\right) \end{aligned}$$

$$\begin{aligned} u'(x)v(x)-u(x)v'(x) &= (1-x)^2\left(1-(n+1)^2x^n+n(n+2)x^{n+1}\right) \\ &\quad +2(1-x)\left(x-(n+1)x^{n+1}+nx^{n+2}\right) \\ &= (1-x)\big[(1-x)\left(1-(n+1)^2x^n+n(n+2)x^{n+1}\right) \\ &\quad +2\left(x-(n+1)x^{n+1}+nx^{n+2}\right)\big] \\ &= (1-x)\left(1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}\right) \\ \frac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2} &= \frac{(1-x)\left(1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}\right)}{(1-x)^4} \\ &= \frac{1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}}{(1-x)^3} \end{aligned}$$

Which means

$$\begin{aligned} x\frac{d}{dx}(T(n,1,x)) &= x\frac{1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}}{(1-x)^3} \\ &= \frac{x+x^2-(n+1)^2x^{n+1}+(2n^2+2n-1)x^{n+2}-n^2x^{n+3}}{(1-x)^3} \qquad \square \end{aligned}$$

ii)

$$T(n,2,x) = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}$$

$$\begin{aligned} T(n,2,1) &= \frac{1 + 1 - (n+1)^2 + (2n^2+2n-1) - n^2}{0} \\ &= \frac{2 - (n^2+2n+1) + (2n^2+2n-1) - n^2}{0} \\ &= \frac{0}{0} \end{aligned}$$

Which is indeterminate. Apply l'Hopital's rule:

$$\begin{aligned} \lim_{x\to 1} T(n,2,x) &= \lim_{x\to 1} \frac{\frac{d}{dx}\left[x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}\right]}{\frac{d}{dx}[(1-x)^3]} \\ &= \lim_{x\to 1} \frac{1 + 2x - (n+1)^3 x^n + (n+2)(2n^2+2n-1)x^{n+1} - (n+3)n^2 x^{n+2}}{-3(1-x)^2} \\ &= \lim_{x\to 1} \frac{2 - n(n+1)^3 x^{n-1} + (n+1)(n+2)(2n^2+2n-1)x^n - (n+2)(n+3)n^2 x^{n+1}}{6(1-x)} \\ &= \lim_{x\to 1} \frac{-(n-1)n(n+1)^3 x^{n-2} + n(n+1)(n+2)(2n^2+2n-1)x^{n-1} - (n+1)(n+2)(n+3)n^2 x^n}{-6} \\ &= \frac{-(n-1)n(n+1)^3 + n(n+1)(n+2)(2n^2+2n-1) - (n+1)(n+2)(n+3)n^2}{-6} \\ &= \frac{n(n+1) -(n-1)(n+1)^2 + n(n+1)\ (n+2)(2n^2+2n-1) + n(n+1) -n(n+2)(n+3)}{-6} \\ &= \frac{n(n+1)[-(n-1)(n+1)^2 + (n+2)(2n^2+2n-1) - n(n+2)(n+3)]}{-6} \\ &= \frac{n(n+1)[-2n-1]}{-6} = \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

iii)

$$\begin{aligned} T(n,3,x) &= x\frac{d}{dx}\left(T(n,2,x)\right) \\ &= x\frac{d}{dx}\left[\frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}\right] \end{aligned}$$

Using the quotient rule

$$\left[\frac{u(x)}{v(x)}\right]\frac{d}{dx} = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

With

$$
\begin{aligned}
u(x) &= x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3} \\
u'(x) &= 1 + 2x - (n+1)^3 x^n + (2n^2+2n-1)(n+2)x^{n+1} - (n+3)n^2 x^{n+2} \\
v(x) &= (1-x)^3 \\
v'(x) &= -3(1-x)^2 \\
[v(x)]^2 &= (1-x)^6
\end{aligned}
$$

$$
\begin{aligned}
u'(x)v(x) &= \left(1 + 2x - (n+1)^3 x^n + (2n^2+2n-1)(n+2)x^{n+1} - (n+3)n^2 x^{n+2}\right)(1-x)^3 \\
&= (1-x)\left(1 + 2x - (n+1)^3 x^n + (n+2)(2n^2+2n-1)x^{n+1} - (n+3)n^2 x^{n+2}\right)(1-x)^2 \\
u(x)v'(x) &= \left(x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}\right) \times -3(1-x)^2 \\
&= \left(-3x - 3x^2 + 3(n+1)^2 x^{n+1} - 3(2n^2+2n-1)x^{n+2} + 3n^2 x^{n+3}\right)(1-x)^2
\end{aligned}
$$

Let $g(x) = (1 + 2x - (n+1)^3 x^n + (n+2)(2n^2+2n-1)x^{n+1} - (n+3)n^2 x^{n+2})$, which means $u'(x)v(x) = (1-x)g(x)\,(1-x)^2$. Here is the calculation of $(1-x)g(x)$ putting onto a grid:

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$g(x)$	1	2		$-(n+1)^3$	$(n+2)(2n^2+2n-1)$	$-(n+3)n^2$	
$xg(x)$		1	2		$-(n+1)^3$	$(n+2)(2n^2+2n-1)$	$-(n+3)n^2$
$(1-x)\,g(x)$	1	1	-2	$-(n+1)^3$	...	...	n^3+3n^2

Because

$$
\begin{aligned}
(n+2)(2n^2+2n-1) - [-(n+1)^3] &= (2n^3+4n^2+2n^2+4n-n-2) \\
&\quad + (n^3+3n^2+3n+1) \\
&= (2n^3+6n^2+3n-2) \\
&\quad + (n^3+3n^2+3n+1) \\
&= 3n^3+9n^2+6n-1
\end{aligned}
$$

and

$$
\begin{aligned}
-(n+3)n^2 - (n+2)(2n^2+2n-1) &= -(n^3+3n^2) \\
&\quad - (2n^3+6n^2+3n-2) \\
&= -3n^3-9n^2-3n+2
\end{aligned}
$$

We have

$$
\begin{aligned}
(1-x)\,g(x) &= 1 + x - 2x^2 - (n+1)^3 x^n + (3n^3+9n^2+6n-1)x^{n+1} \\
&\quad + (-3n^3-9n^2-3n+2)x^{n+2} + (n^3+3n^2)x^{n+3} \\
u'(x)v(x) &= (1-x)\,g(x)\,(1-x)^2 \\
&= [1 + x - 2x^2 - (n+1)^3 x^n + (3n^3+9n^2+6n-1)x^{n+1} \\
&\quad + (-3n^3-9n^2-3n+2)x^{n+2} + (n^3+3n^2)x^{n+3}]\,(1-x)^2
\end{aligned}
$$

Let

$$
\begin{aligned}
h_1(x) &= [1 + x - 2x^2 - (n+1)^3 x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1} \\
&\quad + (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3}] \\
u'(x)v(x) &= h_1(x)\,(1-x)^2 \\
h_2(x) &= -3x - 3x^2 + 3(n+1)^2 x^{n+1} \\
&\quad - 3(2n^2 + 2n - 1)x^{n+2} + 3n^2 x^{n+3} \\
u(x)v'(x) &= h_2(x)\,(1-x)^2
\end{aligned}
$$

$$
u'(x)v(x) - u(x)v'(x) = (h_1(x) - h_2(x))\,(1-x)^2
$$

The calculation can be put into a grid like this:

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$h_1(x)$	1	1	-2	$-(n+1)^3$	$3n^3 + 9n^2 + 6n - 1$	$-3n^3 - 9n^2 - 3n + 2$	$n^3 + 3n^2$
$h_2(x)$		-3	-3		$3(n+1)^2$	$-3(2n^2 + 2n - 1)$	$3n^2$
$h_1(x) - h_2(x)$	1	4	1	$-(n+1)^3$	…	…	n^3

Because

$$
\begin{aligned}
(3n^3 + 9n^2 + 6n - 1) - 3(n+1)^2 &= (3n^3 + 9n^2 + 6n - 1) - (3n^2 + 6n + 3) \\
&= 3n^3 + 6n^2 - 4
\end{aligned}
$$

$$
\begin{aligned}
(-3n^3 - 9n^2 - 3n + 2) - (-3)(2n^2 + 2n - 1) &= (-3n^3 - 9n^2 - 3n + 2) + (6n^2 + 6n - 3) \\
&= -3n^3 - 3n^2 + 3n - 1
\end{aligned}
$$

Then

$$
\begin{aligned}
u'(x)v(x) - u(x)v'(x) &= [1 + 4x + x^2 - (n+1)^3 x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3 x^{n+3}]\,(1-x)^2 \\
\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} &= \frac{\begin{gathered}[1 + 4x + x^2 - (n+1)^3 x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\ (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3 x^{n+3}]\end{gathered}}{(1-x)^4} \\
T(n,3,x) &= x\frac{\begin{gathered}[1 + 4x + x^2 - (n+1)^3 x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\ (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3 x^{n+3}]\end{gathered}}{(1-x)^4} \\
&= \frac{\begin{gathered}[x + 4x^2 + x^3 - (n+1)^3 x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} + \\ (-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3 x^{n+4}]\end{gathered}}{(1-x)^4}
\end{aligned}
$$

—

The second way to solve this

$$\begin{aligned}
F(x) &= \frac{P(x)}{(1-x)^m} \\
&= \frac{u(x)}{v(x)} \\
F'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} \\
&= \frac{P'(x)(1-x)^m - P(x)(-m)(1-x)^{m-1}}{(1-x)^{2m}} \\
&= \frac{P'(x) + \frac{mP(x)}{1-x}}{(1-x)^m} \\
&= \frac{(1-x)P'(x) + mP(x)}{(1-x)^{m+1}}
\end{aligned}$$

Let

$$\begin{aligned}
P(x) &= x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3} \\
P'(x) &= 1 + 2x - (n+1)^3 x^n + (n+2)(2n^2+2n-1)x^{n+1} - (n+3)n^2 x^{n+2}
\end{aligned}$$

Using a grid to calculate $(1-x)\,P'(x)$

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$P(x)$		1	1		$-(n+1)^2$	$2n^2+2n-1$	$-n^2$
$P'(x)$	1	2		$-(n+1)^3$	$(n+2)(2n^2+2n-1)2$	$-(n+3)(2n^2+2n-1)$	
$xP'(x)$		1	2		$-(n+1)^3$	$(n+2)(2n^2+2n-1)2$	$-(n+3)(2n^2+2n-1)$
$(1-x)P'(x)$	1	1	-2	$-(n+1)^3$	$\ldots$	$\ldots$	$(n+3)(2n^2+2n-1)$

iv)

$$\begin{aligned}
T(n,3,x) &= \frac{[x + 4x^2 + x^3 - (n+1)^3 x^{n+1} + (3n^3+6n^2-4)x^{n+2} + (-3n^3-3n^2+3n-1)x^{n+3} + n^3 x^{n+4}]}{(1-x)^4} \\
T(n,3,1) &= \frac{[1+4+1-(n+1)^3 + (3n^3+6n^2-4) + (-3n^3-3n^2+3n-1) + n^3]}{0} \\
&= \frac{[6 - (1+3n+3n^2+n^3) + (-4+6n^2+3n^3) + (-1+3n-3n^2-3n^3) + n^3]}{0}
\end{aligned}$$

Simplifying the numerator using a grid we have:

	1	n	n^2	n^3
	6			
	-1	-3	-3	-1
	-4		6	3
	-1	3	-3	-3

				1
Total	0	0	0	0

Which means the numerator is 0 and $T(n,3,1) = \frac{0}{0}$.

To apply l'Hopital's rule, let $f(x)$ be the numerator and $g(x)$ be the denominator and

$$\begin{aligned}
f(x) &= [x + 4x^2 + x^3 - (n+1)^3 x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3 x^{n+4}] \\
&= [x + 4x^2 + x^3 - (n+1)^3 x^{n+1} + l(n)x^{n+2} + m(n)x^{n+3} + n^3 x^{n+4}] \\
&= f_1(x) + f_{21(x)} + f_{22(x)} + f_{23(x)} + f_{24(x)} \\
g(x) &= (1-x)^4
\end{aligned}$$

where

$$\begin{aligned}
f_1(x) &= x + 4x^2 + x^3 \\
f_{21}(x) &= -(n+1)^3 x^{n+1} \\
f_{22}(x) &= (3n^3 + 6n^2 - 4)x^{n+2} \\
f_{23}(x) &= (-3n^3 - 3n^2 + 3n - 1)x^{n+3} \\
f_{24}(x) &= n^3 x^{n+4}
\end{aligned}$$

Derivatives of $g(x)$ and $f_1(x)$ and other functions:

$$\begin{aligned}
f_1(x) &= x + 4x^2 + x^3 \\
f_1'(x) &= 1 + 8x + 3x^2 \\
f_1^{(2)}(x) &= 8 + 6x \\
f_1^{(3)}(x) &= 6 \\
f_1^{(4)}(x) &= 0
\end{aligned}$$

$$\begin{aligned}
f_{21}(x) &= -(n+1)^3 x^{n+1} \\
f_{21}'(x) &= -(n+1)^4 x^n \\
f_{21}^{(2)}(x) &= -n(n+1)^4 x^{n-1} \\
f_{21}^{(3)}(x) &= -(n-1)n(n+1)^4 x^{n-2} \\
f_{21}^{(4)}(x) &= -(n-2)(n-1)n(n+1)^4 x^{n-3}
\end{aligned}$$

$$
\begin{aligned}
f_{22}(x) &= (3n^3 + 6n^2 - 4)x^{n+2} \\
f'_{22}(x) &= (n+2)(3n^3 + 6n^2 - 4)x^{n+1} \\
f^{(2)}_{22}(x) &= (n+1)(n+2)(3n^3 + 6n^2 - 4)x^{n} \\
f^{(3)}_{22}(x) &= n(n+1)(n+2)(3n^3 + 6n^2 - 4)x^{n-1} \\
f^{(4)}_{22}(x) &= (n-1)n(n+1)(n+2)(3n^3 + 6n^2 - 4)x^{n-2}
\end{aligned}
$$

$$
\begin{aligned}
f_{23}(x) &= (-3n^3 - 3n^2 + 3n - 1)x^{n+3} \\
f'_{23}(x) &= (n+3)(-3n^3 - 3n^2 + 3n - 1)x^{n+2} \\
f^{(2)}_{23}(x) &= (n+2)(n+3)(-3n^3 - 3n^2 + 3n - 1)x^{n+1} \\
f^{(3)}_{23}(x) &= (n+1)(n+2)(n+3)(-3n^3 - 3n^2 + 3n - 1)x^{n} \\
f^{(4)}_{23}(x) &= n(n+1)(n+2)(n+3)(-3n^3 - 3n^2 + 3n - 1)x^{n-1}
\end{aligned}
$$

$$
\begin{aligned}
f_{24}(x) &= n^3 x^{n+4} \\
f'_{24}(x) &= (n+4)n^3 x^{n+3} \\
f^{(2)}_{24}(x) &= (n+3)(n+4)n^3 x^{n+2} \\
f^{(3)}_{24}(x) &= (n+2)(n+3)(n+4)n^3 x^{n+1} \\
f^{(4)}_{24}(x) &= (n+1)(n+2)(n+3)(n+4)n^3 x^{n}
\end{aligned}
$$

$$
\begin{aligned}
g(x) &= (1-x)^4 \\
g'(x) &= -4(1-x)^3 \\
g^{(2)}(x) &= 12(1-x)^2 \\
g^{(3)}(x) &= -24(1-x) \\
g^{(4)}(x) &= 24
\end{aligned}
$$

$$
\begin{aligned}
f_{21}^{(4)}(x) + f_{22}^{(4)}(x) &= -(n-2)(n-1)n(n+1)^4 x^{n-3} \\
&\quad + (n-1)n(n+1)(n+2)(3n^3+6n^2-4)x^{n-2} \\
f_{21}^{(4)}(1) + f_{22}^{(4)}(1) &= -(n-2)(n-1)n(n+1)^4 \\
&\quad + (n-1)n(n+1)(n+2)(3n^3+6n^2-4) \\
&= (n-1)n(n+1)[-(n-2)(n+1)^3] \\
&\quad + (n-1)n(n+1)[(n+2)(3n^3+6n^2-4)] \\
&= (n-1)n(n+1)[-(n-2)(n+1)^3 + (n+2)(3n^3+6n^2-4)] \\
&= (n-1)n(n+1)[(-n+2)(n^3+3n^2+3n+1) \\
&\quad + (n+2)(3n^3+6n^2-4)] \\
&= (n-1)n(n+1)[((-n^4+2n^3)+(-3n^3+6n^2)+(-3n^2+6n)+(-n+2)) \\
&\quad + ((3n^4+6n^3)+(6n^3+12n^2)+(-4n-8))] \\
&= (n-1)n(n+1)[(-1+3)n^4+(2-3+6+6)n^3 \\
&\quad + (6-3+12)n^2+(6-1-4)n+(2-8)] \\
&= (n-1)n(n+1)[2n^4+11n^3+15n^2+n-6]
\end{aligned}
$$

—

$$
\begin{aligned}
f_{23}^{(4)}(x) + f_{24}^{(4)}(x) &= n(n+1)(n+2)(n+3)(-3n^3-3n^2+3n-1)x^{n-1} \\
&\quad + (n+1)(n+2)(n+3)(n+4)n^3 x(n) \\
f_{23}^{(4)}(1) + f_{24}^{(4)}(1) &= n(n+1)(n+2)(n+3)(-3n^3-3n^2+3n-1) \\
&\quad + (n+1)(n+2)(n+3)(n+4)n^3 \\
&= n(n+1)(n+2)(n+3)[-3n^3-3n^2+3n-1] \\
&\quad + n(n+1)(n+2)(n+3)[(n+4)n^2] \\
&= n(n+1)(n+2)(n+3)[(-3n^3-3n^2+3n-1)+(n+4)n^2] \\
&= n(n+1)(n+2)(n+3)[(-3n^3-3n^2+3n-1)+(n^3+4n^2)] \\
&= n(n+1)(n+2)(n+3)[(-3+1)n^3+(-3+4)n^2+3n-1] \\
&= n(n+1)(n+2)(n+3)[-2n^3+n^2+3n-1]
\end{aligned}
$$

Because $f_1^{(4)}(x) = 0$

$$
\begin{aligned}
f^{(4)}(1) &= f_{21}^{(4)}(1) + f_{22}^{(4)}(1) + f_{23}^{(4)}(1) + f_{24}^{(4)}(1) \\
&= (n-1)n(n+1)[2n^4+11n^3+15n^2+n-6] \\
&\quad + n(n+1)(n+2)(n+3)[2n^3+n^2+3n-1] \\
&= n(n+1)[(n-1)(2n^4+11n^3+15n^2+n-6)] \\
&\quad + n(n+1)[(n+2)(n+3)(-2n^3+n^2+3n-1)] \\
&= n(n+1)[(n-1)(2n^4+11n^3+15n^2+n-6) + \\
&\quad + (n+2)(n+3)(-2n^3+n^2+3n-1)]
\end{aligned}
$$

Let

$$
\begin{aligned}
h_1(n) &= (n-1)(2n^4 + 11n^3 + 15n^2 + n - 6) \\
&= (2n^5 - 2n^4) + (11n^4 - 11n^3) + (15n^3 - 15n^2) + (n^2 - n) + (-6n + 6) \\
&= 2n^5 + (-2 + 11)n^4 + (-11 + 15)n^3 + (-15 + 1)n^2 + (-1 - 6)n + 6 \\
&= 2n^5 + 9n^4 + 4n^3 - 14n^2 - 7n + 6 \\
h_2(n) &= (n+2)(n+3)(-2n^3 + n^2 + 3n - 1) \\
&= (n^2 + 5n + 6)(-2n^3 + n^2 + 3n - 1) \\
&= (-2n^5 - 10n^4 - 12n^3) + (n^4 + 5n^3 + 6n^2) \\
&\quad + (3n^3 + 15n^2 + 18n) + (-n^2 - 5n - 6) \\
&= -2n^5 + (-10 + 1)n^4 + (-12 + 5 + 3)n^3 + (6 + 15 - 1)n^2 + (18 - 5)n - 6 \\
&= -2n^5 - 9n^4 - 4n^3 + 20n^2 + 13n - 6
\end{aligned}
$$

$$
\begin{aligned}
h_1(n) + h_2(n) &= (2n^5 + 9n^4 + 4n^3 - 14n^2 - 7n + 6) \\
&\quad + (-2n^5 - 9n^4 - 4n^3 + 22n^2 + 13n - 6) \\
&= 6n^2 + 6n \\
&= 6n(n+1)
\end{aligned}
$$

Which means

$$
\begin{aligned}
f^4(1) &= n(n+1)\,6n(n+1) \\
&= 6(n(n+1))^2 \\
\frac{f^4(1)}{g^4(1)} &= \frac{6(n(n+1))^2}{24} \\
&= \frac{n(n+1)^2}{4} \\
&= \left(\frac{n(n+1)}{2}\right)^2 \\
S(n,3) &= T(n,3,1) \\
&= \lim_{x \to 1} \frac{f(1)}{g(1)} \\
&= \frac{f^4(1)}{g^4(1)} \\
&= \left(\frac{n(n+1)}{2}\right)^2 \quad \square
\end{aligned}
$$

<u>Exercise 3</u>

Compute $S(n,4) = \sum_{k=1}^{n} k^4$ using the recursion formula

$$
S(n,i) = \frac{1}{i+1}\left((n+i)^{i+1} - 1 - \sum_{j=0}^{i-1} \binom{i+1}{j} S(n,j)\right) \forall i \geq 1
$$

The term $\binom{i+1}{j}$ is the binomial coefficient defined as follows:

$$
\binom{i+1}{j} = \frac{(i+1)!}{j!(i+1-j)!}
$$

Answer

$$
\begin{aligned}
S(n,i) &= \frac{1}{i+1}\left((n+i)^{i+1} - 1 - \sum_{j=0}^{i-1}\binom{i+1}{j}S(n,j)\right) \forall i \geq 1 \\
S(n,4) &= \frac{1}{5}\left((n+4)^5 - 1 - \sum_{j=0}^{3}\binom{5}{j}S(n,j)\right) \\
&= \frac{1}{5}\left((n+4)^5 - 1 - \left[\binom{5}{0}S(n,0) + \binom{5}{1}S(n,1)\right.\right. \\
&\quad \left.\left.+ \binom{5}{2}S(n,2) + \binom{5}{3}S(n,3)\right]\right) \\
&= \frac{1}{5}\left((n+4)^5 - 1 - \left[\binom{5}{0}n + \binom{5}{1}\frac{n(n+1)}{2}\right.\right. \\
&\quad \left.\left.+ \binom{5}{2}\frac{n(n+1)(2n+1)}{6} + \binom{5}{3}\left(\frac{n(n+1)}{2}\right)^2\right]\right) \\
&= \frac{1}{5}\left((n+4)^5 - 1 - \left[n + 5\frac{n(n+1)}{2} + 10\frac{n(n+1)(2n+1)}{6} + 10\left(\frac{n(n+1)}{2}\right)^2\right]\right) \\
&= \frac{1}{5}\left((n+4)^5 - 1 - \left[n + 5\frac{n(n+1)}{2} + 10\frac{n(n+1)(2n+1)}{6} + 10\left(\frac{n(n+1)}{2}\right)^2\right]\right) \\
&= \frac{1}{5}\left((n+4)^5 - (1+n) - 5\frac{n(n+1)}{2} - 10\frac{n(n+1)(2n+1)}{6} - 10\left(\frac{n(n+1)}{2}\right)^2\right) \\
&= \frac{(n+4)^5 - (n+1)}{5} - \frac{n(n+1)}{2} - n(n+1)\frac{2n+1}{3} - \frac{(n(n+1))^2}{2} \\
&= \frac{(n+4)^5 - (n+1)}{5} - 3\frac{n(n+1)}{6} - 2n(n+1)\frac{2n+1}{6} - 3\frac{(n(n+1))^2}{6} \\
&= \frac{(n+4)^5 - (n+1)}{5} - \frac{n(n+1)}{6}(3 + 2(2n+1) + 3n(n+1)) \\
&= \frac{(n+4)^5 - (n+1)}{5} - \frac{n(n+1)}{6}(3 + (4n+2) + (3n^2+3n)) \\
&= \frac{(n+4)^5 - (n+1)}{5} - \frac{n(n+1)(3n^2+7n+5)}{6} \\
&= \frac{6((n+4)^5 - (n+1)) - 5(n(n+1)(3n^2+7n+5))}{30}
\end{aligned}
$$

Let

$$
\begin{aligned}
f(x) &= 6((n+4)^5 - (n+1)) \\
&= 6((n^5 + 5n^4 4 + 10n^3 4^2 + 10n^2 4^3 + 5n4^4 + 4^5) - (n+1)) \\
&= 6((n^5 + 20n^4 + 160n^3 + 640n^2 + 1280n + 1024) - (n+1)) \\
&= 6(n^5 + 20n^4 + 160n^3 + 640n^2 + 1279n + 1023) \\
g(x) &= 5(n(n+1)(-3n^2 - n + 1)) \\
&= 5(n^2+n)(-3n^2 - n + 1)
\end{aligned}
$$