

## 05 Exercises

### Exercise 1

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an odd function.

- i) Show that
  - $xf(x)$  is an even function
  - $x^2f(x)$  is an odd function
- ii) Show that
  - The function  $g_1 : \mathbb{R} \rightarrow \mathbb{R}$  given by  $g_1(x) = f(x^2)$  is an even function
  - The function  $g_2 : \mathbb{R} \rightarrow \mathbb{R}$  given by  $g_2(x) = f(x^3)$  is an odd function
- iii) Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $h(x) = x^i f(x^j)$ , where  $i$  and  $j$  are positive integers. When is  $h(x)$  an odd function?

### Answer

$f(x)$  is an odd function, which means:

$$\begin{aligned}f(x) &= -f(-x) \\f(-x) &= -f(x)\end{aligned}$$

- i) Let  $f_1 : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f_1(x) = xf(x)$ .

$$\begin{aligned}f_1(-x) &= -xf(-x) \\&= xf(x) \\&= f_1(x)\end{aligned}$$

Which means  $f_1(x)$  is an even function.

Let  $f_2 : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f_2(x) = x^2f(x)$ .

$$\begin{aligned}f_2(-x) &= (-x)^2f(-x) \\&= x^2f(-x) \\&= -x^2f(x) \\&= -f_2(x)\end{aligned}$$

Which means  $f_2(x)$  is an odd function.

- ii) We have

$$\begin{aligned}g_1(-x) &= f((-x)^2) \\&= f(x^2) \\&= g_1(x)\end{aligned}$$

Which means  $g_1(x)$  is an even function.

$$\begin{aligned}g_2(-x) &= f((-x)^3) \\&= f(-x^3) \\&= -f(x^3) \\&= -g_2(x)\end{aligned}$$

Which means  $g_2(x)$  is an odd function.

iii) Doing some transformation

$$\begin{aligned}
 h(x) &= x^i f(x^j) \\
 h(-x) &= (-x)^i f((-x)^j) \\
 &= (-1)^i x^i f((-x)^j) \\
 &= (-1)^i x^i f((-1)^j x^j) \\
 &= (-1)^i (-1)^j x^i f(x^j) \\
 &= (-1)^{i+j} x^i f(x^j) \\
 &= (-1)^{i+j} h(x)
 \end{aligned}$$

Because

- $(-1)^{i+j} = -1$  when  $i + j$  is odd, and
- $(-1)^{i+j} = 1$  when  $i + j$  is even

Then

- $h(x) = -h(x)$  or  $h(x)$  is an odd function, when  $i + j$  is odd, and
- $h(x) = h(x)$  or  $h(x)$  is an even function, when  $i + j$  is even

### Exercise 2

Let

- $S(n, 2) = \sum_{k=1}^n k^2$  and
  - $S(n, 3) = \sum_{k=1}^n k^3$
- i) Let  $T(n, 2, x) = \sum_{k=1}^n k^2 x^k$ .

Use formula

$$T(n, j+1, x) = x \frac{d}{dx} (T(n, j, x)) \quad \forall j \geq 0$$

for  $j = 1$ , i.e.,

$$T(n, 2, x) = x \frac{d}{dx} (T(n, 1, x))$$

And formula

$$T(n, 1, x) = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

for  $T(n, 1, x)$ , to show that

$$T(n, 2, x) = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}$$

- ii) Note that  $S(n, 2) = T(n, 2, 1)$ . Use l'Hopital's rule to evaluate  $T(n, 2, 1)$ , and conclude that  $S(n, 2) = \frac{n(n+1)(2n+1)}{6}$
- iii) Compute  $T(n, 3, x) = \sum_{k=1}^n k^3 x^k$  using formula

$$T(n, j+1, x) = x \frac{d}{dx} (T(n, j, x)) \quad \forall j \geq 0$$

for  $j = 2$ , i.e.,

$$T(n, 3, x) = x \frac{d}{dx} (T(n, 2, x))$$

iv) Note that  $S(n, 3) = T(n, 3, 1)$ . Use l'Hopital's rule to evaluate  $T(n, 3, 1)$  and conclude that  $S(n, 3) = \left(\frac{n(n+1)}{2}\right)^2$ .

Answer

i)

$$T(n, 2, x) = x \frac{d}{dx} \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

Using the quotient rule

$$\left[ \frac{u(x)}{v(x)} \right] \frac{d}{dx} = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

With

$$u(x) = x - (n+1)x^{n+1} + nx^{n+2}$$

$$u'(x) = 1 - (n+1)^2x^n + n(n+2)x^{n+1}$$

$$v(x) = (1-x)^2$$

$$v'(x) = -2(1-x)$$

$$u'(x)v(x) = (1 - (n+1)^2x^n + n(n+2)x^{n+1})(1-x)^2$$

$$= (1-x)^2(1 - (n+1)^2x^n + n(n+2)x^{n+1})$$

$$u(x)v'(x) = (x - (n+1)x^{n+1} + nx^{n+2}) - 2(1-x)$$

$$= -2(1-x)(x - (n+1)x^{n+1} + nx^{n+2})$$

$$u'(x)v(x) - u(x)v'(x) = (1-x)^2(1 - (n+1)^2x^n + n(n+2)x^{n+1})$$

$$+ 2(1-x)(x - (n+1)x^{n+1} + nx^{n+2})$$

$$= (1-x)[(1-x)(1 - (n+1)^2x^n + n(n+2)x^{n+1})$$

$$+ 2(x - (n+1)x^{n+1} + nx^{n+2})]$$

$$= (1-x)(1+x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - n^2x^{n+2})$$

$$\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} = \frac{(1-x)(1+x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - n^2x^{n+2})}{(1-x)^4}$$

$$= \frac{1+x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - n^2x^{n+2}}{(1-x)^3}$$

Which means

$$x \frac{d}{dx} (T(n, 1, x)) = x \frac{1+x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - n^2x^{n+2}}{(1-x)^3}$$

$$= \frac{x + x^2 - (n+1)^2x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2x^{n+3}}{(1-x)^3} \quad \square$$

ii)

$$\begin{aligned}
 T(n, 2, x) &= \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \\
 T(n, 2, 1) &= \frac{1 + 1 - (n+1)^2 + (2n^2 + 2n - 1) - n^2}{0} \\
 &= \frac{2 - (n^2 + 2n + 1) + (2n^2 + 2n - 1) - n^2}{0} \\
 &= \frac{0}{0}
 \end{aligned}$$

Which is indeterminate. Apply l'Hopital's rule:

$$\begin{aligned}
 \lim_{x \rightarrow 1} T(n, 2, x) &= \lim_{x \rightarrow 1} \frac{\frac{d}{dx} [x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}]}{\frac{d}{dx} [(1-x)^3]} \\
 &= \lim_{x \rightarrow 1} \frac{1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}}{-3(1-x)^2} \\
 &= \lim_{x \rightarrow 1} \frac{2 - n(n+1)^3 x^{n-1} + (n+1)(n+2)(2n^2 + 2n - 1)x^n - (n+2)(n+3)n^2 x^{n+1}}{6(1-x)} \\
 &= \lim_{x \rightarrow 1} \frac{-(n-1)n(n+1)^3 x^{n-2} + n(n+1)(n+2)(2n^2 + 2n - 1)x^{n-1} - (n+1)(n+2)(n+3)n^2 x^n}{-6} \\
 &= \frac{-(n-1)n(n+1)^3 + n(n+1)(n+2)(2n^2 + 2n - 1) - (n+1)(n+2)(n+3)n^2}{-6} \\
 &= \frac{n(n+1) - (n-1)(n+1)^2 + n(n+1)(n+2)(2n^2 + 2n - 1) + n(n+1) - n(n+2)(n+3)}{-6} \\
 &= \frac{n(n+1)[-(n-1)(n+1)^2 + (n+2)(2n^2 + 2n - 1) - n(n+2)(n+3)]}{-6} \\
 &= \frac{n(n+1)[-2n-1]}{-6} = \frac{n(n+1)(2n+1)}{6}
 \end{aligned}$$

iii)

$$\begin{aligned}
 T(n, 3, x) &= x \frac{d}{dx} (T(n, 2, x)) \\
 &= x \frac{d}{dx} \left[ \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \right]
 \end{aligned}$$

Using the quotient rule

$$\left[ \frac{u(x)}{v(x)} \right] \frac{d}{dx} = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

With

$$\begin{aligned}
u(x) &= x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3} \\
u'(x) &= 1 + 2x - (n+1)^3 x^n + (2n^2 + 2n - 1)(n+2)x^{n+1} - (n+3)n^2 x^{n+2} \\
v(x) &= (1-x)^3 \\
v'(x) &= -3(1-x)^2 \\
[v(x)]^2 &= (1-x)^6
\end{aligned}$$

$$\begin{aligned}
u'(x)v(x) &= (1 + 2x - (n+1)^3 x^n + (2n^2 + 2n - 1)(n+2)x^{n+1} - (n+3)n^2 x^{n+2}) (1-x)^3 \\
&= (1-x)(1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}) (1-x)^2 \\
u(x)v'(x) &= (x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}) \times -3(1-x)^2 \\
&= (-3x - 3x^2 + 3(n+1)^2 x^{n+1} - 3(2n^2 + 2n - 1)x^{n+2} + 3n^2 x^{n+3}) (1-x)^2
\end{aligned}$$

Let  $g(x) = (1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2})$ , which means  $u'(x)v(x) = (1-x)g(x)(1-x)^2$ . Here is the calculation of  $(1-x)g(x)$  putting onto a grid:

	1	$x$	$x^2$	$x^n$	$x^{n+1}$	$x^{n+2}$	$x^{n+3}$
$g(x)$	1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)$	$-(n+3)n^2$	
$xg(x)$		1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)$	$-(n+3)n^2$
$(1-x)g(x)$	1	1	-2	$-(n+1)^3$	...	...	$n^3 + 3n^2$

Because

$$\begin{aligned}
(n+2)(2n^2 + 2n - 1) - [-(n+1)^3] &= (2n^3 + 4n^2 + 2n^2 + 4n - n - 2) \\
&\quad + (n^3 + 3n^2 + 3n + 1) \\
&= (2n^3 + 6n^2 + 3n - 2) \\
&\quad + (n^3 + 3n^2 + 3n + 1) \\
&= 3n^3 + 9n^2 + 6n - 1
\end{aligned}$$

and

$$\begin{aligned}
-(n+3)n^2 - (n+2)(2n^2 + 2n - 1) &= -(n^3 + 3n^2) \\
&\quad - (2n^3 + 6n^2 + 3n - 2) \\
&= -3n^3 - 9n^2 - 3n + 2
\end{aligned}$$

We have

$$\begin{aligned}
(1-x)g(x) &= 1 + x - 2x^2 - (n+1)^3 x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1} \\
&\quad + (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3} \\
u'(x)v(x) &= (1-x)g(x)(1-x)^2 \\
&= [1 + x - 2x^2 - (n+1)^3 x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1} \\
&\quad + (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3}] (1-x)^2
\end{aligned}$$

Let

$$\begin{aligned}
h_1(x) &= [1 + x - 2x^2 - (n+1)^3x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1} \\
&\quad + (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3}] \\
u'(x)v(x) &= h_1(x)(1-x)^2 \\
h_2(x) &= -3x - 3x^2 + 3(n+1)^2x^{n+1} \\
&\quad - 3(2n^2 + 2n - 1)x^{n+2} + 3n^2x^{n+3} \\
u(x)v'(x) &= h_2(x)(1-x)^2 \\
u'(x)v(x) - u(x)v'(x) &= (h_1(x) - h_2(x))(1-x)^2
\end{aligned}$$

The calculation can be put into a grid like this:

	1	$x$	$x^2$	$x^n$	$x^{n+1}$	$x^{n+2}$	$x^{n+3}$
$h_1(x)$	1	1	-2	$-(n+1)^3$	$3n^3 + 9n^2 + 6n - 1$	$-3n^3 - 9n^2 - 3n + 2$	$n^3 + 3n^2$
$h_2(x)$		-3	-3		$3(n+1)^2$	$-3(2n^2 + 2n - 1)$	$3n^2$
$h_1(x) - h_2(x)$	1	4	1	$-(n+1)^3$	...	...	$n^3$

Because

$$\begin{aligned}
(3n^3 + 9n^2 + 6n - 1) - 3(n+1)^2 &= (3n^3 + 9n^2 + 6n - 1) - (3n^2 + 6n + 3) \\
&= 3n^3 + 6n^2 - 4 \\
(-3n^3 - 9n^2 - 3n + 2) - (-3)(2n^2 + 2n - 1) &= (-3n^3 - 9n^2 - 3n + 2) + (6n^2 + 6n - 3) \\
&= -3n^3 - 3n^2 + 3n - 1
\end{aligned}$$

Then

$$\begin{aligned}
u'(x)v(x) - u(x)v'(x) &= [1 + 4x + x^2 - (n+1)^3x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3x^{n+3}](1-x)^2 \\
&\quad [1 + 4x + x^2 - (n+1)^3x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3x^{n+3}] \\
\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} &= \frac{(1-x)^4}{[1 + 4x + x^2 - (n+1)^3x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3x^{n+3}] \\
T(n, 3, x) &= x \frac{(1-x)^4}{[1 + 4x + x^2 - (n+1)^3x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3x^{n+4}]} \\
&= \frac{(1-x)^4}{(1-x)^4}
\end{aligned}$$

—

The second way to solve this

$$\begin{aligned}
F(x) &= \frac{P(x)}{(1-x)^m} \\
&= \frac{u(x)}{v(x)} \\
F'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} \\
&= \frac{P'(x)(1-x)^m - P(x)(-m)(1-x)^{m-1}}{(1-x)^{2m}} \\
&= \frac{P'(x) + \frac{mP(x)}{1-x}}{(1-x)^m} \\
&= \frac{(1-x)P'(x) + mP(x)}{(1-x)^{m+1}}
\end{aligned}$$

Let

$$\begin{aligned}
P(x) &= x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3} \\
P'(x) &= 1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}
\end{aligned}$$

Using a grid to calculate  $(1-x)P'(x)$

	1	$x$	$x^2$	$x^n$	$x^{n+1}$	$x^{n+2}$	$x^{n+3}$
$P(x)$		1	1		$-(n+1)^2$	$2n^2 + 2n - 1$	$-n^2$
$P'(x)$	1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)2$	$-(n+3)(2n^2 + 2n - 1)$	
$xP'(x)$			1	2	$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)2$	$-(n+3)(2n^2 + 2n - 1)$
$(1-x)P'(x)$	1	1	-2	$-(n+1)^3$	...	...	$(n+3)(2n^2 + 2n - 1)$

$$\begin{aligned}
\text{iv)} \quad & [x + 4x^2 + x^3 - (n+1)^3 x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} + \\
& (-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3 x^{n+4}] \\
T(n, 3, x) &= \frac{(1-x)^4}{[1 + 4 + 1 - (n+1)^3 + (3n^3 + 6n^2 - 4) + (-3n^3 - 3n^2 + 3n - 1) + n^3]} \\
T(n, 3, 1) &= \frac{[6 - (1 + 3n + 3n^2 + n^3) + (-4 + 6n^2 + 3n^3) + (-1 + 3n - 3n^2 - 3n^3) + n^3]}{0} \\
&= \frac{[6 - (1 + 3n + 3n^2 + n^3) + (-4 + 6n^2 + 3n^3) + (-1 + 3n - 3n^2 - 3n^3) + n^3]}{0}
\end{aligned}$$

Simplifying the numerator using a grid we have:

	1	$n$	$n^2$	$n^3$
	6			
	-1	-3	-3	-1
	-4		6	3
	-1	3	-3	-3

				1
Total	0	0	0	0

Which means the numerator is 0 and  $T(n, 3, 1) = \frac{0}{0}$ .

To apply l'Hopital's rule, let  $f(x)$  be the numerator and  $g(x)$  be the denominator and

$$\begin{aligned}
f(x) &= [x + 4x^2 + x^3 - (n+1)^3x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3x^{n+4}] \\
&= [x + 4x^2 + x^3 - (n+1)^3x^{n+1} + l(n)x^{n+2} + m(n)x^{n+3} + n^3x^{n+4}] \\
&= f_1(x) + f_{21(x)} + f_{22(x)} + f_{23(x)} + f_{24(x)}
\end{aligned}$$

$$g(x) = (1 - x)^4$$

where

$$f_1(x) = x + 4x^2 + x^3$$

$$f_{21}(x) = -(n+1)^3x^{n+1}$$

$$f_{22}(x) = (3n^3 + 6n^2 - 4)x^{n+2}$$

$$f_{23}(x) = (-3n^3 - 3n^2 + 3n - 1)x^{n+3}$$

$$f_{24}(x) = n^3x^{n+4}$$

Derivatives of  $g(x)$  and  $f_1(x)$  and other functions:

$$f_1(x) = x + 4x^2 + x^3$$

$$f_1'(x) = 1 + 8x + 3x^2$$

$$f_1^{(2)}(x) = 8 + 6x$$

$$f_1^{(3)}(x) = 6$$

$$f_1^{(4)}(x) = 0$$

$$f_{21}(x) = -(n+1)^3x^{n+1}$$

$$f_{21}'(x) = -(n+1)^4x^n$$

$$f_{21}^{(2)}(x) = -n(n+1)^4x^{n-1}$$

$$f_{21}^{(3)}(x) = -(n-1)n(n+1)^4x^{n-2}$$

$$f_{21}^{(4)}(x) = -(n-2)(n-1)n(n+1)^4x^{n-3}$$



$$f_{22}(x) = (3n^3 + 6n^2 - 4)x^{n+2}$$

$$f'_{22}(x) = (n+2)(3n^3 + 6n^2 - 4)x^{n+1}$$

$$f_{22}^{(2)}(x) = (n+1)(n+2)(3n^3 + 6n^2 - 4)x^n$$

$$f_{22}^{(3)}(x) = n(n+1)(n+2)(3n^3 + 6n^2 - 4)x^{n-1}$$

$$f_{22}^{(4)}(x) = (n-1)n(n+1)(n+2)(3n^3 + 6n^2 - 4)x^{n-2}$$

$$f_{23}(x) = (-3n^3 - 3n^2 + 3n - 1)x^{n+3}$$

$$f'_{23}(x) = (n+3)(-3n^3 - 3n^2 + 3n - 1)x^{n+2}$$

$$f_{23}^{(2)}(x) = (n+2)(n+3)(-3n^3 - 3n^2 + 3n - 1)x^{n+1}$$

$$f_{23}^{(3)}(x) = (n+1)(n+2)(n+3)(-3n^3 - 3n^2 + 3n - 1)x^n$$

$$f_{23}^{(4)}(x) = n(n+1)(n+2)(n+3)(-3n^3 - 3n^2 + 3n - 1)x^{n-1}$$

$$f_{24}(x) = n^3x^{n+4}$$

$$f'_{24}(x) = (n+4)n^3x^{n+3}$$

$$f_{24}^{(2)}(x) = (n+3)(n+4)n^3x^{n+2}$$

$$f_{24}^{(3)}(x) = (n+2)(n+3)(n+4)n^3x^{n+1}$$

$$f_{24}^{(4)}(x) = (n+1)(n+2)(n+3)(n+4)n^3x^n$$

$$g(x) = (1-x)^4$$

$$g'(x) = -4(1-x)^3$$

$$g^{(2)}(x) = 12(1-x)^2$$

$$g^{(3)}(x) = -24(1-x)$$

$$g^{(4)}(x) = 24$$

$$\begin{aligned}
f_{21}^{(4)}(x) + f_{22}^{(4)}(x) &= -(n-2)(n-1)n(n+1)^4x^{n-3} \\
&\quad + (n-1)n(n+1)(n+2)(3n^3+6n^2-4)x^{n-2} \\
f_{21}^{(4)}(1) + f_{22}^{(4)}(1) &= -(n-2)(n-1)n(n+1)^4 \\
&\quad + (n-1)n(n+1)(n+2)(3n^3+6n^2-4) \\
&= (n-1)n(n+1)[-(n-2)(n+1)^3] \\
&\quad + (n-1)n(n+1)[(n+2)(3n^3+6n^2-4)] \\
&= (n-1)n(n+1)[-(n-2)(n+1)^3 + (n+2)(3n^3+6n^2-4)] \\
&= (n-1)n(n+1)[(-n+2)(n^3+3n^2+3n+1) \\
&\quad + (n+2)(3n^3+6n^2-4)] \\
&= (n-1)n(n+1)[((-n^4+2n^3) + (-3n^3+6n^2) + (-3n^2+6n) + (-n+2)) \\
&\quad + ((3n^4+6n^3) + (6n^3+12n^2) + (-4n-8))] \\
&= (n-1)n(n+1)[(-1+3)n^4 + (2-3+6+6)n^3 \\
&\quad + (6-3+12)n^2 + (6-1-4)n + (2-8)] \\
&= (n-1)n(n+1)[2n^4 + 11n^3 + 15n^2 + n - 6]
\end{aligned}$$

—

$$\begin{aligned}
f_{23}^{(4)}(x) + f_{24}^{(4)}(x) &= n(n+1)(n+2)(n+3)(-3n^3-3n^2+3n-1)x^{n-1} \\
&\quad + (n+1)(n+2)(n+3)(n+4)n^3x(n) \\
f_{23}^{(4)}(1) + f_{24}^{(4)}(1) &= n(n+1)(n+2)(n+3)(-3n^3-3n^2+3n-1) \\
&\quad + (n+1)(n+2)(n+3)(n+4)n^3 \\
&= n(n+1)(n+2)(n+3)[-3n^3-3n^2+3n-1] \\
&\quad + n(n+1)(n+2)(n+3)[(n+4)n^2] \\
&= n(n+1)(n+2)(n+3)[(-3n^3-3n^2+3n-1) + (n+4)n^2] \\
&= n(n+1)(n+2)(n+3)[(-3n^3-3n^2+3n-1) + (n^3+4n^2)] \\
&= n(n+1)(n+2)(n+3)[(-3+1)n^3 + (-3+4)n^2 + 3n-1] \\
&= n(n+1)(n+2)(n+3)[-2n^3+n^2+3n-1]
\end{aligned}$$

Because  $f_1^{(4)}(x) = 0$

$$\begin{aligned}
f^{(4)}(1) &= f_{21}^{(4)}(1) + f_{22}^{(4)}(1) + f_{23}^{(4)}(1) + f_{24}^{(4)}(1) \\
&= (n-1)n(n+1)[2n^4 + 11n^3 + 15n^2 + n - 6] \\
&\quad + n(n+1)(n+2)(n+3)[2n^3 + n^2 + 3n - 1] \\
&= n(n+1)[(n-1)(2n^4 + 11n^3 + 15n^2 + n - 6) \\
&\quad + n(n+1)[(n+2)(n+3)(-2n^3 + n^2 + 3n - 1)]] \\
&= n(n+1)[(n-1)(2n^4 + 11n^3 + 15n^2 + n - 6) + \\
&\quad + (n+2)(n+3)(-2n^3 + n^2 + 3n - 1)]
\end{aligned}$$

Let

$$\begin{aligned}
h_1(n) &= (n-1)(2n^4 + 11n^3 + 15n^2 + n - 6) \\
&= (2n^5 - 2n^4) + (11n^4 - 11n^3) + (15n^3 - 15n^2) + (n^2 - n) + (-6n + 6) \\
&= 2n^5 + (-2 + 11)n^4 + (-11 + 15)n^3 + (-15 + 1)n^2 + (-1 - 6)n + 6 \\
&= 2n^5 + 9n^4 + 4n^3 - 14n^2 - 7n + 6 \\
h_2(n) &= (n+2)(n+3)(-2n^3 + n^2 + 3n - 1) \\
&= (n^2 + 5n + 6)(-2n^3 + n^2 + 3n - 1) \\
&= (-2n^5 - 10n^4 - 12n^3) + (n^4 + 5n^3 + 6n^2) \\
&\quad + (3n^3 + 15n^2 + 18n) + (-n^2 - 5n - 6) \\
&= -2n^5 + (-10 + 1)n^4 + (-12 + 5 + 3)n^3 + (6 + 15 - 1)n^2 + (18 - 5)n - 6 \\
&= -2n^5 - 9n^4 - 4n^3 + 20n^2 + 13n - 6 \\
h_1(n) + h_2(n) &= (2n^5 + 9n^4 + 4n^3 - 14n^2 - 7n + 6) \\
&\quad + (-2n^5 - 9n^4 - 4n^3 + 22n^2 + 13n - 6) \\
&= 6n^2 + 6n \\
&= 6n(n+1)
\end{aligned}$$

Which means

$$\begin{aligned}
f^4(1) &= n(n+1)6n(n+1) \\
&= 6(n(n+1))^2 \\
\frac{f^4(1)}{g^4(1)} &= \frac{6(n(n+1))^2}{24} \\
&= \frac{n(n+1)^2}{4} \\
&= \left(\frac{n(n+1)}{2}\right)^2 \\
S(n, 3) &= T(n, 3, 1) \\
&= \lim_{x \rightarrow 1} \frac{f(1)}{g(1)} \\
&= \frac{f^4(1)}{g^4(1)} \\
&= \left(\frac{n(n+1)}{2}\right)^2 \quad \square
\end{aligned}$$

### Exercise 3

Compute  $S(n, 4) = \sum_{k=1}^n k^4$  using the recursion formula

$$S(n, i) = \frac{1}{i+1} \left( (n+i)^{i+1} - 1 - \sum_{j=0}^{i-1} \binom{i+1}{j} S(n, j) \right) \forall i \geq 1$$

The term  $\binom{i+1}{j}$  is the binomial coefficient defined as follows:

$$\binom{i+1}{j} = \frac{(i+1)!}{j!(i+1-j)!}$$

Answer

$$S(n, i) = \frac{1}{i+1} \left( (n+1)^{i+1} - 1 - \sum_{j=0}^{i-1} \binom{i+1}{j} S(n, j) \right) \forall i \geq 1$$

$$\begin{aligned} S(n, 4) &= \frac{1}{5} \left( (n+1)^5 - 1 - \sum_{j=0}^3 \binom{5}{j} S(n, j) \right) \\ &= \frac{1}{5} \left( (n+1)^5 - 1 - \left[ \binom{5}{0} S(n, 0) + \binom{5}{1} S(n, 1) + \binom{5}{2} S(n, 2) + \binom{5}{3} S(n, 3) \right] \right) \\ &= \frac{1}{5} \left( (n+1)^5 - 1 - \left[ \binom{5}{0} n + \binom{5}{1} \frac{n(n+1)}{2} + \binom{5}{2} \frac{n(n+1)(2n+1)}{6} + \binom{5}{3} \left( \frac{n(n+1)}{2} \right)^2 \right] \right) \\ &= \frac{1}{5} \left( (n+1)^5 - 1 - \left[ n + 5 \frac{n(n+1)}{2} + 10 \frac{n(n+1)(2n+1)}{6} + 10 \left( \frac{n(n+1)}{2} \right)^2 \right] \right) \\ &= \frac{1}{5} \left( (n+1)^5 - \left[ (n+1) + \frac{5n(n+1)}{2} + \frac{10n(n+1)(2n+1)}{6} + \left( 10 \frac{n(n+1)}{2} \right)^2 \right] \right) \\ &= \frac{n+1}{5} \left( (n+1)^4 - \left[ 1 + \frac{5n}{2} + \frac{10n(2n+1)}{6} + \frac{10n^2(n+1)}{4} \right] \right) \\ &= \frac{n+1}{5} \left( (n+1)^4 - \left[ 1 + \frac{5n}{2} + \frac{5n(2n+1)}{3} + \frac{5n^2(n+1)}{2} \right] \right) \\ &= \frac{n+1}{30} (6(n+1)^4 - [6 + 15n + 10n(2n+1) + 15n^2(n+1)]) \\ &= \frac{n+1}{30} (6(n^4 + 4n^3 + 6n^2 + 4n + 1) \\ &\quad - [6 + 15n + (20n^2 + 10n) + 15n^3 + 15n^2]) \\ &= \frac{n+1}{30} ([6n^4 + 24n^3 + 36n^2 + 24n + 6] \\ &\quad - [15n^3 + 35n^2 + 25n + 6]) \\ &= \frac{n+1}{30} (6n^4 + 9n^3 + n^2 - n) \\ &= \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30} \quad \square \end{aligned}$$

Exercise 4

It's easy to see that the sequence  $(x_n)_{n \geq 1}$  given by  $x_n = \sum_{k=1}^n k^2$  satisfies the recursion

$$x_{n+1} = x_n + (n+1)^2, \forall n \geq 1$$

with  $x_1 = 1$

i) By substituting  $n+1$  for  $n$  in the above formula, obtain that

$$x_{n+2} = x_{n+1} + (n+2)^2$$

Subtract  $x_{n+1}$  from  $x_{n+2}$  to find that

$$x_{n+2} = 2x_{n+1} - x_n + 2n + 3 \quad \forall n \geq 1$$

with  $x_1 = 1$  and  $x_2 = 5$ .

ii) Similarly, substitute  $n + 1$  for  $n$  in the above finding and obtain that

$$x_{n+3} = 2x_{n+2} - x_{n+1} + 2(n+1) + 3$$

Subtract  $x_{n+2}$  from  $x_{n+3}$  to find that

$$x_{n+3} = 3x_{n+2} - 3x_{n+1} + x_n + 2 \quad \forall n \geq 1$$

with  $x_1 = 1$ ,  $x_2 = 5$ , and  $x_3 = 14$ .

iii) Use a similar method to prove that the sequence  $(x_n)_{n \geq 0}$  satisfies the linear recursion

$$x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} + x_n = 0$$

The characteristic polynomial associated to the above recursion function is:

$$P(z) = z^4 - 4z^3 + 6z^2 - 4z + 1 = (z - 1)^4$$

Use the fact that  $x_1 = 1$ ,  $x_2 = 5$ ,  $x_3 = 14$ , and  $x_4 = 30$  to show that

$$x_n = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \geq 1$$

and conclude that

$$S(n, 2) = \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \geq 1$$

### Answer

i)

$$x_{n+1} = x_n + (n+1)^2$$

$$x_{n+2} = x_{n+1} + (n+2)^2$$

$$x_{n+2} - x_{n+1} = [x_{n+1} + (n+2)^2] - [x_n + (n+1)^2]$$

$$x_{n+2} - x_{n+1} = x_{n+1} - x_n + (n+2)^2 - (n+1)^2$$

$$x_{n+2} - x_{n+1} = x_{n+1} - x_n + (n^2 + 4n + 4) - (n^2 + 2n + 1)$$

$$x_{n+2} - x_{n+1} = x_{n+1} - x_n + 2n + 3$$

$$x_{n+2} = 2x_{n+1} - x_n + 2n + 3$$

ii)

$$x_{n+2} = 2x_{n+1} - x_n + 2n + 3$$

$$x_{n+3} = 2x_{n+2} - x_{n+1} + 2(n+1) + 3$$

$$x_{n+3} - x_{n+2} = [2x_{n+2} - x_{n+1} + 2(n+1) + 3] - [2x_{n+1} - x_n + 2n + 3]$$

$$= 2x_{n+2} + (-1-2)x_{n+1} + x_n + (2-2)n + (2+3-3)$$

$$= 2x_{n+2} - 3x_{n+1} + x_n + 2$$

$$x_{n+3} = 3x_{n+2} - 3x_{n+1} + x_n + 2$$

iii)

$$\begin{aligned}
x_{n+3} &= 3x_{n+2} - 3x_{n+1} + x_n + 2 \\
x_{n+4} &= 3x_{n+3} - 3x_{n+2} + x_{n+1} + 2 \\
x_{n+4} - x_{n+3} &= [3x_{n+3} - 3x_{n+2} + x_{n+1} + 2] - [3x_{n+2} - 3x_{n+1} + x_n + 2] \\
&= 3x_{n+3} + (-3 - 3)x_{n+2} + (1 + 3)x_{n+1} - x_n + (2 - 2) \\
&= 3x_{n+3} - 6x_{n+2} + 4x_{n+1} - x_n \\
x_{n+4} &= 4x_{n+3} - 6x_{n+2} + 4x_{n+1} - x_n \\
x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} + x_n &= 0
\end{aligned}$$

The characteristic polynomial is:

$$P(z) = z^4 - 4z^3 + 6z^2 - 4z + 1 = (z - 1)^4$$

Which means the only root of  $P(z)$  is  $\lambda = 1$ . The general form of  $x_n$ :

$$x_n = C_1 + C_2n + C_3n^2 + C_4n^3$$

As

$$\begin{aligned}
x_1 &= 1 \\
x_2 &= 5 \\
x_3 &= 14 \\
x_4 &= 30
\end{aligned}$$

We have this linear system:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 14 \\ 30 \end{pmatrix}$$

Solving the system yields

$$\begin{aligned}
C_1 &= 0 \\
C_2 &= \frac{1}{6} \\
C_3 &= \frac{1}{2} \\
C_4 &= \frac{1}{3}
\end{aligned}$$

Which means

$$\begin{aligned}
x_n &= C_1 + C_2n + C_3n^2 + C_4n^3 \\
&= 0 + \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} \\
&= \frac{n(n+1)(2n+1)}{6} \quad \square
\end{aligned}$$

### Exercise 5

Find the general form of the sequence  $x(n)_{n \geq 0}$  satisfying the linear recursion

$$x_{n+3} = 2x_{n+1} + x_n, \quad \forall n \geq 0$$

With  $x_0 = 1$ ,  $x_1 = -1$  and  $x_2 = 1$

Answer

$$x_{n+3} - 2x_{n+1} - x_n = 0$$

The characteristic polynomial is:

$$\begin{aligned} P(z) &= z^3 - 2z - 1 \\ &= (z+1)(z^2 - z - 1) \\ &= (z+1) \left[ z - \frac{1+\sqrt{5}}{2} \right] \left[ z - \frac{1-\sqrt{5}}{2} \right] \end{aligned}$$

Which means the roots of  $P(z)$  are:

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= \frac{1+\sqrt{5}}{2} \\ \lambda_3 &= \frac{1-\sqrt{5}}{2} \end{aligned}$$

The general form of  $x_n$  is:

$$x_n = C_1 \lambda_1^n + C_2 \lambda_2^n + C_3 \lambda_3^n$$

With  $x_0 = 1$ ,  $x_1 = -1$  and  $x_2 = 1$ :

$$\begin{aligned} 1 &= C_1 + C_2 + C_3 \\ -1 &= C_1 + \frac{1+\sqrt{5}}{2} C_2 + \frac{1-\sqrt{5}}{2} C_3 \\ 1 &= C_1 + \frac{(1+\sqrt{5})^2}{4} C_2 + \frac{(1-\sqrt{5})^2}{4} C_3 \\ &= C_1 + \frac{3+\sqrt{5}}{2} C_2 + \frac{3-\sqrt{5}}{2} C_3 \end{aligned}$$

Solving the linear equation system yields:

$$\begin{aligned} C_1 &= -1 \\ C_2 &= \frac{1+\sqrt{5}}{\sqrt{5}} \\ C_3 &= \frac{-1+\sqrt{5}}{\sqrt{5}} \end{aligned}$$

Which means the general form of  $x_n$  is:

$$x_n = -1 + \left[ \frac{1+\sqrt{5}}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n \right] + \left[ \frac{-1+\sqrt{5}}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n \right] \quad \square$$