

05 Exercises

Exercise 1

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an odd function.

- i) Show that
- $xf(x)$ is an even function
 - $x^2f(x)$ is an odd function
- ii) Show that
- The function $g_1 : \mathbb{R} \rightarrow \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function
 - The function $g_2 : \mathbb{R} \rightarrow \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function
- iii) Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $h(x) = x^i f(x^j)$, where i and j are positive integers. When is $h(x)$ an odd function?

Answer

$f(x)$ is an odd function, which means:

$$\begin{aligned} f(x) &= -f(-x) \\ f(-x) &= -f(x) \end{aligned} \tag{1}$$

- i) Let $f_1 : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f_1(x) = xf(x)$.

$$\begin{aligned} f_1(-x) &= -xf(-x) \\ &= xf(x) \\ &= f_1(x) \end{aligned} \tag{2}$$

Which means $f_1(x)$ is an even function.

Let $f_2 : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f_2(x) = x^2f(x)$.

$$\begin{aligned} f_2(-x) &= (-x)^2f(-x) \\ &= x^2f(-x) \\ &= -x^2f(x) \\ &= -f_2(x) \end{aligned} \tag{3}$$

Which means $f_2(x)$ is an odd function.

- ii) We have

$$\begin{aligned} g_1(-x) &= f((-x)^2) \\ &= f(x^2) \\ &= g_1(x) \end{aligned} \tag{4}$$

Which means $g_1(x)$ is an even function.

$$\begin{aligned} g_2(-x) &= f((-x)^3) \\ &= f(-x^3) \\ &= -f(x^3) \\ &= -g_2(x) \end{aligned} \tag{5}$$

Which means $g_2(x)$ is an odd function.

iii) Doing some transformation

$$\begin{aligned}
 h(x) &= x^i f(x^j) \\
 h(-x) &= (-x)^i f((-x)^j) \\
 &= (-1)^i x^i f((-x)^j) \\
 &= (-1)^i x^i f((-1)^j x^j) \\
 &= (-1)^i (-1)^j x^i f(x^j) \\
 &= (-1)^{i+j} x^i f(x^j) \\
 &= (-1)^{i+j} h(x)
 \end{aligned}
 \tag{6}$$

Because

- $(-1)^{i+j} = -1$ when $i + j$ is odd, and
- $(-1)^{i+j} = 1$ when $i + j$ is even

Then

- $h(x) = -h(x)$ or $h(x)$ is an odd function, when $i + j$ is odd, and
- $h(x) = h(x)$ or $h(x)$ is an even function, when $i + j$ is even

Exercise 2

Let

- $S(n, 2) = \sum_{k=1}^n k^2$ and
 - $S(n, 3) = \sum_{k=1}^n k^3$
- i) Let $T(n, 2, x) = \sum_{k=1}^n k^2 x^k$.

Use formula

$$T(n, j+1, x) = x \frac{d}{dx} (T(n, j, x)) \quad \forall j \geq 0 \tag{7}$$

for $j = 1$, i.e.,

$$T(n, 2, x) = x \frac{d}{dx} (T(n, 1, x)) \tag{8}$$

And formula

$$T(n, 1, x) = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} \tag{9}$$

for $T(n, 1, x)$, to show that

$$T(n, 2, x) = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \tag{10}$$

- ii) TBA
- iii) TBA

Answer

i) Replacing Equation 9 into Equation 8, we have:

$$T(n, 2, x) = x \frac{d}{dx} \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} \quad 11.$$

Using the quotient rule

$$\left[\frac{u(x)}{v(x)} \right] \frac{d}{dx} = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} \quad 12.$$

With

$$\begin{aligned} u(x) &= x - (n+1)x^{n+1} + nx^{n+2} \\ u'(x) &= 1 - (n+1)^2x^n + n(n+2)x^{n+1} \\ v(x) &= (1-x)^2 \\ v'(x) &= -2(1-x) \\ u'(x)v(x) &= (1 - (n+1)^2x^n + n(n+2)x^{n+1})(1-x)^2 \\ &= (1-x)^2(1 - (n+1)^2x^n + n(n+2)x^{n+1}) \\ u(x)v'(x) &= (x - (n+1)x^{n+1} + nx^{n+2})-2(1-x) \\ &= -2(1-x)(x - (n+1)x^{n+1} + nx^{n+2}) \\ u'(x)v(x) - u(x)v'(x) &= (1-x)^2(1 - (n+1)^2x^n + n(n+2)x^{n+1}) \\ &\quad + 2(1-x)(x - (n+1)x^{n+1} + nx^{n+2}) \\ &= (1-x)[(1-x)(1 - (n+1)^2x^n + n(n+2)x^{n+1}) \\ &\quad + 2(x - (n+1)x^{n+1} + nx^{n+2})] \\ &= (1-x)(1+x - (n+1)^2x^n + (2n^2+2n-1)x^{n+1} - n^2x^{n+2}) \\ \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} &= \frac{(1-x)(1+x - (n+1)^2x^n + (2n^2+2n-1)x^{n+1} - n^2x^{n+2})}{(1-x)^4} \\ &= \frac{1+x - (n+1)^2x^n + (2n^2+2n-1)x^{n+1} - n^2x^{n+2}}{(1-x)^3} \end{aligned} \quad 13.$$

Which means

$$\begin{aligned} x \frac{d}{dx} (T(n, 1, x)) &= x \frac{1+x - (n+1)^2x^n + (2n^2+2n-1)x^{n+1} - n^2x^{n+2}}{(1-x)^3} \\ &= \frac{x+x^2 - (n+1)^2x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2x^{n+3}}{(1-x)^3} \quad \square \end{aligned} \quad 14.$$