

05 Exercises

Exercise 1

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an odd function.

- i) Show that
 - $xf(x)$ is an even function
 - $x^2f(x)$ is an odd function
- ii) Show that
 - The function $g_1 : \mathbb{R} \rightarrow \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function
 - The function $g_2 : \mathbb{R} \rightarrow \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function
- iii) Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $h(x) = x^i f(x^j)$, where i and j are positive integers. When is $h(x)$ an odd function?

Answer

$f(x)$ is an odd function, which means:

$$\begin{aligned}f(x) &= -f(-x) \\f(-x) &= -f(x)\end{aligned}$$

- i) Let $f_1 : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f_1(x) = xf(x)$.

$$\begin{aligned}f_1(-x) &= -xf(-x) \\&= xf(x) \\&= f_1(x)\end{aligned}$$

Which means $f_1(x)$ is an even function.

Let $f_2 : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f_2(x) = x^2f(x)$.

$$\begin{aligned}f_2(-x) &= (-x)^2f(-x) \\&= x^2f(-x) \\&= -x^2f(x) \\&= -f_2(x)\end{aligned}$$

Which means $f_2(x)$ is an odd function.

- ii) We have

$$\begin{aligned}g_1(-x) &= f((-x)^2) \\&= f(x^2) \\&= g_1(x)\end{aligned}$$

Which means $g_1(x)$ is an even function.

$$\begin{aligned}g_2(-x) &= f((-x)^3) \\&= f(-x^3) \\&= -f(x^3) \\&= -g_2(x)\end{aligned}$$

Which means $g_2(x)$ is an odd function.

iii) Doing some transformation

$$\begin{aligned}
 h(x) &= x^i f(x^j) \\
 h(-x) &= (-x)^i f((-x)^j) \\
 &= (-1)^i x^i f((-x)^j) \\
 &= (-1)^i x^i f((-1)^j x^j) \\
 &= (-1)^i (-1)^j x^i f(x^j) \\
 &= (-1)^{i+j} x^i f(x^j) \\
 &= (-1)^{i+j} h(x)
 \end{aligned}$$

Because

- $(-1)^{i+j} = -1$ when $i + j$ is odd, and
- $(-1)^{i+j} = 1$ when $i + j$ is even

Then

- $h(x) = -h(x)$ or $h(x)$ is an odd function, when $i + j$ is odd, and
- $h(x) = h(x)$ or $h(x)$ is an even function, when $i + j$ is even

Exercise 2

Let

- $S(n, 2) = \sum_{k=1}^n k^2$ and
 - $S(n, 3) = \sum_{k=1}^n k^3$
- i) Let $T(n, 2, x) = \sum_{k=1}^n k^2 x^k$.

Use formula

$$T(n, j+1, x) = x \frac{d}{dx} (T(n, j, x)) \quad \forall j \geq 0$$

for $j = 1$, i.e.,

$$T(n, 2, x) = x \frac{d}{dx} (T(n, 1, x))$$

And formula

$$T(n, 1, x) = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

for $T(n, 1, x)$, to show that

$$T(n, 2, x) = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}$$

- ii) Note that $S(n, 2) = T(n, 2, 1)$. Use l'Hopital's rule to evaluate $T(n, 2, 1)$, and conclude that $S(n, 2) = \frac{n(n+1)(2n+1)}{6}$
- iii) Compute $T(n, 3, x) = \sum_{k=1}^n k^3 x^k$ using formula

$$T(n, j+1, x) = x \frac{d}{dx} (T(n, j, x)) \quad \forall j \geq 0$$

for $j = 2$, i.e.,

$$T(n, 3, x) = x \frac{d}{dx} (T(n, 2, x))$$

iv) Note that $S(n, 3) = T(n, 3, 1)$. Use l'Hopital's rule to evaluate $T(n, 3, 1)$ and conclude that $S(n, 3) = \left(\frac{n(n+1)}{2}\right)^2$.

Answer

i)

$$T(n, 2, x) = x \frac{d}{dx} \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

Using the quotient rule

$$\left[\frac{u(x)}{v(x)} \right] \frac{d}{dx} = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

With

$$u(x) = x - (n+1)x^{n+1} + nx^{n+2}$$

$$u'(x) = 1 - (n+1)^2x^n + n(n+2)x^{n+1}$$

$$v(x) = (1-x)^2$$

$$v'(x) = -2(1-x)$$

$$u'(x)v(x) = (1 - (n+1)^2x^n + n(n+2)x^{n+1})(1-x)^2$$

$$= (1-x)^2(1 - (n+1)^2x^n + n(n+2)x^{n+1})$$

$$u(x)v'(x) = (x - (n+1)x^{n+1} + nx^{n+2}) - 2(1-x)$$

$$= -2(1-x)(x - (n+1)x^{n+1} + nx^{n+2})$$

$$u'(x)v(x) - u(x)v'(x) = (1-x)^2(1 - (n+1)^2x^n + n(n+2)x^{n+1})$$

$$+ 2(1-x)(x - (n+1)x^{n+1} + nx^{n+2})$$

$$= (1-x)[(1-x)(1 - (n+1)^2x^n + n(n+2)x^{n+1})$$

$$+ 2(x - (n+1)x^{n+1} + nx^{n+2})]$$

$$= (1-x)(1+x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - n^2x^{n+2})$$

$$\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} = \frac{(1-x)(1+x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - n^2x^{n+2})}{(1-x)^4}$$

$$= \frac{1+x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - n^2x^{n+2}}{(1-x)^3}$$

Which means

$$x \frac{d}{dx} (T(n, 1, x)) = x \frac{1+x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - n^2x^{n+2}}{(1-x)^3}$$

$$= \frac{x + x^2 - (n+1)^2x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2x^{n+3}}{(1-x)^3} \quad \square$$

ii)

$$\begin{aligned}
 T(n, 2, x) &= \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \\
 T(n, 2, 1) &= \frac{1 + 1 - (n+1)^2 + (2n^2 + 2n - 1) - n^2}{0} \\
 &= \frac{2 - (n^2 + 2n + 1) + (2n^2 + 2n - 1) - n^2}{0} \\
 &= \frac{0}{0}
 \end{aligned}$$

Which is indeterminate. Apply l'Hopital's rule:

$$\begin{aligned}
 \lim_{x \rightarrow 1} T(n, 2, x) &= \lim_{x \rightarrow 1} \frac{\frac{d}{dx} [x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}]}{\frac{d}{dx} [(1-x)^3]} \\
 &= \lim_{x \rightarrow 1} \frac{1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}}{-3(1-x)^2} \\
 &= \lim_{x \rightarrow 1} \frac{2 - n(n+1)^3 x^{n-1} + (n+1)(n+2)(2n^2 + 2n - 1)x^n - (n+2)(n+3)n^2 x^{n+1}}{6(1-x)} \\
 &= \lim_{x \rightarrow 1} \frac{-(n-1)n(n+1)^3 x^{n-2} + n(n+1)(n+2)(2n^2 + 2n - 1)x^{n-1} - (n+1)(n+2)(n+3)n^2 x^n}{-6} \\
 &= \frac{-(n-1)n(n+1)^3 + n(n+1)(n+2)(2n^2 + 2n - 1) - (n+1)(n+2)(n+3)n^2}{-6} \\
 &= \frac{n(n+1) - (n-1)(n+1)^2 + n(n+1)(n+2)(2n^2 + 2n - 1) + n(n+1) - n(n+2)(n+3)}{-6} \\
 &= \frac{n(n+1)[-(n-1)(n+1)^2 + (n+2)(2n^2 + 2n - 1) - n(n+2)(n+3)]}{-6} \\
 &= \frac{n(n+1)[-2n-1]}{-6} = \frac{n(n+1)(2n+1)}{6}
 \end{aligned}$$

iii)

$$\begin{aligned}
 T(n, 3, x) &= x \frac{d}{dx} (T(n, 2, x)) \\
 &= x \frac{d}{dx} \left[\frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \right]
 \end{aligned}$$

Using the quotient rule

$$\left[\frac{u(x)}{v(x)} \right] \frac{d}{dx} = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

With

$$\begin{aligned}
u(x) &= x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3} \\
u'(x) &= 1 + 2x - (n+1)^3 x^n + (2n^2 + 2n - 1)(n+2)x^{n+1} - (n+3)n^2 x^{n+2} \\
v(x) &= (1-x)^3 \\
v'(x) &= -3(1-x)^2 \\
[v(x)]^2 &= (1-x)^6
\end{aligned}$$

$$\begin{aligned}
u'(x)v(x) &= (1 + 2x - (n+1)^3 x^n + (2n^2 + 2n - 1)(n+2)x^{n+1} - (n+3)n^2 x^{n+2}) (1-x)^3 \\
&= (1-x)(1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}) (1-x)^2 \\
u(x)v'(x) &= (x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}) \times -3(1-x)^2 \\
&= (-3x - 3x^2 + 3(n+1)^2 x^{n+1} - 3(2n^2 + 2n - 1)x^{n+2} + 3n^2 x^{n+3}) (1-x)^2
\end{aligned}$$

Let $g(x) = (1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2})$, which means $u'(x)v(x) = (1-x)g(x)(1-x)^2$. Here is the calculation of $(1-x)g(x)$ putting onto a grid:

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$g(x)$	1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)$	$-(n+3)n^2$	
$xg(x)$		1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)$	$-(n+3)n^2$
$(1-x)g(x)$	1	1	-2	$-(n+1)^3$	$n^3 + 3n^2$

Because

$$\begin{aligned}
(n+2)(2n^2 + 2n - 1) - [-(n+1)^3] &= (2n^3 + 4n^2 + 2n^2 + 4n - n - 2) \\
&\quad + (n^3 + 3n^2 + 3n + 1) \\
&= (2n^3 + 6n^2 + 3n - 2) \\
&\quad + (n^3 + 3n^2 + 3n + 1) \\
&= 3n^3 + 9n^2 + 6n - 1
\end{aligned}$$

and

$$\begin{aligned}
-(n+3)n^2 - (n+2)(2n^2 + 2n - 1) &= -(n^3 + 3n^2) \\
&\quad - (2n^3 + 6n^2 + 3n - 2) \\
&= -3n^3 - 9n^2 - 3n + 2
\end{aligned}$$

We have

$$\begin{aligned}
(1-x)g(x) &= 1 + x - 2x^2 - (n+1)^3 x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1} \\
&\quad + (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3} \\
u'(x)v(x) &= (1-x)g(x)(1-x)^2 \\
&= [1 + x - 2x^2 - (n+1)^3 x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1} \\
&\quad + (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3}] (1-x)^2
\end{aligned}$$

Let

$$\begin{aligned}
h_1(x) &= [1 + x - 2x^2 - (n+1)^3x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1} \\
&\quad + (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3}] \\
u'(x)v(x) &= h_1(x)(1-x)^2 \\
h_2(x) &= -3x - 3x^2 + 3(n+1)^2x^{n+1} \\
&\quad - 3(2n^2 + 2n - 1)x^{n+2} + 3n^2x^{n+3} \\
u(x)v'(x) &= h_2(x)(1-x)^2 \\
u'(x)v(x) - u(x)v'(x) &= (h_1(x) - h_2(x))(1-x)^2
\end{aligned}$$

The calculation can be put into a grid like this:

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$h_1(x)$	1	1	-2	$-(n+1)^3$	$3n^3 + 9n^2 + 6n - 1$	$-3n^3 - 9n^2 - 3n + 2$	$n^3 + 3n^2$
$h_2(x)$		-3	-3		$3(n+1)^2$	$-3(2n^2 + 2n - 1)$	$3n^2$
$h_1(x) - h_2(x)$	1	4	1	$-(n+1)^3$	n^3

Because

$$\begin{aligned}
(3n^3 + 9n^2 + 6n - 1) - 3(n+1)^2 &= (3n^3 + 9n^2 + 6n - 1) - (3n^2 + 6n + 3) \\
&= 3n^3 + 6n^2 - 4 \\
(-3n^3 - 9n^2 - 3n + 2) - (-3)(2n^2 + 2n - 1) &= (-3n^3 - 9n^2 - 3n + 2) + (6n^2 + 6n - 3) \\
&= -3n^3 - 3n^2 + 3n - 1
\end{aligned}$$

Then

$$\begin{aligned}
u'(x)v(x) - u(x)v'(x) &= [1 + 4x + x^2 - (n+1)^3x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3x^{n+3}](1-x)^2 \\
&\quad [1 + 4x + x^2 - (n+1)^3x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3x^{n+3}] \\
\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} &= \frac{(1-x)^4}{[1 + 4x + x^2 - (n+1)^3x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3x^{n+3}] \\
T(n, 3, x) &= x \frac{(1-x)^4}{[1 + 4x + x^2 - (n+1)^3x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3x^{n+3}] \\
&= \frac{(1-x)^4}{[x + 4x^2 + x^3 - (n+1)^3x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} + \\
&\quad (-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3x^{n+4}]}
\end{aligned}$$

—

The second way to solve this

$$\begin{aligned}
F(x) &= \frac{P(x)}{(1-x)^m} \\
&= \frac{u(x)}{v(x)} \\
F'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} \\
&= \frac{P'(x)(1-x)^m - P(x)(-m)(1-x)^{m-1}}{(1-x)^{2m}} \\
&= \frac{P'(x) + \frac{mP(x)}{1-x}}{(1-x)^m} \\
&= \frac{(1-x)P'(x) + mP(x)}{(1-x)^{m+1}}
\end{aligned}$$

Let

$$P(x) = x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}$$

$$P'(x) = 1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}$$

Using a grid to calculate $(1-x)P'(x)$

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$P(x)$		1	1		$-(n+1)^2$	$2n^2 + 2n - 1$	$-n^2$
$P'(x)$	1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)2$	$-(n+3)(2n^2 + 2n - 1)$	
$xP'(x)$			1	2	$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)2$	$-(n+3)(2n^2 + 2n - 1)$
$(1-x)P'(x)$	1	1	-2	$-(n+1)^3$	$(n+3)(2n^2 + 2n - 1)$

$$\begin{aligned}
\text{iv) } T(n, 3, 1) &= \frac{[1 + 4 + 1 - (n+1)^3 + (3n^3 + 6n^2 - 4) + (-3n^3 - 3n^2 + 3n - 1) + n^3]}{0} \\
&= \frac{[6 - (1 + 3n + 3n^2 + n^3) + (-4 + 6n^2 + 3n^3) + (-1 + 3n - 3n^2 - 3n^3) + n^3]}{0}
\end{aligned}$$

Simplifying the numerator using a grid we have:

	1	n	n^2	n^3
	6			
	-1	-3	-3	-1
	-4		6	3
	-1	3	-3	-3
				1
Total	0	0	0	0

Which means the numerator is 0 and $T(n, 3, 1) = \frac{0}{0}$. Let

$$f(x) = [x + 4x^2 + x^3 - (n+1)^3x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} + (-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3x^{n+4}]$$

$$g(x) = (1-x)^4$$

Let

$$i(n) = 3n^3 + 6n^2 - 4$$

$$j(n) = -3n^3 - 3n^2 + 3n - 1$$

$$f(x) = [x + 4x^2 + x^3 - (n+1)^3x^{n+1} + i(n)x^{n+2} + j(n)x^{n+3} + n^3x^{n+4}]$$

Derivatives of $f(x)$ and $g(x)$ put onto a grid:

	1	x	x^2	x^3	x^n	x^{n+1}	x^{n+2}	x^{n+3}	x^{n+4}
$f(x)$		1	4	1		$-(n+1)^3$	$i(n)$	$j(n)$	n^3
$f'(x)$	1	8	3		$-(n+1)^4$	$(n+2)i(n)$	$(n+3)j(n)$	$(n+4)n^3$	
$f''(x)$	8	6		$-n(n+1)^4$	$(n+1)(n+2)i(n)$	$(n+2)(n+3)j(n)$	$(n+3)(n+4)n^3$		
$f'''(x)$	6		$-3n(n+1)^4$						