L'Hopital's rule and connections to Taylor expansions

L'Hopital's rule is a method to compute limits when direct computation would give an undefined result of form $\frac{0}{0}$. Informally

- If $\lim_{x \to x_0} f(x) = 0$, and $\lim_{x \to x_0} g(x) = 0$,
- then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$.

Formally, the rule can be stated as follows:

Theorem 1.8. (L'Hopital's Rule.)

Let x_0 be a real number; allow $x_0 = \infty$ and $x_0 = -\infty$ as well. Let $f(x)$ and $g(x)$ be two differentiable functions.

(i) Assume that

- $\lim_{x \to x_0} f(x) = 0$ and
- $\lim_{x \to x_0} g(x) = 0$.

If

- $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists, and if
- there exists an interval (a, b) around x_0 such that $g'(x) \neq 0$ for all $x \in (a, b) \setminus 0$,
- then the limit $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ also exists and

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

(ii) Assume that

- $\lim_{x \to x_0} f(x)$ is either $-\infty$ or ∞, and that
- $\lim_{x \to x_0} g(x)$ is either $-\infty$ or ∞.

If

- the limit $\lim_{x \to x_0}$ exists, and if
- there exists an interval (a, b) around x_0 such that $g'(x) \neq 0 \forall x \in (a, b) \setminus 0$
- then the limit $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ also exists, and

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$