05 Exercises

Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function.

i) Show that
  - $xf(x)$ is an even function
  - $x^2 f(x)$ is an odd function
ii) Show that
  - The function $g_1 : \mathbb{R} \to \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function
  - The function $g_2 : \mathbb{R} \to \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function
iii) Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h(x) = x^i f(x^j)$, where i and j are positive integers. When is $h(x)$ an odd function?

Answer

$f(x)$ is an odd function, which means:

$$f(x) = -f(-x)$$
$$f(-x) = -f(x)$$

i) Let $f_1 : \mathbb{R} \to \mathbb{R}$ be defined as $f_1(x) = xf(x)$.

$$
\begin{aligned}
f_1(-x) &= -xf(-x) \\
&= xf(x) \\
&= f_1(x)
\end{aligned}
$$

Which means $f_1(x)$ is an even function.

Let $f_2 : \mathbb{R} \to \mathbb{R}$ be defined as $f_2(x) = x^2 f(x)$.

$$
\begin{aligned}
f_2(-x) &= (-x)^2 f(-x) \\
&= x^2 f(-x) \\
&= -x^2 f(x) \\
&= -f_2(x)
\end{aligned}
$$

Which means $f_2(x)$ is an odd function.

ii) We have

$$
\begin{aligned}
g_1(-x) &= f((-x)^2) \\
&= f(x^2) \\
&= g_1(x)
\end{aligned}
$$

Which means $g_1(x)$ is an even function.

$$
\begin{aligned}
g_2(-x) &= f((-x)^3) \\
&= f(-x^3) \\
&= -f(x^3) \\
&= -g_2(x)
\end{aligned}
$$

Which means $g_2(x)$ is an odd function.

iii) Doing some transformation

$$h(x) = x^i f(x^j)$$
$$h(-x) = (-x)^i f((-x)^j)$$
$$= (-1)^i x^i f((-x)^j)$$
$$= (-1)^i x^i f((-1)^j x^j)$$
$$= (-1)^i (-1)^j x^i f(x^j)$$
$$= (-1)^{i+j} x^i f(x^j)$$
$$= (-1)^{i+j} h(x)$$

Because

- $(-1)^{i+j} = -1$ when $i + j$ is odd, and
- $(-1)^{i+j} = 1$ when $i + j$ is even

Then

- $h(x) = -h(x)$ or $h(x)$ is an odd function, when $i + j$ is odd, and
- $h(x) = h(x)$ or $h(x)$ is an even function, when $i + j$ is even

Exercise 2

Let

- $S(n, 2) = \sum_{k=1}^{n} k^2$ and
- $S(n, 3) = \sum_{k=1}^{n} k^3$

i) Let $T(n, 2, x) = \sum_{k=1}^{n} k^2 x^k$.

Use formula

$$T(n, j + 1, x) = x \frac{d}{dx}(T(n, j, x)) \quad \forall j \geq 0$$

for $j = 1$, i.e.,

$$T(n, 2, x) = x \frac{d}{dx}(T(n, 1, x))$$

And formula

$$T(n, 1, x) = \frac{x - (n + 1)x^{n+1} + nx^{n+2}}{(1 - x)^2}$$

for $T(n, 1, x)$, to show that

$$T(n, 2, x) = \frac{x + x^2 - (n + 1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1 - x)^3}$$

ii) Note that $S(n, 2) = T(n, 2, 1)$. Use l'Hopital's rule to evaluate $T(n, 2, 1)$, and conclude that $S(n, 2) = \frac{n(n+1)(2n+1)}{6}$

iii) Compute $T(n, 3, x) = \sum_{k=1}^{n} k^3 x^k$ using formula

$$T(n, j + 1, x) = x \frac{d}{dx}(T(n, j, x)) \quad \forall j \geq 0$$

for $j = 2$, i.e,

$$T(n, 3, x) = x\frac{d}{dx}(T(n, 2, x))$$

iv) Note that $S(n, 3) = T(n, 3, 1)$. Use l'Hopital's rule to evaluate $T(n, 3, 1)$ and conclude that $S(n, 3) = \left(\frac{n(n+1)}{2}\right)^2$.

<u>Answer</u>

i)
$$T(n, 2, x) = x\frac{d}{dx}\frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

Using the quotient rule

$$\left[\frac{u(x)}{v(x)}\right]\frac{d}{dx} = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

With

$$u(x) = x - (n+1)x^{n+1} + nx^{n+2}$$
$$u'(x) = 1 - (n+1)^2 x^n + n(n+2)x^{n+1}$$
$$v(x) = (1-x)^2$$
$$v'(x) = -2(1-x)$$

$$u'(x)v(x) = \left(1 - (n+1)^2 x^n + n(n+2)x^{n+1}\right)(1-x)^2$$
$$= (1-x)^2\left(1 - (n+1)^2 x^n + n(n+2)x^{n+1}\right)$$
$$u(x)v'(x) = \left(x - (n+1)x^{n+1} + nx^{n+2}\right)\left(-2(1-x)\right)$$
$$= -2(1-x)\left(x - (n+1)x^{n+1} + nx^{n+2}\right)$$

$$u'(x)v(x) - u(x)v'(x) = (1-x)^2\left(1 - (n+1)^2 x^n + n(n+2)x^{n+1}\right)$$
$$+2(1-x)\left(x - (n+1)x^{n+1} + nx^{n+2}\right)$$
$$= (1-x)\left[(1-x)\left(1 - (n+1)^2 x^n + n(n+2)x^{n+1}\right)\right.$$
$$\left.+2\left(x - (n+1)x^{n+1} + nx^{n+2}\right)\right]$$
$$= (1-x)\left(1 + x - (n+1)^2 x^n + (2n^2 + 2n - 1)x^{n+1} - n^2 x^{n+2}\right)$$
$$\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} = \frac{(1-x)\left(1 + x - (n+1)^2 x^n + (2n^2 + 2n - 1)x^{n+1} - n^2 x^{n+2}\right)}{(1-x)^4}$$
$$= \frac{1 + x - (n+1)^2 x^n + (2n^2 + 2n - 1)x^{n+1} - n^2 x^{n+2}}{(1-x)^3}$$

Which means

$$x\frac{d}{dx}(T(n, 1, x)) = x\frac{1 + x - (n+1)^2 x^n + (2n^2 + 2n - 1)x^{n+1} - n^2 x^{n+2}}{(1-x)^3}$$
$$= \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \qquad \Box$$

ii)
$$T(n, 2, x) = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}$$

$$T(n, 2, 1) = \frac{1 + 1 - (n+1)^2 + (2n^2 + 2n - 1) - n^2}{0}$$

$$= \frac{2 - (n^2 + 2n + 1) + (2n^2 + 2n - 1) - n^2}{0}$$

$$= \frac{0}{0}$$

Which is indeterminate. Apply l'Hopital's rule:

$$\lim_{x \to 1} T(n, 2, x) = \lim_{x \to 1} \frac{\frac{d}{dx}\left[x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}\right]}{\frac{d}{dx}\left[(1-x)^3\right]}$$

$$= \lim_{x \to 1} \frac{1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}}{-3(1-x)^2}$$

$$= \lim_{x \to 1} \frac{2 - n(n+1)^3 x^{n-1} + (n+1)(n+2)(2n^2 + 2n - 1)x^n - (n+2)(n+3)n^2 x^{n+1}}{6(1-x)}$$

$$= \lim_{x \to 1} \frac{-(n-1)n(n+1)^3 x^{n-2} + n(n+1)(n+2)(2n^2 + 2n - 1)x^{n-1} - (n+1)(n+2)(n+3)n^2 x^n}{-6}$$

$$= \frac{-(n-1)n(n+1)^3 + n(n+1)(n+2)(2n^2 + 2n - 1) - (n+1)(n+2)(n+3)n^2}{-6}$$

$$= \frac{n(n+1) - (n-1)(n+1)^2 + n(n+1)\,(n+2)(2n^2 + 2n - 1) + n(n+1) - n(n+2)(n+3)}{-6}$$

$$= \frac{n(n+1)\left[-(n-1)(n+1)^2 + (n+2)(2n^2 + 2n - 1) - n(n+2)(n+3)\right]}{-6}$$

$$= \frac{n(n+1)[-2n - 1]}{-6} = \frac{n(n+1)(2n+1)}{6}$$

iii)
$$T(n, 3, x) = x\frac{d}{dx}\left(T(n, 2, x)\right)$$

$$= x\frac{d}{dx}\left[\frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}\right]$$

Using the quotient rule

$$\left[\frac{u(x)}{v(x)}\right]\frac{d}{dx} = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

With

$$u(x) = x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}$$

$$u'(x) = 1 + 2x - (n+1)^3 x^n + (2n^2 + 2n - 1)(n+2)x^{n+1} - (n+3)n^2 x^{n+2}$$

$$v(x) = (1-x)^3$$

$$v'(x) = -3(1-x)^2$$

$$[v(x)]^2 = (1-x)^6$$

$$u'(x)v(x) = \left(1 + 2x - (n+1)^3 x^n + (2n^2 + 2n - 1)(n+2)x^{n+1} - (n+3)n^2 x^{n+2}\right)(1-x)^3$$

$$= (1-x)\left(1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}\right)(1-x)^2$$

$$u(x)v'(x) = \left(x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}\right) \times -3(1-x)^2$$

$$= \left(-3x - 3x^2 + 3(n+1)^2 x^{n+1} - 3(2n^2 + 2n - 1)x^{n+2} + 3n^2 x^{n+3}\right)(1-x)^2$$

Let $g(x) = \left(1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}\right)$, which means $u'(x)v(x) = (1-x)g(x)(1-x)^2$. Here is the calculation of $(1-x)g(x)$ putting onto a grid:

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$g(x)$	1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)$	$-(n+3)n^2$	
$xg(x)$		1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)$	$-(n+3)n^2$
$(1-x)g(x)$	1	1	-2	$-(n+1)^3$	...	...	$n^3 + 3n^2$

Because

$$(n+2)(2n^2 + 2n - 1) - \left[-(n+1)^3\right] = (2n^3 + 4n^2 + 2n^2 + 4n - n - 2)$$
$$+ (n^3 + 3n^2 + 3n + 1)$$
$$= (2n^3 + 6n^2 + 3n - 2)$$
$$+ (n^3 + 3n^2 + 3n + 1)$$
$$= 3n^3 + 9n^2 + 6n - 1$$

and

$$-(n+3)n^2 - (n+2)(2n^2 + 2n - 1) = -(n^3 + 3n^2)$$
$$- (2n^3 + 6n^2 + 3n - 2)$$
$$= -3n^3 - 9n^2 - 3n + 2$$

We have

$$(1-x)g(x) = 1 + x - 2x^2 - (n+1)^3 x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1}$$
$$+ (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3}$$

$$u'(x)v(x) = (1-x)g(x)(1-x)^2$$
$$= \left[1 + x - 2x^2 - (n+1)^3 x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1}\right.$$
$$\left. + (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3}\right](1-x)^2$$

Let

$$h_1(x) = [1 + x - 2x^2 - (n+1)^3 x^n + (3n^3 + 9n^2 + 6n - 1)x^{n+1}$$
$$+ (-3n^3 - 9n^2 - 3n + 2)x^{n+2} + (n^3 + 3n^2)x^{n+3}]$$
$$u'(x)v(x) = h_1(x)(1-x)^2$$
$$h_2(x) = -3x - 3x^2 + 3(n+1)^2 x^{n+1}$$
$$- 3(2n^2 + 2n - 1)x^{n+2} + 3n^2 x^{n+3}$$
$$u(x)v'(x) = h_2(x)(1-x)^2$$
$$u'(x)v(x) - u(x)v'(x) = (h_1(x) - h_2(x))(1-x)^2$$

The calculation can be put into a grid like this:

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$h_1(x)$	1	1	-2	$-(n+1)^3$	$3n^3 + 9n^2 + 6n - 1$	$-3n^3 - 9n^2 - 3n + 2$	$n^3 + 3n^2$
$h_2(x)$		-3	-3		$3(n+1)^2$	$-3(2n^2 + 2n - 1)$	$3n^2$
$h_1(x) - h_2(x)$	1	4	1	$-(n+1)^3$	...	...	n^3

Because

$$(3n^3 + 9n^2 + 6n - 1) - 3(n+1)^2 = (3n^3 + 9n^2 + 6n - 1) - (3n^2 + 6n + 3)$$
$$= 3n^3 + 6n^2 - 4$$

$$(-3n^3 - 9n^2 - 3n + 2) - (-3)(2n^2 + 2n - 1) = (-3n^3 - 9n^2 - 3n + 2) + (6n^2 + 6n - 3)$$
$$= -3n^3 - 3n^2 + 3n - 1$$

Then

$$u'(x)v(x) - u(x)v'(x) = [1 + 4x + x^2 - (n+1)^3 x^n + (3n^3 + 6n^2 - 4)x^{n+1} +$$
$$(-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3 x^{n+3}](1-x)^2$$

$$\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} = \frac{\begin{array}{c}[1 + 4x + x^2 - (n+1)^3 x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\ (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3 x^{n+3}]\end{array}}{(1-x)^4}$$

$$T(n, 3, x) = x \frac{\begin{array}{c}[1 + 4x + x^2 - (n+1)^3 x^n + (3n^3 + 6n^2 - 4)x^{n+1} + \\ (-3n^3 - 3n^2 + 3n - 1)x^{n+2} + n^3 x^{n+3}]\end{array}}{(1-x)^4}$$

$$= \frac{\begin{array}{c}[x + 4x^2 + x^3 - (n+1)^3 x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} + \\ (-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3 x^{n+4}]\end{array}}{(1-x)^4}$$

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The second way to solve this

$$F(x) = \frac{P(x)}{(1-x)^m}$$

$$= \frac{u(x)}{v(x)}$$

$$F'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

$$= \frac{P'(x)(1-x)^m - P(x)(-m)(1-x)^{m-1}}{(1-x)^{2m}}$$

$$= \frac{P'(x) + \frac{mP(x)}{1-x}}{(1-x)^m}$$

$$= \frac{(1-x)P'(x) + mP(x)}{(1-x)^{m+1}}$$

Let

$$P(x) = x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}$$

$$P'(x) = 1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}$$

Using a grid to calculate $(1-x)\,P'(x)$

	1	x	x^2	x^n	x^{n+1}	x^{n+2}	x^{n+3}
$P(x)$		1	1		$-(n+1)^2$	$2n^2 + 2n - 1$	$-n^2$
$P'(x)$	1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)2$	$-(n+3)(2n^2 + 2n - 1)$	
$xP'(x)$		1	2		$-(n+1)^3$	$(n+2)(2n^2 + 2n - 1)2$	$-(n+3)(2n^2 + 2n - 1)$
$(1-x)P'(x)$	1	1	-2	$-(n+1)^3$	...	...	$(n+3)(2n^2 + 2n - 1)$

iv)

$$T(n, 3, x) = \frac{\begin{array}{l}[x + 4x^2 + x^3 - (n+1)^3 x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} + \\ (-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3 x^{n+4}]\end{array}}{(1-x)^4}$$

$$T(n, 3, 1) = \frac{[1 + 4 + 1 - (n+1)^3 + (3n^3 + 6n^2 - 4) + (-3n^3 - 3n^2 + 3n - 1) + n^3]}{0}$$

$$= \frac{[6 - (1 + 3n + 3n^2 + n^3) + (-4 + 6n^2 + 3n^3) + (-1 + 3n - 3n^2 - 3n^3) + n^3]}{0}$$

Simplifying the numerator using a grid we have:

	1	n	n^2	n^3
	6			
	-1	-3	-3	-1
	-4		6	3
	-1	3	-3	-3

				1
Total	0	0	0	0

Which means the numerator is 0 and $T(n, 3, 1) = \frac{0}{0}$.

To apply l'Hopital's rule, let $f(x)$ be the numerator and $g(x)$ be the denominator and

$$f(x) = [x + 4x^2 + x^3 - (n+1)^3 x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} +$$
$$(-3n^3 - 3n^2 + 3n - 1)x^{n+3} + n^3 x^{n+4}]$$
$$= [x + 4x^2 + x^3 - (n+1)^3 x^{n+1} + l(n)x^{n+2} + m(n)x^{n+3} + n^3 x^{n+4}]$$
$$= f_1(x) + f_{21(x)} + f_{22(x)} + f_{23(x)} + f_{24(x)}$$
$$g(x) = (1-x)^4$$

where

$$f_1(x) = x + 4x^2 + x^3$$
$$f_{21}(x) = -(n+1)^3 x^{n+1}$$
$$f_{22}(x) = (3n^3 + 6n^2 - 4)x^{n+2}$$
$$f_{23}(x) = (-3n^3 - 3n^2 + 3n - 1)x^{n+3}$$
$$f_{24}(x) = n^3 x^{n+4}$$

Derivatives of $g(x)$ and $f_1(x)$ and other functions:

$$f_1(x) = x + 4x^2 + x^3$$
$$f_1'(x) = 1 + 8x + 3x^2$$
$$f_1^{(2)}(x) = 8 + 6x$$
$$f_1^{(3)}(x) = 6$$
$$f_1^{(4)}(x) = 0$$

$$f_{21}(x) = -(n+1)^3 x^{n+1}$$
$$f_{21}'(x) = -(n+1)^4 x^n$$
$$f_{21}^{(2)}(x) = -n(n+1)^4 x^{n-1}$$
$$f_{21}^{(3)}(x) = -(n-1)n(n+1)^4 x^{n-2}$$
$$f_{21}^{(4)}(x) = -(n-2)(n-1)n(n+1)^4 x^{n-3}$$

$$f_{22}(x) = (3n^3 + 6n^2 - 4)x^{n+2}$$

$$f'_{22}(x) = (n+2)(3n^3 + 6n^2 - 4)x^{n+1}$$

$$f_{22}^{(2)}(x) = (n+1)(n+2)(3n^3 + 6n^2 - 4)x^n$$

$$f_{22}^{(3)}(x) = n(n+1)(n+2)(3n^3 + 6n^2 - 4)x^{n-1}$$

$$f_{22}^{(4)}(x) = (n-1)n(n+1)(n+2)(3n^3 + 6n^2 - 4)x^{n-2}$$

$$f_{23}(x) = (-3n^3 - 3n^2 + 3n - 1)x^{n+3}$$

$$f'_{23}(x) = (n+3)(-3n^3 - 3n^2 + 3n - 1)x^{n+2}$$

$$f_{23}^{(2)}(x) = (n+2)(n+3)(-3n^3 - 3n^2 + 3n - 1)x^{n+1}$$

$$f_{23}^{(3)}(x) = (n+1)(n+2)(n+3)(-3n^3 - 3n^2 + 3n - 1)x^n$$

$$f_{23}^{(4)}(x) = n(n+1)(n+2)(n+3)(-3n^3 - 3n^2 + 3n - 1)x^{n-1}$$

$$f_{24}(x) = n^3 x^{n+4}$$

$$f'_{24}(x) = (n+4)n^3 x^{n+3}$$

$$f_{24}^{(2)}(x) = (n+3)(n+4)n^3 x^{n+2}$$

$$f_{24}^{(3)}(x) = (n+2)(n+3)(n+4)n^3 x^{n+1}$$

$$f_{24}^{(4)}(x) = (n+1)(n+2)(n+3)(n+4)n^3 x^n$$

$$g(x) = (1-x)^4$$

$$g'(x) = -4(1-x)^3$$

$$g^{(2)}(x) = 12(1-x)^2$$

$$g^{(3)}(x) = -24(1-x)$$

$$g^{(4)}(x) = 24$$

$$f_{21}^{(4)}(x) + f_{22}^{(4)}(x) = -(n-2)(n-1)n(n+1)^4 x^{n-3}$$
$$+ (n-1)n(n+1)(n+2)(3n^3 + 6n^2 - 4)x^{n-2}$$

$$f_{21}^{(4)}(1) + f_{22}^{(4)}(1) = -(n-2)(n-1)n(n+1)^4$$
$$+ (n-1)n(n+1)(n+2)(3n^3 + 6n^2 - 4)$$
$$= (n-1)n(n+1)\left[-(n-2)(n+1)^3\right]$$
$$+ (n-1)n(n+1)\left[(n+2)(3n^3 + 6n^2 - 4)\right]$$
$$= (n-1)n(n+1)\left[-(n-2)(n+1)^3 + (n+2)(3n^3 + 6n^2 - 4)\right]$$
$$= (n-1)n(n+1)\left[(-n+2)(n^3 + 3n^2 + 3n + 1)\right.$$
$$+ (n+2)(3n^3 + 6n^2 - 4)\Big]$$
$$= (n-1)n(n+1)\left[\left((-n^4 + 2n^3) + (-3n^3 + 6n^2) + (-3n^2 + 6n) + (-n+2)\right)\right.$$
$$+ \left((3n^4 + 6n^3) + (6n^3 + 12n^2) + (-4n - 8)\right)\Big]$$
$$= (n-1)n(n+1)\left[(-1+3)n^4 + (2 - 3 + 6 + 6)n^3\right.$$
$$+ (6 - 3 + 12)n^2 + (6 - 1 - 4)n + (2 - 8)\Big]$$
$$= (n-1)n(n+1)\left[2n^4 + 11n^3 + 15n^2 + n - 6\right]$$

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$$f_{23}^{(4)}(x) + f_{24}^{(4)}(x) = n(n+1)(n+2)(n+3)(-3n^3 - 3n^2 + 3n - 1)x^{n-1}$$
$$+ (n+1)(n+2)(n+3)(n+4)n^3 x(n)$$

$$f_{23}^{(4)}(1) + f_{24}^{(4)}(1) = n(n+1)(n+2)(n+3)(-3n^3 - 3n^2 + 3n - 1)$$
$$+ (n+1)(n+2)(n+3)(n+4)n^3$$
$$= n(n+1)(n+2)(n+3)\left[-3n^3 - 3n^2 + 3n - 1\right]$$
$$+ n(n+1)(n+2)(n+3)\left[(n+4)n^2\right]$$
$$= n(n+1)(n+2)(n+3)\left[(-3n^3 - 3n^2 + 3n - 1) + (n+4)n^2\right]$$
$$= n(n+1)(n+2)(n+3)\left[(-3n^3 - 3n^2 + 3n - 1) + (n^3 + 4n^2)\right]$$
$$= n(n+1)(n+2)(n+3)\left[(-3 + 1)n^3 + (-3 + 4)n^2 + 3n - 1\right]$$
$$= n(n+1)(n+2)(n+3)\left[-2n^3 + n^2 + 3n - 1\right]$$

Because $f_1^{(4)}(x) = 0$

$$f^{(4)}(1) = f_{21}^{(4)}(1) + f_{22}^{(4)}(1) + f_{23}^{(4)}(1) + f_{24}^{(4)}(1)$$
$$= (n-1)n(n+1)\left[2n^4 + 11n^3 + 15n^2 + n - 6\right]$$
$$+ n(n+1)(n+2)(n+3)\left[2n^3 + n^2 + 3n - 1\right]$$
$$= n(n+1)\left[(n-1)(2n^4 + 11n^3 + 15n^2 + n - 6)\right]$$
$$+ n(n+1)\left[(n+2)(n+3)(-2n^3 + n^2 + 3n - 1)\right]$$
$$= n(n+1)\left[(n-1)(2n^4 + 11n^3 + 15n^2 + n - 6) +\right.$$
$$+ (n+2)(n+3)(-2n^3 + n^2 + 3n - 1)\Big]$$

Let

$$h_1(n) = (n-1)(2n^4 + 11n^3 + 15n^2 + n - 6)$$
$$= (2n^5 - 2n^4) + (11n^4 - 11n^3) + (15n^3 - 15n^2) + (n^2 - n) + (-6n + 6)$$
$$= 2n^5 + (-2 + 11)n^4 + (-11 + 15)n^3 + (-15 + 1)n^2 + (-1 - 6)n + 6$$
$$= 2n^5 + 9n^4 + 4n^3 - 14n^2 - 7n + 6$$
$$h_2(n) = (n+2)(n+3)(-2n^3 + n^2 + 3n - 1)$$
$$= (n^2 + 5n + 6)(-2n^3 + n^2 + 3n - 1)$$
$$= (-2n^5 - 10n^4 - 12n^3) + (n^4 + 5n^3 + 6n^2)$$
$$+ (3n^3 + 15n^2 + 18n) + (-n^2 - 5n - 6)$$
$$= -2n^5 + (-10 + 1)n^4 + (-12 + 5 + 3)n^3 + (6 + 15 - 1)n^2 + (18 - 5)n - 6$$
$$= -2n^5 - 9n^4 - 4n^3 + 20n^2 + 13n - 6$$

$$h_1(n) + h_2(n) = (2n^5 + 9n^4 + 4n^3 - 14n^2 - 7n + 6)$$
$$+ (-2n^5 - 9n^4 - 4n^3 + 22n^2 + 13n - 6)$$
$$= 6n^2 + 6n$$
$$= 6n(n+1)$$

Which means

$$f^4(1) = n(n+1)\, 6n(n+1)$$
$$= 6(n(n+1))^2$$
$$\frac{f^4(1)}{g^4(1)} = \frac{6(n(n+1))^2}{24}$$
$$= \frac{n(n+1)^2}{4}$$
$$= \left(\frac{n(n+1)}{2}\right)^2$$
$$S(n,3) = T(n,3,1)$$
$$= \lim_{x \to 1} \frac{f(1)}{g(1)}$$
$$= \frac{f^4(1)}{g^4(1)}$$
$$= \left(\frac{n(n+1)}{2}\right)^2 \quad \square$$

Exercise 3

Compute $S(n,4) = \sum_{k=1}^{n} k^4$ using the recursion formula

$$S(n,i) = \frac{1}{i+1}\left((n+i)^{i+1} - 1 - \sum_{j=0}^{i-1} \binom{i+1}{j} S(n,j)\right) \forall i \geq 1$$

The term $\binom{i+1}{j}$ is the binomial coefficient defined as follows:

$$\binom{i+1}{j} = \frac{(i+1)!}{j!(i+1-j)!}$$

$$S(n, i) = \frac{1}{i+1}\left((n+1)^{i+1} - 1 - \sum_{j=0}^{i-1} \binom{i+1}{j} S(n, j) \right) \forall i \geq 1$$

$$S(n, 4) = \frac{1}{5}\left((n+1)^5 - 1 - \sum_{j=0}^{3} \binom{5}{j} S(n, j) \right)$$

$$= \frac{1}{5}\left((n+1)^5 - 1 - \left[\binom{5}{0} S(n, 0) + \binom{5}{1} S(n, 1) + \binom{5}{2} S(n, 2) + \binom{5}{3} S(n, 3) \right] \right)$$

$$= \frac{1}{5}\left((n+1)^5 - 1 - \left[\binom{5}{0} n + \binom{5}{1}\frac{n(n+1)}{2} + \binom{5}{2}\frac{n(n+1)(2n+1)}{6} + \binom{5}{3}\left(\frac{n(n+1)}{2}\right)^2 \right] \right)$$

$$= \frac{1}{5}\left((n+1)^5 - 1 - \left[n + 5\frac{n(n+1)}{2} + 10\frac{n(n+1)(2n+1)}{6} + 10\left(\frac{n(n+1)}{2}\right)^2 \right] \right)$$

$$= \frac{1}{5}\left((n+1)^5 - \left[(n+1) + \frac{5n(n+1)}{2} + \frac{10n(n+1)(2n+1)}{6} + \left(10\frac{n(n+1)}{2}\right)^2 \right] \right)$$

$$= \frac{n+1}{5}\left((n+1)^4 - \left[1 + \frac{5n}{2} + \frac{10n(2n+1)}{6} + \frac{10n^2(n+1)}{4} \right] \right)$$

$$= \frac{n+1}{5}\left((n+1)^4 - \left[1 + \frac{5n}{2} + \frac{5n(2n+1)}{3} + \frac{5n^2(n+1)}{2} \right] \right)$$

$$= \frac{n+1}{30}\left(6(n+1)^4 - [6 + 15n + 10n(2n+1) + 15n^2(n+1)] \right)$$

$$= \frac{n+1}{30}\left(6(n^4 + 4n^3 + 6n^2 + 4n + 1) \right.$$
$$\left. - [6 + 15n + (20n^2 + 10n) + 15n^3 + 15n^2] \right)$$

$$= \frac{n+1}{30}\left([6n^4 + 24n^3 + 36n^2 + 24n + 6] \right.$$
$$\left. - [15n^3 + 35n^2 + 25n + 6] \right)$$

$$= \frac{n+1}{30}(6n^4 + 9n^3 + n^2 - n)$$

$$= \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30} \quad \square$$

Exercise 4

It's easy to see that the sequence $(x_n)_{n \geq 1}$ given by $x_n = \sum_{k=1}^{n} k^2$ satisfies the recursion

$$x_{n+1} = x_n + (n+1)^2, \forall n \geq 1$$

with $x_1 = 1$

i) By substituting $n + 1$ for n in the above formula, obtain that

$$x_{n+2} = x_{n+1} + (n+2)^2$$

Subtract x_{n+1} from x_{n+2} to find that

$$x_{n+2} = 2x_{n+1} - x_n + 2n + 3 \quad \forall n \geq 1$$

with $x_1 = 1$ and $x_2 = 5$.

ii) Similarly, substitute $n + 1$ for n in the above finding and obtain that

$$x_{n+3} = 2x_{n+2} - x_{n+1} + 2(n + 1) + 3$$

Subtract x_{n+2} from x_{n+3} to find that

$$x_{n+3} = 3x_{n+2} - 3x_{n+1} + x_n + 2 \quad \forall n \geq 1$$

with $x_1 = 1$, $x_2 = 5$, and $x_3 = 14$.

iii) Use a similar method to prove that the sequence $(x_n)_{n \geq 0}$ satisfies the linear recursion

$$x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} + x_n = 0$$

The characteristic polynomial associated to the above recursion function is:

$$P(z) = z^4 - 4z^3 + 6z^2 - 4z + 1 = (z - 1)^4$$

Use the fact that $x_1 = 1$, $x_2 = 5$, $x_3 = 14$, and $x_4 = 30$ to show that

$$x_n = \frac{n(n + 1)(2n + 1)}{6}, \quad \forall n \geq 1$$

and conclude that

$$S(n, 2) = \sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}, \quad \forall n \geq 1$$

Answer

i)
$$x_{n+1} = x_n + (n + 1)^2$$
$$x_{n+2} = x_{n+1} + (n + 2)^2$$
$$x_{n+2} - x_{n+1} = \left[x_{n+1} + (n + 2)^2\right] - \left[x_n + (n + 1)^2\right]$$
$$x_{n+2} - x_{n+1} = x_{n+1} - x_n + (n + 2)^2 - (n + 1)^2$$
$$x_{n+2} - x_{n+1} = x_{n+1} - x_n + (n^2 + 4n + 4) - (n^2 + 2n + 1)$$
$$x_{n+2} - x_{n+1} = x_{n+1} - x_n + 2n + 3$$
$$x_{n+2} = 2x_{n+1} - x_n + 2n + 3$$

ii)
$$x_{n+2} = 2x_{n+1} - x_n + 2n + 3$$
$$x_{n+3} = 2x_{n+2} - x_{n+1} + 2(n + 1) + 3$$
$$x_{n+3} - x_{n+2} = \left[2x_{n+2} - x_{n+1} + 2(n + 1) + 3\right] - \left[2x_{n+1} - x_n + 2n + 3\right]$$
$$= 2x_{n+2} + (-1 - 2)x_{n+1} + x_n + (2 - 2)n + (2 + 3 - 3)$$
$$= 2x_{n+2} - 3x_{n+1} + x_n + 2$$
$$x_{n+3} = 3x_{n+2} - 3x_{n+1} + x_n + 2$$

iii)

$$x_{n+3} = 3x_{n+2} - 3x_{n+1} + x_n + 2$$
$$x_{n+4} = 3x_{n+3} - 3x_{n+2} + x_{n+1} + 2$$
$$x_{n+4} - x_{n+3} = [3x_{n+3} - 3x_{n+2} + x_{n+1} + 2] - [3x_{n+2} - 3x_{n+1} + x_n + 2]$$
$$= 3x_{n+3} + (-3 - 3)x_{n+2} + (1 + 3)x_{n+1} - x_n + (2 - 2)$$
$$= 3x_{n+3} - 6x_{n+2} + 4x_{n+1} - x_n$$
$$x_{n+4} = 4x_{n+3} - 6x_{n+2} + 4x_{n+1} - x_n$$
$$x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} + x_n = 0$$

The characteristic polynomial is:

$$P(z) = z^4 - 4z^3 + 6z^2 - 4z + 1 = (z - 1)^4$$

Which means the only root of $P(z)$ is $\lambda = 1$. The general form of x_n:

$$x_n = C_1 + C_2 n + C_3 n^2 + C_4 n^3$$

As

$$x_1 = 1$$
$$x_2 = 5$$
$$x_3 = 14$$
$$x_4 = 30$$

We have this linear system:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 14 \\ 30 \end{pmatrix}$$

Solving the system yields

$$C_1 = 0$$
$$C_2 = \frac{1}{6}$$
$$C_3 = \frac{1}{2}$$
$$C_4 = \frac{1}{3}$$

Which means

$$x_n = C_1 + C_2 n + C_3 n^2 + C_4 n^3$$
$$= 0 + \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}$$
$$= \frac{n(n + 1)(2n + 1)}{6} \quad \square$$

Exercise 5

Find the general form of the sequence $x(n)_{n \geq 0}$ satisfiying the linear recursion

$$x_{n+3} = 2x_{n+1} + x_n, \quad \forall n \geq 0$$

With $x_0 = 1$, $x_1 = -1$ and $x_2 = 1$

Answer

$$x_{n+3} = 2x_{n+1} + x_n x_{n+3} - 2x_{n+1} - x_n = 0$$

The characteristic polynomial is:

$$P(z) = z^3 - 2z - 1$$
$$= (z+1)(z^2 - z - 1)$$
$$= (z+1)\left[z - \frac{1+\sqrt{5}}{2}\right]\left[z - \frac{1-\sqrt{5}}{2}\right]$$

Which means the roots of $P(z)$ are:

$$\lambda_1 = 1$$
$$\lambda_2 = \frac{1+\sqrt{5}}{2}$$
$$\lambda_3 = \frac{1-\sqrt{5}}{2}$$

The general form of x_n is:

$$x_n = C_1\lambda_1^n + C_2\lambda_2^n + C_3\lambda_3^n$$

With $x_0 = 1$, $x_1 = -1$ and $x_2 = 1$:

$$1 = C_1 + C_2 + C_3$$
$$-1 = C_1 + \frac{1+\sqrt{5}}{2}C_2 + \frac{1-\sqrt{5}}{2}C_3$$
$$1 = C_1 + \frac{\left(1+\sqrt{5}\right)^2}{4}C_2 + \frac{\left(1-\sqrt{5}\right)^2}{4}C_3$$
$$= C_1 + \frac{3+\sqrt{5}}{2}C_2 + \frac{3-\sqrt{5}}{2}C_3$$

Solving the linear equation system yields:

$$C_1 = -1$$
$$C_2 = \frac{1+\sqrt{5}}{\sqrt{5}}$$
$$C_3 = \frac{-1+\sqrt{5}}{\sqrt{5}}$$

Which means the general form of x_n is:

$$x_n = -1 + \left[\frac{1+\sqrt{5}}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right] + \left[\frac{-1+\sqrt{5}}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\right] \quad \square$$

Exercise 6

The sequence $(x_n)_{n \geq 0}$ satisfies the recursion

$$x_{n+1} = 3x_n + 2, \quad \forall n \geq 0$$

with $x_0 = 1$.

i) Show that the sequence $(x_n)_{n \geq 0}$ satisfies the linear recursion

$$x_{n+2} = 4x_{n+1} - 3x_n, \quad \forall n \geq 0$$

with $x_0 = 1$ and $x_1 = 5$.

ii) Find the general formula for $x_n, n \geq 0$.

Answer

i) Replacing $n + 2$ into x_{n+1}, we have

$$x_{n+1} = 3x_n + 2$$
$$x_{n+2} = 3x_{n+1} + 2$$

$$x_{n+2} - x_{n+1} = 3x_{n+1} - 3x_n$$
$$x_{n+2} = 4x_{n+1} - 3x_n$$

ii) From the above transformation, we have:

$$x_{n+2} - 4x_{n+1} + 3x_n = 0$$

The characteristic polynomial is:

$$P(z) = z^2 - 4z + 3 = (z - 3)(z - 1)$$

Which means the roots of $P(z)$ are:

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

The general form of x_n is:

$$x_n = C_1 \lambda_1^n + C_2 \lambda_2^n$$

With $x_0 = 1$ and $x_1 = 5$, we have this system of equations:

$$1 = C_1 + C_2$$
$$5 = 3C_1 + C_2$$

Solving it yields:

$$C_1 = 2$$
$$C_2 = -1$$

Then the general form of x_n is:

$$x_n = 2 \times 3^n - 1$$

Exercise 7

The sequence $(x_n)_{n \geq 0}$ satisfies the recursion

$$x_{n+1} = 3x_n + n + 2, \quad \forall n \geq 0$$

with $x_0 = 1$.

i) Show that the sequence $(x_n)_{n \geq 0}$ satisfies the linear recursion

$$x_{n+3} = 5x_{n+2} - 7x_{n+1} + 3x_n, \quad \forall n \geq 0$$

With $x_0 = 1$, $x_1 = 5$, and $x_2 = 18$.

ii) Find the general formula for x_n, $n \geq 0$.

<u>Answer</u>

i)
$$x_{n+1} = 3x_n + n + 2$$
$$x_{n+2} = 3x_{n+1} + (n+1) + 2$$
$$x_{n+2} - x_{n+1} = [3x_{n+1} + (n+1) + 2] - [3x_n + n + 2]$$
$$= 3x_{n+1} - 3x_n + (1-1)n + (1+2-2)$$
$$= 3x_{n+1} - 3x_n + 1$$
$$x_{n+2} = 4x_{n+1} - 3x_n + 1$$
$$x_{n+3} = 4x_{n+2} - 3x_{n+1} + 1$$
$$x_{n+3} - x_{n+2} = [4x_{n+2} - 3x_{n+1} + 1] - [4x_{n+1} - 3x_n + 1]$$
$$= 4x_{n+2} + (-3 - 4)x_{n+1} + 3x_n + (1-1)$$
$$= 4x_{n+2} - 7x_{n+1} + 3x_n$$
$$x_{n+3} = 5x_{n+2} - 7x_{n+1} + 3x_n \quad \blacksquare$$

ii) From the above transformation, we have:

$$x_{n+3} - 5x_{n+2} + 7x_{n+1} - 3x_n = 0$$

The characteristic polynomial is:

$$P(z) = z^3 - 5z^2 + 7z - 3 = (z-3)(z-1)^2$$

Which means the roots of $P(z)$ are:

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

The general form of x_n is:

$$x_n = C_1 \lambda_1^n + C_2 + C_3 n$$

With $x_0 = 1$, $x_1 = 5$, and $x_2 = 18$, we have this system of equations:

$$1 = C_1 + C_2$$
$$5 = 3C_1 + C_2 + C_3$$
$$18 = 9C_1 + C_2 + 2C_3$$

Solving it yields:

$$C_1 = \frac{9}{4}$$

$$C_2 = -\frac{5}{4}$$

$$C_3 = -\frac{1}{2}$$

Then the general form of x_n is:

$$x_n = \frac{9}{4} \times 3^n - \frac{5}{4} - \frac{n}{2}$$

Exercise 8

Let $P(z) = \sum_{i=0}^{k} a_i z^i$ be the characteristic polynomial corresponding to the linear recursion

$$\sum_{i=0}^{k} a_i x_{n+i} = 0, \quad \forall n \geq 0$$

Assume that λ is a root of multiplicity of 2 of $P(z)$. Show that the sequence $(y_n)_{n \geq 0}$ is given by

$$y_n = Cn\lambda^n, \quad n \geq 0$$

Where C is an arbitrary constant, satistifes the recursion above.

Hint: Show that

$$\sum_{i=1}^{k} a_i y_{n+i} = Cn\lambda^n P(\lambda) + C\lambda^{n+1} P'(\lambda), \quad \forall n \geq 0$$

and recall that λ is a root of multiplicity 2 of the polynomial $P(z)$ if and only if $P(\lambda) = 0$ and $P'(\lambda) = 0$.

Answer

Because λ is a root of multiplicity 2 of $P(z)$

$$P(z) = (z - \lambda)^2 Q(z)$$

Where $Q(z)$ is some polynomial with $Q(\lambda) \neq 0$. It's easily see that:

$$P(\lambda) = (\lambda - \lambda)^2 Q(z) = 0$$

Using product rule:

$$
\begin{aligned}
P'(z) &= \frac{d}{dz}[(z - \lambda)^2 Q(z)] \\
&= 2(z - \lambda)Q(z) + (z - \lambda)^2 Q'(z) \\
&= (z - \lambda)[2Q(z) + (z - \lambda)Q'(z)] \\
P'(\lambda) &= (\lambda - \lambda)[2Q(\lambda) + (\lambda - \lambda)Q'(z)] \\
&= 0[2Q(\lambda) + 0(\lambda - \lambda)Q'(z)] \\
&= 0
\end{aligned}
$$

We then have

$$P(z) = \sum_{i=0}^{k} a_i z^i$$

$$P(\lambda) = \sum_{i=0}^{k} a_i \lambda^i = 0$$

$$P'(z) = \left(\sum_{i=0}^{k} a_i z^i \right)' = \sum_{i=1}^{k} i a_i z^{i-1}$$

$$P'(\lambda) = \sum_{i=1}^{k} i a_i \lambda^{i-1} = 0$$

Because $y_n = Cn\lambda^n$ then $y_{n+i} = C(n+i)\lambda^{n+1}$ and

$$\sum_{i=0}^{k} a_i y_{n+i} = \sum_{i=0}^{k} a_i C(n+i)\lambda^{n+i}$$

$$= Cn \sum_{i=0}^{k} a_i \lambda^{n+i} + C \sum_{i=0}^{k} i a_i \lambda^{n+i}$$

$$= Cn\lambda^n \sum_{i=0}^{k} a_i \lambda^i + C\lambda^{n+1} \sum_{i=0}^{k} i a_i \lambda^{i-1}$$

$$= Cn\lambda^n P(\lambda) + C\lambda^{n+1} P'(\lambda)$$

$$= Cn\lambda^n \times 0 + C\lambda^{n+1} \times 0$$

$$= 0$$

Or $(y_n)_{n \geq 0}$ satisfies the linear recursion.