03 Sequences satsifying linear recursions

A sequence $(x_n)_{n \geq 0}$ satisfies a linear recursion of order k if and only if there exist constants $a_i, i = 0 : k$, with $a_k \neq 0$, such that:

$$\sum_{i=0}^{k} a_i x_{n+i} = 0, \forall n \geq 0$$

Intuition: k is like *how many past element would a new element depend on.*

Example 1: geometric growth

Sequence: $1, 2, 4, 8, 16, 32, \ldots$

Formula: $x_{n+1} = 2x_n$

Parameters: $k = 1 : a_1 = 2$

Example 2: Fibonacci

Sequence: $0, 1, 1, 2, 3, 5, 8, 13, \ldots$

Formula: $x_{n+2} = x_{n+1} + x_n$

Parameters: $k = 1 : a_1 = 1, a_2 = 1$

The recursion is called a linear recursion because of the following linearity properties:

1. If the sequence $(x_n)_{n \geq 0}$ satisfies the linear recursion, then the sequence $(z_n)_{n \geq 0}$ given by

   $$z_n = C x_n, \forall n \geq 0$$

   also satisfies the linear recursion.

2. If the sequences $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ satisfy the linear recursion, then the sequence $(z_n)_{n \geq 0}$ given by

   $$z_n = x_n + y_n, \forall n \geq 0$$

   also satisfies the linear recursion.

If the first k elements of the sequence, i.e., $x_0, x_1, \ldots, x_{k-1}$, are specified, then all entries in the sequence are uniquely determined by the recursion formula: since $a_k \neq 0$, we can solve linear recursion formula, we can solve for x_{n+k}

$$x_{n+k} = -\frac{1}{a_k} \sum_{i=0}^{k-1} a_i x_{n+i} \quad \forall n \geq 0$$

Using Fibonacci as an example, with $x_0 = 0$, and $x_1 = 1$, we then find out:

- $x_2 = x_0 + x_1 = 0 + 1 = 1$
- $x_3 = x_1 + x_2 = 1 + 1 = 2$
- $x_4 = x_2 + x_3 = 1 + 2 = 3$
- $x_5 = x_3 + x_4 = 2 + 3 = 5$
- …

The characteristic polynomial $P(z)$ corresponding to the linear recursion is defined as:

$$P(z) = \sum_{i=0}^{k} a_i z^i$$

Note that P(z) is a polynomial of degree k. Recall that every polynomial of degree k with real coefficients has extractly k roots (which could be complex numbers), when counted with their multiplicities.

More precisely, if $P(z)$ has

- p different roots $\lambda_j, j = 1 : p$, with
- $p \leq k$, and if
- $m(\lambda_j)$ denotes the multiplicity of the root λ_j

Then $\sum_{j=1}^{p} m(\lambda_j) = k$.

Let

- $(x_n)_n$ be a sequence satisfying the linear recursion with
- $P(z) = \sum_{i=0}^{k} a_i z^i$ be the associated characteristic polynomial

The general form of the sequence $(x_n)_{n \geq 0}$ satisfying the linear recursion is:

$$x_n = \sum_{j=1}^{p} \left(\sum_{i=0}^{m(\lambda_j - 1)} C_{i,j} n^i \right) \lambda_j^n \quad \forall n \geq 0$$

Example question: find the general formula for the terms of the Fibonacci sequence.

Answer:

- The linear recursion formula of the Fibonacci sequence:

$$x_{n+2} = x_{n+1} + x_n$$
$$x_{n+2} - x_{n+1} - x_n = 0$$

- The characteristic polynomial $P(z)$:

$$z^2 - z - 1 = 0$$

- The roots of $P(z)$:

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

- The general form of x_n:

$$x_n = C_1 \lambda_1^n + C_2 \lambda_2^n$$

- Suppose the sequence is $0, 1, 1, 2, \ldots$, we pick $x_n = 1$ and $x_{n+1} = 1$

$$C_1 + C_2 = 1$$
$$C_1 \lambda_1 + C_2 \lambda_2 = 1$$

- The solution to the above is:

$$C_1 = \frac{\sqrt{5}+1}{2\sqrt{5}}$$
$$C_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

- Replace it back to the general form, we have:

$$
\begin{aligned}
x_n &= \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\sqrt{5}-1\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \quad \forall n \geq 0
\end{aligned}
$$