05 Exercises

Exercise 1

Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function.

i) Show that
   - $xf(x)$ is an even function
   - $x^2 f(x)$ is an odd function

ii) Show that
   - The function $g_1 : \mathbb{R} \to \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function
   - The function $g_2 : \mathbb{R} \to \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function

iii) Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h(x) = x^i f(x^j)$, where i and j are positive integers. When is $h(x)$ an odd function?

Answer

$f(x)$ is an odd function, which means:

$$\begin{aligned} f(x) &= -f(-x) \\ f(-x) &= -f(x) \end{aligned} \tag{1.}$$

i) Let $f_1 : \mathbb{R} \to \mathbb{R}$ be defined as $f_1(x) = xf(x)$.

$$\begin{aligned} f_1(-x) &= -xf(-x) \\ &= xf(x) \\ &= f_1(x) \end{aligned} \tag{2.}$$

Which means $f_1(x)$ is an even function.

Let $f_2 : \mathbb{R} \to \mathbb{R}$ be defined as $f_2(x) = x^2 f(x)$.

$$\begin{aligned} f_2(-x) &= (-x)^2 f(-x) \\ &= x^2 f(-x) \\ &= -x^2 f(x) \\ &= -f_2(x) \end{aligned} \tag{3.}$$

Which means $f_2(x)$ is an odd function.

ii) We have

$$\begin{aligned} g_1(-x) &= f((-x)^2) \\ &= f(x^2) \\ &= g_1(x) \end{aligned} \tag{4.}$$

Which means $g_1(x)$ is an even function.

$$\begin{aligned} g_2(-x) &= f((-x)^3) \\ &= f(-x^3) \\ &= -f(x^3) \\ &= -g_2(x) \end{aligned} \tag{5.}$$

Which means $g_2(x)$ is an odd function.

iii) Doing some transformation

$$
\begin{aligned}
h(x) &= x^i f(x^j) \\
h(-x) &= (-x)^i f((-x)^j) \\
&= (-1)^i x^i f((-x)^j) \\
&= (-1)^i x^i f((-1)^j x^j) \\
&= (-1)^i (-1)^j x^i f(x^j) \\
&= (-1)^{i+j} x^i f(x^j) \\
&= (-1)^{i+j} h(x)
\end{aligned} \qquad 6.
$$

Because

- $(-1)^{i+j} = -1$ when $i + j$ is odd, and
- $(-1)^{i+j} = 1$ when $i + j$ is even

Then

- $h(x) = -h(x)$ or $h(x)$ is an odd function, when $i + j$ is odd, and
- $h(x) = h(x)$ or $h(x)$ is an even function, when $i + j$ is even

<u>Exercise 2</u>

Let

- $S(n, 2) = \sum_{k=1}^{n} k^2$ and
- $S(n, 3) = \sum_{k=1}^{n} k^3$

i) Let $T(n, 2, x) = \sum_{k=1}^{n} k^2 x^k$.

Use formula

$$T(n, j+1, x) = x \frac{d}{dx}(T(n, j, x)) \quad \forall j \geq 0 \qquad 7.$$

for $j = 1$, i.e.,

$$T(n, 2, x) = x \frac{d}{dx}(T(n, 1, x)) \qquad 8.$$

And formula

$$T(n, 1, x) = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} \qquad 9.$$

for $T(n, 1, x)$, to show that

$$T(n, 2, x) = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \qquad 10.$$

ii) Note that $S(n, 2) = T(n, 2, 1)$. Use l'Hopital's rule to evaluate $T(n, 2, 1)$, and conclude that $S(n, 2) = \frac{n(n+1)(2n+1)}{6}$

iii) TBA

Answer

i) Replacing Equation 9 into Equation 8, we have:

$$T(n,2,x) = x\frac{d}{dx}\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2} \tag{11.}$$

Using the quotient rule

$$\left[\frac{u(x)}{v(x)}\right]\frac{d}{dx} = \frac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2} \tag{12.}$$

With

$$\begin{aligned}
u(x) &= x-(n+1)x^{n+1}+nx^{n+2}\\
u'(x) &= 1-(n+1)^2x^n+n(n+2)x^{n+1}\\
v(x) &= (1-x)^2\\
v'(x) &= -2(1-x)
\end{aligned}$$

$$\begin{aligned}
u'(x)v(x) &= \left(1-(n+1)^2x^n+n(n+2)x^{n+1}\right)(1-x)^2\\
&= (1-x)^2\left(1-(n+1)^2x^n+n(n+2)x^{n+1}\right)\\
u(x)v'(x) &= \left(x-(n+1)x^{n+1}+nx^{n+2}\right)-2(1-x)\\
&= -2(1-x)\left(x-(n+1)x^{n+1}+nx^{n+2}\right)\\
u'(x)v(x)-u(x)v'(x) &= (1-x)^2\left(1-(n+1)^2x^n+n(n+2)x^{n+1}\right)\\
&\quad +2(1-x)\left(x-(n+1)x^{n+1}+nx^{n+2}\right)\\
&= (1-x)\left[(1-x)\left(1-(n+1)^2x^n+n(n+2)x^{n+1}\right)\right.\\
&\quad \left.+2\left(x-(n+1)x^{n+1}+nx^{n+2}\right)\right]\\
&= (1-x)\left(1+x-(n+1)^2x^n+\left(2n^2+2n-1\right)x^{n+1}-n^2x^{n+2}\right)\\
\frac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2} &= \frac{(1-x)\left(1+x-(n+1)^2x^n+\left(2n^2+2n-1\right)x^{n+1}-n^2x^{n+2}\right)}{(1-x)^4}\\
&= \frac{1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}}{(1-x)^3}
\end{aligned} \tag{13.}$$

Which means

$$\begin{aligned}
x\frac{d}{dx}(T(n,1,x)) &= x\frac{1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}}{(1-x)^3}\\
&= \frac{x+x^2-(n+1)^2x^{n+1}+(2n^2+2n-1)x^{n+2}-n^2x^{n+3}}{(1-x)^3} \quad \square
\end{aligned} \tag{14.}$$

ii)

$$T(n,2,x) = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}$$

$$T(n,2,1) = \frac{1 + 1 - (n+1)^2 + (2n^2+2n-1) - n^2}{0}$$

15.

$$\begin{aligned}
&= \frac{2 - (n^2+2n+1) + (2n^2+2n-1) - n^2}{0} \\
&= \frac{0}{0}
\end{aligned}$$

Which is indeterminate. Apply l'Hopital's rule:

$$\begin{aligned}
\lim_{x\to 1} T(n,2,x) &= \lim_{x\to 1} \frac{\frac{d}{dx}[x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}]}{\frac{d}{dx}[(1-x)^3]} \\
&= \lim_{x\to 1} \frac{1 + 2x - (n+1)^3 x^n + (n+2)(2n^2+2n-1)x^{n+1} - (n+3)n^2 x^{n+2}}{-3(1-x)^2} \\
&= \lim_{x\to 1} \frac{2 - n(n+1)^3 x^{n-1} + (n+1)(n+2)(2n^2+2n-1)x^n - (n+2)(n+3)n^2 x^{n+1}}{6(1-x)} \\
&= \lim_{x\to 1} \frac{-(n-1)n(n+1)^3 x^{n-2} + n(n+1)(n+2)(2n^2+2n-1)x^{n-1} - (n+1)(n+2)(n+3)n^2 x^n}{-6}
\end{aligned}$$

16.

$$\begin{aligned}
&= \frac{-(n-1)n(n+1)^3 + n(n+1)(n+2)(2n^2+2n-1) - (n+1)(n+2)(n+3)n^2}{-6} \\
&= \frac{n(n+1)\,-(n-1)(n+1)^2 + n(n+1)\,(n+2)(2n^2+2n-1) + n(n+1)\,-n(n+2)(n+3)}{-6} \\
&= \frac{n(n+1)[-(n-1)(n+1)^2 + (n+2)(2n^2+2n-1) - n(n+2)(n+3)]}{-6} \\
&= \frac{n(n+1)[-2n-1]}{-6} = \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$