1.2 Brief review of integration

Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Recall that $F(x)$ is the antiderivative of $f(x)$ if and only if $F'(x) = f(x)$, i.e.,

$$F(x) = \int f(x)dx \quad \Leftrightarrow \quad F'(x) = f(x)$$

Theorem 1.4. (Fundamental Theorem of Calculus.)

Let $f(x)$ be a continuous function on the interval $[a, b]$, and let $F(x)$ be the antiderivative of $f(x)$. Then

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

Theorem 1.5. (Integration by parts.)

Let $f(x)$ and $g(x)$ be continuous functions. Then

$$\int f(x)g(x)dx = F(x)g(x) - \int F(x)g'(x)dx$$

Where $F(x) = \int f(x)dx$ is the antiderivative of $f(x)$. For definite integrals,

$$\int_a^b f(x)g(x)dx = F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)dx$$

Theorem 1.6. (Integration by substitution.)

Let $f(x)$ be an integrable function. Assume that $g(u)$ is an invertible and continuously differentiable function. The substitution $x = g(u)$ changes the integration variable from x to u as follows:

$$\int f(x)dx = \int f(g(u))g'(u)du$$

For definite integrals,

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)du$$

Examples:

$$\int \ln(1+x)dx = \int \ln(1+x)\,1\,dx$$

Apply the Integral by parts formula:

$$\int f(x)g(x) = F(x)g(x) - \int F(x)g'(x)dx$$

With

$$\begin{aligned}
f(x) &= 1 \\
F(x) &= x \\
g(x) &= \ln(1+x) \\
g'(x) &= \frac{1}{1+x}
\end{aligned}$$

$$\begin{aligned}
\int \ln(1+x)dx &= x\ln(1+x) - \int \frac{x}{1+x}dx \\
&= x\ln(1+x) - \int 1 - \frac{1}{1+x}dx \\
&= x\ln(1+x) - \int dx + \int \frac{1}{1+x}dx \\
&= x\ln(1+x) - x + \ln(1+x) + C \\
&= (1+x)\ln(1+x) - x + C
\end{aligned}$$

—

$$\int_1^3 xe^x dx$$

Apply Integration by parts formula:

$$\int_a^b f(x)g(x)dx = F(x)g(x)|_a^b - \int_a^b F(x)g'(x)dx$$

With

$$\begin{aligned}
a &= 1 \\
b &= 3 \\
f(x) &= e^x \\
g(x) &= x \\
F(x) &= e^x \\
g'(x) &= 1
\end{aligned}$$

We have

$$\begin{aligned}
F(x)g(x)|_a^b &= xe^x \ |_1^3 \\
&= 3e^3 - e \\
\int_a^b F(x)g'(x)dx &= \int_1^3 e^x dx \\
&= e^x \ |_1^3 \\
&= e^3 - e
\end{aligned}$$

$$\int_1^3 xe^x dx = (3e^3 - e) - (e^3 - e) = 2e^3$$

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$$\int x^2 \ln(x)dx$$

Apply Integration by parts formula:

$$\int f(x)g(x)dx = F(x)g(x) - \int F(x)g'(x)dx$$

With

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= \ln(x) \\ F(x) &= \frac{x^3}{3} \\ g'(x) &= \frac{1}{x} \end{aligned}$$

$$\begin{aligned} F(x)g(x) &= \frac{x^3}{3} \cdot \ln(x) \\ &= \frac{x^3 \ln(x)}{3} \\ \int F(x)g'(x)dx &= \int \frac{x^3}{3} \cdot \frac{1}{x} \cdot dx \\ &= \frac{1}{3} \int x^2 dx \\ &= \frac{1}{9}x^3 + C \end{aligned}$$

$$\begin{aligned} \int x^2 \ln(x)dx &= \frac{x^3 \ln(x)}{3} - \frac{1}{9}x^3 - C \\ &= \frac{x^3}{3}\left(\ln(x) - \frac{1}{3}\right) - C \end{aligned}$$

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$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$$

Apply integration by substitution formula:

$$\int f(x)dx = \int f(g(u))g'(u)du$$

With

$$\begin{aligned} u &= \sqrt{x} \\ du &= \left(\sqrt{x}\right)' dx = \frac{dx}{\sqrt{x}} \end{aligned}$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int e^u du = e^u + C = e^{\sqrt{x}} + C$$

—

$$\int_{-1}^{0} x^2 \left(x^3 - 1\right)^4 dx$$

Let

$$u = x^3 - 1 du = 3x^2 dx$$

Then with

$$\begin{aligned} x = -1 &\quad u = -2 \\ x = 0 &\quad u = -1 \end{aligned}$$

$$\begin{aligned} \int_{-1}^{0} x^2 \left(x^3 - 1\right)^4 dx &= \frac{1}{3} \int_{-1}^{0} \left(x^3 - 1\right)^4 3x^2 dx \\ &= \frac{1}{3} \int_{-2}^{-1} u^4 du \\ &= \frac{1}{3} \frac{u^5}{5} \Big|_{-2}^{-1} \\ &= \frac{1}{15} u^5 \big|_{-2}^{-1} \\ &= \frac{1}{15} \left((-1)^5 - (-2)^5\right) \\ &= \frac{31}{15} \end{aligned}$$

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$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$$

Let

$$\begin{aligned} u &= e^x - e^{-x} \\ du &= \left(e^x - e^{-x}\right)' dx = \left(e^x + e^{-x}\right) dx \end{aligned}$$

$$\begin{aligned} \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx &= \int \frac{1}{u} du \\ &= \ln(u) + C \\ &= \ln\left(e^x - e^{-x}\right) + C \end{aligned}$$