1.1 Brief review of differentiation

The function $f : R \to R$ is differentiable at the point $x \in \mathbb{R}$ if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, in which case the derivative $f'(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The function $f(x)$ is called differentiable if it is differentiable at all points x.

Theorem 1.1. (Product Rule.)

The product $f(x)g(x)$ of two differentiable functions $f(x)$ and $g(x)$ is differentiable, and

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

Theorem 1.2. (Quotient Rule.)

The quotient $\frac{f(x)}{g(x)}$ of two differentiable functions $f(x)$ and $g(x)$ is differentiable at every point x where the function $\frac{f(x)}{g(x)}$ is well defined, and

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Theorem 1.3. (Chain Rule.)

The composite function $(g \circ f) = g(f(x))$ of two differentiable functions $f(x)$ and $g(x)$ is differentiable at every point x where $g(f(x))$ is well defined and

$$(g(f(x)))' = g'(f(x))f'(x)$$

Example: Chain Rule is often used for power functions, exponential functions, and logarithmic functions.

$$\frac{d}{dx}((f(x))^n) = n(f(x))^{n-1}f'(x)$$

$$\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)}f'(x)$$

$$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{\ln(f(x))}$$

Lemma 1.1.

Let $f : [a,b] \to [c,d]$ be a differentiable function, and assume that $f(x)$ has an inverse function denoted by $f^{-1}(x)$ with $f^{-1} : [c,d] \to [a,b]$. The function $f^{-1}(x)$ is differentiable at every point $x \in [c,d]$ where $f'(f^{-1}(x)) \neq 0$ and

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

$$\left(\frac{1}{f(x)}\right)' = \frac{1}{f'(f^{-1}(x))}$$

Reuse the Chain Rule

$$(g(f(z)))' = g'(f(z))f'(z)$$

With $g = f^{-1}$, we have

$$\begin{aligned}\left(f^{-1}(f(z))\right)' &= \left(f^{-1}\right)'(f(z))f'(z) \\ z' &= \left(f^{-1}\right)'(f(z))f'(z) \\ 1 &= \left(f^{-1}\right)'(f(z))f'(z)\end{aligned}$$

Let $z = f^{-1}(x)$, then $f(z) = f(f^{-1}(x)) = x$ and

$$\begin{aligned}1 &= \left(f^{-1}\right)'(x)f'\left(f^{-1}(x)\right) \\ \frac{1}{f'(f^{-1}(x))} &= \left(f^{-1}\right)'(x) \\ &= \left(f^{-1}(x)\right)'\end{aligned}$$

Examples:

$$\begin{aligned}\frac{d}{dx}\left(xe^{3x^2-1}\right) &= x'e^{3x^2-1} + x\left(e^{3x^2-1}\right)' \\ &= e^{3x^2-1} + x\left(3x^2-1\right)'\left(e^{3x^2-1}\right) \\ &= e^{3x^2-1} + 6x^2\left(e^{3x^2-1}\right) \\ &= \left(1+6x^2\right)\left(e^{3x^2-1}\right)\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sqrt{3x^2-1}}{\sqrt{3x^2-1}+4}\right) &= \frac{d}{dx}\left(\frac{\sqrt{3x^2-1}+4-4}{\sqrt{3x^2-1}+4}\right) \\ &= \frac{d}{dx}\left(1-\frac{4}{\sqrt{3x^2-1}+4}\right) \\ &= \left(\frac{4}{\sqrt{3x^2-1}+4}\right)'\end{aligned}$$

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$$\frac{d}{dx}\left(\frac{\sqrt{3x^2-1}}{\sqrt{3x^2-1}+4}\right) = \frac{d}{dx}\left(1-\frac{4}{\sqrt{3x^2-1}+4}\right)$$

Apply Quotient Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

With $f(x) = -4$, $g(x) = \sqrt{3x^2-1} + 4$:

$$
\begin{aligned}
f'(x) &= 0 \\
g'(x) &= \left(\sqrt{3x^2-1}\right)' \\
&= \frac{1}{2} \cdot \left(3x^2-1\right)^{-\frac{1}{2}} \cdot 6x \\
&= \frac{3x}{\sqrt{3x^2-1}} \\
(g(x))^2 &= \left(\sqrt{3x^2-1}+4\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2} &= \frac{0 \cdot \left(\sqrt{3x^2-1}+4\right)+4\left(3x \div \sqrt{3x^2-1}\right)}{\left(\sqrt{3x^2-1}+4\right)^2} \\
&= \frac{\left(12x \div \sqrt{3x^2-1}\right)}{\left(\sqrt{3x^2-1}+4\right)^2} \\
&= \frac{12x}{\sqrt{3x^2-1}\left(\sqrt{3x^2-1}+4\right)^2}
\end{aligned}
$$

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To calculate:

$$
\frac{d}{dx}\left(\frac{e^{x^2}+1}{x-1}\right)
$$

Apply Quotient Rule:

$$
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}
$$

With

$$
\begin{aligned}
f(x) &= e^{x^2}+1 \\
f'(x) &= \left(x^2\right)' e^{x^2} \\
&= 2xe^{x^2} \\
g(x) &= x-1 \\
g'(x) &= 1 \\
f'(x)g(x)-f(x)g'(x) &= 2xe^{x^2}(x-1)-\left(e^{x^2}+1\right) \\
(g(x))^2 &= (x-1)^2
\end{aligned}
$$

We have

$$
\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2} = \frac{2xe^{x^2}(x-1)-\left(e^{x^2}+1\right)}{(x-1)^2}
$$

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$$
\begin{aligned}
\frac{d}{dx}\left(\ln\left(\frac{x}{2x^2+1}\right)\right) &= \frac{d}{dx}(\ln(x) - \ln(2x^2+1)) \\
&= \ln(x)' - \ln\big(2x^2+1\big)' \\
&= \frac{1}{x} - \frac{\big(2x^2+1\big)'}{2x^2+1} \\
&= \frac{1}{x} - \frac{4x}{2x^2+1} \\
&= \frac{2x^2+1}{x(2x^2+1)} - \frac{4x^2}{x(2x^2+1)} \\
&= \frac{-2x^2+1}{x(2x^2+1)}
\end{aligned}
$$