05 Exercises

Exercise 1

Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function.

i) Show that
   - $xf(x)$ is an even function
   - $x^2 f(x)$ is an odd function

ii) Show that
   - The function $g_1 : \mathbb{R} \to \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function
   - The function $g_2 : \mathbb{R} \to \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function

iii) Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h(x) = x^i f(x^j)$, where i and j are positive integers. When is $h(x)$ an odd function?

Answer

$f(x)$ is an odd function, which means:

$$\begin{aligned} f(x) &= -f(-x) \\ f(-x) &= -f(x) \end{aligned}$$

i) Let $f_1 : \mathbb{R} \to \mathbb{R}$ be defined as $f_1(x) = xf(x)$.

$$\begin{aligned} f_1(-x) &= -xf(-x) \\ &= xf(x) \\ &= f_1(x) \end{aligned}$$

   Which means $f_1(x)$ is an even function.

   Let $f_2 : \mathbb{R} \to \mathbb{R}$ be defined as $f_2(x) = x^2 f(x)$.

$$\begin{aligned} f_2(-x) &= (-x)^2 f(-x) \\ &= x^2 f(-x) \\ &= -x^2 f(x) \\ &= -f_2(x) \end{aligned}$$

   Which means $f_2(x)$ is an odd function.

ii)
$$\begin{aligned} g_1(-x) &= f((-x)^2) \\ &= f(x^2) \\ &= g_1(x) \end{aligned}$$

   Which means $g_1(x)$ is an even function.

$$\begin{aligned} g_2(-x) &= f((-x)^3) \\ &= f(-x^3) \\ &= -f(x^3) \\ &= -g_2(x) \end{aligned}$$

   Which means $g_2(x)$ is an odd function.

iii)
$$
\begin{aligned}
h(x) &= x^i f(x^j) \\
h(-x) &= (-x)^i f((-x)^j) \\
&= (-1)^i x^i f((-x)^j) \\
&= (-1)^i x^i f((-1)^j x^j) \\
&= (-1)^i (-1)^j x^i f(x^j) \\
&= (-1)^{i+j} x^i f(x^j) \\
&= (-1)^{i+j} h(x)
\end{aligned}
$$

Because

- $(-1)^{i+j} = -1$ when $i + j$ is odd, and
- $(-1)^{i+j} = 1$ when $i + j$ is even

Then

- $h(x) = -h(x)$ or $h(x)$ is an odd function, when $i + j$ is odd, and
- $h(x) = h(x)$ or $h(x)$ is an even function, when $i + j$ is even

<u>Exercise 2</u>

Let

- $S(n, 2) = \sum_{k=1}^{n} k^2$ and
- $S(n, 3) = \sum_{k=1}^{n} k^3$

i) Let $T(n, 2, x) = \sum_{k=1}^{n} k^2 x^k$.

   Use formula

$$T(n, j + 1, x) = x \frac{d}{dx}(T(n, j, x)) \quad \forall j \geq 0$$

   for $j = 1$, i.e.,

$$T(n, 2, x) = x \frac{d}{dx}(T(n, 1, x))$$

   And formula

$$T(n, 1, x) = \frac{x - (n + 1)x^{n+1} + nx^{n+2}}{(1 - x)^2}$$

   for $T(n, 1, x)$, to show that

$$T(n, 2, x) = \frac{x + x^2 - (n + 1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1 - x)^3}$$

ii) Show that
   - The function $g_1 : \mathbb{R} \to \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function
   - The function $g_2 : \mathbb{R} \to \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function

iii) Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h(x) = x^i f(x^j)$, where i and j are positive integers. When is $h(x)$ an odd function?

<u>Answer</u>