05 Exercises

<u>Exercise 1</u>

Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function.

i) Show that
- $xf(x)$ is an even function
- $x^2 f(x)$ is an odd function

ii) Show that
- The function $g_1 : \mathbb{R} \to \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function
- The function $g_2 : \mathbb{R} \to \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function

iii) Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h(x) = x^i f(x^j)$, where i and j are positive integers. When is $h(x)$ an odd function?

<u>Answer</u>

$f(x)$ is an odd function, which means:

$$f(x) = -f(-x)$$
$$f(-x) = -f(x)$$

1.

i) Let $f_1 : \mathbb{R} \to \mathbb{R}$ be defined as $f_1(x) = xf(x)$.

$$f_1(-x) = -xf(-x)$$
$$= xf(x)$$
$$= f_1(x)$$

2.

Which means $f_1(x)$ is an even function.

Let $f_2 : \mathbb{R} \to \mathbb{R}$ be defined as $f_2(x) = x^2 f(x)$.

$$f_2(-x) = (-x)^2 f(-x)$$
$$= x^2 f(-x)$$
$$= -x^2 f(x)$$
$$= -f_2(x)$$

3.

Which means $f_2(x)$ is an odd function.

ii) We have

$$g_1(-x) = f((-x)^2)$$
$$= f(x^2)$$
$$= g_1(x)$$

4.

Which means $g_1(x)$ is an even function.

$$g_2(-x) = f((-x)^3)$$
$$= f(-x^3)$$
$$= -f(x^3)$$
$$= -g_2(x)$$

5.

Which means $g_2(x)$ is an odd function.

iii) Doing some transformation

$$h(x) = x^i f(x^j)$$
$$h(-x) = (-x)^i f((-x)^j)$$
$$= (-1)^i x^i f((-x)^j)$$
$$= (-1)^i x^i f((-1)^j x^j)$$
$$= (-1)^i (-1)^j x^i f(x^j)$$
$$= (-1)^{i+j} x^i f(x^j)$$
$$= (-1)^{i+j} h(x)$$

6.

Because

- $(-1)^{i+j} = -1$ when $i + j$ is odd, and
- $(-1)^{i+j} = 1$ when $i + j$ is even

Then

- $h(x) = -h(x)$ or $h(x)$ is an odd function, when $i + j$ is odd, and
- $h(x) = h(x)$ or $h(x)$ is an even function, when $i + j$ is even

Exercise 2

Let

- $S(n, 2) = \sum_{k=1}^{n} k^2$ and
- $S(n, 3) = \sum_{k=1}^{n} k^3$

i) Let $T(n, 2, x) = \sum_{k=1}^{n} k^2 x^k$.

Use formula

$$T(n, j+1, x) = x \frac{d}{dx}(T(n, j, x)) \quad \forall j \geq 0$$

7.

for $j = 1$, i.e.,

$$T(n, 2, x) = x \frac{d}{dx}(T(n, 1, x))$$

8.

And formula

$$T(n, 1, x) = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

9.

for $T(n, 1, x)$, to show that

$$T(n, 2, x) = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}$$

10.

ii) Note that $S(n, 2) = T(n, 2, 1)$. Use l'Hopital's rule to evaluate $T(n, 2, 1)$, and conclude that $S(n, 2) = \frac{n(n+1)(2n+1)}{6}$

iii) TBA

<u>Answer</u>

i) Replacing Equation 9 into Equation 8, we have:

$$T(n, 2, x) = x \frac{d}{dx} \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

11.

Using the quotient rule

$$\left[\frac{u(x)}{v(x)} \right] \frac{d}{dx} = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

12.

With

$$u(x) = x - (n+1)x^{n+1} + nx^{n+2}$$
$$u'(x) = 1 - (n+1)^2 x^n + n(n+2)x^{n+1}$$
$$v(x) = (1-x)^2$$
$$v'(x) = -2(1-x)$$

$$u'(x)v(x) = \left(1 - (n+1)^2 x^n + n(n+2)x^{n+1}\right)(1-x)^2$$
$$= (1-x)^2 \left(1 - (n+1)^2 x^n + n(n+2)x^{n+1}\right)$$
$$u(x)v'(x) = \left(x - (n+1)x^{n+1} + nx^{n+2}\right)(-2(1-x))$$
$$= -2(1-x)\left(x - (n+1)x^{n+1} + nx^{n+2}\right)$$

$$u'(x)v(x) - u(x)v'(x) = (1-x)^2 \left(1 - (n+1)^2 x^n + n(n+2)x^{n+1}\right)$$
$$+ 2(1-x)\left(x - (n+1)x^{n+1} + nx^{n+2}\right)$$
$$= (1-x)\left[(1-x)\left(1 - (n+1)^2 x^n + n(n+2)x^{n+1}\right)\right.$$
$$\left. + 2\left(x - (n+1)x^{n+1} + nx^{n+2}\right)\right]$$
$$= (1-x)\left(1 + x - (n+1)^2 x^n + (2n^2 + 2n - 1)x^{n+1} - n^2 x^{n+2}\right)$$

$$\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} = \frac{(1-x)\left(1 + x - (n+1)^2 x^n + (2n^2 + 2n - 1)x^{n+1} - n^2 x^{n+2}\right)}{(1-x)^4}$$
$$= \frac{1 + x - (n+1)^2 x^n + (2n^2 + 2n - 1)x^{n+1} - n^2 x^{n+2}}{(1-x)^3}$$

13.

Which means

$$x \frac{d}{dx}(T(n, 1, x)) = x \frac{1 + x - (n+1)^2 x^n + (2n^2 + 2n - 1)x^{n+1} - n^2 x^{n+2}}{(1-x)^3}$$
$$= \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \quad \square$$

14.

ii)

$$T(n, 2, x) = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}$$

$$T(n, 2, 1) = \frac{1 + 1 - (n+1)^2 + (2n^2 + 2n - 1) - n^2}{0}$$

$$= \frac{2 - (n^2 + 2n + 1) + (2n^2 + 2n - 1) - n^2}{0}$$

$$= \frac{0}{0}$$

15.

Which is indeterminate. Apply l'Hopital's rule:

$$\lim_{x \to 1} T(n, 2, 1) = \lim_{x \to 1} \frac{\frac{d}{dx}\left[x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}\right]}{\frac{d}{dx}\left[(1-x)^3\right]}$$

$$= \lim_{x \to 1} \frac{1 + 2x - (n+1)^3 x^n + (n+2)(2n^2 + 2n - 1)x^{n+1} - (n+3)n^2 x^{n+2}}{-3(1-x)^2}$$

$$= \frac{1 + 2 - (n+1)^3 + (n+2)(2n^2 + 2n - 1) - (n+3)n^2}{0}$$

$$= \frac{3 - (n^3 + 3n^2 + 3n + 1) + (2n^3 + 6n^2 + 3n - 2) - (n^3 + 3n^2)}{0}$$

$$= \frac{0}{0}$$

16.

Let $h = x - 1$, so $x = 1 + h$, $\lim_{x \to 1} = \lim_{h \to 0}$.

$$T(n, 2, x) = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}$$

$$= \frac{(1+h) + (1+h)^2 - (n+1)^2(1+h)^{n+1} + (2n^2 + 2n - 1)(1+h)^{n+2} - n^2(1+h)^{n+3}}{h^3}$$

17.