Newton's Fractal for Quadratic and Cubic Polynomials

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December 10, 2021

Abstract

Newton's Method is a great method to approximate a root of a function. The method works well not only in the real line but also in the complex plane. When drawing those basins of attractions, we get some interesting fractals, which are called Newton's Fractals. In this paper, we will explain how Newton's Method works, and then examines the Newton's Fractals from the Quadratic and Cubic polynomials.

1 Introduction

Newton's Method is a method to approximate a root of a function using the tangent line. We can get the Newton's Fractal when extending the method to the complex plane. The paper focuses on the Newton's Fractals from the Quadratic and Cubic polynomials.

In preparing this paper, most of my numerical exploration was based on the interaction [5] from 3Blue1Brown.

2 Newton's Method

A method to approximate a root of $g(x) = 0$ is to find a function $A(x)$ and a closed interval I in such a way that

$$f(x) = x - A(x)g(x)$$

is a contraction mapping $f : I \to I$. The Newton's method uses $A(x) = \frac{1}{g'(x)}$.

Definition 2.1. Newton's Method: If the equation $f(x) = 0$ has a root x^* somewhere in an open interval J where $f'(x)$ and $f''(x)$ are continuous, and where $f'(x)$ never vanishes, then J contains a sub-interval $I = [a, b]$ that

1. $a < x^* < b$
2. the function $N(x) = x - \frac{f(x)}{f'(x)}$ is a contraction mapping from I to itself, with the only fixed point x^*.

Using the Bounded Derivative Condition, the conditions mean that for all $x \in I$, $|N'(x)| = |1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}| = |\frac{f(x)f''(x)}{(f'(x))^2}| \leq K < 1$. K is known as the Lipschitz constant of N.

If all the conditions satisfy, then every solution of (I, N) will converge to x^*.

Remark 2.2. Extending Newton's Method: as you may see, the union of the sub-intervals I_i for each root r_i may not always equal to the set of all real numbers $\mathbb{R}$. It does not mean if we choose a point that does not belong to any sub-intervals, it will not converge to any root using the Newton's Method. There are points that actually do converge, but the result (the root that they converge to) might not be the closest root. There are also points that fall into cycles, or have "chaotic" orbits, $\cdots$. Those uncertainties are what make Newton's Fractals fractals. Therefore, we will extend Newton's Method space to include these points.

However, if we include all real numbers, there may be some numbers p that $f'(p) = 0$, which means p must not in the domain of N.

Define $S_f = \{p \,|\, f'(p) = 0\}$.

Because of those reasons, for the rest of the paper, when we talk about the Newton's Method in the real line, we are talking about $(\mathbb{R} \setminus S_f, N)$.

Remark 2.3. Intuitively, the Newton's Method uses the tangent line to approximate the root. For example, look at the function $f(x) = x^2 - 1$ below:

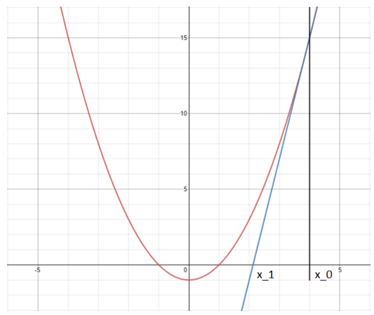

Figure 1: $f(x) = x^2 - 1$ - picture by Desmos

I first choose a point $x_0 = 4$. We can see that when we draw the tangent line at that point, it gives us x_1, which is much closer to the root. If we keep iterating it, it would give us a good approximation of the root.

Using some geometry, we can figure out the formula for the Newton's Method:

$\frac{f(x_0)}{x_0 - x_1} = f'(x_0)$

$\iff x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

Because we will iterate it again and again, we can come up with the feedback function mentioned in the definition 2.1.

* Example: $f(x) = x^3 - 2x^2 - 5x + 6$.

1. Define $N(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 2x^2 - 5x + 6}{3x^2 - 4x - 5}$.

2. Approximate the roots of f by iterating x_0 using the feedback function $N(x)$ with different initial conditions x_0: $-4, 1.5$, and 2.5:

   - $x_0 = -4$:
     $x_1 = -2.813559$
     $x_2 = -2.2123812$
     $x_3 = -2.020504$
     ...
   - $x_0 = 1.5$:
     $x_1 = 0.8823529$
     $x_2 = 0.99829$
     $x_3 = 0.9999995$
     ...
   - $x_0 = 2.5$:
     $x_1 = 3.4$
     $x_2 = 3.07761194$
     $x_3 = 3.003881288$
     ...

   Those solutions seem to converge to either $-2, 1$, or 3, which implies these can be the roots of $f(x)$. In fact, that is the case since $g(x) = x^3 - 2x^2 - 5x + 6 = (x-1)(x+2)(x-3)$.

3. We have $N'(x) = \frac{(x^3-2x^2-5x+6)(6x-4)}{(3x^2-4x-5)^2}$. Solving $|N'(x)| < 1$, we have N as a contraction mapping which self-maps in some intervals $(-\infty, -1.5458857)$, $(-0.0103768, 1.5616)$, and $(2.6613, \infty)$.

   That means under the feedback function N, if we choose any $x \in (-\infty, -1.5458857)$, it will converge to -2; if we choose any $x \in (-0.0103768, 1.5616)$, it will converge to 1; and if we choose any $x \in (2.6613, \infty)$, it will converge to 3.

   Outside of those intervals, it may not converge, or it may converge to not a closest root. For example, if we choose the initial condition to be $x_0 = 2$, it will converge to -2.

3 Newton's Fractals

As it turns out, Newton's Method also works well in the complex plane.

* Example: $f(x) = x^2 + 1$.

1. Define $N(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2+1}{2x} = \frac{x^2-1}{2x}$.

2. Approximate the roots of f by iterating x_0 using the feedback function $N(x)$ with different initial conditions x_0: $1 + 0.5i$ and $0.5 - i$:

   - $x_0 = 1 + 0.5i$:
     $x_1 = 0.1 + 0.45i$
     $x_2 = -0.185 + 1.284i$
     $x_3 = -0.038 - 1.023i$
     $x_4 = i$
     ...
   - $x_0 = 0.5 - i$:
     $x_1 = 0.05 - 0.9i$
     $x_2 = -i$
     ...

   Those solutions seem to converge to either i or $-i$, which implies these can be the roots of $f(x)$. In fact, that is the case since $f(x) = x^2 + 1 = (x - i)(x + i)$.

Definition 3.1. Newton's Fractals: Newton's Fractals are pictures of the basins of attraction of all the fixed points of N in the complex plane.

Remark 3.2. For the same reasons mentioned in Remark 2.2, for the rest of the paper, when we talk about the Newton's Method in the complex plane, we are talking about $(\mathbb{C} \setminus S_f, N)$.

* Example: $f(x) = x^3 + 2x^2 + x + 2 = (x + 2)(x^2 + 1)$.

The Figure 2 says that for for every point with color purple, green, or blue, when use it as the initial condition x_0 for $(\mathbb{C} \setminus S_f, N)$, the solution will converge to the root (the circle) in the same color.

4 Newton's Fractals for Quadratic polynomials

* Example: $f(x) = x^2 - 1 = (x - 1)(x + 1)$.

Here is a picture (Figure 3) of the Newton's Fractal for $f(x) = x^2 - 1 = (x - 1)(x + 1)$. As we can see, fractal does not look as interesting as the above; that does not even look like a fractal. This property, in fact, holds for all quadratic polynomials $f(x)$. This was proven by Cayley in 1879.

Theorem 4.1 (Cayley 1879). *Let the complex quadratic polynomial $f(x) = ax^2 + bx + c$ have zeros α and β in the complex plane. Let L be the perpendicular bisector of the line segment from α to β. Then, when Newton's Method is applied to $f(x)$, the half-planes into which L divides the complex plane are exactly $B(\alpha)$ and $B(\beta)$, the basins of attraction to α and β.*

Remark 4.2. The set $T = \{p \,|\, p \text{ is on the line } L\}$ is self-mapping under N.

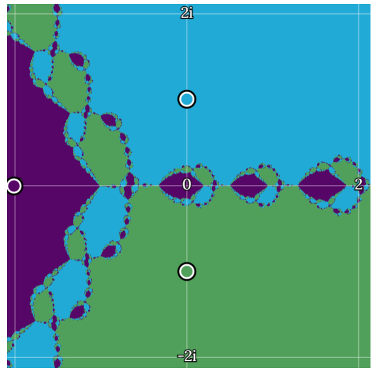

Figure 2: Newton's Fractal for $f(x) = x^3 + 2x^2 + x + 2 = (x+2)(x^2+1)$

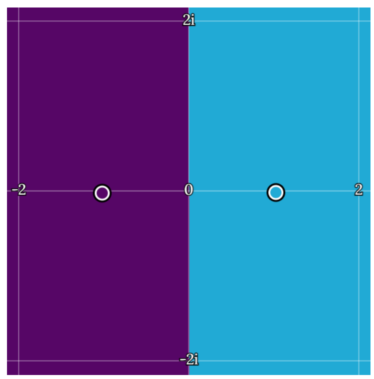

Figure 3: Newton's Fractal for $f(x) = x^2 - 1 = (x-1)(x+1)$

5 Newton's Fractals for Cubic polynomials

As we can see in figure 1, the Newton's Fractals for Cubic polynomials are more interesting. Firstly, I want to take a look at some specific cases of the cubic polynomials $f(x)$. Particularly, I want to look at two cases:

a. $f(x) = a(x-b)^3$ for some $a, b \in$.
b. $f(x) = a(x-b)(x-c)(x-d)$ for some $a, b, c, d \in$.

(a) $f(x) = a(x-b)^3$ for some $a, b \in$.

Proposition 5.1. *For $f(x) = a(x-b)^3$ for some $a, b \in$, all solutions of $(\mathbb{C}, N)$ converge to b.*

PROOF. $f(x) = a(x-b)^3$ for some $a, b \in$
$\implies f'(x) = 3a(x-b)^3$
$\implies N(x) = x - \frac{a(x-b)^3}{3a(x-b)^2} = \frac{2}{3}x + \frac{b}{3}$.
Notice that $f'(x) = 0 \iff x = b$, which is the only root of $f(x)$. So, $S_f = \emptyset$ and $(\mathbb{C}, N)$ is defined.
For any $x, y \in \mathbb{C}$,

$$\begin{aligned} |N(x) - N(y)| &= \left|\frac{2}{3}x + \frac{b}{3} - \frac{2}{3}y - \frac{b}{3}\right| \\ &= \left|\frac{2}{3}x - \frac{2}{3}y\right| \\ &= \frac{2}{3}|x - y|. \end{aligned}$$

We can see $N(x)$ is a contraction mapping with a Lipschitz constant of $\frac{2}{3} < 1$.

Theorem 5.2 (Contraction Mapping Principle)**.** *Suppose that X is compact and $f : X \to X$ is a contraction mapping. Then f has a unique fixed point p, and all solutions of (X, f) converge to p.*

Using the theorem above, we can conclude that all solutions of $\mathbb{C}, N$ will converge to b, the only root of f. $\square$

(b) $f(x) = a(x-b)(x-c)(x-d)$ for some $a, b, c, d \in$.

* Example: $f(x) = x^3 - x = x(x-1)(x+1)$

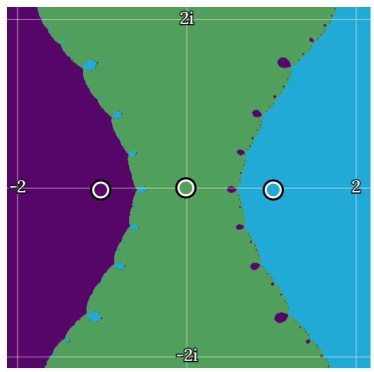

Figure 4: Newton's Fractal for $f(x) = x^3 - x = x(x-1)(x+1)$

We can see that there is some kind of symmetry over the x-axis:

Proposition 5.3. *The Newton's Fractal for $f(x) = a(x-b)(x-c)(x-d)$ for some $a, b, c, d \in$ is symmetric over the x-axis. That is, for all $e, f \in \mathbb{R}$, the two points $e + fi$ and $e - fi$ will have the same behavior (converge to the same root, fall into cycles, $\cdots$).*

PROOF. $f(x) = a(x-b)(x-c)(x-d) = x^3 - a(b+c+d)x^2 + a(bc+bd+cd)x - abcd$ for some $a, b, c, d \in$

$\iff f'(x) = 3ax^2 - 2a(b+c+d)x + a(bc+bd+cd)$

$\implies N(x) = x - \frac{x^3 - a(b+c+d)x^2 + a(bc+bd+cd)x - abcd}{3ax^2 - 2a(b+c+d)x + a(bc+bd+cd)} = \frac{2x^3 - (b+c+d)x^2 + bcd}{3x^2 - 2(b+c+d)x + bc + bd + cd}$.

Let $e, f \in \mathbb{R}$ be given. Then,

$$
\begin{aligned}
N(e+fi) &= \frac{2(e+fi)^3 - (b+c+d)(e+fi)^2 + bcd}{3(e+fi)^2 - 2(b+c+d)(e+fi) + bc + bd + cd} \\
&= \frac{(2e^3 - 6ef^2 + (b+c+d)(f^2 - e^2) + bcd) + (6e^2 f - 2f^3 - 2ef(b+c+d))i}{(3e^2 - 3f^2 - 2e(b+c+d) + bc + bd + cd) + (6ef - 2f(b+c+d))i}
\end{aligned}
$$

and

$$
\begin{aligned}
N(e-fi) &= \frac{2(e-fi)^3 - (b+c+d)(e-fi)^2 + bcd}{3(e-fi)^2 - 2(b+c+d)(e-fi) + bc + bd + cd} \\
&= \frac{(2e^3 - 6ef^2 + (b+c+d)(f^2 - e^2) + bcd) - (6e^2 f - 2f^3 - 2ef(b+c+d))i}{(3e^2 - 3f^2 - 2e(b+c+d) + bc + bd + cd) - (6ef - 2f(b+c+d))i}
\end{aligned}
$$

Let

$A = 2e^3 - 6ef^2 + (b+c+d)(f^2 - e^2) + bcd$,

$B = e^2 f - 2f^3 - 2ef(b+c+d)$,

$C = 3e^2 - 3f^2 - 2e(b+c+d) + bc + bd + cd$, and

$D = 6ef - 2f(b+c+d)$.

Then,

$N(e+fi) = \frac{A+Bi}{(C+Di)} = \frac{AC+BD}{C^2+D^2} + \frac{BC-AD}{C^2+D^2}i$, and
$N(e-fi) = \frac{A-Bi}{(C-Di)} = \frac{AC+BD}{C^2+D^2} - \frac{BC-AD}{C^2+D^2}i$.
After an iteration, $e+fi$ and $e-fi$ will map to two points that still have the same real part and opposite imaginary part. By induction, we can conclude $e+fi$ and $e-fi$ will have the same behavior. □

I also want to discuss something about the fixed points and cycles of those Newton's Fractals.

Proposition 5.4. *All the fixed points of N are asymptotically stable.*

PROOF. If p is a fixed point of N, p is a root of f, so $f(p) = 0$.
$N'(x) = \left[x - \frac{f(x)}{f'(x)}\right]' = 1 - \frac{\left(f'(x)\right)^2 - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$.
$N'(p) = \frac{f(p)f''(p)}{(f'(p))^2} = 0 \implies |N'(p)| < 1$.
Therefore, by linearization, we can conclude that the fixed point p is asymptotically stable. □

About cycles, there are some cubic polynomials f that either have no cycle (besides the fixed points), have a repelling cycle, or have an attracting cycle.

* Example: $f(x) = (x-1)^3$. By Proposition 5.1, we know that $N(x)$ for this function has no n-cycle for all $n > 1$.

* Example: $f(x) = x^3 - 2x^2 - 5x + 6$
$\implies N(x) = x - \frac{x^3-2x^2-5x+6}{3x^2-4x-5}$, which has a 2-cycle $\{-0.33756\cdots, 1.9059856\cdots\}$.

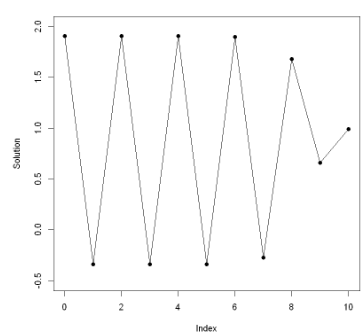

Figure 5: A solution of $N(x)$ for $x_0 = 1.9059856$

As in Figure 5, if we choose an initial condition really close but not equal to an element in the 2−cycle, after some iterations, the solution seems to go off the cycle, with may imply that the cycle is repelling.

* Example: $f(x) = x^3 - 2x + 2$
$\implies N(x) = x - \frac{x^3-2x+2}{3x^2-2}$, which has a 2-cycle $\{0, 1\}$.

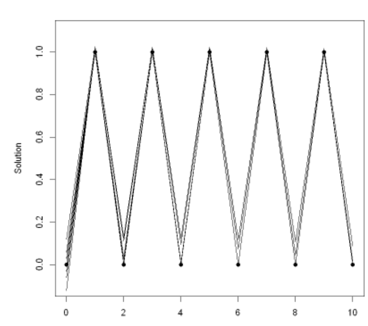

Figure 6: A solution of $N(x)$ for some x_0 around 0

As in Figure 6, I choose a lot of initial conditions x_0 around 0. After some iterations, all solutions seems to merge into one line, with may imply that the cycle is attracting.

As it turns out, there is an easy way to test if a Newton's Fractal has an attracting cycle or not:

Theorem 5.5 (Fatou 1919)**.** *If $R(z)$, a rational function, has an attracting periodic cycle, then the orbit of at least one critical point will converge to it.*

For $N'(x) = \frac{f(x)\cdot f''(x)}{(f'(x))^2}$, we can see that if $N(x)$ has an attracting cycle with an element in the orbit p, $f(p) = 0$ or $f''(p) = 0$. However, if $f(p) = 0$, p is a root of f.

$\Longrightarrow$ If $N(x)$ has an attracting cycle, the value $p \in \mathbb{C}$ that $f''(p) = 0$ will converge to that cycle.

Proposition 5.6. *Suppose $f(x)$ is a cubic polynomial. If its Newton's Fractal has a attracting cycle, then the average of all the root of f will converge to it using the Newton's Method.*

PROOF. Because $f(x)$ is a cubic polynomial, we can write it as

$f(x) = a(x-b)(x-c)(x-d) = ax^3 - a(b+c+d)x^2 + a(bc+bd+cd)x - abcd$ for some $a, b, c, d \in$

$\iff f'(x) = 3ax^2 - 2a(b+c+d)x + a(bc+bd+cd)$

$\iff f''(x) = 6ax - 2a(b+c+d)$.

$f''(x) = 0 \iff x = \frac{b+c+d}{3}$.$\square$

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