Chapter 10

Overlapping Generations Models with Individual and Aggregate Uncertainty

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Overview In this chapter, we introduce both idiosyncratic and aggregate uncertainty into the overlapping generations (OLG) model. The methods that we will apply for the computation of these models are already familiar to you from previous chapters on the (stochastic) neoclassical growth model and will only be modified to allow for the more complex age structure of OLG models. In particular, we will apply the perturbation methods from Chapter 3, the algorithm for the computation of the stationary distribution from Chapter 7, and the algorithm by Krussell and Smith (1998) from Chapter 8 for the solution of the non-stochastic steady state and the business cycle dynamics of the OLG model.

In the following, we will first introduce individual stochastic productivity in the standard OLG model and, subsequently, aggregate stochastic productivity. In the first section, agents have different productivity types. Different from the traditional Auerbach-Kotlikoff models, agents are subject to idiosyncratic shocks and may change their productivity types at random. As a consequence, the direct computation of policies — i.e. solving the first-order and equilibrium conditions with the help of the Newton-Rhapson algorithm or similar methods — is no longer feasible, and we will resort to value function iteration. As an interesting application, we will attempt to explain the empirically observed income and wealth heterogeneity. In the second section, we introduce aggregate uncertainty and study the business cycle dynamics of the OLG model.

10.1 Overlapping Generations Models with Individual Uncertainty

One of the main aims of the heterogeneous-agent literature in the 1990s was to explain the high concentration of wealth. In the US economy, the distribution of wealth is characterized by a Gini coefficient equal to 0.78 according to estimates by Díaz-Giménez et al. (1997) and Budría Rodríguez et al. (2002). In the period 1990-2010, wealth inequality increased. For example, Quadrini and Ríos-Rull (2015) report a rise in the Gini coefficient of wealth from 0.80 to 0.85 during the period 1998-2010; during the Great Recession between 2006 and 2007, however, the top wealth quintile experienced the largest fall in net worth among all quintiles as documented by Krueger et al. (2016). In particular, net worth of the top quintile fell by an annual rate of 4.9% during 2006-2010, while it actually increased for the second through fourth quintiles.

One main explanatory factor of the inequality in the wealth distribution, of course, is the unequal distribution of earnings. However, when we added heterogeneous productivity into the Ramsey model in Section 7.4.2, the model failed to replicate the observed wealth concentration. In the present chapter, we add another important determinant of the wealth distribution in addition to heterogeneous individual productivity: life-cycle savings. Agents accumulate wealth to finance consumption in old age. For this reason, we will consider an overlapping generations model in the following.[1]

Our OLG model for the study of the wealth distribution builds upon the model presented in Chapter 9.3. It is characterized by the following features:

1. life-cycle savings,
2. uncertain earnings,
3. uncertain lifetime,
4. pay-as-you-go pensions and
5. endogenous labor supply.

[1] As an alternative way to model life-cycle savings, Castañeda et al. (2003) consider the standard Ramsey model with heterogeneous productivity. In addition, they assume that agents retire and die with certain probabilities. In the former case, agents receive pensions that are lower than labor income. Krueger et al. (2016) also introduce a constant probability of retiring into the model of Krussell and Smith (1998) (using their model variant with preference heterogeneity) to study the impact of the Great Recession on the US wealth distribution and consumption expenditures.

Uncertain earnings also generate additional wealth heterogeneity because income-rich agents increase their precautionary savings to ensure against the misfortune of a fall in individual earnings. As a consequence, the discount rate, $\beta^{-1}-1$, increases relative to the real interest rate r.[2] Therefore, if the lifetime is certain, consumption increases over the lifetime, even into the final years of life. Empirically, however, the consumption-age profile is hump-shaped in the US. For this (and other) reasons, we also introduce stochastic survival to improve the model's quality.[3] If agents have lower survival probabilities in old age, consumption is hump-shaped again because utility in future periods of life are discounted at a higher (effective) rate.

Uncertain lifetime also help us to explain the empirical distribution of wealth because households have an additional motive to accumulate precautionary savings to insure against the risk of longevity. In the presence of public pension, however, the incentive to accumulate savings is particularly reduced among the low-income households as their earnings are closer to the public pensions than in the case of the income-rich households. However, the quantitative effect of the public pay-as-you-go pensions system on savings and, hence, wealth inequality depends sensitively on the progressiveness of the pension schedule with respected to the individuals' accumulated pension contributions. Finally, endogenous labor supply help us to explain why wage inequality is smaller than earnings inequality in the US economy.

In the following OLG model, we first analyze the case with lump-sum pensions such that the dimension of the individual state space in continuous variables only amounts to one. We will compute the steady state for this model. The computational time of this problem is much higher than that in Chapter 9.3.1 due to the presence of idiosyncratic uncertainty and the use of value function iteration. In the second part, we introduce contribution-dependent pensions that will effectively expand the individual state space by one dimension. For these two economies, we will compute the stationary equilibrium only and compare the computa-

[2] Wickens (2011) analyzes the two-period OLG model and shows that capital accumulation can be either higher or lower than in the Ramsey model, meaning that the discount rate, $1/\beta-1$, may also be lower or higher than the real interest rate r in the OLG model. In the model of secular stagnation in Eggertsson et al. (2019), for example, the discount rate, $1/\beta-1$, amounts to 2%, while the real interest rate ranges between -1.5% and -2.0%.

[3] Uncertain lifetime was already introduced in the model of the demographic transition in Section 9.3.1.

tional speed of the different programming languages: PYTHON, GAUSS and JULIA.

10.1.1 The Model

The model builds upon that described in Section 9.1.3 and extends it to idiosyncratic income risk. For your convenience, we present the full model in the following.

Demographics

A period, t, corresponds to one year. At each period t, a new generation of households is born. Newborns have a real-life age of 21 denoted by $s = 1$. All generations retire at the end of age $s = T^W = 45$ (corresponding to a real-life age of 65) and live up to a maximum age of $s = T = 70$ (real-life age 90). The number of periods during retirement is equal to $T^R = T - T^W = 25$.

Let $N_t(s)$ denote the number of agents of age s at t. We denote the total population at t by N_t. At t, all agents of age s survive until age $s+1$ with probability ϕ_t^s, where $\phi_t^0 = 1$ and $\phi_t^T = 0$. In period t, the newborn cohort grows at rate n_t as described in Section 9.1.3:

$$N_{t+1}(1) = (1+n_t)N_t(1). \tag{10.1}$$

Households

Each household comprises one (possibly retired) worker. Households maximize expected intertemporal utility at the beginning of age 1 in period t

$$\max \sum_{s=1}^{T} \beta^{s-1} \left(\Pi_{j=1}^{s} \phi_{t+j-1}^{j-1} \right) \mathbb{E}_t \left[u(c_{t+s-1}^s, l_{t+s-1}^s) + \omega(g_{t+s-1}) \right], \tag{10.2}$$

where $\beta > 0$ denotes the discount factor. Instantaneous utility $u(c, 1-l)$ is specified as a function of consumption c and leisure $1-l$:

$$u(c, 1-l) = \frac{\left(c^{\gamma}(1-l)^{1-\gamma}\right)^{1-\eta}}{1-\eta}, \tag{10.3}$$

where $1/\eta$ denotes the intertemporal elasticity of substitution (IES) and γ is the share of consumption in utility.

During working life, the labor supply of the s-year-old household amounts to $0 \leq l^s \leq l^{max}$, $s = 1, \ldots, T^W$, where we impose a constraint on the minimum and maximum working hours, 0 and l^{max}, respectively. During retirement, $l_t^s \equiv 0$ for $s = T^W + 1, \ldots, T$.

Utility from government consumption, $\omega(g_t)$, is additive, so government consumption per capita, g_t, does not have any direct effect on household behavior (only indirectly through its effects on transfers and taxes).

Total gross labor income of the s-year-old worker in period t, $\epsilon(s, \theta, e) A_t w_t l_t^s$, consists of the product of his idiosyncratic productivity $\epsilon(s, \theta, e)$, aggregate productivity A_t, the wage per efficiency unit w_t and his working time l_t^s. Aggregate productivity A_t grows at rate g_A. The household's idiosyncratic labor productivity $\epsilon(s, \theta, e)$ is stochastic and also depends on his age s according to $\epsilon(s, \theta, e) = \theta e \bar{y}^s$. The shock θ follows a Markov process and takes only a finite number n_θ of possible values in the set $\Theta = \{\theta_1 = \underline{\theta}, \ldots, \theta_{n_\theta} = \overline{\theta}\}$ with $\theta_i < \theta_{i+1}$ for $i = 1, \ldots, n_\theta - 1$. Let $prob(\theta'|\theta)$ denote the transition probability of the household with productivity θ in this period to become a household with productivity θ' in the next period for households aged $s = 1, \ldots, T^W - 1$. In old age, we assume without loss of generality that productivity remains constant. Further note that we assume that the transition matrix is independent of age $s < T^W$ and time-invariant. The shocks θ are independent across agents, and the law of large numbers holds (there is an infinite number of agents) so that there is no aggregate uncertainty. The labor productivity process is calibrated in detail below. In addition, the individual's productivity depends on his permanent productivity type $e \in \{e_1, e_2\}$, which is chosen to reflect the difference in earnings that stems from different levels of education (high school/college) and an age component $\bar{y}^s$, which is hump-shaped over the life-cycle.

All households receive transfers tr_t from the government. The worker pays labor income taxes τ^l and social security contribution τ^p proportional to his labor income, $\epsilon(s, \theta, e) A_t w_t l_t^s$. The average accumulated earnings of the $s+1$-year old household at the beginning of period $t+1$ are summarized by the accounting variable at age $s+1$, x_{t+1}^{s+1} as follows:

$$x_{t+1}^{s+1} = \begin{cases} \frac{(s-1)x_t^s + \epsilon(s,\theta,e) A_t w_t l_t^s}{s}, & s = 1, \ldots, n_w \\ x_t^s (1 + g_A), & s = n_w + 1, \ldots, T, \end{cases} \tag{10.4}$$

with initial accumulated earnings equal to zero at the beginning of life, $x_t^1 = 0$. Note that workers do not accrue interest on their social security payments but that accumulated contributions in old age grow at the rate of aggregate productivity g_A.[4]

In old age, the retired worker receives a pension. We distinguish two cases: 1) In the first case, pensions $pen(x_t^s)$ are independent of the individual's contribution (or rather earnings), x_t^s, $pen(x_t^s) \equiv pen_t$. 2) In the second case, pensions $pen(x_t^s)$ depend on the average contributions over the life-cycle, and the variable x_t^s enters the individual state vector as a second continuous state variable. In the United States, the pension system is (imperfectly) indexed to contributions but redistributes from those with high contributions to those with low contributions. Such a pension system is called a 'progressive' system and will be specified in greater detail in the next section. Pensions are assumed to not be subject to income taxes.

Accordingly, net non-capital income y_t^s is represented by

$$y_t^s = \begin{cases} (1-\tau_t^l-\tau_t^p)\epsilon(s,\theta,e)A_t w_t l_t^s & s=1,\ldots,T^W, \\ \\ pen(x_t^s) & s=T^W+1,\ldots,T. \end{cases} \tag{10.5}$$

The budget constraint of the household at age $s=1,\ldots,T$ is then given by

$$(1+\tau_t^c)c_t^s+k_{t+1}^{s+1}+b_{t+1}^{s+1} = y_t^s+(1-\tau_t^k)(r_t-\delta)k_t^s+k_t^s+R_t^b b_t^s+tr_t, \tag{10.6}$$

where k_t^s and b_t^s denote the capital stock and government bonds of the s-year-old agent at the beginning of period t. The household is born without assets and leaves no bequests at the end of its life, implying $k_t^1 = k_t^{T+1} = 0$ and $b_t^1 = b_t^{T+1} = 0$. The agent receives interest income r_t and $r_t^b = R_t^b - 1$ on capital and government bonds and pays income taxes on labor and capital income at the rates of τ_t^l and τ_t^k, respectively. Capital depreciation δk_t^s is tax exempt. Consumption is taxed at the rate τ_t^c.

In addition, we impose a non-negativity constraint on assets, $b_t^s \geq 0$ and $k_t^s \geq 0$. If the size of households deceases, accidental bequests are collected by the government.

[4] As a consequence, stationary contributions $\tilde{x}_t^s = x_t/A_t$ and, hence, pensions during old age, $s = nw+1,\ldots,T$, will be constant in steady state and do not decline during retirement.

Technology

Output is produced with the help of capital K_t and effective labor L_t according to the standard Cobb-Douglas function:

$$Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha}. \tag{10.7}$$

Firms are competitive and maximize profits $\Pi_t = Y_t - r_t K_t - w_t A_t L_t$ such that factor prices are given by:

$$w_t = (1-\alpha) K_t^{\alpha} (A_t L_t)^{-\alpha}, \tag{10.8a}$$

$$r_t = \alpha K_t^{\alpha-1} (A_t L_t)^{1-\alpha}. \tag{10.8b}$$

Government and Social Security

The government levies income taxes τ_t^l and τ_t^k on labor and capital income and taxes τ_t^c on consumption in period t. In addition, the government confiscates all accidental bequests Beq_t. It pays aggregate transfers Tr_t, provides a certain level G_t of total public expenditures, and pays interest r_t^b on the accumulated public debt B_t. In each period, the government budget is financed by issuing government debt:

$$Tr_t + G_t + r_t^b B_t - Tax_t - Beq_t = B_{t+1} - B_t, \tag{10.9}$$

where taxes Tax_t are given by

$$Tax_t = \tau_t^l A_t L_t w_t + \tau_t^k (r_t - \delta) K_t + \tau_t^c C_t, \tag{10.10}$$

and C_t denotes aggregate consumption.

The government provides pay-as-you-go pensions to retirees that it finances with the contributions of workers. Let Pen_t denote aggregate pension payments. The social security budget is assumed to balance:

$$Pen_t = \tau_t^p A_t L_t w_t. \tag{10.11}$$

Households' Optimization Problem

In the following, we formulate the household optimization problem in recursive form. Therefore, we first make an assumption with respect to the portfolio choice of individuals. In equilibrium, as argued in Section 9.3, the allocation to the two assets is indeterminate because we have many agents, and the after-tax returns on both assets, K and B, are equal. Therefore, we make the innocuous assumption that all households hold both assets in the same proportion, which is equal to $K/(K+B)$ and $B/(K+B)$. In addition, we define household assets a_t^s as the sum of the two individual assets, $k_t^s + b_t^s$.

Let the state vectors be given by $z = (s, \theta, e, a)$ in case 1 and $z = (s, \theta, e, a, x)$ in case 2. In the following, we will use the abbreviation $z_s = (\theta, e, a)$ (case 1) and $z_s = (\theta, e, a, x)$ (case 2) for the individual state vector of the s-year old.[5] $v_t(z_s)$ denotes the value function of the s-year old household in period t.[6] The optimization problem of the household is given by

$$v_t(z_s) = \max_{c,l,a'} \left\{ u(c, 1-l) + \omega(g) + \beta \phi_t^s \sum_{\theta'} prob(\theta'|\theta)\, v_{t+1}(z_{s+1}) \right\}, \tag{10.12}$$

subject to (10.4) and:

$$(1+\tau_t^c)c = y + \left[1 + (1-\tau_t^k)(r_t - \delta)\right] a + tr - a', \tag{10.13a}$$

$$a \geq 0, \tag{10.13b}$$

with the terminal condition $v_t(z_{T+1}) = 0$. a' denotes next-period wealth.

Stationary Equilibrium

In the following, we consider the equilibrium for case 1 in which pensions are lump-sum, and the individual state vector z does not include average accumulated contributions x. To express the equilibrium in terms of stationary variables, we have to divide individual variables (with the exception of individual labor supply l_t^s) by aggregate productivity A_t and aggregate variables (with the exception of effective labor L_t) by the product of aggregate productivity A_t and the measure of the total population N_t. Therefore, we define the following stationary individual variables $\tilde{x}_t^s$ for $x \in \{c, y, k, b\}$:

[5] For notational convenience, we drop the time-period index t and the age index s in the following whenever appropriate. For example, c, l, y and a will denote individual consumption, labor, net income and wealth at age s in period t.

[6] Note that outside the steady state, the value function is not independent of the time period t.

$$\tilde{x}_t^s \equiv \frac{x_t^s}{A_t}$$

and stationary aggregate variables $\tilde{X}_t$ for $X \in \{Pen, Tr, G, B, Beq, Tax, Y, K, C, \Omega\}$:

$$\tilde{X}_t \equiv \frac{X_t}{A_t N_t}.$$

Stationary aggregate labor is defined as $\tilde{L}_t = L_t / N_t$. Moreover, individual and aggregate government transfers are identical given that transfers are distributed lump-sum and in equal amount to all households:

$$\widetilde{Tr}_t = \tilde{tr}_t.$$

To formulate this optimization problem in stationary variables, define

$$\tilde{v}_t = \frac{v_t}{A_t^{\gamma(1-\eta)}}, \tag{10.14}$$

$$\tilde{z}_s = \begin{cases} (\theta, e, \tilde{a}) & \text{case 1} \\ (\theta, e, \tilde{a}, \tilde{x}) & \text{case 2,} \end{cases} \tag{10.15}$$

$$u(\tilde{c}, 1-l) = \frac{\left(\tilde{c}^{\gamma}(1-l)^{1-\gamma}\right)^{1-\eta}}{1-\eta} \tag{10.16}$$

$$= \frac{u(c, 1-l)}{A_t^{\gamma(1-\eta)}}. \tag{10.17}$$

Accordingly, the Bellman equation can be rewritten in stationary form:

$$\tilde{v}_t(\tilde{z}_s) = \max_{\tilde{c}, l, \tilde{a}'} \left\{ u(\tilde{c}, 1-l) + \omega(\tilde{g}) + (1+g_A)^{\gamma(1-\eta)} \beta \phi_t^s \sum_{\theta'} prob(\theta'|\theta)\, \tilde{v}_{t+1}(\tilde{z}_{s+1}) \right\}, \tag{10.18}$$

with the terminal condition $\tilde{v}_t(\tilde{z}_{T+1}) = 0$.

The budget constraint of the household in stationary variables is given by

$$(1+\tau_t^c)\tilde{c} = \tilde{y} + \left[1 + (1-\tau_t^k)(r_t - \delta)\right]\tilde{a} + \tilde{tr} - (1+g_A)\tilde{a}' \tag{10.19}$$

with

$$\tilde{y} = \begin{cases} (1-\tau_t^l - \tau_t^p)\epsilon(s, \theta, e)\, l w_t & s = 1, \ldots, T^W, \\ \widetilde{pen} & s = T^W + 1, \ldots, T. \end{cases} \tag{10.20}$$

To define the measures of households, we introduce the concept of the density function $f(.)$ (with the associated distribution function $F(.)$) on the individual state space. For the sake of simplicity, we assume that we have already discretized the state space on which we compute the distribution function. In particular, we have chosen grids over the asset space $\tilde{a} \in \mathscr{A} = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_{n_a}\}$ with n_a grid points.[7] Let $f_t(s, e, \theta, \tilde{a})$ denote the (discretized) density function in period t associated with the state $\tilde{z} = (s, e, \theta, \tilde{a})$.

DEFINITION. In a stationary equilibrium with constant survival probabilities ϕ^s, a constant population growth rate, g_N, and constant public policy $\{\tau^c, \tau^k, \tau^l, \tau^p, \tilde{G}, \widetilde{Tr}, \widetilde{Pen}\}$, prices and the distribution of the individual state variables $f(s, \theta, e, \tilde{a})$ are constant and satisfy the following conditions:

1. Total population N_t is equal to the sum of all cohorts:

$$N_t = \sum_{s=1}^{T} N_t(s)$$

   with associated constant shares of the s-year-old cohorts

$$\mu^s = \frac{N_t(s)}{N_t}.$$

2. Population N_t and the youngest cohort $N_t(1)$ grow at the same rates $g_{N,t} = \frac{N_{t+1}}{N_t} - 1$ and $n_t = \frac{N_{t+1}(1)}{N_t(1)} - 1$, respectively, implying:

$$\frac{N_{t+1} - N_t}{N_t} = n.$$

3. Households maximize their lifetime utility subject to their budget constraint (10.19) and the non-negative constraint on wealth, $\tilde{a} \geq 0$, as described by the solution to the Bellman equation (10.18), implying the optimal policy functions $\tilde{a}'(\tilde{z})$, $\tilde{c}(\tilde{z})$ and $l(\tilde{z})$ for next-period wealth, consumption and labor supply.

[7] In the computation of the stationary equilibrium, we will use a finer grid on the asset space for the discretization of the value function than that for the policy function. For example, this numerical device will allow us to assess the accuracy of the policy function off grid points when we weight the Euler residuals by the measure of the households. We already described the choice of finer asset grids over the distribution than for the policy function (which follows Ríos-Rull (1999)) in Chapter 7.

4. Aggregate effective labor supply is equal to the sum of the individual effective labor supplies:

$$\tilde{L}_t = \sum_{s=1}^{T^w} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_a=1}^{n_a} \epsilon(s, \theta_{i_\theta}, e_j)\, l(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a})\, f(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a}). \tag{10.21}$$

5. Aggregate wealth $\tilde{\Omega}$ is equal to the sum of the individual wealth levels:

$$\tilde{\Omega} = \sum_{s=1}^{T} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_a=1}^{n_a} \tilde{a}_{i_a}\, f(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a}). \tag{10.22}$$

6. Firms maximize profits, implying the factor prices w and r:

$$w = (1-\alpha)\tilde{K}^{\alpha}\tilde{L}^{-\alpha}, \tag{10.23a}$$
$$r = \alpha\tilde{K}^{\alpha-1}\tilde{L}^{1-\alpha}. \tag{10.23b}$$

7. The after-tax returns on the two assets $\tilde{K}$ and $\tilde{B}$ are equal:

$$r^b = (1-\tau_r^k)(r-\delta). \tag{10.24}$$

8. In capital market equilibrium,

$$\tilde{\Omega} = \tilde{B} + \tilde{K}. \tag{10.25}$$

9. At the beginning of period $t+1$, the government collects accidental bequests from the s-year old households that do not survive from period t until period $t+1$:

$$\frac{\widetilde{Beq}'}{(1+n)(1+g_A)} = \sum_{s=2}^{T} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_a=1}^{n_a} (1-\phi^s)\left[1+(1-\tau^k)(r-\delta)\right] \tilde{a}'(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a})\, f(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a}), \tag{10.26}$$

where $\widetilde{Beq}'$ denotes next-period accidental bequests.

10. The goods markets clear:

$$\tilde{Y} = \tilde{C} + \tilde{G} + (1+g_A)(1+n)K' - (1-\delta)K, \tag{10.27}$$

where aggregate consumption $\tilde{C}$ is the sum of individual consumption values:

$$\tilde{C} = \sum_{s=1}^{T} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_a=1}^{n_a} \tilde{c}(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a})\, f(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a}). \tag{10.28}$$

11. The density function $f(s,\theta,e,\tilde{a})$ (and the associated distribution function $F(s,\theta,e,\tilde{a})$) of the per-capita variables (detrended by aggregate productivity A_t) are constant, $f(.) = f'(.)$ $(F(.) = F'(.))$. The dynamics of the distribution function $F(s,\theta,e,\tilde{a})$ evolves according to

$$F'(s+1,\theta',e,\tilde{a}') = \sum_{\theta'} \sum_{\tilde{a}=\tilde{a}'^{-1}(s,\theta,e,\tilde{a}')} Prob(\theta'|\theta) \frac{\phi^s}{1+n} F(s,\theta,e,\tilde{a}),$$

where, on the right-hand side of the equation, we sum over all the productivity types θ' in period $t+1$ in the outer sum and the wealth levels $\tilde{a}$ in period t that imply a next-period level $\tilde{a}'$ for given (s,θ,e) in the inner sum.[8]
The distribution of $\tilde{z} = (s,\theta,e,\tilde{a})$ among the newborn cohort is constant and is represented by:

$$F(1,\theta,e,\tilde{a}) = \begin{cases} \mu^1 \times \nu(\theta) \times \pi(e) & \text{if } \tilde{a} = 0 \\ 0 & \text{else,} \end{cases}$$

where $\nu(\theta)$ and $\pi(e)$ denote the shares of the θ and e productivity types in the cohorts (assumed to be constant over age s).[9]

Calibration

Our model is calibrated for the US economy for the year 2015. We assume that the US economy is in steady state in this period.[10] Periods correspond to years. Agents are born at real lifetime age 21, which corresponds to

[8] In the definition of $\tilde{a}'^{-1}$, we refer to the inverse function of $\tilde{a}'(s,\theta,e,\tilde{a})$ with respect to the argument $\tilde{a}' > 0$. For $\tilde{a}' = 0$, the inverse function may not be well defined because there may be an interval $[0,\tilde{a}^0]$ on which the optimal next-period wealth is equal to $\tilde{a}' = 0$ due to the credit constraint. In this case, we choose the upper boundary $\tilde{a}^0$ as the value of the function $\tilde{a}'^{-1}$ (with a slight abuse of notation). Note that $\tilde{a}'(s,\theta,e,\tilde{a})$ is a strictly monotone function of $\tilde{a}$ for values of $\tilde{a}' > 0$.

[9] In our definition of the stationary distribution dynamics, we choose to use 1) the continuous rather than the discretized version of $F(.)$ and 2) apply the concept of the distribution rather than the density function. 1) Using the continuous rather than the discretized state variables helps us to avoid the problem in the formulation of the dynamics if the optimal next-period asset level $\tilde{a}'(s,\theta,e,\tilde{a})$ at a particular grid point $\tilde{a} = a_{i_a}$ is not a grid point. We will discuss the numerical implementation of the dynamics in the next section. 2) By using $F(.)$ rather than $f(.)$, we can avoid the problem that, due to the credit constraint $\tilde{a} \geq 0$, the measure of the households with $\tilde{a} = 0$ is non-zero.

[10] Of course, this is not an innocuous assumption given that the Great Recession just took place during 2007-08.

age $s = 1$. As stated above, they work $T^w = 45$ years, corresponding to a real lifetime age of 65, and live a maximum life of 70 years ($T^R = 25$), meaning that agents do not become older than real lifetime age 90. Our survival probabilities ϕ_t^s and population growth rates n_t are taken from the UN (2015), which provides 5-year forecasts until the year 2100. We interpolate population data using cubic splines and assume that survival probabilities and the population growth rate are constant after 2100. In 2015, the population growth rate n amounted to 0.754%.

For the discount factor, we choose the parameter value $\beta = 1.011$ in accordance with the empirical estimates of Hurd (1989), who explicitly accounts for mortality risk. In addition, we choose the intertemporal elasticity of substitution $1/\eta = 1/2.0$. The preference parameter $\gamma = 0.33$ is calibrated such that the average labor supply $\bar{l}$ is approximately 0.30.[11] The model parameters are presented in Table 10.1.

Table 10.1 Calibration of OLG model with idiosyncratic uncertainty

Parameter	Value	Description
α	0.35	production elasticity of capital
δ	8.3%	depreciation rate of capital
g_A	2.0%	growth rate of output
$1/\eta$	1/2	intertemporal elasticity of substitution
γ	0.33	preference parameter for utility weight of consumption
β	1.011	discount factor
n	0.754%	population growth rate
$\tau^l + \tau^p$	28%	tax on labor income
τ^k	36%	tax on capital income
τ^c	5%	tax on consumption
G/Y	18%	share of government spending in steady-state production
B/Y	63%	debt-output ratio
$repl$	35.2%	gross pension replacement rate
$\{e_1, e_2\}$	$\{0.57, 1.43\}$	permanent productivity types

As in Section 9.1.3, the tax parameters are set to $\tau^l + \tau^p = 28\%$, $\tau^k = 36\%$, and $\tau^c = 5\%$, and the government consumption share in GDP is set equal to $G/Y = 18\%$ following Trabandt and Uhlig (2011). The

[11] We found this value by trial and error. As an initial guess, we took $\gamma = 0.338$ from p. 28 of Heer (2019) who considers the corresponding neoclassical growth model.

debt-output level amounts to $B/Y = 63\%$. The gross replacement rates of pensions with respect to wage income (of the household with unitary individual efficiency and average labor supply $\bar{l}$), $\theta = pen/(w\bar{l}) = 35.2\%$, for 2014 are taken from the OECD (2015) (series: gross pension replacement rates for men, % of pre-retirement earnings). In case 1 with lump-sum pensions, the endogenous social security rate τ^p amounts to 7.58%. In addition, we find a government-transfer-to-GDP ratio equal to $Tr/Y = 4.93\%$.

Finally, we calibrate the labor efficiency of the s-year old household, $\epsilon(s,\theta,e) = \theta e \bar{y}^s$. Following Krueger and Ludwig (2007), we choose the permanent efficiency types of the workers $\{e_1, e_2\} = \{0.57, 1.43\}$.[12] The mean efficiency index $\bar{y}^s$ of the s-year-old worker is taken from Hansen (1993b) and illustrated in Fig. 9.4. The logarithm of the idiosyncratic productivity shock, $\ln\theta$, follows a Markov process. The first-order autoregressive process is given by:

$$\ln\theta' = \rho \ln\theta + \xi, \tag{10.29}$$

where $\xi \sim N(0, \sigma_\xi)$ is distributed independently of age s. Huggett uses $\rho = 0.96$ and $\sigma_\xi^2 = 0.045$. Furthermore, we follow Huggett (1996) and choose a log-normal distribution of earnings for the 21-year old with $\sigma_{y_1} = 0.38$ and mean $\overline{y}^1$. As the log endowment of the initial generation of agents is normally distributed, the log efficiency of subsequent agents will continue to be normally distributed. This is a useful property of the earnings process, which has often been described as log-normal in the literature.[13]

We discretize the state space $\Theta = \{\theta_1, \ldots, \theta_{n_\theta}\}$ using $n_\theta = 5$ values. The logarithm of the states θ_{i_θ}, $i_\theta = 1, \ldots, n_\theta$, are equally spaced and range from $-m\sigma_{y_1}$ to $m\sigma_{y_1}$. We choose $m = 1.0$ such that the Gini coefficient of hourly wages amounts to 0.374, which we estimated for the US during the calibration period using PSID data. Our grid Θ, therefore, is presented by:

$$\Theta = (0.4688, 0.6847, 1.0000, 1.4605, 2.1332)$$

[12] This calibration is in accordance with recent evidence presented by Heer and Rohrbacher (2021) according to which the weekly hourly earnings of college graduates are approximately 2.5 times higher than those of the high school graduates since 1994.

[13] The log-normal distribution, however, has an unrealistically thin top tail. As argued by Saez (2001), this property of the income distribution has important implications for the design of optimal income tax progressivity. In addition, Mankiw et al. (2009) note that the derivation of the ability distribution, which is central to the parameterization of our model, from the income distribution is 'fraught with perils'.

with corresponding logarithmic values

$$\ln\Theta = (-0.7576, -0.3788, 0.0000, 0.3788, 0.7576).$$

The probability of having productivity θ_{i_θ} in the first period of life is computed by integrating the area under the normal distribution, implying the initial distribution among the 21-year-old agents for each permanent productivity type e_i, $i = 1, 2$:

$$\nu(\theta) = \begin{pmatrix} 0.1783 \\ 0.2010 \\ 0.2413 \\ 0.2010 \\ 0.1783 \end{pmatrix}.$$

Each permanent efficiency type e_i, $i = 1, 2$, has a share of 50% in each cohort.

The transition probabilities are computed using Tauchen's method as described in Algorithm 14.2.1. As a consequence, the efficiency index θ follows a finite Markov-chain with transition matrix:

$$Prob(\theta'|\theta) = \begin{pmatrix} 0.7734 & 0.2210 & 0.0056 & 0.0000 & 0.0000 \\ 0.1675 & 0.6268 & 0.2011 & 0.0046 & 0.0000 \\ 0.0037 & 0.1823 & 0.6281 & 0.1823 & 0.0033 \\ 0.0000 & 0.0046 & 0.2011 & 0.6268 & 0.1675 \\ 0.0000 & 0.0000 & 0.0056 & 0.2210 & 0.7734 \end{pmatrix}. \tag{10.30}$$

The pension function $\widetilde{pen}(.)$ and the social security parameters are calibrated as follows. In case 1, we simply assume that pensions are lump-sum and that the ratio of pensions to average labor income is equal to the constant replacement rate of 35.2%. In case 2, we calibrate the PAYG pension system in closer accordance with the US pension system. Following Huggett and Ventura (2000), pensions $\widetilde{pen}(\tilde{x})$ are modeled as a piecewise linear function of average past earnings:

$$\widetilde{pen}(\tilde{x}) = \begin{cases} \widetilde{pen}^{min} + 0.9\tilde{x} & \text{if } \tilde{x} \leq 0.2\bar{x} \\ \widetilde{pen}^{min} + 0.9(0.2\bar{x}) + 0.32(\tilde{x} - 0.2\bar{x}) & \text{if } 0.2\bar{x} < \tilde{x} \leq 1.24\bar{x} \\ \widetilde{pen}^{min} + 0.9(0.2\bar{x}) + 0.32(1.24\bar{x} - 0.2\bar{x}) + 0.15(\tilde{x} - 1.24\bar{x}) & \text{if } \tilde{x} > 1.24\bar{x} \end{cases} \tag{10.31}$$

where $\bar{x}$ denotes the average value of accumulated earnings $\tilde{x}$ among the retired in the economy. The lump-sum benefit $\widetilde{pen}^{min}$ is set equal to 12.42% of GDP per capita in the model economy. Depending on the bracket in which the retired agent's average earnings $\tilde{x}$ are situated, she receives 90% of the first 20% of $\bar{x}$, 32% of the next 104% of $\bar{x}$, and 15% of the remaining earnings $(\tilde{x} - 1.24\bar{x})$. Therefore, the marginal benefit rate declines with average earnings. We find that, in this case, the average replacement rate of pensions with respect to gross wage is higher than in the case of lump-sum pensions and amounts to 50.5%.

10.1.2 Computation of the Stationary Equilibrium

In the following, we compute the stationary equilibrium for the economy with lump-sum pensions where the population parameters ϕ^s and n are constant. We calibrate the model to match the parameters displayed in Table 10.1. The solution algorithm closely follows Algorithm 9.1.1 and consists of the following steps:

Algorithm 10.1.1 (OLG Model with Idiosyncratic Uncertainty)

Purpose: *Computation and calibration of the stationary equilibrium in the OLG model with idiosyncratic productivity shocks.*

Steps:

Step 1: *Parameterize the model, and choose asset grids for the individual state space.*

Step 2: *Make initial guesses of the steady-state values of the aggregate capital stock $\tilde{K}$, labor $\tilde{L}$, mean working hours $\bar{l}$, labor income taxes τ^l, the social security contribution rate τ^p and government transfers $\widetilde{tr}$.*

Step 3: *Compute the values w and r that solve the firm's Euler equation, and compute $\widetilde{pen}$.*

Step 4: *Compute the household's policy functions by backward induction using value function iteration.*

Step 5: *Compute the optimal path for consumption, savings and labor supply for the new-born generation by forward induction given the initial asset level $\tilde{a}^1 = 0$ and distribution of idiosyncratic productivities e and θ.*

Step 6: *Compute the aggregate savings $\tilde{\Omega}$, labor supply $\tilde{L}$, mean working hours $\bar{l}$, aggregate taxes $\widetilde{Tax}$ and transfers $\widetilde{tr}$.*

Step 7: Update the aggregate variables, and return to step 3 until convergence.
Step 8: Update the asset grid of the individual state space if necessary, and return to step 3 until convergence.

The algorithm is implemented in the program `AK70_stochastic_income`. The computer code is available on our download page in the programming languages PYTHON, JULIA and GAUSS.[14] As presented in Table 10.2, the computational times vary between 27 minutes and 55 hours depending on 1) the computer language and 2) the interpolation mode for the policy function. In general, we find in our heterogeneous-agent applications that PYTHON code is much slower than JULIA or GAUSS; PYTHON code is slower the more (nested) loops are part of the program. We will discuss computational speed in more detail below.

In the first step of the algorithm, we parameterize the model as presented in Table 10.1. In addition, we choose an equispaced grid of $n_a = 500$ points for the policy function over the individual asset space $\tilde{\Omega} = \{\tilde{a}_0 = \tilde{a}^{min}, \ldots, \tilde{a}_{n_a} = \tilde{a}^{max}\}$ with the minimum and maximum asset levels equal to $\tilde{a}^{min} = 0$ and $\tilde{a}^{max} = 20.0$. The upper boundary point $\tilde{a}^{max}$ is found with some trial and error such that the measure of households with wealth level equal and close to a^{max} is zero, but some households hold wealth in the top quintile of the asset space. Since our algorithm restricts the optimal policy function $\tilde{a}'(.)$ to lie on the interval $[\tilde{a}^{min}, \tilde{a}^{max}]$, the behavior of the policy functions displays abrupt changes at the upper boundary of the asset space interval, and we want to restrict the evaluations of policy functions to $\tilde{a} \ll \tilde{a}^{max}$. For the distribution function, we will choose a finer grid of $n_{ag} = 1{,}000$ points as discussed in Chapter 7. In case 2, we also have to specify a grid of n_x points for accumulated earnings $\tilde{x}$. We will consider this case separately in the next section and concentrate on case 1 with lump-sum pensions in the following. We also choose the discretization of the individual idiosyncratic productivity space Θ and the transition matrix of the Markov process, $prob(\theta'|\theta)$, as described in the previous section on calibration.

With the help of the stationary survival probabilities ϕ^s and the population growth rate n, we can compute the stationary age distribution in the

[14] The address of the computer code download page is as follows: https://www.uni-augsburg.de/de/fakultaet/wiwi/prof/vwl/maussner/dgebook/. In addition to the code, you can find an extensive Jupyter manuscript on our web pages that explain the PYTHON code in detail (line by line).

Table 10.2 Comparison of run time and accuracy

Interpolation	linear	cubic	cubic
Grid points			
n_a	500	500	300
n_{ag}	1,000	1,000	1,000
Aggregates			
$\tilde{K}$	1.486	1.486	1.484
$\tilde{L}$	0.3097	0.3097	0.3096
Accuracy			
Young	0.00065	0.000074	0.000088
Old	0.00196	0.000084	0.000118
Run time			
Julia	1h:29m:56s	1h:32m:43s	45m:37s
Gauss	27m:38s	1h:16m:34s	51m:56s
Python	32h:49m:37s	55h:30m:33s	48h:17m:04s

Notes: Accuracy is measured by the mean absolute value of the Euler residuals for the young and old households. The Euler residual is evaluated at all $n_{ag} = 1,000$ asset grid points over the distribution. Run time is given in hours:minutes:seconds on an Intel(R) Xeon(R), 2.90 GHz.

population. Let N_t denote total population and $N_t(s)$ denote the number of s-year old households in period t. We define the measure of the s-year old in period t, μ_t^s, as

$$\mu_t^s \equiv \frac{N_t(s)}{N_t}. \tag{10.32}$$

The sum of all measures μ_t^s is equal to one by definition:

$$\sum_{s=1}^{70} \mu_t^s = 1.0. \tag{10.33}$$

To compute the stationary measure $\{\mu^s\}_{s=1}^{70}$, we simply set up a vector with 70 entries in the computer program 'AK70_stochastic_inomce', initialize the mass of 1-year olds, μ^1, equal to one and iterate over age $s = 1, \ldots, T-1$ as follows:

$$\mu^{s+1} = \frac{\phi^s}{1+n}\mu^s. \tag{10.34}$$

This formula is derived from the following two equations:

$$N_t(1) = (1+n)N_{t-1}(1), \tag{10.35}$$

and

$$N_t(s) = \phi^{s-1}N_{t-1}(s-1). \tag{10.36}$$

Division of both equation by population size N_t and noting that $1+n = \frac{N_t}{N_{t-1}}$ implies our formula.

After we have computed all μ^s, $s = 1, \ldots, T$, we normalize the measures by dividing each measure μ^s by the sum of all measures, $\sum_s \mu^s$, meaning that their sum is equal to one and the μ^s also present the share of s-year olds in the total population. The measure of the cohorts in our calibration declines monotonically with age s as presented in Fig. 10.1.

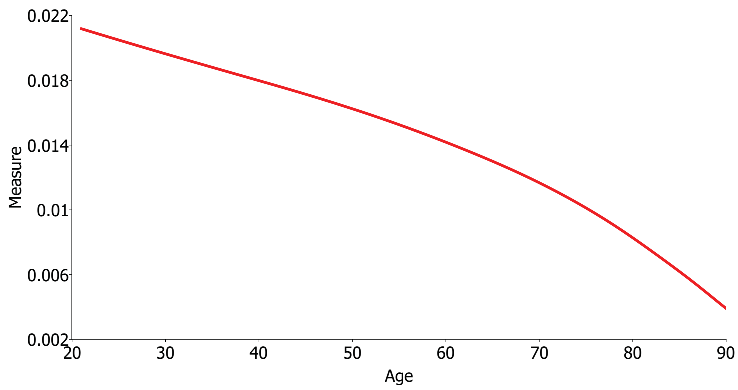

Figure 10.1 Measure μ^s of the s-year-old cohort

In the second step, we provide an initial guess of the endogenous aggregate variables: the aggregate capital stock $\tilde{K}$, efficient labor $\tilde{L}$, mean working hours $\bar{l}$, labor income taxes τ^l, the social security contribution rate τ^p and government transfers $\widetilde{tr}$. We assume that households work 30% of their time, $\bar{l} = 0.30$. We also simply assume that aggregate efficient labor is also equal to $\tilde{L} = 0.30$. Of course, this is only an approximation in our economy. On the one hand, only 78% of the households are working.

Since we normalize the total measure of households to one, $\tilde{L}$ should be lower for this reason. On the other hand, we observe that workers with higher productivity also work longer hours. Therefore, $\tilde{L}$ should be higher than the number of working hours. As it turns out, our initial guess of $\tilde{L}$ of 0.30 is close to its final value (=0.304). The convergence and the final result of our computation is insensitive to the initial guesses; however, computational time until convergence to the final values may depend on our initial choice.

Using a real interest rate (net of depreciatoin) equal to $r-\delta = 3\%$ and efficient labor supply $\tilde{L} = 0.30$, we can compute the implied steady-state value of the capital stock from the first-order condition of the firm, implying $\tilde{K} = 1.708$. In our model, aggregate capital is lower than aggregate savings, $\tilde{K} < \tilde{\Omega}$, as part of the savings is invested in government bonds $\tilde{B}$. In equilibrium, government bonds amount to 63% of GDP. Since the wealth-GDP ratio is approximately 3.0 in the United States, government bonds amount to approximately 21% of wealth, and hence, physical capital is approximately 79% of wealth. Therefore, we obtain an initial guess for our wealth $\tilde{\Omega}$ by using the approximation $\tilde{\Omega} \approx 1.26\tilde{K}$. For the final remaining aggregate quantities, we need to provide a guess for government transfers $\widetilde{tr}$, which we initialize with a small number, $\widetilde{tr} = 0.01$.

In Step 3, we use the firm's first-order conditions (10.8) with respect to labor and capital to compute the wage w and interest rate r. In stationary variables, these equations can be expressed as follows:

$$w_t = (1-\alpha)\tilde{K}_t^{\alpha}\left(\tilde{L}_t\right)^{-\alpha}, \tag{10.37a}$$

$$r_t = \alpha\tilde{K}_t^{\alpha-1}\left(\tilde{L}_t\right)^{1-\alpha}. \tag{10.37b}$$

Stationary pensions are computed with the help of our calibration for the replacement rate $repl$ with the help of $\widetilde{pen} = repl \times w\bar{l}$. The social security tax τ^p follows from the social security budget. Noting that the share of retirees in total population is equal to 21.9%, we find the equilibrium social security tax using $\tau^p = 0.219\widetilde{pen}/(w\tilde{L})$. Moreover, we can compute τ^l from our calibration $\tau^l + \tau^p = 28\%$. As a result, we have the values of all variables, $\{w, \tau^l, \tau^p, r, r^b, \widetilde{tr}, \widetilde{pen}\}$, that are needed as input for the computation of the individual policy functions $\tilde{a}(.)$, $\tilde{c}(.)$ and $l(.)$.

In Step 4, we need to solve a finite-time dynamic programming problem to find the optimal policy function. Again, we use value function iteration with linear (or, alternatively, cubic) interpolation between grid points. The optimization step of the rhs of the Bellman equation (10.18) is performed using the golden section search algorithm described in Section 13.6.1.

The optimizing function in PYTHON is called using the command 'GoldenSectionMax(f,ay,by,cy,tol)' where 'f' is a one-dimensional function, the maximum is bracketed by the points 'ay' and 'cy', 'by' is an intermediate point and 'tol' describes the maximum difference between the upper and lower boundary, 'cy-ay'.[15] In the above parameterization of the program, we have chosen the tolerance 'tol=1e-5'.

The finite value function problem is solved by starting in the last period of life, $s=T$, and iterating backwards in age until the first year of life, $s=1$. First, we initialize the value function of the retirees 'vr[:,:]' ('vr[:,:,:]') as an array with $n_a \times n_r$ ($n_a \times n_x \times n_r$) zero entries in case 1 (case 2). Note that, in old age, the value functions (and the policy functions) of the retirees do not depend on the productivity type of the worker. The pensions are either lump-sum (case 1) or dependent on accumulated earnings $\tilde{x}$ (case 2) but not on θ or e during retirement. Similarly, we initialize the arrays containing the policy functions for consumption $\tilde{c}(.)$ and next-period wealth $\tilde{a}'(.)$.

The value function in the last period of life, $T = T^W + T^R = 70$, with given exogenous pensions $\widetilde{pen}$ (case 1) and interest rate r is represented by

$$\tilde{V}^T(\tilde{a}^T) = \tilde{u}(\tilde{c}^T, 1) \tag{10.38}$$

with

$$\tilde{c}_t^T = \frac{\widetilde{pen} + [1 + (1-\tau^k)(r-\delta)]\tilde{a}^T + \widetilde{tr}}{1+\tau^c}. \tag{10.39}$$

Retired agents at age $T=70$ consume their total income consisting of pensions and interest income plus government transfers. We also store the optimal policy function $\tilde{c}()$ and optimal next-period assets $\tilde{a}'()$ (=0) of the T-year-old household for all asset points $\tilde{a}_{i_a}$, $i_a = 1, \ldots, n_a$.

In the next step of the value function iteration over age $s = T, \ldots, 1$, we consider the retiree in his second-to-last period at retirement age $T-1$. We use golden section search to compute the optimal solution to the right-hand side of the Bellman equation:

$$\tilde{v}^{T-1}(\tilde{a}^{T-1}) = \max_{\tilde{a}^T} \left\{ \tilde{u}\left(\frac{\widetilde{pen} + [1+(1-\tau^k)(r-\delta)]\tilde{a}^{T-1} + \widetilde{tr} - (1+g_A)\tilde{a}^T}{1+\tau^c}, 1 \right) + (1+g_A)^{\gamma(1-\eta)} \beta \phi^{T-1}\, \tilde{v}^T(\tilde{a}^T) \right\}.$$

[15] In the GAUSS codes, the golden section search routine is invoked by the command 'GoldenSectionMax(&f,ay,by,cy,tol)' providing the function '&f'. In JULIA, we use the package 'optimize' and specify a function 'f(x,y)' where the optimization with the help of golden section search takes place with respect to the argument 'x,' and global variables are provided by the variable 'y'. The optimization step is performed by the command 'optimize(x− >value1(x,y),ax,bx, GoldenSection(),abs_tol=tol1,rel_tol=tol1)'.

(10.40)

In our computer code, the right-hand side of the Bellman equation is stored as a function of $\tilde{a}'$ and age s. To optimize it, we need to evaluate the value function at age $s+1$ for the asset level $\tilde{a}'$, which may not be a grid point stored in the array 'vr[:]'. For this reason, we need to use interpolation to find the value. We will use either linear or cubic interpolation and compare them with respect to speed and accuracy below.

As a second input of the golden section search routine, we need to bracket the maximum by the interval $[ay, cy]$ with an intermediate value by. Therefore, we first initialize the value 'v0' of the value function at this grid point $\tilde{a} = \tilde{a}_{i_a}$ with a large negative value 'neg=-1e10'. Next, we start to search over the grid points $\tilde{a}'$ on the grid $\tilde{\Omega} = \{\tilde{a}_1, \ldots, \tilde{a}_{n_a}\}$. Whenever we find a higher value for the rhs of the Bellman equation with the help of $\tilde{a}_j$, we update 'v0' accordingly. Once the rhs of the Bellman equation starts to decline, we know that we have bracketed the maximum. Recall that the rhs of the Bellman equation is a concave function of $\tilde{a}'$. We also make use of the monotonicity of the value function $\tilde{v}(.)$ and the policy function $\tilde{a}'(.)$. Since both functions increase monotonically with $\tilde{a}$, we also speed up our algorithm when we search for $[ay, cy]$. Assume that we have found the lower boundary ay for the grid point $\tilde{a}_{i_a}$. When we start searching for the lower boundary ay for the next grid point, $\tilde{a}_{i_a+1} > \tilde{a}_{i_a}$, we simply start at the value of ay found in the last iteration.

In the optimization, we also need to take special care if ay (cy) happens to be the boundary point $\tilde{a}^{min}$ $(\tilde{a}^{max})$. In this case, we have to evaluate whether we have a corner solution $\tilde{a}' = \tilde{a}^{min}$ $(\tilde{a}' = \tilde{a}^{max})$. For example, we compare the rhs of the Bellman equation at $\tilde{a}^{min}$ with that evaluated at $\tilde{a}^{min} + eps$, where 'eps' is a small constant. If the value at $\tilde{a}^{min}$ is larger than that at $\tilde{a}^{min} + eps$, we have a corner solution. Otherwise, we may apply golden section search again.

At this point, let us mention that it is always a good idea to print intermediate results during the coding process. For example, you should start programming the inner loop over the value function prior to the outer loop over the aggregate variables. Once you have computed the value function and policy functions of the retiree for the first time, you should analyze their graphs for different ages and check whether their shapes are monotone and increasing and whether the policy functions are well-behaved at the boundaries of the state space where $\tilde{a}' \geq 0$ or $\tilde{a}' \leq a^{max}$ might be binding. In our web documentation of the PYTHON code, for example, you will find the illustrations of $\tilde{c}()$, $\tilde{a}'()$ and $\tilde{v}(.)$ as functions of $\tilde{a}$ at the first year of retirement at age $s = T + 1$, and all functions are smooth and monotone.

Next, we turn to the value function iteration for the worker. The procedure is almost the same as in the case of the retiree. The worker also maximizes the right-hand side of the Bellman equation. However, different from the retiree, he also chooses optimal labor supply.

There are different ways in which we can implement this two-dimensional optimization problem. We have chosen to break it up into two nested optimization problems. In the outer loop, we optimize over the next-period wealth $\tilde{a}'$ as in the case of the retiree using golden section search. In the inner loop, we compute the optimal labor supply given this and next-period wealth, $\tilde{a}$ and $\tilde{a}'$, using the first-order condition of the worker with respect to his labor supply:

$$\frac{(1-\tau^l-\tau^p)\epsilon(s,\theta,e)w}{1+\tau^c} = \frac{1-\gamma}{\gamma}\frac{\tilde{c}}{1-l}. \tag{10.41}$$

After the substitution of $\tilde{c}$ from the budget constraint of the worker, we can solve for the labor supply l of the worker:

$$l = \gamma - \frac{1-\gamma}{(1-\tau^l-\tau^p)\epsilon(s,\theta,e)w}\Big(\big[1+(1-\tau^k)(r-\delta)\big]\tilde{a} + \widetilde{tr} - (1+g_A)\tilde{a}'\Big). \tag{10.42}$$

We also have to impose the constraints $l \geq 0$ and $l \leq l^{max}$ if l happens to lie outside the admissible space.

Note that, in the present case, it is easy to compute the optimal labor supply since we can explicitly solve for optimal labor l. If we use a different utility function or consider contribution-based pensions, labor supply may only be computed implicitly, and we will have to use more sophisticated methods. In the case of a different utility function, we may have to solve a non-linear equation problem. If we also consider contribution-based pensions, the computation of the optimal labor supply is more complicated, and we will discuss it in the next section of this chapter.

The implementation of this nested optimization is straightforward. Different from the case of the retired household, however, we need to solve the value function iteration problem for the different productivity types θ and e because the worker's labor supply and income depend upon them. Of course, the additional loops over the variables θ_{i_θ}, $i_\theta = 1, \ldots, 5$, and e_j, $j = 1, 2$, considerably slow the computation. Therefore, the better modeling of the workers' heterogeneity comes at the cost of reduced speed.

To graphically evaluate the computation of the policy functions for the worker is difficult. Given that the number of all productivities and ages of the workers alone amounts to $2 \times n_\theta \times T^w = 450$, we need to restrict our attention to the study of a random choice. In Fig. 10.2, we graph the labor supply function of skilled and unskilled workers at age $s = 21$ with idiosyncratic productivity $\theta_4 = 0.378$. Labor supply falls with higher wealth $\tilde{a}$. The high-skilled (blue curve) has a much higher labor supply than the low-skilled (red curve) for given wealth $\tilde{a}$ (but since the high-skilled workers hold higher wealth on average, the difference in working hours observed empirically is smaller).[16] In addition, the lower boundary $l \geq 0$ starts to bind for the low-skilled worker for a wealth level $\tilde{a}$ in excess of 12.4.

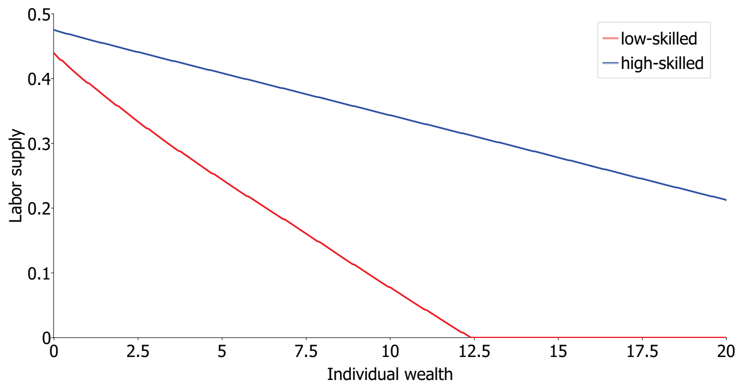

Figure 10.2 Labor supply of the low- and high-skilled workers with idiosyncratic productivity θ_4

To assess the accuracy of our policy function approximation, we study the so-called 'Euler equation residual'. Therefore, we compute the value of the Euler equation in implicit form using the policy functions for consumption $\tilde{c}$ and labor l.[17] For the s-year old worker with wealth level $\tilde{a}$ and productivity type $\epsilon(s, \theta, e)$, the Euler equation residual is defined by:

[16] Blundell et al. (2018) find a significant difference of both male and female employment at the intensive margin across skill groups. According to their Fig. 4, for example, men aged 25-55 with college, high school, and no high school worked approximately 45, 43, and 41 hours per week on average during the period 1978-2007.

[17] The Euler equation residual is considered in many others chapters of this book, for example in Chapter 6 on projection methods.

$$R(\tilde{a}) = 1 - \frac{\tilde{u}_c(\tilde{c}, 1-l)}{\beta(1+r^b)(1+g_A)^{\gamma(1-\eta)-1}\phi^s \mathbb{E}\left\{\tilde{u}_c(\tilde{c}', 1-l')\right\}}, \qquad (10.43)$$

where $\tilde{c}'$ and l' are the next-period consumption level and labor supply and expectations $\mathbb{E}$ are conditional on information at age s in period t (after observing the idiosyncratic productivity shock in this period). The right-hand side of (10.43) presents the first-order condition of the household with respect to next-period wealth $\tilde{a}'$ and follows from the envelope condition[18] and the derivation of the Bellman equation (10.18) with respect to $\tilde{a}'$ (after the substitution of the budget constraint):

$$\frac{d\tilde{v}}{d\tilde{a}} = \frac{\partial \tilde{u}(\tilde{c}, 1-l)}{\partial \tilde{c}} \frac{1+(1-\tau^k)(r-\delta)}{1+\tau^c}, \qquad (10.44a)$$

$$\frac{\partial \tilde{u}(\tilde{c}, 1-l)}{\partial \tilde{c}} \frac{1+g_A}{1+\tau^c} = (1+g_A)^{\gamma(1-\eta)}\beta\phi^s \mathbb{E}\left\{\frac{d\tilde{v}'}{d\tilde{a}'}\right\}. \qquad (10.44b)$$

Shifting the time (and age) index of (10.44a) into the next period and substituting the expression for $\frac{d\tilde{v}'}{d\tilde{a}'}$ in (10.44b), we derive the Euler equation (using the arbitrage condition $r^b = (1-\tau^k)(r-\delta)$):

$$\frac{\partial \tilde{u}(\tilde{c}, 1-l)}{\partial \tilde{c}} = (1+g_A)^{\gamma(1-\eta)-1}\beta(1+r^b)\phi^s \mathbb{E}\left\{\frac{\partial \tilde{u}(\tilde{c}', 1-l)}{\partial \tilde{c}'}\right\}. \qquad (10.45)$$

If $R(\tilde{a}) = 0$, the Euler equation error is zero, and we have a perfect fit.

As emphasized in previous chapters, the Euler equation residual is a popular measure of accuracy. In particular, Santos (2000) has shown that the accuracy of the numerical approximation of the policy function is of the same order of magnitude as the numerical error of the Euler equation residual.[19]

Therefore, we apply the Euler equation error as our criterion to get an idea about the goodness of fit. We use a finer grid of the asset level for the Euler equation residual than that for the policy function. We want to consider points off the policy function grid where we have to interpolate between grid points and accuracy might be lower. As our measure of fit, we use the average absolute Euler equation residual. Alternatively, you

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[18] Please compare this with the envelope condition in the Ramsey model presented in Chapter 3.

[19] Note, however, that his results are derived for the stochastic neoclassical growth model where households are infinitely lived.

may weigh the Euler residuals with the measure of the s-year-old households with asset level $\tilde{a}_{i_a}$ and productivity $\epsilon(e_j, \theta_{i_\theta}, s)$, $f(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a})$.

To compute the Euler equation residual, we also need to define functions for the marginal utility of consumption for the present and the next period in our computer code. The evaluation of next-period marginal utility is very time-consuming because we need to interpolate the next-period policy functions for consumption and labor supply to derive c' and l'. Therefore, you should compute the Euler residual only when you start to set up the program to check for the accuracy of your computer code in the development stage and after the final outer loop over the aggregate variables when you want to determine whether the number of grid points and the mode of interpolation is sufficient.

In the case of linear interpolation of the policy function, the Euler equation residuals are equal to 0.065% and 0.196% for the workers and retirees, respectively, as presented in Table 10.2. Given the accuracy of our golden section search routine (1e-5), this accuracy is admissible, and we are fairly certain that we have correctly programmed the value function iteration problem. The solution to the dynamic programming problem and that from the Euler equation (almost) coincide.

If you want to increase accuracy, you need to either 1) consider more grid points n_a and/or 2) use cubic interpolation instead of linear interpolation. As is often the case in computational economics, you are faced with the trade-off between speed and accuracy. In the case of cubic interpolation and $n_a = 500$ grid points, the accuracy measure in form of the mean absolute Euler equation residual improves and decreases to 0.0074% and 0.0085% for the young and old workers, respectively, but computational time increases considerably in GAUSS and PYTHON and only to a small extent in JULIA (compare the first and second entry columns in Table 10.2). Evidently, we can achieve higher accuracy by exploiting the curvature of the value function. For a smaller number of grid points, $n_a = 300$, we find that, in the case of JULIA, cubic rather than linear interpolation does not only provide higher accuracy as measured by the Euler equation residual (amounting to 0.0088% and 0.0118% for the worker and retiree) but also allows for a faster computation. As displayed in the third entry-column of Table 10.2, computational time drops from 1 hour and 29 minutes (linear interpolation with $n_a = 500$) to 45 minutes (cubic interpolation with $n_a = 300$). This observation, however, does not hold unanimously for all computer languages, e.g., for the computer languages PYTHON and GAUSS in the present example. We, therefore, recommend that you program your code flexibly enough that you can easily switch between linear and cubic interpolation.

In Step 5, we compute the endogenous wealth distribution $f(s,e,\theta,\tilde{a})$ in each cohort s over the asset space $[\tilde{a}^{min},\tilde{a}^{max}]$ for the n_θ idiosyncratic and n_e permanent productivity types. Over the asset space, we use an equispaced grid of $n_{ag}=1,000$ points, resulting in a total number of grid points (in case 1 with lump-sum pensions):

$$n_{ag}\times n_e\times n_\theta\times T^W+n_{ag}\times T^R=1000\times 2\times 5\times 45+1000\times 25=475,000. \tag{10.46}$$

In the computer program '*AK70_stochastic_income*', the distribution is stored in the variables 'gkw[$\tilde{a}$,θ,e,s]' and 'gkr[$\tilde{a}$,s]' for the worker and retiree, respectively. In the case of the worker, 'gkw' is a four-dimensional array, while, in the case of the retired household, 'gkr' is only two-dimensional because the productivity type does not affect behavior in old age.

We start the computation of the distribution function with the newborn generation at age $s=1$ with zero wealth. The newborn generation has measure $\mu^1=0.02118$. Furthermore, we know the distribution of the idiosyncratic productivity at age 1, which is given by $\nu(\theta)$ (see the calibration section). Each permanent productivity has measure 1/2 in each cohort of workers. The distribution in the first period is initialized as follows:

$$gkw[\tilde{a},\theta,e,1]=\begin{cases}\frac{1}{2}\mu^1\nu(\theta) & \text{if } \tilde{a}=0\\ 0 & \text{else.}\end{cases}$$

Given the distribution of the wealth level $\tilde{a}$ and productivity $\epsilon(s,\theta,e)$ at age $s=1$, we can compute the distribution of $(\tilde{a},\theta,e)$ at age $s=2$ by using the optimal decision functions of the agents with respect to labor supply l, consumption $\tilde{c}$ and next-period wealth $\tilde{a}'$ and the transition probabilities for the idiosyncratic productivities. The policy functions at age $s=1$ were stored in the arrays 'lopt[$\tilde{a}$,θ,e,1]', 'cwopt[$\tilde{a}$,θ,e,1]' and 'awopt[$\tilde{a}$,θ,e,1]' in Step 4.

For expositional reasons, let us consider a specific example, e.g., a low-skilled worker with idiosyncratic productivity $\theta_4=1.4605$ and zero wealth $\tilde{a}=0.0$. His measure is equal to 0.00213 and given by the product of 1) the measure of the 1-year old, μ^1, 2) the share of workers with idiosyncratic productivity θ_4, $\nu(\theta_4)$, and 3) the share of low-skilled workers among all workers (equal to 1/2). His optimal next-period wealth is represented by 'awopt[0,θ_4,e_1,1]=0.008365'. Therefore, we know that the households at age $s=1$ in period t with measure equal to 0.00213 will choose to have next-period wealth $\tilde{a}' = 0.008365$. Some households will die, so only $\phi^1 = 99.92\%$ of the 1-year-old households survive. In addition, the share of each cohort decreases between period t and period $t+1$ by the factor $1/(1+n) = 0.9925$ because the population grows. As a consequence, the measure of households in the next period that is implied by this type of worker amounts to

$$0.002129 \frac{\phi^1}{1+n} = 0.002111. \tag{10.47}$$

In the computation of the dynamics, we also have to take care of the Markov transition matrix $prob(\theta'|\theta)$ that describes the behavior of the stochastic part of idiosyncratic productivity. For example, for θ_4, we know from inspection of (10.30) that a household with productivity θ_4 at age 1 becomes a household with productivity θ_{i_θ} at age 2 with probabilities 0.0000, 0.0046, 0.2011, 0.6268 and 0.1675 for $i_\theta = 1,\ldots,5$. Therefore, we have to add the above measure to the five productivity levels $\theta_1 - \theta_5$ weighted by the respective probabilities.[20] Similarly, we compute the dynamics of the households at age 1 over the complete productivity and asset space. In the next outer iteration, we increase the age s by one.

We also have to discuss how we handle the case when the next-period asset $\tilde{a}'$ is not a grid point on the grid 'ag' for the distribution function. For example, in our example above, the next-period wealth amounts to $\tilde{a}' = 0.008365$ and lies between the first two grid points $ag_1 = 0$ and $ag_2 = 0.01998$. In this case, we proceed as described in Chapter 7.2 and introduce a simple lottery: If the optimal next-period wealth $\tilde{a}'$ happens to lie between ag_{ia-1} and ag_{ia}, $ag_{ia-1} < \tilde{a}' < ag_{ia}$, we simply assume that the next-period wealth level will be ag_{ia} with probability $(\tilde{a}' - ag_{ia-1})/(ag_{ia} - ag_{ia-1})$ and ag_{ia-1} with the complementary probability $(ag_{ia} - \tilde{a}')/(ag_{ia} - ag_{ia-1})$.

The computational time for Step 5 diverges significantly across the different computer languages. In essence, we have to compute five nested iterations in the five variables age s, permanent productivity e, idiosyncratic productivity θ, individual wealth $\tilde{a}$ and next-period idiosyncratic productivity θ' (from the outer to the inner loop) to compute the measures at the 475,000 grid points. We programmed the computation of the

[20] As a consequence, we have to iterate over five loops (s, e, θ, $\tilde{a}$, θ') to compute the distribution $f(s,e,\theta,\tilde{a})$ at age $s = 2,\ldots,T^w$.

distribution in exactly the same way in the three programming languages PYTHON, JULIA and GAUSS, meaning that we executed exactly the same number of operations (products and sums) and stored the numbers with the same accuracy (with a single precision of 8 digits). PYTHON, in particular, proved to be particularly slow in the computation of Step 5 and needed 33 minutes. In comparison, JULIA and GAUSS (8.2 and 11.3 seconds) were considerably faster.[21]

In Step 6, we update the aggregate variables $\tilde{\Omega}$, which is simply equal to the sum of all savings, weighted by the measure of the household with state vector $(s, \theta, e, \tilde{a})$. To compute the new aggregate capital stock $\tilde{K}$, we need to compute production $\tilde{Y}$ first (using the old values of the capital stock and aggregate labor) and use the calibration that debt $\tilde{B}$ is equal to $0.63\tilde{Y}$ to derive the value of $\tilde{B}$. Aggregate capital $\tilde{K}$ is then implied by the capital market equilibrium:

$$\tilde{K} = \tilde{\Omega} - \tilde{B}. \tag{10.48}$$

The labor market variables $\tilde{L}$ and $\bar{l}$ are simply equal to the sum of the individual variables weighted by their measures 'gkw' of the distribution function. The pension contribution rate τ^p is computed with the help of the social security budget. To derive government transfers, we need to compute accidental bequests and total taxes first. The former variable is derived from the sum of all individual accidental bequests, weighted by the measure of the s-year old individual with wealth $\tilde{a}$ and productivity $\epsilon(\theta, e, s)$. Total taxes can be derived with the help of

$$\widetilde{Tax} = \tau^l w \tilde{L} + \tau^k (r - \delta)\tilde{K} + \tau^c \tilde{C},$$

where aggregate consumption $\tilde{C}$ is equal to the sum of all individual consumption values (weighted by their measure).

Given Walras's law, we have one redundant equation in our model, which is represented by the goods market equilibrium (10.27):

$$\tilde{Y} = \tilde{K}^{\alpha}\tilde{L}^{1-\alpha} = \tilde{C} + \tilde{G} + [(1+n)(1+g_A) - 1 + \delta]\tilde{K}.$$

We have not yet used this equation in the computation of our solution. As an additional accuracy check of our computation, we should evaluate and compare the left-hand and right-hand sides of this equation. In our case, the deviation is less than 0.001 after the final iteration $q = 28$, and we can be quite confident that our computation is correct and has converged.

[21] In the case of PYTHON and JULIA, we also sped up the code using multi-threading.

In Step 7, we use this new value for the capital stock ('knew') to update 'kbar' using a weighted average of the new and the old values of the capital stock (with weights 0.2 and 0.8, respectively) to derive $\tilde{K}$ for the next outer loop over the aggregate variables. The use of this extrapolation (using the value from the last iteration in the update) helps to stabilize the sequence so that the outer loop converges. We proceed in a similar way with the other aggregate variables $\tilde{L}$, $\bar{l}$, τ^p, τ^l and $\widetilde{\text{tr}}$ and return to Step 2. We also save the values of the aggregate variables in each outer loop and find that the convergence of the capital stock $\tilde{K}$ is hump-shaped and smooth.

Finally, in Step 8, we use the distribution function $f(s, e, \theta, \tilde{a})$ to evaluate whether the upper boundary $\tilde{a}^{max}$ is chosen reasonably. If very close to $\tilde{a}^{max}$, all measures are equal to zero, we know that we have bracketed the ergodic distribution. If the highest wealth level $\tilde{a}$ with a strictly positive measure is reasonably close to $\tilde{a}^{max}$ we have also chosen an efficient upper boundary. In addition, we study the Euler equation residuals to gauge the accuracy of the interpolation scheme (linear or cubic) and the number of grid points for the asset grid of the policy functions. If not, we have to adjust $\tilde{a}^{max}$ and n_a and restart the outer loop in Step 3.

In summary, we find that the coding for the computation of a large-scale overlapping generations model with idiosyncratic productivity is a non-trivial task. The computer code may easily amount to several hundred lines. Often, we find a trade-off between accuracy and speed. The difference in speed is only caused to a small extent by our different coding techniques in the programming languages JULIA, GAUSS and PYTHON.[22] For example, in GAUSS and PYTHON, we used our own routine for the golden section search, while, in JULIA, we used a code provided by the package 'optim'. The differences in speed become most evident in the computation of the distribution in Step 6 where we iterate over five nested loops (in the individual variables age s, permanent productivity θ, idiosyncratic productivity θ, wealth $\tilde{a}$ and next-period idiosyncratic productivity θ'). Here, the number of operations (additions and multiplications) is exactly the same in the code of all three programming languages, as is the accuracy that we use to store the numbers (1e-8). The difference in speed is impressive. In particular, GAUSS and JULIA display approximately the same computational speed, while PYTHON is considerably slower. In our experience, we find that in many applications of heterogeneous-agent models with overlapping generations, GAUSS and JULIA outperform PYTHON in

[22] In fact, the syntax of most commands in the three languages is very similar.

terms of speed and that the difference is often crucial given the extensive run times.[23] Consider the policy experiment where we would like to find the optimal steady-state pension replacement rate in the present model (e.g., the highest lifetime utility of the average newborn) by searching over a fine grid of $repl$. In PYTHON, the computational time may become prohibitive in this particular example.

RESULTS. Fig. 10.3 displays the average wealth of the cohorts over the lifetime. As you know by now, the hump-shaped profile is typical for the life-cycle model. Households save for old age when their non-wealth income shrinks (pensions are below earnings). In the case of stochastic survival, however, individual wealth peaks prior to retirement because next-period utility is discounted at an increasing rate, $1/(\phi^s \beta)-1$, given that the survival probabilities ϕ^s fall with age s. Average wealth amounts to $\tilde{\Omega} = 1.824$, of which 81.5% is held in the form of physical capital, $\tilde{K} = 1.486$, and the remaining part is held in the form of government debt, $\tilde{B} = 0.338$. The real interest on government bonds amounts to 2.77%.

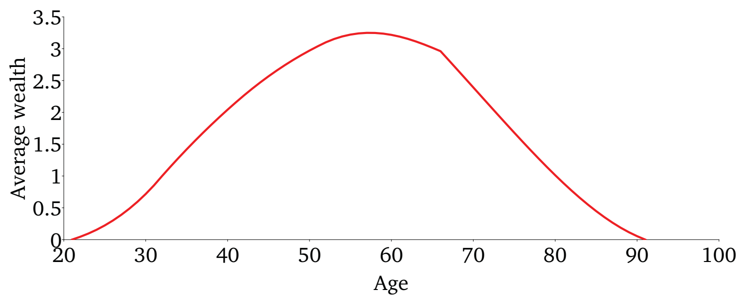


Figure 10.3 Age-wealth profile

[23] For the same reason, we are also reluctant to write MATLAB code for the computation of heterogeneous-agent OLG models (Heer (2019) provides MATLAB code for small- and medium-sized OLG models). As a faster alternative, Fortran and C++ code may be considered. However, these languages come at a cost. 1) PYTHON and JULIA are freeware, while the Fortran and C++ compilers are not available for free. 2) Learning the languages Fortran or C++ is more time-consuming than learning PYTHON, JULIA or GAUSS. 3) Writing your code in Fortran or C++ is also more difficult than the composition of the code with the help of a high-productivity language such as PYTHON or JULIA. If the reader is interested in how to write Fortran code for the computation of large-scale OLG models, we recommend consulting the textbook by Fehr and Kindermann (2018).

Consumption as presented in Fig. 10.4 peaks in the last period of the working life. At the first year of retirement, leisure jumps to 1.0 (retirees do not work), and to smooth utility over time, consumption falls. Aggregate consumption $\tilde{C}$ amounts to 0.275, and its share in total demand is equal to 51.3% (government consumption and investment take up 18.0% and 30.7% of GDP, respectively).

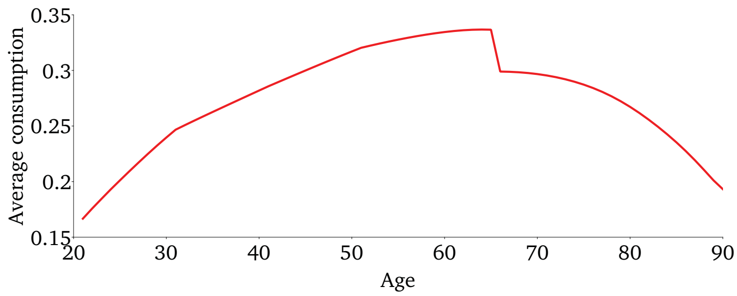


Figure 10.4 Age-consumption profile

Fig. 10.5 presents average working hours (averages among the s-year-old workers), which peak at age 32 and decline thereafter. The age-labor-supply profile displays kinks at the same ages as the underlying age-efficiency profile $\bar{y}^s$ displayed in Fig. 9.4. The initial increase in labor supply is caused by the increase in the age productivity $\bar{y}^s$; however, $\bar{y}^s$ peaks at age 52, while labor starts to decline prior to this age because of the wealth effect. Workers become wealth-richer with increasing age and, for this reason, reduce their labor supply ceteris paribus. As we noted above, the labor supply of high-skilled workers is larger than that of low-skilled workers so that efficient working hours also exceed working hours. Given a workforce share in total population equal to 78%, aggregate efficient labor amounts to $\tilde{L} = 0.310$.

While the process for the individual's hourly wage rate is exogenous in our model, her labor supply and, hence, earnings are endogenous. The Lorenz curve for the earnings in the steady state of our model (red line) and for the US economy (blue line) are displayed in Fig. 10.6.[24] Evidently, the inequality in earnings for the model economy matches the empirical value in the US, although earnings are slightly less concentrated in the

[24] The empirical data displayed in Figs. 10.6 and 10.7 are taken from Tables 5 and 7 in Budría Rodríguez et al. (2002).

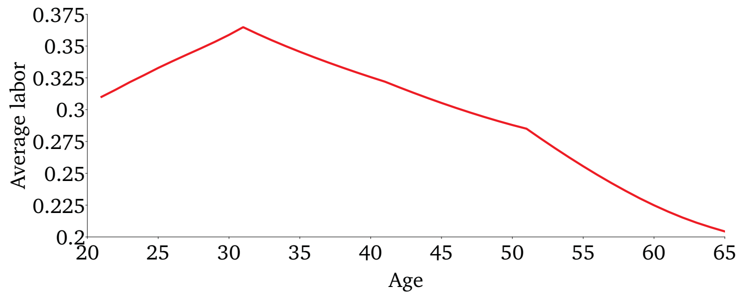


Figure 10.5 Age-labor profile

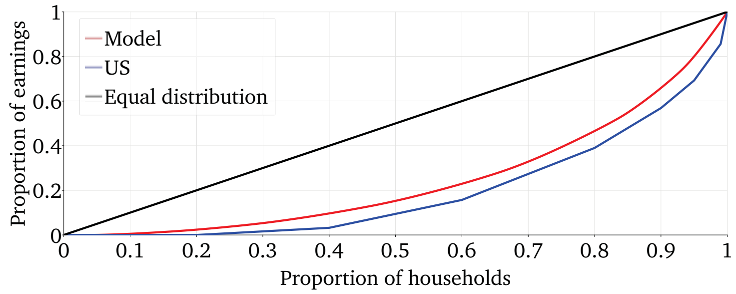


Figure 10.6 Lorenz curve of US and model earnings

model than in the US economy. In our model, the (earnings-) poorest quintile of the workers receives only 2.5% of total earnings, while the top quintile earns 53.4% of the total. According to the estimates of Budría Rodríguez et al. (2002), the corresponding quintiles received 0% and 60.2% in the US in 1996. The Gini coefficient of earnings amounts to 0.51 in the model compared to a value of 0.66 in the US economy in 1996.[25] Earnings are more concentrated than hourly wages (with a Gini coefficient equal to 0.37) because the more productive workers supply more labor than the less productive workers in our model (as observed empirically).

[25] Krueger et al. (2016) find lower inequality of earnings for the US economy in 2006 than Budría Rodríguez et al. (2002). Different from Budría Rodríguez et al. (2002), they use data from the Panel of Survey of Income Dynamics (PSID) rather than data from the Survey of Consumer Finances (SCF). For example, Krueger et al. (2016) find a Gini coefficient of earnings equal to 0.43. Therefore, our results for the inequality of earnings are in between those implied by the two different data sets in 1996 and 2006.

Fig. 10.7 presents the Lorenz curves of wealth in the model (red line) and in the US economy (blue line). Wealth is more concentrated than earnings in our model. The (wealth-) poorest quintile holds no wealth at all, and 20% of the model population is credit-constrained. The top quintile of the wealth distribution holds 67.6% of total wealth. Both Budría Rodríguez et al. (2002) and Krueger et al. (2016) report that the lowest quintile does not hold any wealth in the US economy in 1996 and 2006, while the top quintile holds 81-82% in both years. The Gini coefficient of wealth amounts to 0.66 in our model, while empirically, the Gini coefficient is even higher and amounts to values close to 0.80 as reported by Budría Rodríguez et al. (2002) and Krueger et al. (2016).

In summary, our model is able to replicate the fact that wealth is distributed more unequally than earnings. The OLG model also generates more wealth heterogeneity than the neoclassical model with heterogeneous productivity presented in Section 7.4.2. For this reason, the OLG model is a predominant prototype model when the researcher would like to study wealth distribution effects of economic and fiscal policies.[26] The life-cycle savings motive seems to be a very natural and intuitive way to generate heterogeneity and also makes the wealth distribution dependent on demographic effects. For example, the present model helps to shed light on the question of how wealth inequality will behave in times of aging as is presently underway in the advanced industrial countries.[27]

There are numerous reasons why the endogenous wealth heterogeneity of our model is smaller than observed empirically:[28]

1. **Earnings-related pensions:**
   Pensions are not related to the earnings history of the recipient. If the earnings-rich agents receive larger pensions, one might suppose that

[26] You have already encountered methods to create more wealth inequality in the neoclassical growth model with heterogeneous agents. Usually, however, these modeling choices impose exogenous sources of inequality such as differences in preference parameters to generate sufficient wealth heterogeneity. For example, the original paper by Krussell and Smith (1998) uses the assumptions that households are heterogeneous with respect to their discount factor β, which follows an AR(1) process; however, the underlying parameter β cannot be observed empirically.

[27] You are asked to compute the distribution effects of aging in Problem 1.

[28] For a more comprehensive survey of wealth heterogeneity in quantitative general equilibrium models, see De Nardi (2015). Quadrini and Ríos-Rull (1997) present an early review of studies of wealth heterogeneity in computable general equilibrium models with uninsurable idiosyncratic exogenous shocks to earnings, distinguishing business ownership, higher rates of return on high asset levels, and changes in health and marital status, among others, as possible explanatory factors of wealth inequality.

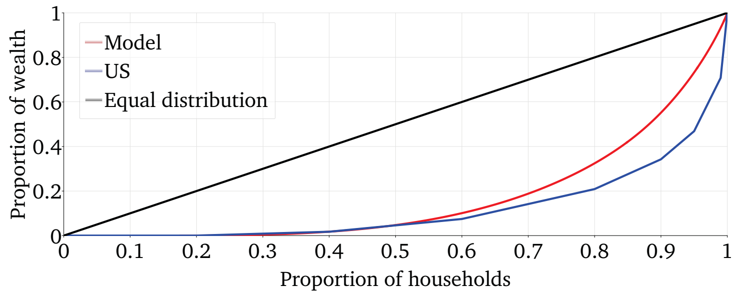


Figure 10.7 Lorenz curve of US and model wealth

wealth heterogeneity would also be higher. However, as earnings-poor agents also know that they will only receive small pensions, they will also save more for precautionary reasons.[29]

2. **Asset-based means test of social security:**
   We neglect any asset-based means tests of social security. Hubbard et al. (1995b) show that, in the presence of social insurance programs with means tests, low-income households are likely to hold virtually no wealth across their lifetime. Unemployment and asset-based social insurance would imply a much higher proportion of agents with zero or near-zero wealth.
3. **Unemployment:**
   Furthermore, agents are subject to employment risk. Heer (2003b) studies a life-cycle economy with endogenous search unemployment. Working agents may lose their job at an exogenous rate; higher search effort increases the job finding probability, but searching for a job also causes a disutility for the agent. Heer (2003b) shows that the replacement rate of unemployment insurance has only a very small effect on wealth heterogeneity. Although income is redistributed from income-rich agents to income-poor workers with the help of unemployment insurance, higher unemployment insurance also increases endogenous unemployment, so that the number of unemployment recipients increases. As a consequence, the wealth Gini coefficient changes by less than one percentage point if the replacement ratio of unemployment

[29] In our own research, we have only encountered applications where the introduction of earnings-related benefits decreased wealth heterogeneity (as measured by the Gini coefficient). In the next section where we introductions earnings-related pensions, we will confirm this result.

insurance increases from 0% to 50% or even to 100%; for a replacement ratio exceeding 70%, wealth heterogeneity actually starts to increase again.

4. **Bequests:**
   In our model, we assumed that the government collects accidental bequests and that the households do not have any altruistic bequest motive. Empirically, however, parents care for their children and, in particular, affluent individuals leave bequests to their descendants. For example, Kessler and Masson (1989), considering France, find that only 36% of the households receive any inheritances and those who do are approximately 2.4 times richer than the representative household. Heer (2001b) considers an OLG model where parents leave altruistic and accidental bequests to their children. He, however, finds that bequests are able to explain only a small fraction of observed wealth heterogeneity. The main reasons are that i) poor agents may also receive bequests and ii) agents who expect a high inheritance in the future also spend more on consumption. Importantly, however, Heer (2001b) only considers intergenerational transfers of physical wealth, not transfers of human wealth. Rich parents may have rich children because they may invest in their college education, for example. Loury (1981) analyzes parental human capital investment in their offspring. The allocation of training and hence the earnings of the children depend on the distribution of earnings among the parents. Becker and Tomes (1979) present a model framework comprising both human and non-human capital transfers from parents to children. The introduction of human capital transfers in an OLG model to explain the observed wealth heterogeneity is analyzed in a general equilibrium by De Nardi and Yang (2016). In their model, parents pass on both bequests of wealth and inheritance of abilities. Importantly, they are able to match the skewness and the long tail of the distribution of wealth and bequests.
5. **Credit limits in financial markets:**
   In our model, agents are not allowed to borrow against anticipated bequests, implying a credit limit $a \geq 0$. For lower binding constraints, $a < 0$, wealth heterogeneity increases as demonstrated by Huggett (1996). In particular, the proportion of agents holding zero and negative assets increases.
6. **Entrepreneurship:**
   In our model, entrepreneurship is missing. Quadrini (2000) provides a comparative analysis of differences in income and wealth between

workers and entrepreneurs in the US economy. According to his results, the top wealth group contains a high share of entrepreneurs. Prominent examples of successful entrepreneurs who earned considerable wealth in present times include persons such as Bill Gates, Elon Musk or Jeff Bezos who are associated with the firms Microsoft, Tesla and Amazon, among others.

As one of the first quantitative studies in dynamic general equilibrium, Quadrini (2000) introduces entrepreneurship into the neoclassical growth model to explain the high concentration of wealth among the very rich agents. Cagetti and de Nardi (2009) introduce endogenous entrepreneurship in a dynamic life-cycle general equilibrium model to study the distribution and welfare effects of lower estate taxation.[30] They find that, if other taxes were raised to compensate for the shortfall in fiscal revenues, most households would lose.

7. **Stochastic health:**

   Using data from the Assets and Health Dynamics of the Oldest (AHEAD), De Nardi et al. (2010) estimate that the average out-of-pocket annual medical expenditures increase from \$ 1,100 at age 75 to \$ 9,2000 at age 95. In addition, both health expenditures and longevity are highly uncertain in old age, so households accumulate precautionary savings. One possible way to model the heterogeneity in health and, hence, precautionary savings, is presented by Jung and Tran (2016). In their OLG model, households also hold health capital that is modeled in a similar way as physical capital. In particular, health capital increases with medical services and depreciates with age. In addition, it is subject to a health shock. As a consequence, they are able to match the variance in health capital among the old in the US.[31] Nevertheless, wealth inequality falls short of empirical values.

   As a policy application, Jung and Tran (2016) consider the U.S. health care reform 2010 in the form of the Affordable Care Act (ACA) also known as 'Obamacare'. As the main goal of this policy, the health coverage rate, especially among low-income groups, is targeted to in-

[30] Cagetti and de Nardi (2009) like Krueger et al. (2016) use an adaption of the neoclassical growth model that is based upon work by Blanchard (1985) and Gertler (1999). Households are born as workers. With certain probabilities they first become retired before they decease. Deceased households are replaced by young (working) households. See also Footnote 1 in Chapter 9.

[31] In old age, for example, the health capital in the bottom three quartiles of the US health capital distribution diverges by -5.9%, -27.8% and -59.1% from that of the top quartile. We would like to thank Juergen Jung and Chung Tran for the provision of their estimates which is based upon data from the Medical Expenditure Panel Survey (MEPS).

crease. The ACA is found to redistribute income both from low to high health risk types and from high to low income groups.

10.1.3 Multi-Dimensional Individual State Space

In this section, we add another dimension to the individual state space and study how the computational time is affected by the so-called 'curse of dimensionality'. Computational speed increases non-linearly with the number of continuous state variables. Therefore, with present computer technology, it is difficult to solve large-scale heterogeneous-agent OLG models with uncertainty once the number of continuous (individual) state variables exceeds two or three.

In the following, we add contribution-dependent pensions following Huggett and Ventura (1999) and Kitao (2014). Therefore, accumulated earnings $\tilde{x}$ as described in (10.4) enter the individual state space as a new (continuous) variable. We will show below that the value and policy functions do not display substantial curvature with respect to the variable $\tilde{x}$. As a consequence, it suffices to use only a few grid points n_x over the asset space $\mathscr{X} = \{\tilde{x}_1, \ldots, \tilde{x}_{n_x}\}$ in the approximation of the functions. Evidently, the severity of the curse of dimensionality critically depends on the functional relationship between the policy functions and the individual state variables and implies high computational costs in the presence of high curvature or discontinuity of the policy functions or its derivatives. Sources of discontinuity in the behavior of the policy functions are frequent and include, among others, credit constraints on assets, asset-based means tests, tax exemptions, or income eligibility thresholds for social insurance programs such as Medicaid. These kinds of constraints might result in kinks of or jumps in the policy functions that are difficult to approximate numerically.

With the enlarged state space, Algorithm 10.1.1 for the computation of the stationary equilibrium in the large-scale OLG model still applies. However, in some steps, our implementation will slightly diverge from the model with lump-sum pensions, which we discuss in the following.

In Step 1, we need to provide the parameters of the pension schedule as described in the calibration section. Note that in the model with contribution-dependent pensions, we need to update an additional aggregate variable, that for average accumulated earnings $\bar{x}$. As an initial value of $\bar{x}$, we compute the average wage income per worker with the help of the initial values of $\tilde{K}$ and $\tilde{L}$ in Step 2.

A noted above, we also need to compute the value and policy functions over an additional dimension of a continuous state variable in the form of accumulated earnings $\tilde{x}$ in our multi-dimensional optimization problem. For this reason, we discretize the state space with n_x gridpoints with respect to the variable $\tilde{x}$, $\mathcal{X} = \{\tilde{x}_1, \ldots, \tilde{x}_{n_x}\}$. The computation of this higher-dimensional problem, of course, is much slower than the problem studied in the previous section, and we need to be careful in the choice of the number of grid points. We choose $n_a = 200$ and $n_x = 10$, which captures the curvature of the policy functions with respect to the state variables $\tilde{a}$ and $\tilde{x}$. As we noted above, the policy functions are rather linear with respect to $\tilde{x}$, and a coarse grid on the accumulated earnings $\tilde{x}$ is sufficient. Of course, you will only know this from experience or after the execution of the first test runs with the computer program for a new model. Thorough study of your policy functions in the setup of the code is therefore a vital component of your programming technique and implies substantial trial and error. In total, we compute the solution of the value function iteration in the inner loop the following number of times:

$$nw \times n_a \times n_x \times n_y \times 2 + nr \times n_a \times n_x = 950,000.$$

In the present model specification, we also have to specify another grid for labor supply. Different from the model with lump-sum pensions, our first-order condition with respect to labor supply is now represented by

$$\begin{aligned} -\frac{\partial u(\tilde{c},l)}{\partial l} = & \frac{\partial u(\tilde{c},l)}{\partial c} \frac{(1-\tau^l-\tau^p)\epsilon(s,e,\theta)w}{1+\tau^c} \\ & + (1+g_A)^{\gamma(1-\eta)} \beta \phi^s \mathbb{E}\left\{ \frac{\partial \tilde{v}'(\tilde{a}',\tilde{x}')}{\partial \tilde{x}'} \frac{\epsilon(s,e,\theta)w}{s} \right\}. \end{aligned} \tag{10.49}$$

In comparison with (10.41), the term with the derivative of the value function with respect to accumulated earnings, $\partial v'(.)/\partial \tilde{x}'$, enters as a new additive term on the right-hand-side of equation (10.49). The first-order condition is derived from the derivative of the value function with respect to labor l at age $s = 1, \ldots, nw$:

$$\begin{aligned} \tilde{v}(\tilde{a},\tilde{x}) = \max_{\tilde{a}',l} \Bigg\{ & u\left(\frac{(1-\tau^l-\tau^p)\epsilon(s,e,\theta)wl + [1+(1-\tau^k)(r-\delta)]\tilde{a} + \widetilde{tr} - (1+g_A)\tilde{a}'}{1+\tau^c}, l \right) \\ & + (1+g_A)^{\gamma(1-\eta)} \beta \phi^s\, \tilde{v}'(\tilde{a}',\tilde{x}') \Bigg\}. \end{aligned}$$

(10.50)

and noting from (10.4) that accumulated earnings in the next period at age $s+1$, $\tilde{x}'$, increase with labor l at age s according to

$$\frac{\partial \tilde{x}'}{\partial l} = \frac{\epsilon(s,e,\theta)w}{s}.$$

In essence, we have two options to compute the optimal labor supply in our model. 1) We may solve (10.49) using a non-linear equations routine such as the Newton-Rhapson algorithm. Therefore, we have to evaluate the derivative of the value function with respect to its second argument, accumulated earnings $\tilde{x}$. If we store the value function at n_x grid points, this involves finding the derivative between grid points $\tilde{x}$ and, therefore, a higher-order interpolation method along this dimension. To see this point, assume instead that we only use linear interpolation of the value function between two neighboring grid points, $\tilde{x}_i$ and $\tilde{x}_{i+1}$. As a consequence, the derivative $\partial v'(.)/\partial \tilde{x}'$ is no longer continuous (it is constant between grid points and jumps at grid points) so that a solution to (10.49) may not be found. You are asked in problem 2 to study this method. 2) We may specify a grid over labor, $\mathscr{L} = \{l_1, \ldots, l_{n_l}\}$ with $l_1 = 0$ and $l_{n_l} = l^{max}$ and n_l grid points. We then use a simple search mechanism to compute the labor supply that implies the maximum value of the value function. We adopt this procedure in the following. For this reason, we choose a grid over labor with $n_l = 30$ equispaced points.

In Step 4, we, again use value function iteration to compute optimal policy functions and the value function at grid points. Different from the computational method in the previous section, we do not evaluate the right-hand side of the Bellman equation (10.50) between grid points but only use the simplest computational technique and choose the value $\tilde{a}'$ as the grid point $\tilde{a}_{i_a}$ on $\mathscr{A}$ that implies the maximum value. We introduced this technique as 'Simple Iterative Procedure' in Chapter 4.[32] While this technique is easy to program, the computational speed for given accuracy is usually much slower than value function iteration that optimizes between grid points. Therefore, you have to consider the trade-off between the time you need to write the program and the execution time of the program. In addition, if you want to obtain a quick first impression about the solution, high accuracy may not be the most important criterion for your choice, and you may prefer a computational method that is fast to pro-

[32] You are asked to solve this problem using intermediate values of the grid in Problem 2.

gram. If the (slow) solution with low accuracy yields promising results, you can still switch to a more accurate and fast method.

We implemented the maximization step at a grid point $(\tilde{a}_{i_a}, \tilde{x}_{i_x})$ for an s-year old worker with permanent and idiosyncratic productivities, e_j and θ_{i_θ}, as follows.[33] First, we compute the next-period values of accumulated contributions $\tilde{x}'$ for all points on the labor grid $\mathscr{L}$. If any of the values for $\tilde{x}'$ happens to lie above the upper boundary $\tilde{x}^{max}$, we set it equal to $\tilde{x}^{max}$.[34] Next, we compute present consumption $\tilde{c}$ implied by the next-period wealth $\tilde{a}' \in \mathscr{A}$ and the n_l possible values of the labor supply $l \in \mathscr{L}$ using the budget constraint (10.19). The result consists of a matrix with $(n_a \times n_l)$ entries. To compute the first additive term on the right-hand side of the Bellman equation (10.50), we pass the matrix on to our function 'utility(c,l)' (in PYTHON or JULIA) or procedure 'utility(c,l)' (in GAUSS) in the computer program. Note that, for this reason, you have to program the function/procedure 'utility(.,.)' in such a way that it can use matrices as arguments and also returns a matrix. Usually, this command requires a special syntax depending on the computer language that you are using.[35] In addition, we need to generate a matrix for the second input argument l by constructing a matrix with n_a rows where the row vectors are equal to the grid points on $\mathscr{L}$. Next, we compute the second additive term on the right-hand side of the Bellman equation (10.50). Therefore, we need to interpolate the next-period value function at the values of $\tilde{x}'$ that do not lie on the grid $\mathscr{X}$. Again, the computed values of the expected next-period value function (discounted by the factor $(1+g_A)^{\gamma(1-\eta)}\beta\phi^s$) are stored in a matrix. We add the two matrices that contain the values of $u(\tilde{c}, l)$ and the discounted next-period value function $\tilde{v}'(\tilde{a}', \tilde{x}')$ and call the resulting matrix 'bellman' in our programs. Finally, we simply allocate the maximum value in the matrix 'bellman' using the particular command provided in the computer language, e.g., 'xmax = np.where(bellman == np.max(bellman))' in PYTHON.

[33] The optimization step for the retiree is comparatively simple because we do not need to optimize over the labor supply. For this reason, we do not describe it here in detail.

[34] Of course, if the upper boundary $\tilde{x}^{max}$ keeps binding during the final iteration over the aggregate variables, we need to increase $\tilde{x}^{max}$ and rerun the program. To check for this problem, we plot the distribution of $\tilde{x}$ at the first year of retirement in the program *AK70_prog_pension*.

[35] For example, if you are using JULIA and specified a function 'u(c,l)' for instantaneous utility from consumption c and labor l, you need to apply the command 'u.(c0,l0)' to compute the matrix of instantaneous utilities for your $(n_a \times n_l)$-matrices 'c0' and 'l0'.

Let us pause at this point to reflect on our methodological choices. Why did we not consider the grid points on accumulated earnings, $\tilde{x}' \in \mathscr{X}$, rather than the grid points on labor supply, $l \in \mathscr{L}$, to find the maximum of the right-hand side of the Bellman equation? This way, we would have avoided the time-consuming step to interpolate the next-period value function $\tilde{v}'(\tilde{a}', \tilde{x}')$ along the $\tilde{x}'$-dimension because all points would lie on the grid space. To illustrate this point, let us consider a particular numerical example from our program *AK70_prog_pension.py*. In particular, let us consider a 40-year old with permanent productivity $e_1 = 0.57$ and idiosyncratic stochastic component $\theta_1 = 0.4688$. The age component of this 40-year old amounts to $\bar{y}^{40} = 1.0590$. For a wage rate approximately equal to $w = 1.3$ in our stationary equilibrium, the wage per hour amounts to $\epsilon(e, \theta, s)w \approx 0.3679$. Assume that we would like to compute the optimal labor supply with an accuracy of at least $\Delta l = 0.02$ (as for the present choice of numerical parameters). An increase in labor supply by 0.02 results in an increase of accumulated average earnings $\tilde{x}$ at age 41 by the amount

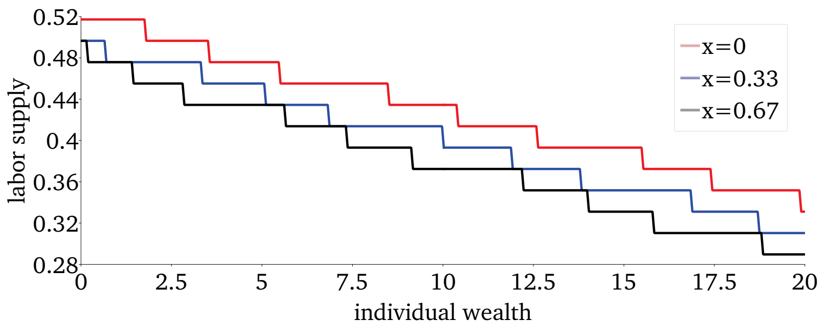
$$\frac{\epsilon(e, \theta, s)w}{40} \Delta l = 4.60 \cdot 10^{-5}.$$

As a consequence, we would have to choose a grid for $\tilde{x}$ with a distance between grid points equal to $4.60 \cdot 10^{-5}$ or, given the lower and upper boundaries $\tilde{x}^{min} = 0$ and $\tilde{x}^{max} = 3.0$, a total number of grid points equal to $n_x = 16,310$. Of course, this high number would result in the breakdown of our algorithm since we would have to compute and store the solution to the value function iteration problem at the following number of points:

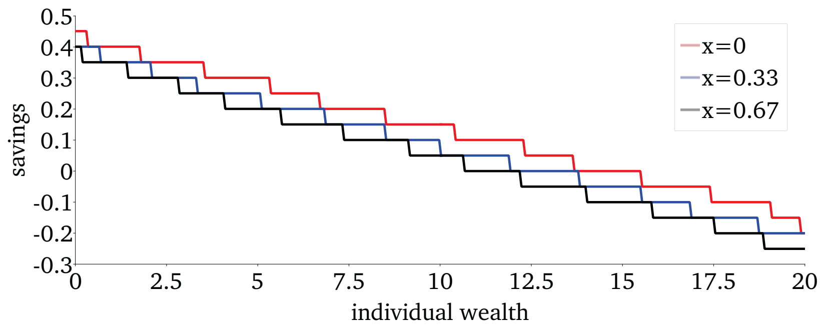
$$nw \times n_a \times n_x \times n_y \times 2 + nr \times n_a \times n_x = 1.6 \cdot 10^9.$$

Fig. 10.8 displays the behavior of the policy functions labor supply, $l(.)$, and savings, $\tilde{a}'(.) - \tilde{a}$, as a function of individual wealth, $\tilde{a}$, for different levels of accumulated earnings, $\tilde{x} \in \{0, 0.33, 0.66\}$.[36] We randomly select a 20-year old with permanent productivity $e_2 = 1.43$ and stochastic productivity $\theta_4 = 1.4605$. Evidently, both labor supply and savings decrease monotonically with wealth $\tilde{a}$. In Fig. 10.9, we plot the policy functions as a function of accumulated earnings $\tilde{x}$ for three different levels of wealth, $\tilde{a} \in \{0, 0.45, 0.95\}$, where the upper two values correspond to 26.3% and

[36] Average accumulated earnings amount to $\bar{x} = 0.4551$.



(a) Labor supply l

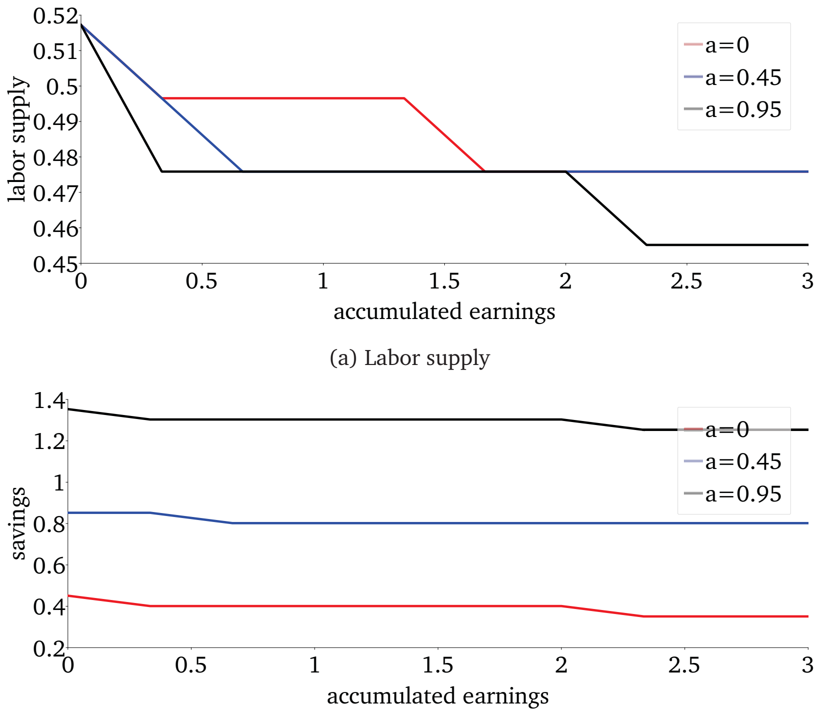


(b) Savings $\tilde{a}' - \tilde{a}$

Figure 10.8 Policy functions as a function of wealth $\tilde{a}$

55.4% of the average wealth $\tilde{\Omega}$ in the economy. Clearly, the policy functions are rather flat with respect to the state variable $\tilde{x}$, and we can be quite confident that a coarse grid for accumulated earnings $\tilde{x}$ is sufficient.

The accuracy of the policy functions as measured by the Euler residuals is much smaller than in the case of methods that search over the optimum between grid points. For a number of grid points on the asset space $\mathscr{A}$ and labor $\mathscr{L}$ with $n_a = 200$ and $n_l = 30$ grid points, respectively, we find that the mean absolute Euler residuals of the young and old amount to 3.62% and 3.63% and, therefore, are higher by a factor of approximately 10-50 than those from the model with lump-sum pensions above. Obviously, the accuracy is low and may be prohibitive if we would like to derive policy implications. To improve upon the accuracy, we increase the number of grid points on $\mathscr{A}$ and $\mathscr{L}$ to $n_a = 400$ and $n_l = 60$ once the aggregate state variables $\{\tilde{K}, \tilde{L}, \widetilde{tr}, \bar{x}\}$ are close to convergence. In particular, we increase the number of grid points in Step 7



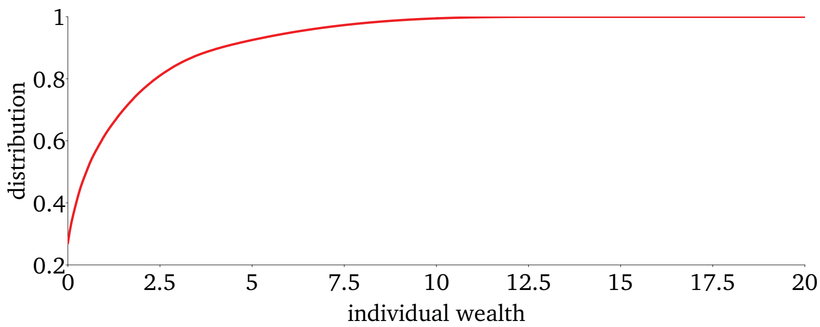
(a) Labor supply

(b) Savings

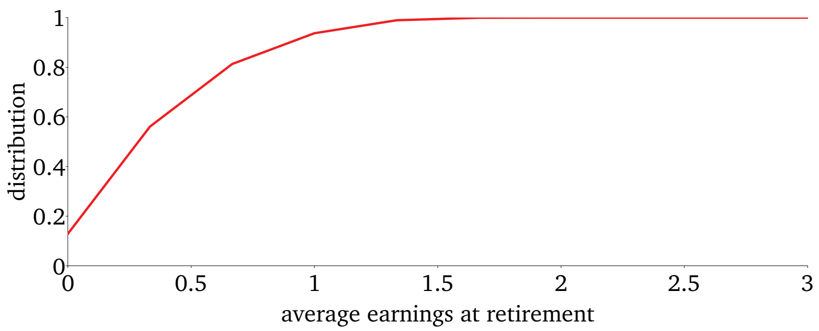
Figure 10.9 Policy functions as a function of accumulated average earnings $\tilde{x}$

as soon as the percentage deviation of the aggregate capital stock $\tilde{K}$ falls below 0.1%. We continue to iterate over the aggregate capital stock until the capital stock again converges and deviates by less than 0.1% from its value in the previous outer iteration. By starting our program with a low accuracy that we only increase shortly prior to convergence, we considerably economize on computational speed. In the final iteration over the aggregate variables, the absolute Euler residuals still average 3.27% and 1.65% for the worker and retiree, respectively.

In Step 5, we compute the endogenous distribution of wealth and accumulated earnings distribution, $f(s, e, \theta, \tilde{a}, \tilde{x})$, in each cohort s for the n_e permanent and n_θ idiosyncratic productivity types. Different from the model with lump-sum pensions where we interpolated between grid points to find the optimal levels of $\tilde{a}'$, we choose the same number of grid points n_a for individual wealth as in the computation of the policy functions. Since we require the next-period wealth $\tilde{a}'$ to be a grid point on $\mathscr{A}$, there is no need to chose a finer grid. The cumulative distributions of individual wealth $\tilde{a}$ of the total population and accumulated average earnings $\tilde{x}$ of a 46-year old at the start of retirement are illustrated in Fig. 10.10. The measure of the households with wealth $\tilde{a} \geq 17.5$ is zero, so the upper boundary $\tilde{a}^{max} = 20.0$ is chosen sufficiently high. Similarly, the measure of 46-year-old households with accumulated earnings $\tilde{x} \geq 2.33$ is equal to zero, so the upper boundary $\tilde{x}^{max} = 3.0$ is also chosen adequately. In addition, we checked the distribution of accumulated average earnings at different working ages (it is constant during retirement) and made the same observation. When you set up the program, it is advisable that you examine the distribution of the individual states at the beginning and in the final run of the program.



(a) Distribution of $\tilde{a}$



(b) Distribution of $\tilde{x}$ at age $s = 46$

Figure 10.10 Cumulative distribution functions of individual state variables

Computational time. The computational time is summarized in Table 10.3. Using GAUSS, the computation of the optimal policy for given aggregates amounts to 32 minutes, 47 seconds, while the computation of the distribution is much faster and only requires 1 minute, 13 seconds. Total computational time is considerable, amounting to more than 1 day. Comparing the three languages, we find that JULIA is approximately twice as slow as GAUSS, while PYTHON is by far the slowest among the three computer languages considered. In fact, total computational time for the PYTHON code amounts to more than 6 days! This ranking is in good accordance with our own personal experience from the computation of large-scale OLG models with heterogeneous agents. In some applications, however, we have also experienced JULIA code being faster than GAUSS code (for example, in the present model, in the computation of the distribution) so that a unanimous overall ranking of the two programming languages GAUSS and JULIA with respect to computational speed above is only tentative.

Table 10.3 Comparison of run time

	policy	distribution	total
Julia	1h:17m:3s	30s	3d:10h:41m:56s
Gauss	31m:47s	1m:13s	1d:11h:13m:2sm:s
Python	2h:28m:12s	24m:38s	6d:1h:17m:04s

Notes: Run time is given in days:hours:minutes:seconds on an Intel(R) Xeon(R), 2.90 GHz. The first- and second entry-columns report the time for one execution of Steps 4 and 5 of Algorithm 10.1.1, respectively, using $n_a = 200$, $n_x = 10$ and $n_l = 30$ grid points.

Results. Aggregate savings amount to $\tilde{\Omega} = 1.713$, implying an aggregate capital stock equal to $\tilde{K} = 1.376$. Both values are slightly bit lower than the corresponding values found in the model with lump-sum pensions (with $\tilde{\Omega} = 1.824$ and $\tilde{K} = 1.486$). There are many opposing effects of contribution-based pensions on aggregate savings. First, workers with high income will receive a higher pension and, hence, need to save less to provide for old age, so aggregate savings fall. Second, workers with low income need to increase their old-age savings. Third, with contribution-based pensions, the incentive to supply labor increases for all workers such that income and, hence, savings increase. In fact, aggregate labor in efficiency units amounts to $\tilde{L} = 0.327$ and is higher than in the case of lump-sum pensions above (with $\tilde{L} = 0.310$). Fourth, with our new calibration of the pension schedule, the average replacement rate of pensions increases from 35% (with lump-sum pensions) to 53%. As a consequence, all households have to save less for old age ceteris paribus. In our calibration, we find that the negative effects 1) and 4) dominate the positive effects 2) and 3) on savings.

Our main interest in the study of this multi-dimensional optimization problem is motivated by the question of whether contribution-based pensions help to explain the high wealth inequality observed empirically. Our hope was that the provision of higher pensions to the retirees with high incomes during working life also increases wealth inequality. However, the Gini coefficient of wealth even decreases from 0.66 (with lump-sum pensions) to 0.64 (with earnings-dependent pensions) as the low- (high-) income workers have to save a higher (lower) share of their income for old age.

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