efficiency units amounts to $\tilde{L} = 0.327$ and is higher than in the case of lump-sum pensions above (with $\tilde{L} = 0.310$). Fourth, with our new calibration of the pension schedule, the average replacement rate of pensions increases from 35% (with lump-sum pensions) to 53%. As a consequence, all households have to save less for old age ceteris paribus. In our calibration, we find that the negative effects 1) and 4) dominate the positive effects 2) and 3) on savings.

Our main interest in the study of this multi-dimensional optimization problem is motivated by the question of whether contribution-based pensions help to explain the high wealth inequality observed empirically. Our hope was that the provision of higher pensions to the retirees with high incomes during working life also increases wealth inequality. However, the Gini coefficient of wealth even decreases from 0.66 (with lump-sum pensions) to 0.64 (with earnings-dependent pensions) as the low- (high-) income workers have to save a higher (lower) share of their income for old age.

10.2 Overlapping Generations with Aggregate Uncertainty

In this section, we introduce aggregate uncertainty into the standard OLG model. One prominent application of OLG models with aggregate uncertainty is to consider the effects of aging and the public pension system on the equity premium.[39] Having a higher share of older agents is likely to increase the returns from stocks relative to those from bonds. Old agents prefer to hold a large part of their wealth in the form of safe assets such as bonds because their (relatively safe) nonfinancial wealth in the form of discounted pensions is relatively small. Younger agents may prefer to invest predominantly in risky assets such as stocks because their total wealth mainly consists of discounted lifetime labor income that is characterized by relatively little risk.[40] Aging may now increase the demand for bonds relative to stocks and thus increase the equity premium. If, however, public pension systems are changed from pay-as-you-go to fully funded, total savings may increase, and if the pension funds invest the additional savings primarily in the stock market, the equity premium may fall. Brooks (2002) quantitatively explores the impact of the baby boom

[39] Please see also Section **6.3.4** of this book on the equity premium puzzle.

[40] There are numerous other variables than age that influence the portfolio decision of the households such as housing.

on stock and bond returns in a model with 4 overlapping generations. He predicts a sharp rise in the equity premium when the baby boomers retire. As an important step to answering this question in a more realistic setting, Storesletten et al. (2007) consider an OLG model with annual periods. They analyze the effects of idiosyncratic risk and life-cycle aspects of asset pricing.

Obviously, in models with aggregate uncertainty, we can no longer study the transition dynamics because the time path is stochastic. To see this point, assume that the US is in a steady state at present (period $t = 0$) and that there is a sudden, unexpected and permanent decline in the fertility rate. If aggregate technology were a deterministic variable, we could compute the transition path just as in Section 9.2. In this case, agents would have to predict the time path for the factor prices. We would know that after some 200–300 periods, the economy is close to the new steady state and that we may stop the computation. If technology is stochastic, however, agents can only form expectations about the time path of factor prices, and the number of possible time paths becomes infinite. Assume that technology can only take two different values. Even in this case, we would have to compute some 2^n different transition paths with n denoting the number of periods. Given our experience, it is safe to assume that in a model with annual periods, n should be in the range of 200–300. The computational costs become unbearable.[41] Therefore, we will confine the analysis of OLG models with aggregate uncertainty to the study of the dynamics around the steady state.

In the following, we will describe two methods for the computation of the dynamics close to the steady state that you have already encountered in previous chapters of the book. First, we will consider models without idiosyncratic uncertainty. In these models, we can compute the steady state by solving directly for the individual state variables with the help of nonlinear equation solvers. The method has been described in Section 9.1. Although the state space may be quite large and include hundreds of variables, it is often possible to apply log-linearization or even higher-dimensional perturbation methods as described in Chapters 2–4 to compute the solution. In the following Section 10.2.1, we will use linear perturbation to study the business cycle dynamics of an OLG model with 280 overlapping generations. Second, we consider an OLG model with both idiosyncratic and aggregate uncertainty in Section 10.2.2.1. As

[41] As one possible solution to this problem, one can use Monte Carlo simulation techniques to compute multiple possible transition paths and the associated distribution for the factor prices during the transition. We will not pursue this method here.

the most practical approach to the solution of such a problem, we advocate the algorithm of Krussell and Smith (1998) from Section 8.4.2. We will use the latter method to compute the business cycle dynamics of the income distribution and compare our results with those from the model with infinitely lived households from Section 8.4.2.

10.2.1 Perturbation Methods

There are few studies that apply linear (or quadratic) perturbation methods to OLG models of large-scale economies. In his pioneering work, Ríos-Rull (1996) considers the dynamics in a stochastic life-cycle model and compares the real business cycle statistics in the OLG model with those in the standard neoclassical growth model.[42] He finds that in the two models, the aggregate variables such as output, investment and consumption are characterized by similar second moments. In our own work (Heer and Maußner (2012)), we consider the redistributive effects of inflation following an unanticipated monetary expansion. We find that unanticipated inflation that is caused by a monetary shock reduces income inequality. Heer and Scharrer (2018) consider the redistributive effects of a government demand shock and show that an unexpected increase in government spending decreases both income and wealth inequality. In particular, and contrary to conventional wisdom, they show that debt rather than tax financing of additional government spending may also harm retirees.

10.2.1.1 The Stochastic OLG Model with Quarterly Periods

In the following, we will illustrate the numerical and analytical methods with the help of a 280-period OLG model that is described in Example 10.2.1. The model builds upon that considered in Section 9.1.3. However, while we neglect government debt, we subsequently consider quarterly rather than annual periods to study business cycle dynamics. In addition, we introduce both a technology shock ϵ_t^Z and a government demand shock ϵ_t^G into the model. In particular, both the (logarithmic) aggregate technology level Z_t and government consumption G_t follow AR(1)-processes.

[42] Different from our algorithm in this section, he concentrates on the analysis of a pareto-optimal economy and studies the problem of a central planner. In particular, he uses the LQ-approximation presented in Section **2.3** to compute the solution of this model.

Example 10.2.1

280-Period Overlapping Generations Model with Aggregate Uncertainty. The demographics are modeled with respect to the characteristics of the US economy in the year 2015. The model consists of three sectors: households, firms and the government.

Demographics. Total population N_t grows at the constant rate n:

$$N_{t+1} = (1+n)N_t. \tag{10.51}$$

Households live a maximum of $T = 280$ periods (=quarters) corresponding to 70 years (real-life ages 21 to 90). They survive from age (quarter) s to age (quarter) $s+1$ with probability ϕ^s (with $\phi^0 \equiv 1.0$). Let μ_t^s denote the share of generation s in the total population in period t. We consider a stationary population that is characterized by constant μ^s, $s = 1, \dots, T$. For the first $T^W = 180$ periods (=45 years), households are working, for the last $T - T^W = 100$ periods (=25 years), they are retired and receive pensions.

Households. Households maximize expected lifetime utility at age 1 in period t:

$$\mathbb{E}_t \sum_{s=1}^{T} \beta^{s-1} \left(\prod_{j=1}^{s} \phi^{s-1} \right) u(c_{t+s-1}^s, 1 - l_{t+s-1}^s).$$

Instantaneous utility is a function of both consumption c and leisure $1-l$:

$$u(c, 1-l) = \frac{\left(c^\gamma (1-l)^{1-\gamma}\right)^{1-\eta} - 1}{1-\eta}.$$

The working agent of age s with age-dependent productivity $\bar{y}^s$ faces the following budget constraint in period t:

$$k_{t+1}^{s+1} = \left(1 + (1-\tau^k) r_t\right) k_t^s + (1 - \tau^L - \tau_t^p) w_t A_t \bar{y}^s l_t^s + tr_t - (1+\tau^c) c_t^s, \quad s = 1, \dots, 180, \tag{10.52}$$

where r_t, w_t and A_t denote the interest rate, the wage rate and aggregate labor productivity in period t. The government imposes constant taxes τ^c, τ^k and τ^l on consumption as well as capital and labor income, while social security contributions τ_t^p are time-dependent. In addition, the households receive government transfers tr_t. Households are born with zero wealth, $k_t^1 \equiv 0$.

Retirees receive pensions pen_t and government transfers tr_t. Accordingly, their budget constraint is given by

$$k_{t+1}^{s+1} = (1 + (1-\tau^k) r_t) k_t^s + pen_t + tr_t - (1+\tau^c) c_t^s, \quad s = 181, \dots, 280, \tag{10.53}$$

with $k_t^{T+1} \equiv 0$ and $l_t^{T^W+1} = l_t^{T^W+2} = \dots = l_t^T \equiv 0$.

PRODUCTION. Goods and factor markets are competitive. Production Y_t is characterized by constant returns to scale and assumed to be Cobb-Douglas:

$$Y_t = Z_t (A_t L_t)^{1-\alpha} K_t^{\alpha}, \tag{10.54}$$

where aggregate labor productivity A_t grows at an exogenous rate g_A:

$$A_{t+1} = (1 + g_A)A_t, \tag{10.55}$$

and technology $\ln Z_t$ follows the AR(1)-process:

$$\ln Z_t = \rho \ln Z_{t-1} + \epsilon_t^Z.$$

The technology innovation ϵ_t is i.i.d., $\epsilon_t^Z \sim N(0, \sigma_Z^2)$.

In a factor market equilibrium, factors are rewarded with their marginal product:

$$w_t = (1-\alpha) Z_t (A_t L_t)^{-\alpha} K_t^{\alpha}, \tag{10.56}$$
$$r_t = \alpha Z_t (A_t L_t)^{1-\alpha} K_t^{\alpha-1} - \delta, \tag{10.57}$$

where capital K_t depreciates at the quarterly rate δ and L_t denotes aggregate effective labor.

PUBLIC SECTOR. The government receives taxes from consumption and (labor and capital) income and collects accidental bequests Beq_t:[43]

$$Beq_t = \sum_{s=1}^{T-1} (1-\phi^s) \mu_t^s N_t k_{t+1}^{s+1}. \tag{10.58}$$

Total government revenues are spent on government consumption G_t and lump-sum transfers to all households Tr_t. Stationary public consumption, $\tilde{G}_t \equiv G_t/(A_t N_t)$, is stochastic and follows an AR(1) Markov process:

$$\ln \tilde{G}_t = (1-\rho^G) \ln \tilde{G} + \rho^G \ln \tilde{G}_{t-1} + \epsilon_t^G, \tag{10.59}$$

where ϵ_t^G is i.i.d., $\epsilon_t^G \sim N(0, \sigma_G^2)$.

The government budget is balanced in every period t:

$$\tau^L w_t A_t L_t + \tau^k r_t K_t + \tau^c C_t + Beq_t = G_t + Tr_t, \tag{10.60}$$

where C_t denotes aggregate consumption.

The social security authority pays pen_t to all retirees and adjusts the social security contribution rate τ_t^p on wage income such that its budget is balanced in every period t:

$$Pen_t = \sum_{s=T^W+1}^{T} \mu_t^s N_t pen_t = \tau_t^p w_t A_t L_t. \tag{10.61}$$

[43] (10.58) is derived in Appendix A.10.

EQUILIBRIUM. In equilibrium, individual and aggregate behavior are consistent:

$$L_t = \sum_{s=1}^{T^w} \mu_t^s N_t \bar{y}^s l_t^s, \tag{10.62a}$$

$$K_t = \sum_{s=1}^{T} \mu_t^s N_t k_t^s, \tag{10.62b}$$

$$\bar{l}_t = \frac{\sum_{s=1}^{T^W} \mu_t^s N_t l_t^s}{\sum_{s=1}^{T^W} \mu_t^s N_t}, \tag{10.62c}$$

$$C_t = \sum_{s=1}^{T} \mu_t^s N_t c_t^s, \tag{10.62d}$$

$$Tr_t = \sum_{s=1}^{T} \mu_t^s N_t tr_t, \tag{10.62e}$$

and the goods market clears:

$$Z_t (A_t L_t)^{1-\alpha} K_t^\alpha = C_t + G_t + I_t, \tag{10.63}$$

with investment I_t:

$$I_t = K_{t+1} - (1-\delta) K_t. \tag{10.64}$$

CALIBRATION. The discount factor $\beta = 0.99$ is chosen in accordance with its value in the business cycle models presented in the first part of the book, while $\gamma = 0.29$ is set such that average labor supply amounts to approximately $\bar{l} = 0.30$. The remaining preference, production and tax parameter values are chosen as in Section 9.1.3 (adjusted to quarterly rates): $\eta = 2.0$, $\alpha = 0.35$, $\delta = 2.075\%$, $g_A = 0.50\%$, $\tau^c = 5.0\%$, $\tau^k = 36.0\%$, and $\tau^l + \tau^p = 28.0\%$. Government consumption in steady state is equal to 18% of GDP. Pensions pen are constant and calibrated such that the nonstochastic replacement rate of pensions relative to net wage earnings is equal to $repl = \frac{pen}{(1-\tau^l-\tau^p)wA\bar{l}} = 49.4\%$ in accordance with OECD (2019), where $\bar{l}$ is the average labor supply in the nonstochastic steady state of the economy. The stochastic survival probabilities ϕ^s are calibrated for the US economy in the year 2015 using UN (2015); in addition, we set the annual population growth equal to $n = 0.754\%$. The age-efficiency profile is taken from Hansen (1993a) and interpolated to values in between years.

The parameters of the AR(1) process for (logarithmic) technology Z_t are set equal to $\rho = 0.95$ and $\sigma = 0.00763$ as in Prescott (1986b). Following Schmitt-Grohé and Uribe (2007b), we use $\rho^G = 0.87$ and $\sigma_G = 0.016$ for the parameters of the autoregressive process for (logarithmic) government consumption G_t.

10.2.1.2 Computation

We solve for the local dynamics of the model with the help of perturbation methods as described in Chapters 2–4. In particular, we use the toolbox **CoRRAM**. Therefore, we, first, have to solve the nonstochastic steady state and, second, provide the contemporaneous and dynamic equilibrium conditions in a procedure/function to **CoRRAM**. The GAUSS computer code *AK70_perturbation.gss* that is available as a download form our homepage computes Example 10.2.1.[44] The total runtime amounts to 41 minutes on an Intel(R) Xeon(R), 2.90 GHz, of which 39 minutes are required for the computation of the steady state versus 2 minutes for the computation of the dynamics.

NONSTOCHASTIC STEADY STATE. For the economy described in Example 10.2.1, we can compute the nonstochastic steady state with the help of the methods described in Section 9.1 by solving a large-scale nonlinear system of equations (in 464 variables). For this reason, we transform the model equations using the following individual stationary variables

$$\tilde{k}_t^s = \frac{k_t^s}{A_t}, \;\; \tilde{c}_t^s = \frac{c_t^s}{A_t}, \;\; \tilde{tr}_t = \frac{tr_t}{A_t}, \;\; \widetilde{pen}_t = \frac{pen_t}{A_t},$$

and aggregate stationary variables

$$\tilde{G}_t = \frac{G_t}{A_tN_t}, \;\; \widetilde{Tr}_t = \frac{Tr_t}{A_tN_t}, \;\; \widetilde{Pen}_t = \frac{Pen_t}{A_tN_t}, \;\; \tilde{K}_t = \frac{K_t}{A_tN_t}, \;\; \widetilde{Beq}_t = \frac{Beq_t}{A_tN_t},$$
$$\tilde{L}_t = \frac{L_t}{N_t},$$

with $\tilde{tr}_t = \widetilde{Tr}_t$ in equilibrium.

Using stationary variables, we can express the budget constraint of the households with the help of

$$\begin{aligned}(1+\tau^c)\tilde{c}_t^s + (1+g_A)\tilde{k}_{t+1}^{s+1} - (1+(1-\tau^k))r_t)\tilde{k}_t^s \\ = \begin{cases} (1-\tau^L-\tau_t^p)w_t\bar{y}^s l_t^s + \tilde{tr}_t, & s=1,\ldots,T^W, \\ \widetilde{pen}_t + \tilde{tr}_t, & s = T^W+1,\ldots,T. \end{cases}\end{aligned} \tag{10.65}$$

[44] To run program *AK70_perturbation.gss*, you first need to install the CoRRAM package. For a description of the package and its installation, please see our documentation *CoRRAM user guide* that is also available on our homepage.

To solve the individual optimization model, we also need to derive the stationary first-order conditions of the household presented by

$$(1-\tau^l-\tau_t^p)w_t\bar{y}^s(1-l_t^s)=\frac{1-\gamma}{\gamma}(1+\tau^c)\tilde{c}_t^s,\ \ s=1,\ldots,T^W, \quad (10.66)$$

$$\begin{aligned}&\left((1+g_A)\tilde{c}_t^s\right)^{\gamma(1-\eta)-1}\left(1-l_t^s\right)^{(1-\gamma)(1-\eta)} \quad (10.67)\\&=\beta\phi^s\mathbb{E}_t\left\{\left(1+(1-\tau^k)r_{t+1}\right)\left(\tilde{c}_{t+1}^{s+1}\right)^{\gamma(1-\eta)-1}\left(1-l_{t+1}^{s+1}\right)^{(1-\gamma)(1-\eta)}\right\},\\&s=1,\ldots,T-1.\end{aligned}$$

(10.66) describes the optimal labor supply of the worker at age $s=1,\ldots,T^W$, while (10.67) presents the Euler condition of the household at age $s=1,\ldots,T-1$. According to (10.66), the marginal utility from an additional unit of leisure is equal to the utility gain from consumption that results from the wage of an additional labor unit. (10.67) equates the marginal utility gain from consuming one additional unit in the present period and the gain that results from saving and consuming it in the next period.

The nonstochastic steady state is characterized by a constant technology level and government demand, $Z_t=Z=1$ and $\tilde{G}_t=\tilde{G}$. Furthermore, all individual and aggregate variables are constant and are denoted by a variable without a time index. For example, $\tilde{k}^s$ and $\tilde{K}$ denote the nonstochastic steady state capital stock of the individual at age s and the nonstochastic steady-state aggregate capital stock, respectively.

The nonstochastic steady state is described by the 464 equations consisting of the 180 first-order conditions with respect to labor, (10.66), the 279 Euler equations, (10.67), the fiscal budget constraint, (10.60), the social security budget, (10.61), and the 3 aggregate consistency conditions, (10.62a)–(10.62c). The 464 endogenous variables consist of the 279 individual capital stocks $\tilde{k}^s$, $s=2,\ldots,T$, (recall that $\tilde{k}^1=0$), the 180 individual labor supplies l^s, $s=1,\ldots,T^W$, government transfers $\widetilde{tr}$, social security contributions τ^p and the 3 aggregate variables $\tilde{K}$, $\tilde{L}$, and $\bar{l}$. In addition, we eliminate consumption $\tilde{c}^s$ from these equilibrium conditions with the help of the individual budget constraints, (10.65), the factor prices w and r with the help of the marginal products of capital and labor, (10.56) and (10.57), and pensions with the calibration of their replacement rate with respect to net wages, $\widetilde{pen}=repl\times(1-\tau^l-\tau^p)w\bar{l}$. The equilibrium conditions of the steady state are provided in the procedure *steady_state_public(x)* in the computer program *AK70_perturbation.gss*.

The main challenge in the solution of the nonlinear system of equations for the steady state is searching for a good initial value such that the Newton-Rhapson algorithm is able to find the root. There is no general rule for how you should find such a value. Often, you need to start with a simpler problem and apply a trial-and-error procedure to obtain to the initial value of the final problem.[45] This is how we proceed in the following. We start with a simple OLG model where households only live for a maximum of 20 periods (=quarters) and work all their life. In addition, we assume that transfers are zero and neglect bequests and government consumption but include the tax rates in the household optimization problem. As a consequence, we do not derive a general but only a partial equilibrium model.[46]

To compute the solution to this initial model, we first compute the measures of the 20 cohorts by setting $\mu^1 = 1$ and iterating over $s = 1, \ldots, 19$ as follows:

$$\mu^{s+1} = \frac{\phi^s}{1+n}\mu^s.$$

We normalize the measures $\{\mu^s\}_{s=1}^{20}$ by the division of the sum of all measures such that $\sum_s \mu^s = 1.0$. Next, we use an outer and an inner loop to compute the steady state in this partial equilibrium model (where the fiscal budget is not balanced). In the outer loop, we iterate over the aggregate variables $\{\tilde{K}, \tilde{L}, \tau^p, \bar{l}\}$. The aggregate variables imply the factor prices that we need for the computation of the individual optimal policies. We initialize the average and aggregate effective labor supply with $\bar{l} = 0.30$ and $\tilde{L} = 0.30$, respectively. Using the first-order condition of the firms with respect to their capital demand, (10.57), we can derive the initial guess for the aggregate capital stock $\tilde{K} = 1.256$. The social security rate is set to $\tau^p = 0\%$ because we do not consider any retirees yet. In the inner loop, we solve the individual household optimization problem for given exogenous factor prices w and r and tax rates $(\tau^c, \tau^l, \tau^p, \tau^k)$. The optimal policy functions c^s, l^s and k^{s+1} are computed as the solution of the procedure *ss_exog_lab(x)* in the program *AK70_perturbation.gss*. The 19 nonlinear equations in this procedure consists of the 19 Euler conditions, (10.67). The 20 budget constraints, (10.65), are used to substitute for the variables c^s, $s = 1, \ldots, 20$, from these Euler equations. The exogenous labor supplies are all set equal to $l^s = 0.30$, $s = 1, \ldots, 20$. As an initial guess

[45] Please also read Section 9.1.3 for more on this topic.

[46] We also experimented with other initial specifications of the economy consisting of models with a larger number of periods or endogenous labor supply. However, in this case, the Newton-Rhapson algorithm did not converge for our initial guesses of the endogenous variables.

of the 19 endogenous variables k^s, $s = 2, \ldots, 20$, we assume a simple life-cycle profile of savings. In particular, savings k^s increase from $k^1 = 0$ to a maximum k^{19} at age 19 and then fall by half at age 20. This profile is founded in our experience on life-cycle savings in OLG models from this and the previous chapter. We set the maximum individual capital stock to twice the average (=aggregate) capital stock $\tilde{K}$. Once we have computed the solution for the individual problem, we compute the new aggregate capital stock $\tilde{K}^s$ by aggregating over the individual capital stocks k^s using (10.62b). We update the old aggregate capital stock by 20% and continue to iterate over the outer loop. In the inner loop, we use the solution from the previous iteration as an initial guess for $\{k^s\}_{s=2}^{20}$. At this point, we do not need high accuracy (which is only needed in the final steady state with 280 cohorts and a fully specified general equilibrium model with a balanced government budget). Therefore, we only use 5 iterations over the aggregate variables to save on computational time.

Next, we add another period of life in each step and recompute the steady state. Each time, we continue to iterate 5 times over the aggregate variables and use the optimal policy functions found in the previous iteration. To provide a new guess for the capital stock in the added new period at the end of the life, we simply assume it to be half the value of the end-of-life wealth found in the last iteration with a shorter life time. Once we have completed the computation of the model with 180 cohorts of workers, we start to add cohorts of retirees one at a time. In the outer loop over the aggregate variables, we therefore also need to compute the pensions that are calibrated with the help of the net replacement rate. In addition, we compute the social security contribution rate τ^p that balances the budget of the social security authority, (10.61).

In the next step, we compute the same model but with endogenous labor supply. The nonlinear system of equations now additionally contains the 180 first-order conditions with respect to labor, (10.66). If we attempt to compute the solution with the help of the guess $l^s = 0.30$, $s = 1, \ldots, 280$, the Newtwon-Rhapson algorithm does not converge. We therefore have to use a more incremental approach. We first endogenize l^1 and maintain the assumption of exogenous labor supply $l^s = 0.30$ for the other cohorts $s = 2, \ldots, 280$. Next, we endogenize the labor supply of the second cohort and so.[47] In the final model type, we include the government budget in the nonlinear system of equations and endogenize

[47] The nonlinear system of equations is specified in the procedure *ss_endog_lab(x)* in the program *AK70_perturbation.gss*.

government transfers $\widetilde{tr}$ as a new variable. We initialize the variable $\widetilde{tr}$ at zero. We solve the complete nonlinear system of equations in all individual and aggregate variables in one loop.

Our solutions for the aggregate variables are as follows: $\tilde{K} = 8.477$, $\tilde{L} = 0.235$, $\bar{l} = 0.300$, $\tilde{Y} = 0.824$, $\tilde{C} = 0.442$, $\tilde{I} = 0.234$, $\tilde{G} = 0.148$, $\tau^l = 18.21\%$, $\tau^p = 9.79\%$ and $\tilde{t}r = 0.0311$.[48] The optimal policy functions of the cohorts $s = 1, \ldots 280$ are displayed in Fig. 10.11, which presents individual wealth k^s (upper-left panel), consumption c^s (upper-right panel) and labor supply l^s (bottom panel). Individual wealth k^s is hump-shaped and peaks around age 150 (corresponding to real-life age 57). Households start to save for old age beginning in period $s = 35$. Prior to this age, wage income is too low due to low age productivity $\bar{y}^s$, so they instead accumulate debt to finance consumption. Individual consumption c^s is also hump-shaped but peaks only in the last period of the working life. The drop in consumption during retirement is explained by the increase in leisure and the utility-smoothing behavior of individuals. Working hours l^s are also hump-shaped but peak prior to age efficiency $\bar{y}^s$ due to the wealth effect on labor supply.

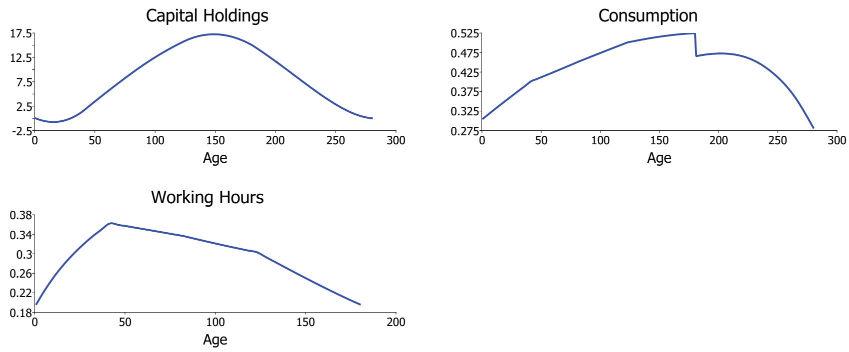


Figure 10.11 Steady-state age profiles of individual policy functions in the OLG model

With the help of the individual policy functions and the measures μ^s of cohort $s = 1, \ldots, 280$, we can also compute inequality measures of the wealth and income distribution. We will use the Gini coefficient that, for

[48] We calibrated $\gamma = 0.29$ such that average labor supply is equal to $\bar{l} = 0.300$. We found this value of γ by using a grid over the parameter and computing the nonstochastic steady state in each case.

example, is defined as follows in the case of the wealth distribution:

$$Gini = 1 - \frac{\sum_{s=1}^{T} \mu^s \left(S_{s-1} + S_s\right)}{S_T}, \tag{10.68}$$

where the accumulated wealth is defined by

$$S_s = \sum_{j=1}^{s} \mu^j k^j$$

with $S_0 \equiv 0$. We also compute the Gini coefficient for gross income and earnings. Gross income amounts to wage income (pensions) plus interest income for the workers (retirees) and transfers. Earnings are simply equal to wage income.[49] The Gini coefficients of wealth, income and earnings amount to 0.418, 0.201 and 0.121 in the nonstochastic steady state of the model and fall short of empirical values, as we only consider the inter-cohort and not the intra-cohort inequality.[50]

Computation of the Dynamics. Our main research question concerns the cyclical behavior of the aggregate variables and the (income and wealth) distribution in the economy. We use perturbation methods (linear approximation) to solve for the dynamics and apply the notation described in Chapters 2–4. In the set of control variables in our dynamic system of equations, we also include GDP $\tilde{Y}$ together with the aggregate demand components consumption $\tilde{C}$, investment $\tilde{I}$ and government consumption $\tilde{G}$ as well as the factor prices and the Gini coefficients of wealth, income and earnings because we would like to study their behavior in response to technology and government demand shocks.

As input into the solver package *CoRRAM*, we specify a procedure *OLG_Eqs1(.)* in our computer program *AK70_perturbation.gss* that contains a total of 755 (static and dynamic) equations in 280 state variables, 473 control variables and two exogenous shocks, $\ln Z_t$ and ϵ_t^G. The state variables consist of the 279 individual capital stocks $\tilde{k}_t^s$, $s = 2, \ldots, 280$, and public consumption in the previous period, $\tilde{G}_{t-1}$. The inclusion of the latter variable among the state variables results from the stochastic autoregressive behavior of government consumption, (10.59). The control

[49] In the computation of the Gini for earnings, we have to modify the computation of the Gini coefficient such that the shares of the workers sum up to 1.0.

[50] Your are asked to introduce within-cohort inequality in Problem 3.

variables consist of the 280 individual consumptions $\tilde{c}^s$, $s = 1, \ldots, 280$, the 180 individual labor supplies, l^s, $s = 1, \ldots, 180$, and the 13 aggregate variables $\tilde{Y}_t$, $\tilde{K}_t$, $\tilde{L}_t$, $\tilde{C}_t$, $\tilde{I}_t$, $\tilde{G}_t$, w_t, r_t, τ_t^p, $\tilde{tr}_t$, $Gini_t^{wealth}$, $Gini_t^{income}$ and $Gini_t^{earnings}$. In addition, we need to provide the information to CoRRAM that there are 194 static equations among the 755 equations consisting of the 180 first-order conditions with respect to individual labor supply, the aggregate consistency conditions, the first-order conditions of the firms, the production function, the definitions of the Gini coefficients, the goods market equilibrium,[51] the budget constraint of a household in its last period of life, $s = 280$, and the budget constraints of the fiscal and social security authorities.

The dynamic system characterizing our model has 282 eigenvalues with absolute value less than one, which is exactly the number of predetermined variables (the number of state variables and exogenous shocks). Therefore, our economy is locally stable.[52]

10.2.1.3 Business Cycle Dynamics of Aggregates and Inequality

Following the procedure presented in Chapter 3, we study impulse responses and second moments in our business cycle model. In particular, we are interested in the cyclical distribution effects on income and wealth.

Effects of a Technology Shock. Fig. 10.12 illustrates the effects of a technology shock in period 2 in the amount of one standard deviation, $\epsilon_2^Z = 0.76\%$. As a consequence, production $\tilde{Y}_t$ increases by 1.1% (black line in the upper-right panel). In addition, wages w_t (black line in the bottom-left panel) and the interest rate r_t (not illustrated) both increase, and the households increase their individual labor supplies such that aggregate labor $\tilde{L}_t$ increases by 0.6% (blue line, bottom left). Due to the higher income, households increase both consumption and savings such that consumption $\tilde{C}_t$ (blue line, top left) and investment $\tilde{I}_t$ (red line, top right) increase by 0.5% and 3.8%, respectively. Note that both the quali-

[51] We use the goods market equilibrium to compute investment $\tilde{I}_t = \tilde{Y}_t - \tilde{C}_t - \tilde{G}_t$. As you know from *Walras' law*, one equilibrium equation in the model is redundant. In our case, we do not specify the aggregate capital accumulation, $(1+g_A)(1+n)\tilde{K}_{t+1} = (1-\delta)\tilde{K}_t + \tilde{I}_t$, in the procedure *OLG_EQ1()*.

[52] See Laitner (1990) for a detailed analysis of local stability and determinacy in Auerbach-Kotlikoff models.

tative and quantitative responses are in very close accordance with those for the benchmark model presented in Fig. **1.6** in Section 1.6. For example, the immediate responses of consumption and investment relative to that of output are approximately 0.5% and 3.5% in both model types.

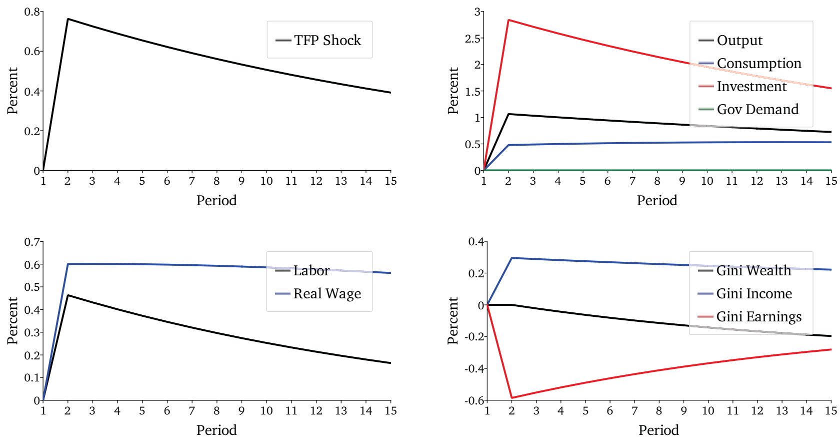


Figure 10.12 Impulse responses to a technology shock in the OLG model

The distribution effects of a technology shock are depicted in the bottom-right panel of Fig. 10.12. The Gini coefficient of earnings (red line) falls by 0.6% on impact. The decline in inequality is explained by the stronger labor response of younger households to a wage increase than that of the older households. Households at young age have lower wealth and a longer remaining period to substitute labor intertemporally. As a consequence, earnings increase by a larger percentage in young than in old age. The younger households subsequently also increase their savings to a larger extent than the older workers such that wealth inequality slowly starts to decline. Note that wealth is predetermined, and hence the shock in period 2 does not affect the wealth distribution until period 3. Income inequality, however, increases. This property of our OLG business cycle model results from our assumption that pensions are constant. As the older retirees are among those with the lowest income (recall that pensions are below average net earnings as expressed in the replacement rate of 49.4%), the increase in income of the working households increases the income gap. In sum, the Gini coefficient of the income distribution rises by 0.3% on impact.

Effects of a Government Demand Shock. Fig. 10.13 presents the impulse responses of the aggregate demand components output $\tilde{Y}_t$, $\tilde{C}_t$, $\tilde{G}_t$ and $\tilde{I}_t$ (upper-right panel), the wage rate w_t and aggregate labor $\tilde{L}_t$ (bottom-left panel), and the Gini coefficients (bottom-right panel) to a government demand shock ϵ_t^G (upper-left panel) of one standard deviation in period 2, $\epsilon_2^G = 1.6\%$. Given the fiscal budget constraint, higher government consumption results in smaller government transfers $\tilde{tr}_t$. As a consequence of this negative income effect, households increase their labor supply such that aggregate labor $\tilde{L}_t$ rises by 0.2%. Note that the quantitative effect on labor supply is relatively small. The marginal product of labor falls with higher aggregate labor, and the wage declines by 0.03%. On impact, the capital stock $\tilde{K}_t$ is constant and production increases by an even smaller percentage (equal to 0.1%) than labor. Private consumption $\tilde{C}_t$ and savings fall due to the decline in income (lower transfers $\tilde{tr}_t$). Therefore, government consumption (partially) crowds out investment, which falls by 0.7%.

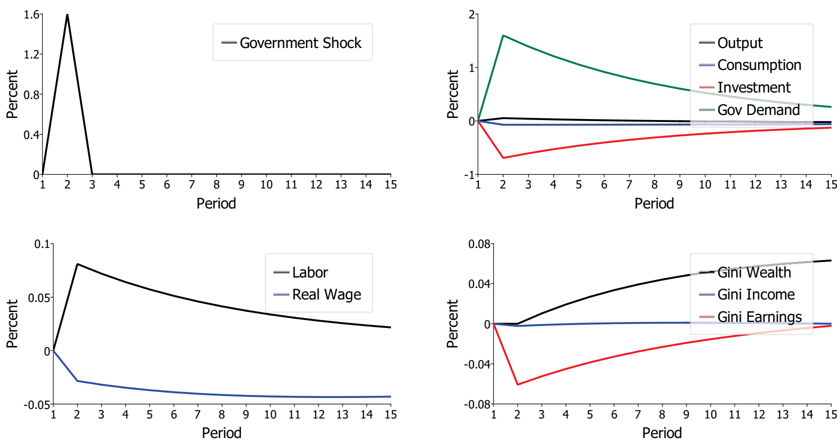


Figure 10.13 Impulse responses to a government demand shock in the OLG model

The impulse responses are in close accordance with empirical observations. For example, Blanchard and Perotti (2002) find evidence in their VAR analysis of the US economy that a positive government consumption shock increases GDP $\tilde{Y}_t$ and employment $\tilde{L}_t$, while investment $\tilde{I}_t$ declines strongly. With respect to the responses of private consumption and real wages to a positive government demand shock, the empirical evidence is more mixed. Most studies, however, find a positive effect on consumption $\tilde{C}_t$, e.g., Blanchard and Perotti (2002), Galí and Lopez-Salido (2007), and Ravn, Schmitt-Grohé, and Uribe (2012).[53] While Rotemberg and Woodford (1992) find evidence that real wages also increase after a government spending shock, Monacelli, Perotti, and Trigari (2010) only find a statistically insignificant increase in the real wage for men.

The effect of a government demand shock on inequality is quantitatively much smaller (in absolute value) than that of a technology shock. As illustrated in the bottom-right panel of Fig. 10.13, the Gini coefficient of earnings only falls by 0.06% as younger workers increase their labor supply to a larger extent than older workers. Government transfers constitute a larger share of disposable income among younger workers than among older workers. Therefore, wealth inequality also increases because younger workers are able to accumulate less life-cycle savings over time. The effect on the income distribution is mixed. On the one hand, earnings among younger workers (who have lower age-dependent productivity $\bar{y}^s$ and, hence, labor income) increase. On the other hand, transfers are reduced by an equal amount to all households, which affects income-poor households more significantly. In addition, the decline in wages and the corresponding increase in interest rates[54] favors the wealth rich, who are also the income rich in our economy (age efficiency $\bar{y}^s$ peaks at real-life age 52, while wealth peaks at real-life age 57). As a consequence, income inequality is basically unaffected by a government demand shock, and the Gini coefficient of income remains almost constant.[55]

[53] Heer (2019) shows that the qualitative response of private consumption to a government demand shock depends on the elasticity of substitution between private and public consumption in utility. For example, if private and public consumption are complements, $\tilde{C}_t$ also increases in response to a positive government demand shock. In the present model, we simply assume that public consumption $\tilde{G}_t$ does not enter utility.

[54] Note , however, that the rise in interest rate in our model is not in accordance with empirical evidence from the US economy. In particular, Auerbach et al. (2020) study regional effects of public defense spending and find declining interest rates in response to an expansive government demand shock. Moreover, Heer et al. (2018) show that stock returns in the consumption and investment goods sectors are uncorrelated with GDP growth.

[55] Heer and Scharrer (2018) introduce adjustment costs of capital into a similar large-scale OLG model and show that, in this case, the return on capital may fall and income inequality declines after a shock to government consumption. They decompose the income responses for the individual age types and study the debt versus tax financing of additional government expenditures, among other topics.

Second Moments. Table 10.4 presents the second moments of our aggregates and Gini coefficients in the form of the volatility s_x, the cross-correlations with output and labor, r_{xY} and r_{xL}, and the autocorrelation r_x. While the standard deviations of the aggregate demand components are in line with empirical evidence, e.g., investment is approximately three times as volatile as output and consumption and labor are less volatile than output, we are unable to model the correlations of our main variables with output and employment. For example, consumption and investment are almost perfectly correlated with labor, while the empirical values reported by Heer (2019) are equal to 0.88 and 0.68 for the US economy during 1953–2014, respectively. In addition, real wages are empirically negatively correlated with employment and output, e.g., the correlation of real wages and output amounted to -0.27 and -0.36 for the US economy, while the correlation is almost perfect in our model.

Table 10.4 Second moments of OLG business cycle model

Variable	s_x	r_{xY}	r_{xL}	r_x
Output $\tilde{Y}_t$	1.39	1.00	0.98	0.72
Labor $\tilde{L}_t$	0.62	0.98	1.00	0.71
Consumption $\tilde{C}_t$	0.64	0.97	0.90	0.74
Investment $\tilde{I}_t$	3.80	0.96	0.92	0.71
Real wage w_t	0.79	0.99	0.95	0.73
Gini wealth	0.11	0.01	0.14	0.96
Gini income	0.38	1.00	0.97	0.72
Gini earnings	0.76	-0.99	-1.00	0.71
Gov demand $\tilde{G}_t$	2.02	0.05	0.17	0.67

Notes: s_x:=standard deviation of HP-filtered simulated series of variable x, r_{xY}:=cross correlation of variable x with output $\tilde{Y}$, r_{xL}:=cross correlation of variable x with aggregate labor $\tilde{L}$, r_x:=first order autocorrelation of variable x. Time series are detrended using the Hodrick-Prescott filter with weight $\lambda = 1600$.

As documented in Section 8.4.2, the low (high) income quintiles display procyclical (anticyclical) income shares such that the Gini coefficient of income is procyclical,[56] whereas Guvenen et al. (2015) report that the last two recessions had little effect on the income distribution with the exception of the top percentile, who were characterized by persistent

[56] In the next paragraph, we will also highlight that the Gini coefficient of income has a positive correlation with output ($r_{Gini^{income},Y} = 0.25$) at an annual frequency.

and enormous losses. In our model, the inequality of income is perfectly procyclical, while the inequality of earnings is strongly anticyclical. The model also fails to replicate the cyclical pattern of the wealth distribution. As documented in Table 3 of Krueger et al. (2016), the shares in net worth increased among the bottom four quintiles of the wealth distribution and fell in the top quintile during the Great Recession between 2006 and 2010. This procyclical behavior of wealth inequality cannot be replicated with our simple model. The Gini coefficient of wealth is uncorrelated with output in our model.

Comparison of OLG Models with Quarterly and Annual Periods. When we attempt to study questions of business cycle dynamics of the income and wealth distribution, we are often confronted with the problem of obtaining high-frequency data. Most data on inequality measures and percentiles of the income and wealth distribution in the US economy (and other countries) are only available at an annual frequency, if at all, for example in the form of the PSID panel data or data from the tax authorities for the case of the US economy. Therefore, it often makes sense to use a business cycle model with annual periods to map it to the appropriate data.

In the following, we will study our OLG business cycle model for annual frequencies.[57] Therefore, the parameterization has to be adjusted for the new period length. For example, we set the population growth rate to 0.754% and 0.754%/4=0.1885% in the cases of annual and quarterly periods, respectively.[58] In the case of the quarterly autoregressive MA(1) processes for technology, we change the quarterly autoregressive parameters, $\rho^Z = 0.96$ and $\sigma^Z = 0.00763$, to $\rho^{Z,annual} = (\rho^z)^4 = 0.815$ and $\sigma^{Z,annual} = \sqrt{1+(\rho^Z)^2+(\rho^Z)^4+(\rho^Z)^6}\sigma^Z = 0.0142$.[59] Similarly, we adjust the parameters of the autoregressive MA(1) process for government consumption, $\rho^G = 0.573$ and $\sigma^G = 0.0266$. In addition, we choose the HP filter weight $\lambda = 100$ for annual data. For the lower frequency, the computational time of the program *AK70_perturbation.gss* decreases considerably, from 32 minutes (quarterly periods) to 1 minute (annual periods).

[57] In the GAUSS program *AK70_perturbation.gss*, you simply have to set the period length to quarters in line 23 of the code.
[58] Alternatively, one could choose $1.00754^{1/4} - 1 = 0.1880\%$ instead.
[59] The derivation of the formulas is relegated to Appendix A.11.

Table 10.5 Comparison of second moments across studies

Variable	US data	Ríos-Rull (1996)	Example 10.2.1
Output $\tilde{Y}$			
s_Y	2.23	1.65	1.78
r_Y	0.50	0.47	0.46
Consumption $\tilde{C}$			
s_C	1.69	0.74	0.95
s_C/s_Y	0.76	0.45	0.53
r_{CY}	0.86	0.97	0.95
Investment $\tilde{I}$			
s_I	8.66	4.45	4.73
s_I/S_Y	3.88	2.71	2.66
r_{IY}	0.87	0.99	0.96
Labor $\tilde{L}$			
s_L	1.72	0.59	0.74
s_L/s_Y	0.77	0.36	0.42
r_{LY}	0.93	0.95	0.96
Gini Income			
s_{Gini}	1.05		0.48
s_{Gini}/s_Y	0.47		0.27
$r_{Gini,Y}$	0.25		0.99

Notes: The entries for the US data are taken from Table 4 of Ríos-Rull (1996) with the exception of the Gini index of income (own calculations). s_x:=standard deviation of HP-filtered simulated series of variable x, r_{xY}:=cross correlation of variable x with output Y, r_x:=first-order autocorrelation of variable x. Time series are detrended using the Hodrick-Prescott filter with weight $\lambda = 100$.

Table 10.5 reports the second moments of annual time series from the US economy, the model of Ríos-Rull (1996) and our own model in this section. The empirical data for the aggregates $\tilde{Y}_t$, $\tilde{C}_t$, $\tilde{I}_t$, and $\tilde{L}_t$ for the US economy are taken from Table 4 in Ríos-Rull (1996), who provides HP-filtered data for the period 1956–1987. The estimates for the second moments of the Gini coefficient of income are calculated with the help of World Bank data for the period 1991–2018.[60] The second entry column

[60] We retrieved the GINI Index for the United States [SIPOVGINIUSA, World Bank] from FRED, Federal Reserve Bank of St. Louis.

replicates the results of Table 5 in Ríos-Rull (1996). The model of Ríos-Rull (1996) is specified without a government sector, so there are no taxes, government spending or public pensions. In addition, he assumes perfect annuities markets. Evidently, the model is able to explain a large part of the fluctuations in GDP with the help of only a technology shock because the standard deviation of output s_Y is equal to 1.65% in the model compared to 2.25% in the US economy. The model is also able to generate the same relative volatilities of consumption and investment with output, the autocorrelation of output r_Y, and the high correlations of the aggregates consumption $\tilde{C}_t$, investment $\tilde{I}_t$ and labor $\tilde{L}_t$ with output $\tilde{Y}_t$.

Our model with a government only mildly improves the results on the matching of the business-cycle facts by Ríos-Rull (1996). In particular, the government helps to increase the volatility of output $\tilde{Y}_t$, consumption $\tilde{C}_t$ and labor $\tilde{L}_t$ to a small extent. In our model, we also analyze the cyclical behavior of the income distribution. Empirically, the Gini coefficient of income is half as volatile as output, $s_{Gini}/s_Y = 0.47$, while we are only able to replicate 56% of this relative volatility. Apparently, allowing only for inter- but not intra-generational inequality and neglecting stochastic individual productivity results in too little volatility in income inequality. In addition, the Gini coefficient of income in the model is too procyclical as measured by the correlation with output ($r_{Gini,Y} = 0.99$) compared to its empirical value ($r_{Gini,Y} = 0.25$).

From this exercise, we might conclude that the business cycle dynamics of the OLG model compare closely with those of the standard Ramsey model. We might even go one step further and conjecture that the OLG model or even other heterogeneous agent models have a limited role in studying business cycles because the much simpler neoclassical growth model is able to do so except for the obvious questions of distributional effects on income, earnings or wealth. However, we should be careful drawing this conclusion. For example, Krueger et al. (2016) set up a model that adds retirees to the model of Krussell and Smith (1998) and is able to more closely replicate the US wealth distribution. In particular, they model the large share of the population at the bottom of the wealth distribution with zero or close-to-zero wealth. For this reason, they introduce preference heterogeneity, unemployment insurance and social security. As a consequence, aggregate consumption declines much more strongly in response to a large shock (as during the Great Recession) and is 0.5 percentage points higher (in absolute value) than that in the representative-agent economy. There is also empirical evidence that highlights the importance of considering the distribution in business cycle models. For example, Brinca et al. (2016) study the effects of wealth inequality on fiscal multipliers in a VAR and find, in a sample of 15 OECD countries, that fiscal multipliers increase with the country Gini of wealth.

10.2.2 The Krusell-Smith Algorithm and Overlapping Generations

The Algorithm proposed by Krussell and Smith (1998) that you learned about in Section 8.3 can also be applied to economies with finite lifetimes with some minor modifications.[61] The individual state space is simply characterized by an additional dimension, which is age. Therefore, the simulation step becomes more time-consuming. However, we have not encountered any other limitations in the application of the Krusell-Smith algorithm to finite-lifetime economies in our experience. In particular, the goodness of fit for the law of motion for a particular functional form is almost identical to that in infinite-lifetime models. Given current computer technology, however, the algorithm is still very time-consuming. Below, we will present a simple example that takes us some 5 hours (41 hours) to compute with an Intel Pentium(R) M, 319 MHz machine using GAUSS (PYTHON), even though technology as the only stochastic aggregate variable is assumed to take only two different values. Therefore, adding more stochastic aggregate variables may seriously exacerbate computational time.

The economic analysis in this section is very closely related to that of the infinite-lifetime model in Section 8.4.2. Hereinafter, we study the business cycle dynamics of the income distribution in an OLG model with aggregate uncertainty. For this reason, let us briefly review our results from Chapter 8. In Table 8.2, we present the empirical correlation between output and income shares as estimated by Castañeda et al. (1998). The US income distribution is highly, but not perfectly, procyclical for the low income quintiles, countercyclical for the top 60-95%, and acyclical for the top 5%. In their model, cyclical fluctuations result from the stochastic technology level. During a boom, the number of unemployed workers decreases. As a consequence, the relative income share of the lower income

[61] In a more recent application, Kaplan et al. (2020), for example, study the effects of the housing boom and bust during the Great Recession. Therefore, they introduce multiple aggregate shocks to income, housing finance conditions and beliefs about future housing demand in a large-scale OLG model with home ownership. The main driver of the volatility in house prices is found to be the shift in beliefs. The main transmission mechanism to the real economy is a wealth effect through the household balance sheet.

quintiles rises relative to that of the higher income quintiles. However, the income shares are almost perfectly correlated with output, either positively or negatively. Therefore, we also fail to replicate the income dynamics of the very income rich, which is acyclical.

In the following, we use a simple business cycle model with overlapping generations and elastic labor supply and improve upon the modeling of the cyclical income distribution dynamics in some aspects. Therefore, we consider a minor modification of Example 10.2.1. In particular, household productivity also includes a stochastic component. As we noted, this element of a model is difficult to integrate in the solution with perturbation methods, and thus we need to apply the Krusell-Smith algorithm for the solution of the model. Moreover, we also simplify the model of Example 10.2.1 with respect to three assumptions. 1) We consider only annual periods and 2) assume a simple two-state process for technology, while 3) government consumption is constant. The rest of the model is unchanged.

In our model, the almost perfect correlation of the lower income quintiles with output is reduced as the high-productivity agents have a more elastic labor supply than their low-productivity counterparts.[62] In addition, the share of the top quintile of the income earners is less anticyclical as in Castañeda et al. (1998) and, for this reason, in better accordance with empirical observations because many of the income-rich agents in our model are wealth-rich agents close to and in early retirement. The economic mechanism is as follows. During an economic expansion, both wages and pensions increase. Pensions are tied to the current wage rate. However, workers also increase their labor supply, which is not possible for retired workers. Therefore, the income share of workers increases and is procyclical. Since the top income quintile contains both workers and retirees, the opposing cyclical effects on these groups result in a lower correlation of this income quintile with GDP.

10.2.2.1 An OLG Model of the Income Distribution Business Cycle Dynamics

In the following, we consider the model of Example 10.2.1 with annual periods and 70 overlapping generations subject to the following modifications:

[62] Heer and Maußner (2012) show that this does not need to be the case in the presence of progressive income taxation.

1. Worker labor productivity $\epsilon(s,e,\theta) = e\theta\bar{y}^s$ depends on the agent's permanent efficiency type $e \in \mathscr{E} = \{e_1, e_2\}$, his idiosyncratic stochastic productivity $\theta \in \Theta = \{\theta_1, \theta_2\}$, and his age $s \in \mathscr{S}$. This modeling of labor productivity has often been applied in DGE analysis for the following reasons: i) Differences in the permanent efficiency type e help to generate the wage heterogeneity that is observed empirically. Often, the wage gap between the two different productivity types e_2 and e_1, $e_2 > e_1$, is interpreted as the skill premium of college versus high-school graduates. According to recent estimates by Heer and Rohrbacher (2021), the skill premium in the US amounts to approximately 150%. ii) Workers will accumulate precautionary savings if they face idiosyncratic productivity risk θ. Therefore, the wealth distribution becomes more heterogenous in better accordance with reality. iii) The age-dependent component $\bar{y}^s$ helps to explain differences in the age-income distribution that is important to explain the movement of the cross-sectional factor shares.
   In each period t, an equal measure of 1-year-old workers (corresponding to real-life age 21) of productivity types $\epsilon(1, e_j, \theta_{i_\theta})$, $i_\theta = 1,2$, $j = 1,2$, is born. During working age, $s = 1, \ldots, 45$, the process for idiosyncratic productivity θ^s is described by a Markov chain:

$$\pi(\theta'|\theta) = Prob\left\{\theta^{s+1} = \theta'|\theta^s = \theta\right\} = \begin{pmatrix} \pi_{11}^{\theta} & \pi_{12}^{\theta} \\ \pi_{21}^{\theta} & \pi_{22}^{\theta} \end{pmatrix}. \tag{10.69}$$

   Depending on his permanent efficiency type e, the agent receives pensions $pen_t(e)$ in old age that are financed by a social security tax τ_t^p on the workers' wage income. Net labor income of a worker of type (s,e,θ) amounts to $(1-\tau_t^l-\tau_t^p)e\theta\bar{y}^s A_t w_t l_t$, so his budget constraint reads as:

$$\begin{aligned} k_{t+1}^{s+1} = &(1-\tau^L-\tau_t^p)e\theta\bar{y}^s A_t w_t l_t^s + \left(1+(1-\tau^k)r_t\right)k_t^s \\ &+ tr_t - (1+\tau^c)c_t^s, \quad s = 1,\ldots,45, \end{aligned}$$

   where, again, τ_t^l, τ_t^k and τ_t^c denotes the wage, capital income and consumption tax rates in period t.
   At age 46 (corresponding to real-life age 66), households retire and face the budget constraint

$$\begin{aligned} k_{t+1}^{s+1} = &pen_t(e) + \left(1+(1-\tau^k)r_t\right)k_t^s + tr_t - (1+\tau^c)c_t^s, \\ &s = 46,\ldots,70. \end{aligned} \tag{10.70}$$

2. Aggregate production is represented by (10.54) where the stochastic component Z_t, $Z_t \in \{Z^b, Z^g\}$, follows a 2-state Markov process:

$$\pi(Z'|Z) = Prob\left\{Z_{t+1} = Z'|Z_t = Z\right\} = \begin{pmatrix} \pi_{11}^Z & \pi_{12}^Z \\ \pi_{21}^Z & \pi_{22}^Z \end{pmatrix}. \tag{10.71}$$

   We associate Z^b and Z^g with a negative shock (recession) and a positive shock (boom), respectively.
3. Government consumption (relative to aggregate labor productivity A_t and population N_t) is constant, $\tilde{G}_t \equiv G_t/(A_t N_t) = \tilde{G}$.[63]
4. The government provides pensions to the retired agents. Pensions are proportional to the current-period net wage rate with the replacement rate being denoted by ζ. In addition, we distinguish between two cases. Pensions are either lump-sum or depend on the permanent efficiency type e:

$$pen_t(e) = \begin{cases} \zeta(1-\tau_t^l - \tau_t^p)w_t\bar{l} & \text{lump-sum}, \\ \zeta(1-\tau_t^l - \tau_t^p)w_t e\bar{l} & \text{efficiency-dependent}, \end{cases}$$

   where $\bar{l}$ denotes the average labor supply in the economy in the non-stochastic steady state (with $Z \equiv 1$). Therefore, pensions of the retired agents do not increase if the contemporary workers increase their labor supply.[64]

Calibration. The calibration of the model parameters is summarized in Table 10.6. We choose the same values for the production, tax and demographic parameters as in the previous section. The production elasticity of capital amounts to $\alpha = 0.35$. Capital depreciates at the rate $\delta = 8.3\%$, and economic growth amounts to 2.0%. The tax rates on labor income (including social security), $\tau^l + \tau^p$, capital income, τ^k, and consumption, τ^c are set at 28%, 36% and 5%, respectively. Population grows at 0.754% annually.

With respect to the preference parameters, we choose the intertemporal elasticity of substitution $1/\eta = 0.5$ and the weight of consumption in

[63] In Problem 5, you are asked to solve the model with stochastic government demand.

[64] This rather innocuous assumption simplifies the computation of the individual policy functions. The household has to forecast next-period wages, interest rates and pensions to derive its optimal savings, consumption and labor supply. For our specification of the model, it does not need to forecast the average working hours in the next period.

Table 10.6

Parameter	Value	Description
α	0.35	production elasticity of capital
δ	8.3%	depreciation rate of capital
g_A	2.0%	growth rate of output
$\tau^l + \tau^p$	28%	tax on labor income
τ^k	36%	tax on capital income
τ^c	5%	tax on consumption
n	0.754%	population growth rate
$1/\eta$	1/2	intertemporal elasticity of substitution
γ	0.29	preference parameter for utility weight of consumption
β	1.011	discount factor
G/Y	18%	share of government spending in steady-state production
$repl$	49.4%	net pension replacement rate
$\{e_1, e_2\}$	$\{0.57, 1.43\}$	permanent productivity types

utility $\gamma = 0.29$ such that average labor supply is approximately equal to 1/3. We set the discount factor β equal to 1.011 in accordance with estimates by Hurd (1989).

The share of government consumption in GDP amounts to 18%. The replacement rate of average pensions relative to net wage earnings is equal to $\zeta = \frac{\bar{p}en_t}{(1-\tau_t^l-\tau_t^p)w_t\bar{l}} = 49.4\%$.

The Markov process (10.71) of the aggregate technology level is calibrated such that the average duration of one cycle is equal to 6 years:

$$\pi(Z'|Z) = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}. \tag{10.72}$$

Aggregate technology is chosen such that the mean $\bar{Z}$ is equal to one and the annual standard deviation of output is approximately equal to 2%, implying $\{Z^b, Z^g\} = \{0.98, 1.02\}$.[65]

The calibration of individual productivity $\epsilon(s, e, \theta)$ is chosen in accordance with Krueger and Ludwig (2007). In particular, we pick $\{e_1, e_2\} = \{0.57, 1.43\}$ such that the average productivity is one and the implied variance of labor income for the new entrants at age $s = 1$ is equal

[65] The standard deviation of annual hp-filtered output amounts to 2.4% in our model.

to the value reported by Storesletten et al. (2007). The annual persistence of the idiosyncratic component θ is chosen to be 0.98. In addition, idiosyncratic productivity has a conditional variance of 8%, implying $\{\theta_1, \theta_2\} = \{0.727, 1.273\}$, and

$$\pi(\theta'|\theta) = \begin{pmatrix} 0.98 & 0.02 \\ 0.02 & 0.98 \end{pmatrix}. \tag{10.73}$$

Note that individual income is highly persistent. The age-efficiency $\bar{y}^s$ profile is taken from Hansen (1993a).[66]

The calibration implies an average labor supply approximately equal to $\bar{l} = 0.333$ and a Gini coefficient of income (wealth) equal to 0.40 (0.61) in good accordance with empirical observations, although the values are lower than those in most recent studies on the empirical wealth and income distribution. As noted previously, Budría Rodríguez et al. (2002) find a value of 0.55 (0.80) for the income Gini (wealth Gini) for the US economy. As the main reason for the underestimation of inequality in our model, we do not model the top percentile of the income distribution.

10.2.2.2 Computation

To compute the OLG model with both individual and aggregate uncertainty, we use the algorithm of Krussell and Smith (1998). The PYTHON and GAUSS programs `OLG_krusell_smith.py` and `OLG_krusell_smith.gss` implement the algorithm that is described by the following steps:

Algorithm 10.2.1 (Krusell-Smith Algorithm for OLG Models)

Purpose: *Computation of the OLG model with individual and aggregate uncertainty*

Steps:

Step 1: *Compute the nonstochastic steady state with $Z \equiv 1$. Store the policy functions and the steady-state distribution of $(s, e, \theta, \tilde{k})$.*

Step 2: *Choose an initial parameterized functional form for the law of motion for the aggregate next-period capital stock $\tilde{K}' = g^K(Z, \tilde{K})$ and aggregate present-period employment $\tilde{L} = g^L(Z, \tilde{K})$ and transfers $\widetilde{Tr} = g^{Tr}(Z, \tilde{K})$.*

[66] See Section 9.3.3.

Step 3: *Solve the consumer's optimization problem as a function of the individual and aggregate state variables,* $(s, e, \theta, \tilde{k}; Z, \tilde{K})$.
Step 4: *Simulate the dynamics of the distribution function.*
Step 5: *Use the time path for the distribution to estimate the law of motion for* $\tilde{K}'$, $\tilde{L}$ *and* $\widetilde{Tr}$.
Step 6: *Iterate until the parameters converge.*
Step 7: *Test the goodness of fit for the functional form* $g = (g^K, g^L, g^{Tr})'$ *using, for example,* R^2. *If the fit is satisfactory, stop; otherwise, choose a different functional form for* $g(.)$ *and return to Step 3.*

STEP 1: COMPUTATION OF THE NONSTOCHASTIC STEADY STATE. The nonstochastic steady state can be computed with two nested loops. In the outer loop, we iterate over the aggregate variables $\tilde{K}$, $\tilde{L}$, $\widetilde{Tr}$ and $\bar{l}$. In the inner loop, we solve the individual's utility maximization problem for given factor prices w and r and pensions $\widetilde{pen}$. In Section 10.1, we applied value function iteration to derive optimal savings, consumption and labor supply in the overlapping generations model with idiosyncratic uncertainty. In the following, however, we use a nonlinear equations solver and apply it to the first-order condition in the form of the Euler equation at each grid point of the discretized state space instead. We find this method to be much faster in the present case, and speed is essential in the computation of this heterogeneous agent model.

Let us start with the description of the outer loop. We initially pick a value for average working hours equal to $\bar{l} = 0.30$. Since the share of workers in the total population is equal to 78% in our calibration with respect to the characteristics of the US economy in 2015, we use a value of aggregate labor equal to $\tilde{L} = 0.78 \cdot 0.30 = 0.234$. Recall that the average (permanent, idiosyncratic and age-dependent) productivities of workers, ϵ, θ and y^s, have been normalized to 1.0. We compute the aggregate capital stock with the help of the marginal product of capital, which we set equal to $r = 2.0\%$ as our initial guess. Finally, we set $\widetilde{Tr} = 0$. In the outer loop, we update $(\tilde{K}, \tilde{L}, \widetilde{Tr}, \bar{l})$ by 30% with respect to their new values. With the help of the aggregate variables, we can compute factor prices w and r from the first-order conditions of the firms and $\widetilde{pen}$ with the help of the net replacement rate ζ.

Next, we compute the optimal policy functions over the individual state space $\tilde{z} = (s, e, \theta, \tilde{k})$ for given w, r and $\widetilde{pen}$. Therefore, we have to discretize the state space with respect to the only continuous variable, individual wealth (or capital) $\tilde{k}$. We specify an equispaced grid with $n_k = 50$ points over the individual asset space $\tilde{k}$ with lower and upper boundaries $\tilde{k}^{min} = 0$ and $\tilde{k}^{max} = 10.0$. The upper value is approximately equal to five times the average wealth in the economy (in the nonstochastic steady state). We have to ascertain that in neither the nonstochastic steady state nor in the simulation do households choose a next-period capital stock at the upper boundary k^{max} or above. If this were the case in our computation, we would have to respecify the boundaries and choose a wider interval.

We iterate backwards over age starting in the last period of life where the next-period capital stock is equal to zero: $\tilde{k}'(70, e, \theta, \tilde{k}) \equiv 0$ for all grid points $(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k})$, for $j = 1, 2$, $i_\theta = 1, 2$ and $i_k = 1, \ldots, 50$. We iterate backwards over ages $s = 69, 68, \ldots, 1$, solving for the nonlinear Euler equation residual at each grid point:

$$rf(\tilde{z}) = u_c(\tilde{c}(\tilde{z}, 1 - l(\tilde{z}))) - \beta \mathbb{E}_t u_c(c'(\tilde{z}', 1 - l(\tilde{z}')))(1 + (1 - \tau^k)r) \stackrel{!}{=} 0 \tag{10.74}$$

with

$$\tilde{l} = \begin{cases} \gamma - (1-\gamma)\frac{\widetilde{tr} + (1+(1-\tau^k))r\tilde{k} - (1+g_A)\tilde{k}'}{(1-\tau^l-\tau^p)e\theta\bar{y}^s w} & s \leq 45, \\ 0 & s > 45, \end{cases} \tag{10.75a}$$

$$\tilde{c} = \frac{(1-\tau^l-\tau^p)e\theta\bar{y}^s wl + \widetilde{tr} + (1+(1-\tau^k))r\tilde{k} - (1+g_A)\tilde{k}'}{1+\tau^c}. \tag{10.75b}$$

If our solution of the residual function $rf(\tilde{z}) = 0$ implies a negative labor supply for the worker, $l < 0$, we set $l = 0$. We store the optimal policy functions $\tilde{k}'(.)$, $\tilde{c}(.)$ and $l(.)$ at each grid point for each age $s = 69, \ldots, 1$.

The computation of the solution for nonlinear equation (10.74) merits some comments. First, we have to check whether our solution for the next-period capital stock is not the lower boundary value $\tilde{k}' = 0$. Therefore, we evaluate (10.74) at $\tilde{k}' = 0$. If $rf(.) < 0$, we know that $\tilde{k}' = 0$. Otherwise, we continue to find the inner solution for $\tilde{k}'$.

Second, we find that the nonlinear equation solver is very sensitive with respect to the choice of the initial value $\tilde{k}'$ supplied to the root-finding routine for $rf(\tilde{z}) = 0$. Depending on the particular nonlinear equation solver applied in our programs (for example, in our PYTHON or GAUSS codes, we use the modified Newton-Rhapson algorithm described in Section 13.5.1) we might have to test different starting values. In the present application, we specified five initial guesses for the optimal policy $\tilde{k}'(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k})$:

1. $\tilde{k}' = 0$,
2. solution for $\tilde{k}_{i_k-1}$ found in the last iteration if $i_k > 1$,
3. solution $\tilde{k}'(s+1, e_j, \theta_{i_\theta}, \tilde{k}_{i_k})$ at age $s+1$,
4. $\tilde{k}' = \tilde{k}$,
5. computation of maximum possible value $\tilde{k}'$ for small $\tilde{c} \approx 0.001$ and using half of this value.

From these values, we pick the one with the lowest absolute value for the residual function $rf(\tilde{z})$.

Third, in the computation of $rf(.)$ with the help of (10.74), we need to evaluate next-period consumption and labor supply at wealth level $\tilde{k}'$. In general, the next-period capital stock $\tilde{k}'$ will not be a grid point. We used the stored policy functions of consumption and labor, $\tilde{c}'(.)$ and $\tilde{l}'(.)$, at age $s+1$ in the previous iteration and interpolate linearly between grid points. The computation is carried out in the functions/procedures `rfold` and `rfyoung` of the GAUSS/PYTHON program `OLG_krusell_smith` for the retired and working households, respectively.

Finally, we have to aggregate the individual capital levels $\tilde{k}$, consumption $\tilde{c}$ and labor l to derive the aggregates $(\tilde{K}, \tilde{L}, \widetilde{Tr}, \bar{l})$. Therefore, we have to compute the (nonstochastic) steady-state distribution $f(s, e, \theta, \tilde{k})$. For the computation of distribution $f(.)$, we start at the newborn generation with age $s = 1$ and iterate forward over age. At age $s = 1$, the newborn cohort has measure $\mu^1 = 0.02118$ (recall that we normalized total population to have a measure of one).[67] Each productivity type (e_j, θ_{i_θ}), $j = 1, 2$, $i_\theta = 1, 2$, has equal measure such that

$$f(1, e_j, \theta_{i_\theta}, \tilde{k}_{i_k}) = \begin{cases} \frac{\mu^1}{4} & \text{if } i_k = 1, \\ 0 & \text{else.} \end{cases}$$

Given the distribution at age s, we update the distribution at age $s+1$ using the optimal policy function $\tilde{k}'(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k})$ and summing over all grid points at age s. If next-period wealth $\tilde{k}'(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k}) = k_{i_{kk}}$ happens to be a grid point $(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k})$ (this will be the case if the credit constraint $\tilde{k}' \geq 0$ is binding), we simply add the respective measure at age s to the measure at age $s+1$ noting that only ϕ^s of the households survive and

[67] We derive this measure using the US population growth rate and survival probabilities of the year 2015 and computing the stationary population.

that the size of the cohort in the total population shrinks by a factor of $(1+n)$ due to population growth:

$$f(s+1, e_j, \theta', \tilde{k}_{i_{kk}}) = \pi(\theta'|\theta_{i_\theta}) \times f(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k}) \times \frac{\phi^s}{1+n},$$

for $\theta' \in \{\theta_1, \theta_2\}$.

If next-period wealth $\tilde{k}'(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k})$ lies between the asset grid points $k_{i_{kk}}$ and $k_{i_{kk}+1}$, we add the measure in period s to the measures of these two points at age $s+1$ according to:[68]

$$f(s+1, e_j, \theta', \tilde{k}_{i_{kk}}) = \frac{\tilde{k}_{i_{kk}+1} - \tilde{k}'}{\tilde{k}_{i_{kk}+1} - \tilde{k}_{i_{kk}}} \pi(\theta'|\theta_{i_\theta}) \times f(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k}) \times \frac{\phi^s}{1+n},$$

$$f(s+1, e_j, \theta', \tilde{k}_{i_{kk}+1}) = \frac{\tilde{k}' - k_{i_{kk}}}{k_{i_{kk}+1} - k_{i_{kk}}} \pi(\theta'|\theta_{i_\theta}) \times f(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k}) \times \frac{\phi^s}{1+n},$$

for $\theta' \in \{\theta_1, \theta_2\}$.

Of course, this division of the measures is necessary due to our discretization of the distribution function and is only an approximation of the true distribution. Only if next-period capital $\tilde{k}'(.)$ were a linear function of present-period capital $\tilde{k}$ would we find the mean capital stock of the discretized distribution $f(.)$ to coincide with its exact value. At low levels of the capital stock $\tilde{k}$, however, we find substantial curvature of the savings function $\tilde{k}'(.)$.[69]

Fig. 10.14 presents the nonstochastic steady-state distribution of the individual capital stock $\tilde{k}$. Note that the upper boundary $k^{max} = 10.0$ is not binding.

Now, we are able to sum up the capital stocks, consumption and labor supply of the individual households weighted by their measures $f(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k})$ to determine the aggregate variables $(\tilde{K}, \tilde{L}, \bar{l})$. We derive aggregate transfers $\widetilde{Tr}$ with the help of the fiscal budget (10.60).

The optimal policy functions for the steady state are stored to use them as an initial guess for the policy functions in Step 3. Similarly, we save the nonstochastic steady state distribution and use it as the initial distribution for the simulation of the stochastic economy in Step 4.

[68] Note that the code can easily be parallelized in the individual loop over ages $s = 1, \ldots, 69$.

[69] For this reason, some economists use more grid points at the lower boundary of the interval $[k^{min}, k^{max}]$.

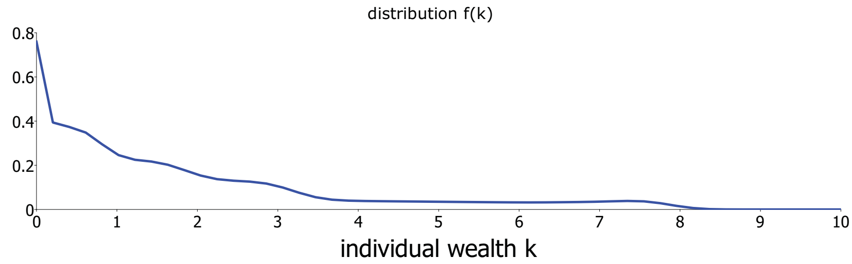


Figure 10.14 Nonstochastic steady-state distribution of $\tilde{k}$

Table 10.7 Runtime: Krusell-Smith algorithm and OLG models

	PYTHON	GAUSS
Nonstochastic steady state	11m:53s	2m:19s
Policy functions	39m:03	14m:18s
Simulation (200 periods)	1h:13m:52s	3m:52s
Total	1d:21h:40m:18s	5h:18m:40s

The computation of the nonstochastic steady state with this Euler equation method is very fast. As presented in Table 10.7, the runtime amounts to approximately 2 minutes (12 minutes) in our GAUSS (PYTHON) program. If you compare these runtimes with those from Table 10.2 where we used value function iteration, the difference is dramatic (hours versus minutes). As is typical, we find that GAUSS is much faster than PYTHON, in this particular application by a factor of 6.

Step 2: Choose an initial parameterized functional form for the aggregates $\tilde{K}'$, $\tilde{L}$ and $\widetilde{Tr}$. In the second step, we need to postulate the laws of motion for the next-period capital stock, employment and transfers:[70]

$$\tilde{K}_{t+1} = g^K(Z_t, \tilde{K}_t), \tag{10.76a}$$
$$\tilde{L}_t = g^L(Z_t, \tilde{K}_t), \tag{10.76b}$$
$$\widetilde{Tr}_t = g^{Tr}(Z_t, \tilde{K}_t). \tag{10.76c}$$

In light of the results obtained by Krussell and Smith (1998), we use a log-log specification for the dynamics of the aggregate capital stock $\tilde{K}_{t+1}$ as a function of stochastic aggregate technology Z_t and present-period capital stock $\tilde{K}_t$. For the contemporaneous employment $\tilde{L}$, we copy this functional relationship. With respect to aggregate transfers $\widetilde{Tr}$, however, we use a function in levels rather than in logs given the observation that aggregate transfers are close to zero. Accordingly, we choose the following specification of $g = (g^K, g^L, g^{Tr})'$:

$$\ln K_{t+1} = \omega_{K,0} + \omega_{K,1} \ln \tilde{K}_t + \omega_{K,2} \mathbf{1}_{Z_t = Z^b} + \omega_{K,3} \mathbf{1}_{Z_t = Z^b} \ln \tilde{K}_t, \tag{10.77}$$
$$\ln \tilde{L}_t = \omega_{L,0} + \omega_{L,1} \ln \tilde{K}_t + \omega_{L,2} \mathbf{1}_{Z_t = Z^b} + \omega_{L,3} \mathbf{1}_{Z_t = Z^b} \ln \tilde{K}_t, \tag{10.78}$$
$$\widetilde{Tr}_t = \omega_{Tr,0} + \omega_{Tr,1} \tilde{K}_t + \omega_{Tr,2} \mathbf{1}_{Z_t = Z^b} + \omega_{Tr,3} \mathbf{1}_{Z_t = Z^b} \tilde{K}_t. \tag{10.79}$$

As initial values, we set $\omega_{x,1}$-$\omega_{x,3}$, $x \in \{\tilde{K}, \tilde{L}, \widetilde{Tr}\}$, equal to zero and choose $\omega_{x,0}$, $x \in \{\tilde{K}, \tilde{L}, \widetilde{Tr}\}$, so that the values of $\tilde{K}_{t+1}$, $\tilde{L}_t$ and $\widetilde{Tr}_t$ are equal to their nonstochastic steady-state counterparts $\tilde{K}$, $\tilde{L}$ and $\widetilde{Tr}$. In the final iteration in Step 6, we find the following solution:

$$\omega_K = \begin{pmatrix} 0.0386 \\ 0.0176 \\ 0.9257 \\ -0.0075 \end{pmatrix}, \quad \omega_L = \begin{pmatrix} -1.2663 \\ 0.0227 \\ 0.0748 \\ -0.0046 \end{pmatrix}, \quad \omega_{Tr} = \begin{pmatrix} 0.00224 \\ 0.00555 \\ -0.00173 \\ 0.00149 \end{pmatrix}.$$

Step 3: Solve the consumer's optimization problem as a function of the individual and aggregate state variables, $(s, e, \theta, \tilde{k}; Z, \tilde{K})$. In Step 3, we compute the individual policy functions as functions of the individual and aggregate state variables for a given law of motion for $\tilde{K}_{t+1}$, $\tilde{L}_t$ and $\widetilde{Tr}_t$. For this reason, we choose a rather loose equispaced grid for the aggregate capital stock $\tilde{K}$ because the curvature of the policy function

[70] Note, in particular, that $\tilde{K}_{t+1}$ is only a function of present-period capital stock $\tilde{K}_t$ and aggregate stochastic technology Z_t. Therefore, employment $\tilde{L}_t$ is not an aggregate state variable. This would be different if we assumed sticky employment, e.g., as resulting from employment search or other frictions in the labor market.

with respect to this argument is rather low. We find that $n_K = 7$ points are sufficient. Furthermore, we choose 80% and 120% of the nonstochastic steady-state aggregate capital stock, $\tilde{K} = 1.820$, as the lower and upper boundary for this interval. In our simulations, the aggregate capital stock always remains within these boundaries, $\tilde{K}_t \in [1.456, 2.184]$.

We compute the policy functions in the same way as described in Step 2 above. However, we have to consider that factor prices are no longer constant but depend on the aggregate states in periods t and $t+1$. Consider the case in which we would like to compute the optimal policy functions at grid point $(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k}; Z^{i_Z}, \tilde{K}_{i_K},)$ with $Z_t = Z^{i_Z}$, $i_Z \in \{b, g\}$, and $\tilde{K}_t = K_{i_K}$. Aggregate employment in period t, $\tilde{L}_t$, can be computed with the help of the function $g^L(Z_t, \tilde{K}_t)$. Given $(Z_t, \tilde{K}_t, \tilde{L}_t)$, we are able to compute factor prices w_t and r_t from the first-order conditions of the firms. In addition, we can compute contemporaneous pensions $\widetilde{pen}_t$ with the help of the replacement ratio.

When we compute the residual function $rf(.)$ with the help of (10.74), we need to evaluate the expected next-period marginal utility of consumption c_{t+1}^{s+1} and the interest rate r_{t+1}. Since both aggregate and individual productivity, Z_{t+1} and θ_{t+1}^{s+1},[71] are stochastic, we need to evaluate c_{t+1}^{s+1} and r_{t+1} in each case and weight the marginal utility of consumption by probability $\pi(Z_{t+1}|Z_t) \times \pi(\theta_{t+1}^{s+1}|\theta_t^s)$. We know that next-period capital stock $\tilde{K}_{t+1}$ is predetermined from our choice in period t and can be computed with the help of $g^K(.)$. Given the realization of Z_{t+1}, we are able to compute $\tilde{L}_{t+1}$ with the help of $g^L(.)$ and, hence, interest rates r_{t+1}. Finally, to compute $\tilde{c}_{t+1}^{s+1}$ and l_{t+1}^{s+1} at age $s+1$ with productivity θ_{t+1}^{s+1} and aggregates Z_{t+1} and $\tilde{K}_{t+1}$, we interpolate $\tilde{c}(.)$ and $l()$ bilinearly at $(\tilde{k}_{t+1}, \tilde{K}_{t+1})$.

The computation of the optimal policies as functions of the individual and aggregate states is quite time-consuming because the total number of grid points amounts to:

$$T \times n_\theta \times n_e \times n_k \times n_K \times n_Z = 70 \times 2 \times 2 \times 50 \times 7 \times 2 = 28,000.$$

As presented in Table 10.7, computational time of Step 4 amounts to 14 minutes (GAUSS) and 39 minutes (PYTHON).

STEP 4: SIMULATE THE DYNAMICS OF THE DISTRIBUTION FUNCTION. Starting with the nonstochastic steady-state distribution as our initial dis-

[71] We distinguish between θ_t^s and θ_{i_θ} in our notation. θ_t^s is the idiosyncratic stochastic productivity of an s-year-old worker in period t, while θ_1 and θ_2 are the elements of the state space Θ that constitute the possible set of realizations.

tribution $f_0(s,e,\theta,\tilde{k})$, we compute the dynamics of $f_t(.)$ over 200 periods $t = 1,\ldots,200$, using Algorithm 7.2.3. In period 0, the aggregates $\tilde{K}_0$, $\tilde{L}_0$, $\widetilde{Tr}_0$ are equal to their nonstochastic steady-state values. To choose Z_0, we use a random number generator of the uniform distribution on the interval $[0,1]$ and assign $Z_0 = Z^b$ if the realization is below 0.5 and $Z_0 = Z^g$ otherwise. We use a pseudo-random number generator to simulate the technology level $\{Z_t\}_{t=0}^{200}$ over 200 periods. Given the distribution in period t, $f_t(s,e,\theta,\tilde{k})$, we can compute the next-period distribution, $f_{t+1}(s,e,\theta,\tilde{k})$, with the help of the policy functions $k'(s,e,\theta,\tilde{k};Z,\tilde{K})$ and $l(s,e,\theta,\tilde{k};Z,\tilde{K})$.

To compute the distribution in period $t+1$, $f_{t+1}(s,e,\theta,\tilde{k})$, given the distribution in period t, $f_t(s,e,\theta,\tilde{k})$, we use the same method as described under Step 1 above. In particular, we initialize the measure of the 1-year-old cohort for each productivity type (e_j,θ_{i_θ}), $j=1,2$, $i_\theta = 1,2$:

$$f_{t+1}(1,e_j,\theta_{i_\theta},\tilde{k}_{i_k}) = \begin{cases} \frac{\mu^1}{4} & \text{if } i_k = 1, \\ 0 & \text{else.} \end{cases}$$

Next, we iterate over all ages $s = 1,\ldots,69$ in period t and compute the next-period wealth $k'(s,e_j,\theta_{i_\theta},\tilde{k}_{i_k})$. If next-period wealth $\tilde{k}'(s,e_j,\theta_{i_\theta},\tilde{k}_{i_k}) = k_{i_{kk}}$ happens to be a grid point, we simply add the respective measure at age s to the measure at age $s+1$, noting that only ϕ^s of the households survive and that the size of the cohort in the total population declines by a factor of $(1+n)$ due to population growth:

$$f_{t+1}(s+1,e_j,\theta',\tilde{k}_{i_{kk}}) = \pi(\theta'|\theta_{i_\theta}) \times f_t(s,e_j,\theta_{i_\theta},\tilde{k}_{i_k}) \times \frac{\phi^s}{1+n},$$

for $\theta' \in \{\theta_1,\theta_2\}$.

If next-period wealth $\tilde{k}'(s,e_j,\theta_{i_\theta},\tilde{k}_{i_k})$ lies between the asset grid points $k_{i_{kk}}$ and $k_{i_{kk}+1}$, we add the measure in period s to the measures of these two points at age $s+1$ according to:

$$f_{t+1}(s+1,e_j,\theta',\tilde{k}_{i_{kk}}) = \frac{\tilde{k}_{i_{kk}+1} - \tilde{k}'}{\tilde{k}_{i_{kk}+1} - \tilde{k}_{i_{kk}}}\pi(\theta'|\theta_{i_\theta}) \times f_t(s,e_j,\theta_{i_\theta},\tilde{k}_{i_k}) \times \frac{\phi^s}{1+n},$$

$$f_{t+1}(s+1,e_j,\theta',\tilde{k}_{i_{kk}+1}) = \frac{\tilde{k}' - k_{i_{kk}}}{k_{i_{kk}+1} - k_{i_{kk}}}\pi(\theta'|\theta_{i_\theta}) \times f_t(s,e_j,\theta_{i_\theta},\tilde{k}_{i_k}) \times \frac{\phi^s}{1+n},$$

for $\theta' \in \{\theta_1,\theta_2\}$.

We also use the distribution in period t, $f_t(s,e,\theta,\tilde{k})$, to compute distributional measures for our results, i.e., the Gini coefficients of income and the income shares of the bottom four quintiles, the 80%–95% percentile and the top 5% as well as GDP $\tilde{Y}_t$. We save the time series for three aggregates $\tilde{K}_t$, $\tilde{L}_t$ and $\widehat{Tr}_t$ for which we specify the law of motion $g = (g^K, g^L, g^{Tr})'$. The series $\{(\tilde{K}_t, \tilde{L}_t, \widehat{Tr}_t)\}_{t=0}^{200}$ will be used in the next step. After these computations, we no longer need to save $f_t(.)$ and can delete it to save memory space.

The computational time for the simulation of 200 periods again depends crucially on the computer language that you apply. As presented in Table 10.7, it only took us 4 minutes to simulate 200 periods with GAUSS, while the PYTHON code took more than 1 hour for the exact same number of operations.[72]

Let us emphasize one point here. Many studies on OLG models with aggregate uncertainty consider a sample of approximately 1,000 households for each generation and simulate their behavior. We find that this method has several disadvantages. First, it is very time-consuming. We instead advocate storing the actual distribution at the grid points $(s,e,\theta,\tilde{k})$ as in Algorithm 7.2.3. This procedure requires less storage capacity. Importantly, the computation of the next-period distribution is much faster than the simulation of some 1,000 households in each generation. Second, if we simulate the behavior of the household sample for each generation, we will have to use a random number generator to switch the agent's type from θ to θ'. As we are only using some 1,000 agents, the law of large numbers does not need to hold, and the percentage of the agents with $\theta' = \theta_1$ or $\theta' = \theta_2$ is not equal to 50%. Therefore, during our simulation, we always have to adjust the number of agents with productivity $\theta' = \theta_1$ ($\theta' = \theta_2$) to one half in each generation, which involves some arbitrariness because we have to select some households whose productivity is changed ad hoc.[73]

STEPS 5-7: ESTIMATE THE LAW OF MOTION FOR $\tilde{K}'$, $\tilde{L}$ AND $\widetilde{Tr}$ AND GOODNESS OF FIT. We simply apply ordinary least squares (OLS) to estimate the ω coefficients in (10.77)–(10.79). We update the coefficients by 30% in each outer loop and stop the algorithm as soon as the maximum absolute change in $\omega_{.,K}$ is below 0.001. In our last iteration, the R^2 in the

[72] We encourage a reader who uses PYTHON to apply methods of parallelization, multithreading and/or Numba to accelerate program execution.

[73] Please compare this with Section 8.3.

three regressions of $\tilde{K}'$, $\tilde{L}$ and $\widetilde{Tr}$ exceeds 0.9999. Therefore, we can be confident that our postulated law of motion $g(.)$ is satisfactory. The total computation of `OLG_krusell_smith.gss` (`OLG_krusell_smith.py`) takes some 5 hours (2 days) on an Intel Pentium(R) M, 319 MHz machine in the case of GAUSS (PYTHON).

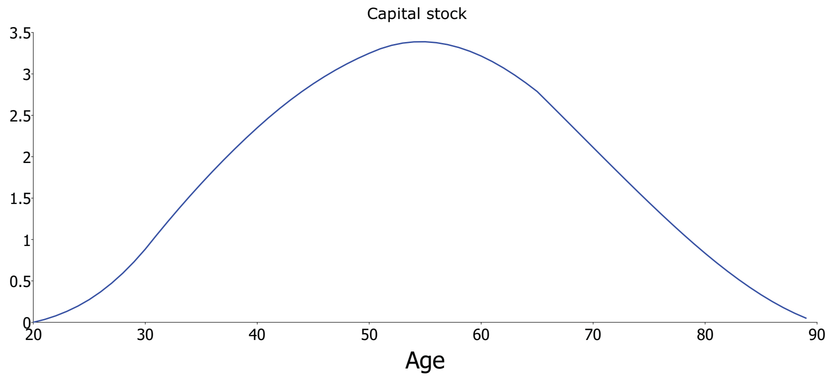
10.2.2.3 Business Cycle Dynamics of the Income Distribution

Figure 10.15 describes the behavior of our economy in the nonstochastic steady state. We graph the average wealth of the cohorts in the top row, average labor supply of the workers in the medium row and average gross income in the bottom row. Agents accumulate savings until the age 36 (corresponding to real lifetime age 56) and dissave thereafter. Labor supply peaks prior to wealth (and age-specific productivity) at real lifetime age 30. With increasing wealth, households decrease their labor supply ceteris paribus. Total income which is defined as the sum of wage and interest income before taxes plus pensions and transfers peaks at real lifetime age 40.

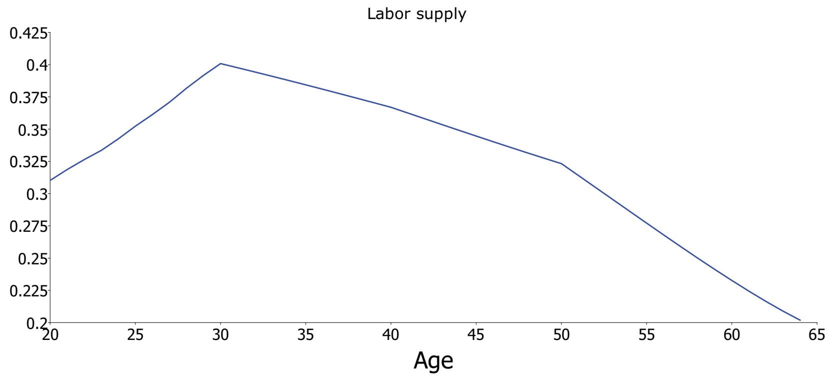
Our average-age profiles do not completely accord with empirical observations in Budría Rodríguez et al. (2002). Based on the 1998 data from the Survey of Consumer Finances they find that US household income, earnings, and wealth peak around ages 51-55, 51-55, and 61-65, respectively. As one possible explanation why households accumulate savings over a shorter time period and supply less labor in old age in our model than empirically, we conjecture that some important aspects are missing, e.g. the risk of large old-age medical expenses which might motivate households to save more and longer. Another possible reason for the shorter saving period in our model is the absence of an operative bequest motive.

In order to compute the correlation of the income distribution with output, we simulate the dynamics of our economy over 150 periods in the outer loop over the law of motion for $\tilde{K}$ (Step 6).[74] The time series of production and the income shares of the bottom and fourth quintiles over the periods 51-100 that resulted from the final simulation are illustrated

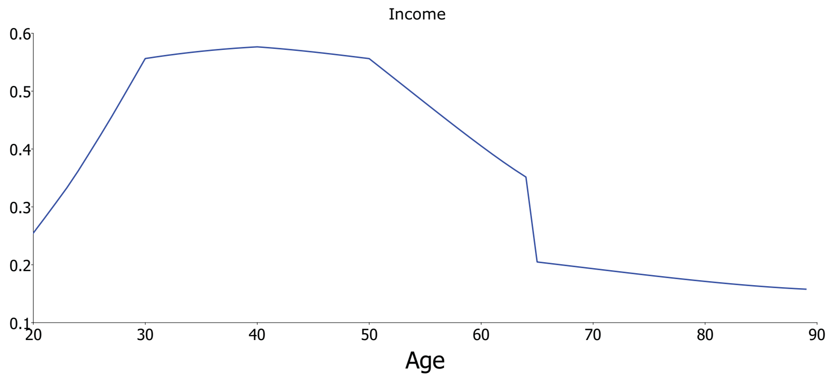
[74] We drop the first 50 periods from our time series so that the initialization of the distribution in period 0 has no effect on our results.



(a) Average cohort wealth $\tilde{k}$



(b) Average cohort labor supply l



(c) Average cohort gross income

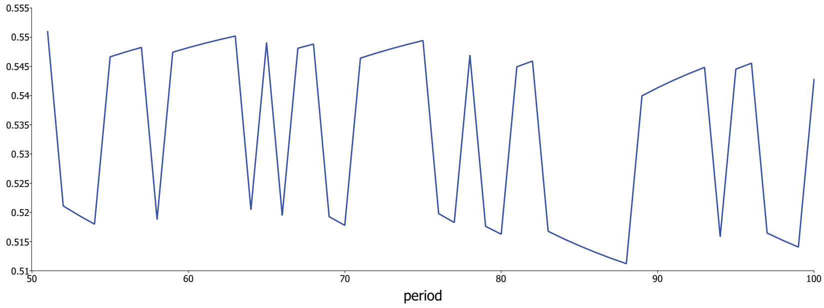
Figure 10.15 Nonstochastic steady-state age profiles in Figure 10.16.[75] In the top row, we graph the dynamics of output.[76] If the technology level jumps from Z^b to Z^g or vice versa, this is also instantaneously reflected in the movement of the production level. Of course, these abrupt changes in production are a direct consequence of our assumption that technology Z_t can only take two different values. Notice that after an initial upward (downward) jump, production continues to increase (fall) as capital accumulates (decumulates) gradually.

In the medium row, we graph the behavior of the bottom quintile of the income distribution. Evidently, this income share mimics the behavior of output to some extent and with a lag. The lowest income quintile is composed of the young workers with the lowest idiosyncratic productivity and the low-productivity retirees with little wealth. For example, productivity of the 1-year old worker with $\theta = \theta_1$ and $e = e_1$ amounts to only 25% of average productivity among workers, $e_1 \cdot \theta_1 \cdot \bar{y}^1 = 0.727 \cdot 0.57 \cdot 0.596 = 0.247$. Since pensions are proportional to permanent productivity e, the retirees with $e = e_1$ only receive 57% of average pensions. If during a boom, stochastic aggregate technology Z_t jumps from 0.98 to 1.02, wages, interest rates and pensions increase by 4% on impact for this reason. Since the capital stock is predetermined, but employment increases as well, interest rates react a little stronger than wages and pensions. This effect increases capital income versus labor income, favoring the higher income shares. Since workers will also increase their labor supply (the substitution effect is stronger than the income effect) some of the households in the lowest quintile increase their income by more than the increase in wages, while the retirees cannot increase their pension income any more. The total effect of the correlation of the bottom income share with GDP is strongly positive and amounts to 0.85 as presented in Table 10.8.

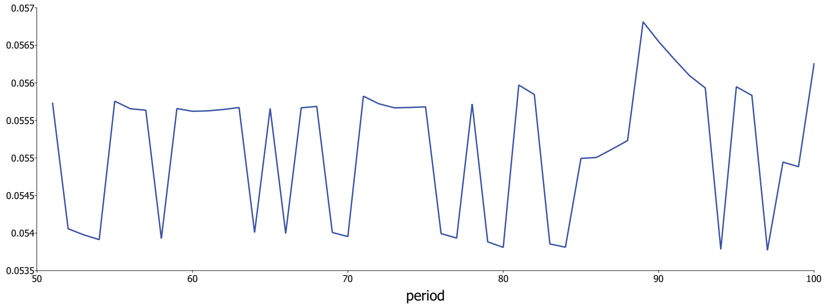
In the case of the fourth income share (60%-80%) illustrated in the bottom row of Fig. 10.16, the comovement with GDP is less evident. Let us consider the period 82-88 where, apparently, the stochastic aggregate technology $Z_t = Z^b$ prevailed and the economy was subject to a lasting recession. During this period, GDP is below average and falling. The fourth income share initially keeps on rising during this period resulting in a negative correlation of this income share (60%-80%) with GDP before it starts to fall after period 86. As presented in Table 10.8, the correlation

[75] If you run the program `OLG_krusell_smith`, of course, the outcome might look differently since it depends on the realizations of Z_t as the outcome of your random number generator.

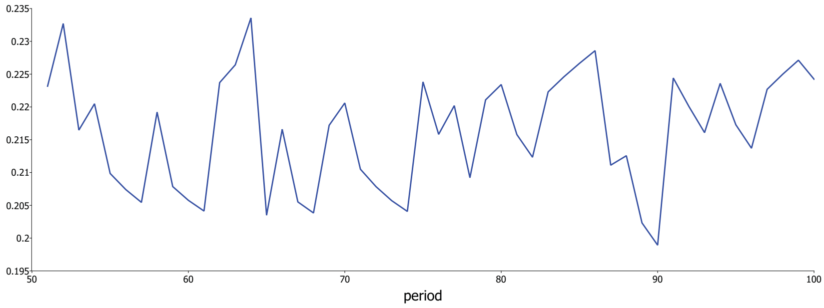
[76] Logarithmic output has been detrended using the Hodrick-Prescott filter with smoothing parameter $\lambda = 100$.



(a) Production $\tilde{Y}_t$



(b) Income share bottom quintile



(c) Income share fourth quintile

Figure 10.16 Simulation results

Table 10.8 Cyclical behavior of the income distribution

	0-20%	20-40%	40-60%	60-80%	80-95%	95-100%
US	0.53	0.49	0.31	-0.29	-0.64	0.00
Castañeda et al. (1998)	0.95	0.92	0.73	-0.56	-0.90	-0.84
our model						
i) $pen_t(\epsilon) = \epsilon \cdot \bar{p}en_t$	0.85	0.54	0.60	-0.30	0.23	-0.44
ii) $pen_t(\epsilon) = \bar{p}en_t$	0.75	-0.48	0.41	0.08	0.06	-0.29

Notes: Entries in rows 1 and 2 are reproduced from Table 4 in Castañeda et al. (1998). Annual logarithmic output has been detrended using the Hodrick-Prescott filter with smoothing parameter $\lambda = 100$.

amounts to -0.30. As one explanation for the initial rise of the income share, this quintile contains many retirees with high productivity $e = e_2$. While the workers decrease their labor supply during the recession, the rich retirees cannot adjust their behavior along this margin. Accordingly, their relative income increases. As the recession continues, however, the composition of the households in the quintile changes.[77]

Table 10.8 shows in detail the behavior of all income quintiles. In the first entry row, we display the empirical correlations of output with the 1st, 2nd, 3rd, and 4th income quintiles, and the 80-95% and 95-100% income groups for the US economy, respectively.[78] In the second row, you find the values as resulting from the simulation of the most preferred model of Castañeda et al. (1998). The last two lines display the values obtained from simulating our economy for the two cases that pensions are either proportional to the individual efficiency e or lump-sum. Obviously, both our specifications (i) and (ii) have their strengths and weaknesses. The model (i) with pensions proportional to productivity e apparently works better in the explanation of the behavior of the second and fourth income quintiles, while the lump-sum case (ii) implies correlations of the third and fifth quintiles in better accordance with empirical observations.

[77] For example, the workers with the highest productivity prior to retirement are among the top income earners and may move to the fourth quintile during retirement. Since their savings are relatively smaller due to the long recession, their interest income is also much less than that of households who enter retirement after a boom period.

[78] The estimates are reproduced from Table 4 in Castañeda et al. (1998).

Comparing the results for our overlapping generations model with those from Castañeda et al. (1998), we conclude that a combination of these two models seems to be a promising starting point to get a better explanation of the cyclical income distribution. Both unemployment and the demographic structure are important explanatory factor of the income distribution which need to be taken in consideration. The focus of the average cyclical patterns of the income shares, however, might hide the importance of specific historic episodes. Income is composed of both labor and capital income. As we noticed during the two most recent recessions, the Great Recession during 2007-2009 and the COVID-19 crisis during 2019-2021, capital income may behave very differently during a recession depending on the cause of the recession. For example, the stock market index SP 500 fell by 38% over the year 2008 during the housing crisis, but increased by 28% over the year 2019. In the former case, the crisis started in the financial markets (the subprime crisis), while the latter was caused by an epidemic affecting both (global) supply and demand. Therefore, capital income behaved diametrically different between these two recessions and an analysis that only considers averages might be misleading.

A.10 Derivation of Aggregate Bequests in Example 10.2.1

To derive accidental bequests, sum up individual budget constraints (10.52) and (10.53) weighted by the measure of s-year old households to get

$$(1+\tau^c)\underbrace{\sum_{s=1}^{T}\mu_t^s N_t c_t^s}_{=C_t} = (1-\tau^l-\tau_t^p)A_t w_t \underbrace{\sum_{s=1}^{T^W}\mu_t^s N_t \bar{y}^s l_t^s}_{=L_t} + \underbrace{\sum_{s=1}^{T}\mu_t^s N_t tr_t}_{=Tr_t} + \underbrace{\sum_{s=T^W+1}^{T}\mu_t^s N_t pen_t}_{=Pen_t}$$
$$+\big(1+(1-\tau^k)r_t\big)\underbrace{\sum_{s=1}^{T}\mu_t^s N_t k_t^s}_{=K_t} - \sum_{s=1}^{T-1}\mu_t^s N_t k_{t+1}^{s+1}.$$

With the help of the social security authority's budget constraint, (10.61), we can eliminate pensions from this equation. Division by $(A_t N_t)$ results in

$$(1+\tau^c)\tilde{C}_t = (1-\tau^l)w_t\tilde{L}_t + \widetilde{Tr}_t + \big(1+(1-\tau^k)r_t\big)\tilde{K}_t - (1+g_A)\sum_{s=1}^{T-1}\mu_t^s\tilde{k}_{t+1}^{s+1}. \quad \text{(A.10.1)}$$

Since we assume constant returns to scale and perfect competition in factor and product markets, the Euler theorem holds:

$$\tilde{Y}_t - \delta K_t = w_t\tilde{L}_t + r_t\tilde{K}_t,$$

such that

$$\tilde{C}_t = \widetilde{Tr}_t + \tilde{Y}_t + (1-\delta)\tilde{K}_t - \tau^L w_t\tilde{L}_t - \tau^k r_t\tilde{K}_t - \tau^c\tilde{C}_t - (1+g_A)\sum_{s=1}^{T-1}\mu_t^s\tilde{k}_{t+1}^{s+1}.$$

Inserting the goods market equilibrium

$$\tilde{Y}_t = \tilde{C}_t + \tilde{G}_t + (1+n)(1+g_A)\tilde{K}_{t+1} - (1-\delta)\tilde{K}_t,$$

in the equation above, we derive

$$\tau^L w_t\tilde{L}_t + \tau^k r_t\tilde{K}_t + \tau^c\tilde{C}_t = \widetilde{Tr}_t + \tilde{G}_t + (1+n)(1+g_A)\tilde{K}_{t+1} - (1+g_A)\sum_{s=1}^{T-1}\mu^s\tilde{k}_{t+1}^{s+1}.$$

Inserting the fiscal budget constraint (10.60), we can solve for accidental bequests

$$
\begin{aligned}
\widetilde{Beq}_t &= (1+g_A)\sum_{s=1}^{T-1}\mu_t^s\tilde{k}_{t+1}^{s+1} - (1+n)(1+g_A)\tilde{K}_{t+1},\\
&= (1+g_A)\sum_{s=1}^{T-1}\mu_t^s\tilde{k}_{t+1}^{s+1} - (1+n)(1+g_A)\sum_{s=1}^{T}\mu_{t+1}^s\tilde{k}_{t+1}^s,\\
&= (1+g_A)\sum_{s=1}^{T-1}\mu_t^s\tilde{k}_{t+1}^{s+1} - (1+n)(1+g_A)\sum_{s=1}^{T-1}\mu_{t+1}^{s+1}\tilde{k}_{t+1}^{s+1},\\
&= (1+g_A)\sum_{s=1}^{T-1}\mu_t^s\tilde{k}_{t+1}^{s+1} - (1+g_A)\sum_{s=1}^{T-1}\mu_t^s\phi^s\tilde{k}_{t+1}^{s+1},\\
&= (1+g_A)\sum_{s=1}^{T-1}\mu_t^s(1-\phi^s)\tilde{k}_{t+1}^{s+1}
\end{aligned}
$$

where we used $k_{t+1}^1 = 0$ to derive the third from the second equation and

$$
N_{t+1}\mu_{t+1}^{s+1} = \phi^s N_t\mu_t^s, \;\; s = 1,\ldots,T-1,
$$

or, equivalently,

$$
\mu_{t+1}^{s+1} = \frac{\mu_t^s\phi^s}{1+n}.
$$

to derive the fourth from the third equation.

The last equation is the stationary pendant of (10.58).

A.11 Parameters of the AR(1) Process with Annual Periods

In this Appendix, we derive the parameters (ρ, σ) that we were using for the AR(1) process with annual periods in Section 10.2.1. In particular, we choose (ρ, σ) so that they correspond to the parameters of the AR(1) with quarterly periods, $(\rho^q, \sigma^q) = (0.95, 0.00763)$.

Let z_t^q denote the logarithm of the technology level in the model with quarterly periods that follows the AR(1) process:

$$z_{t+1}^q = \rho^q z_t^q + \epsilon_{t+1}^q,$$

where $\epsilon_t^q \sim N\left(0, (\sigma^q)^2\right)$. Similarly,

$$\begin{aligned}
z_{t+2}^q &= \rho^q z_{t+1}^q + \epsilon_{t+2}^q,\\
z_{t+3}^q &= \rho^q z_{t+2}^q + \epsilon_{t+3}^q,\\
z_{t+4}^q &= \rho^q z_{t+3}^q + \epsilon_{t+4}^q.
\end{aligned}$$

Let z_T^a denote the logarithm of the technology level in the corresponding model with annual periods that follows the AR(1) process:

$$z_{T+1}^a = \rho z_T^a + \epsilon_{T+1},$$

where $\epsilon_T \sim N\left(0, \sigma^2\right)$.

If we identify the technology level z^q at the beginning of the quarters t, $t+4$, $t+8$ with the annual technology level z^a at the beginning of the periods T, $T+1$, $T+2$, we find:

$$\begin{aligned}
z_{T+1}^a = z_{t+4}^q &= \rho^q z_{t+3}^q + \epsilon_{t+4}^q,\\
&= \rho^q \left(\rho^q z_{t+2}^q + \epsilon_{t+3}^q\right) + \epsilon_{t+4}^q,\\
&= (\rho^q)^2 z_{t+2}^q + \rho^q \epsilon_{t+3}^q + \epsilon_{t+4}^q,\\
&= (\rho^q)^3 z_{t+1}^q + (\rho^q)^2 \epsilon_{t+2}^q + \rho^q \epsilon_{t+3}^q + \epsilon_{t+4}^q,\\
&= (\rho^q)^4 z_t^q + (\rho^q)^3 \epsilon_{t+1}^q + (\rho^q)^2 \epsilon_{t+2}^q + \rho^q \epsilon_{t+3}^q + \epsilon_{t+4}^q,\\
&= (\rho^q)^4 z_T^a + (\rho^q)^3 \epsilon_{t+1}^q + (\rho^q)^2 \epsilon_{t+2}^q + \rho^q \epsilon_{t+3}^q + \epsilon_{t+4}^q.
\end{aligned}$$

Accordingly, we can make the following identifications:

$$\begin{aligned}
\rho &= (\rho^q)^4\\
\epsilon_T &= (\rho^q)^3 \epsilon_{t+1}^q + (\rho^q)^2 \epsilon_{t+2}^q + \rho^q \epsilon_{t+3}^q + \epsilon_{t+4}^q.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{var}(\epsilon) = \sigma^2&\\
&= \operatorname{var}\left(\epsilon^q (1 + \rho^q + (\rho^q)^2 + (\rho^q)^3)\right)\\
&= \left(1 + (\rho^q)^2 + (\rho^q)^4 + (\rho^q)^6\right)(\sigma^q)^2.
\end{aligned}$$

For $(\rho^q, \sigma^q) = (0.95, 0.00763)$, we get $\rho = 0.814$ and $\sigma = 0.0142$.

Problems

10.1 **Concentration of Wealth and the Natural Real Interest Rate**

Recompute the model described in Section 10.1 for the following changes:

a) **Borrowing constraint:**

Recompute the model for a less stricter borrowing constraint where the agent can borrow up to the average net wage in the economy, $(1-\tau^l-\tau^p)w\bar{l}$. How does this affect the Gini coefficient of wealth?

b) **Pay-as-you-go pensions:**

Compute the effect of higher public pensions on the wealth heterogeneity. For this reason, increase the replacement rate of pensions with respect to average gross wages to 50%.

c) **Demographics:**

What is the effect of aging on the inequality of earnings, income and wealth? Recompute the model for the population parameters for the year 2100 as estimated by UN (2015).

d) **Perfect annuities markets:**

Assume that there exist perfect annuities markets in the economy. Financial intermediaries collect accidental bequests without transactions costs and redistribute them to those agents of the same age as the deceased. Consequently, the budget constraints of the households adjust as follows:

$$(1+\tau_t^c)c_t^s = y_t^s + \frac{1+(1-\tau_t^k)(r_t-\delta)}{\phi_{t,s}}k_t^s + \frac{R_t^b}{\phi_{t,s}}b_t^s + tr_t - k_t^{s+1} - b_{t+1}^{s+1}. \quad (10.1)$$

The government no longer collects accidental bequests, $Beq_t = 0$. Recompute the model of Section 10.1 and show that wealth inequality increases.

e) **Bequests:**

Introduce bequests into the model following Eggertsson et al. (2019). Accordingly, lifetime utility (10.2) is replaced by

$$\max \sum_{s=1}^{J} \beta^{s-1}\left(\Pi_{j=1}^{s}\phi_{t+j-1}^{j-1}\right)\mathbb{E}_t\left[u(c_{t+s-1}^s, l_{t+s-1}^s) + v(g_{t+s-1})\right] + \beta^T\left(\Pi_{j=1}^{T}\phi_{t+j-1}^{j-1}\right)\mathbb{E}_t\,\omega(beq_{t+T}^T),$$

where beq_{t+T}^T denotes the amounts of bequests left per descendent at the end of age T. Assume that only bequests left after the maximum period of life T is considered in utility and that all bequests are left to the cohorts at age $T-24$.

As only some of the households survive until the age T, not all members of the $(T-24)$-year-old would receive a bequest. For simplicity assume that the cohort of the $(T-24)$-year-old participates in a bequest insurance market and all household in this cohort receive the same bequest. In addition, assume that the utility from bequests is given by

$$\omega(beq) = \omega_0 \frac{beq^{1-\eta}}{1-\eta}.$$

Calibrate the parameter ω_0 such that total bequests are equal to 3.0% of output. In addition, assume that there exists perfect annuities market as above.

Recompute the model and evaluate the effects of perfect annuities markets and the bequest motive on 1) wealth inequality and 2) the real interest rate r^b. Show that these two model elements help to establish a much lower real interest rate r^b; a negative natural real interest rate serves as the central ingredient in the modeling of secular stagnation in Eggertsson et al. (2019).

f) **Utility function**

Consider the following function of instantaneous utility from consumption and leisure, (10.3), taken from Trabandt and Uhlig (2011):

$$u(c,l) = \begin{cases} \ln c - \kappa l^{1+1/\varphi} & \text{if } \eta = 1, \\ \frac{1}{1-\eta}\left(c^{1-\eta}\left[1-\kappa(1-\eta)l^{1+1/\varphi}\right]^{\eta} - 1\right) & \text{if } \eta > 0 \text{ and } \eta \neq 1. \end{cases}$$

Choose calibration parameters $\kappa = 3.63$ and $\varphi = 1.0$ as in Section 9.1.3. What are the effects of the utility function on wealth inequality (as measured by the Gini coefficient) and the real interest rate on bonds r^b? In light of these results, would you suggest that researchers should study the sensitivity of their dynamic general equilibrium results with respect to specification of the instantaneous utility function?

10.2 Recompute the model in Section 10.1.3. Different from the program *AK70_prog_pensions.gss* optimize the right-hand side of the Bellman equation by choosing optimal next-period wealth $\tilde{a}'$ and labor supply l between grid points. Use golden section search for the optimization with respect to wealth $\tilde{a}'$ and compute the optimal labor supply with the help of the first-order condition (10.49) using the Newton-Rhapson algorithm.

10.3 **Business Cycle Dynamics of Distribution Measures**
Recompute the dynamics of Example 10.2.1 adding the following modifications:

a) In each cohort, there is an equal share of unskilled and skilled workers with permanent productivity $e \in \{0.53, 1.47\}$ so that the individual hourly gross wage of the s-year old with permanent productivity amounts to $w_t e \bar{y}^s A_t$.

b) Assume that pensions are adjusted to the wage deviations from its steady state value with a lag of one period according to

$$\frac{\widetilde{pen}_t - \widetilde{pen}}{\widetilde{pen}} = \frac{w_{t-1} - w}{w}.$$

c) Assume that the skilled and unskilled supply labor at the amounts of $l_t^{low,s}$ and $l_t^{high,s}$ at age s in period t, respectively. Use the nested CES production function suggested by Krussell et al. (2000):

$$Y_t = \psi^t \left[\mu \left(L_t^{low} \right)^{\sigma} + (1-\mu) \left(\alpha K_t^{\rho} + (1-\alpha) \left(L_t^{high} \right)^{\rho} \right)^{\frac{\sigma}{\rho}} \right]^{1/\sigma}, \quad (10.2)$$

where σ and ρ govern the substitution elasticities between unskilled labor L_t^{low}, capital (equipment) K_t and skilled labor L_t^{high}. If $\sigma > \rho$, capital is more complementary with skilled labor than with unskilled labor. Apply the parameterization of Krussell et al. (2000) with $1/(1-\rho) = 0.67$ and$1/(1-\sigma) = 1.67$. Calibrate μ and α so that the wage share in GDP is equal to 64% and the skill premium of the high-skilled amounts to 150%.

Re-compute the impulse response functions and second moments of the Gini coefficients for wealth, income and earnings in each case.

10.4 **Business Cycle Dynamics of the Income Distribution**
Consider the model described in Section 10.2.2.1. Recompute the model for quarterly frequencies. Be careful to recalibrate β and δ. What are the effects on business cycle statistics for the income shares?

10.5 **Redistributive Effects of Cyclical Government Spending**
Introduce stochastic government spending G_t into the model in Section 10.2.2.1. Assume that government spending follows the AR(1)-process

$$\ln \tilde{G}_t = \rho \ln \tilde{G}_{t-1} + (1-\rho) \ln \tilde{G} + \varepsilon_t,$$

with $\varepsilon \sim N(0, \sigma^2)$, $\rho = 0.7$, and $\sigma = 0.007$. Assume further that government expenditures are financed with a proportional tax on factor income and that the government budget balances in each period.

a) Reformulate the model.
b) Compute the nonstochastic steady state assuming that government expenditures amount to 18% of total production. What are the values for the nonstochastic steady state transfers?
c) Discretize the AR(1)-process for government consumption choosing three values. Let the middle point correspond to the one in the nonstochastic steady state. Use the Markov-chain Approximation algorithm from Section 14.2.

d) Compute the business cycle dynamics for the model. The state space consists of $\{s, e, \theta, \tilde{k}; Z, \tilde{K}, \tilde{G}\}$.

e) How do cyclical government spending affect the income distribution? Simulate a time series where the government expenditure are increased above the steady state level for one period and fall back to the steady state level thereafter. Plot the impulse response functions of the gross income Gini index.

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