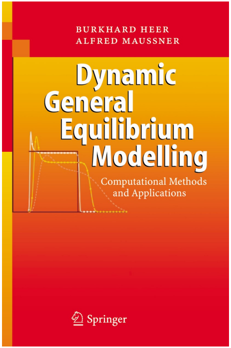
Dynamic General Equilibrium Modeling (3rd ed.)
Chapter 10: The OLG Model with Income Uncertainty

PYTHON — JULIA — GAUSS



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University of Augsburg, Germany
August 19, 2021

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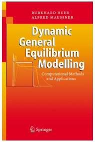
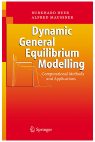

BURKHARD HEER
ALFRED MAUSSNER
Dynamic General Equilibrium Modelling
Computational Methods and Applications
Springer

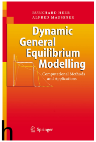
Motivation



Research questions:

1. What is the prototype OLG model for the study of redistributive economic policies?
2. How do we compute it?
3. Trade-off speed and accuracy: Which algorithm/computer language should we use?
   PYTHON — JULIA — GAUSS

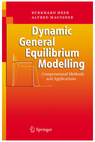
Motivation



Overlapping generations model with endogenous income/wealth distribution:

1. life-cycle savings
2. uncertain lifetime
3. uncertain earnings
4. pay-as-you-go pensions
5. endogenous labor supply

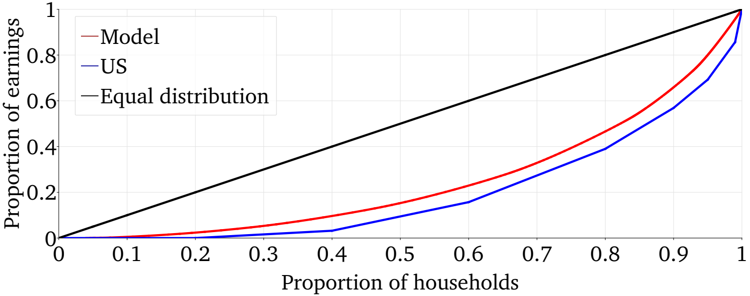
Motivation



	Gini Coefficient	
	US	OLG Model
wages	0.375	0.375
earnings	0.43-0.66	0.505
wealth	0.80	0.66

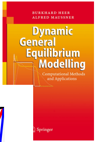
Note: US Data is taken from Budría Rodríguez et al. (2002) and Krueger et al. (2016).

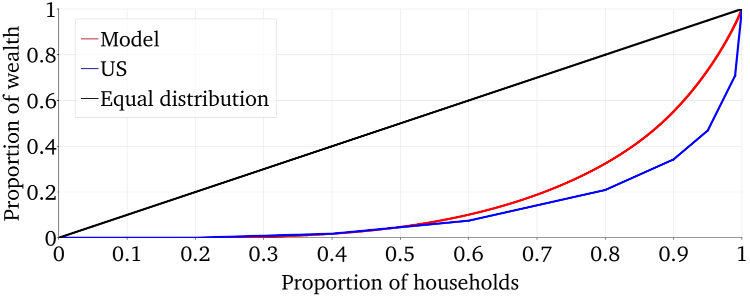
Lorenz Curve: Earnings



Note: US Data is taken from Budría Rodríguez et al. (2002).

Lorenz Curve: Wealth





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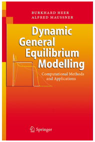
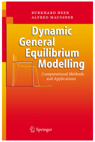
Computation: PYTHON, JULIA or GAUSS?



Computational Methods

Interpolation	linear	cubic	cubic
Grid points			
n_a	500	500	300
n_{ag}	1,000	1,000	1,000
Accuracy			
Young	0.00085	0.00018	0.00032
Old	0.00231	0.00052	0.00140
Run time			
Julia	1h:29m:56s	1h:32m:43s	45m:37s
Gauss	27m:38s	1h:16m:34s	51m:56s
Python	32h:49m:37s	55h:30m:33s	48h:17m:04s

DOWNLOADS: PYTHON, JULIA or GAUSS



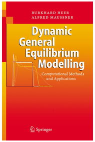
- PYTHON, JULIA and GAUSS CODES (with slides):

  https://www.uni-augsburg.de/de/fakultaet/wiwi/prof/vwl/maussner/dgebook/.

- PYTHON CODE explained in detail:

  PYTHON program tutorial

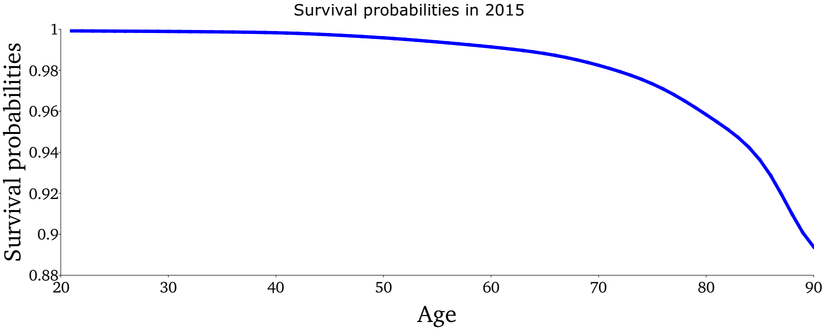
OLG Model: Demographics



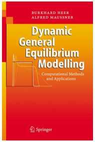
1. Demographics

- Every year, a new generation of equal measure is born at real-life age 21 corresponding to age $s = 1$ in the model.
- Households live a maximum of $T = 70$ years (corresponding to real-life age 90).
- Survival from age s to age $s + 1$ is stochastic with probability ϕ_t^s.
- During their first $T^W = 45$ years as workers, agents supply labor l_t^s at age s in period t enjoying leisure $1 - l_t^s$.
- After T^W years, retirement is mandatory ($l_t^s = 0$ for $s > T^W$).
- The maximum number of retirement periods amounts to T^R.

OLG Model: Demographics



OLG Model: Demographics



- $N_t(s)$ — number of households of age s at t.
- N_t — total population
- Population growth rate $g_{n,t}$:

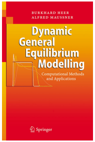
$$N_{t+1} = (1 + g_{n,t})N_t$$

- Newborn cohort growth rate n_t:

$$N_{t+1}(1) = (1 + n_t)N_t(1)$$

- **In the stationary equilibrium**:
  - ϕ^s constant
  - $n = g_n = 0.754\%$

OLG Model: Households

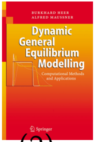


2. Households

- Households maximize expected intertemporal utility:

$$\max \sum_{s=1}^{T} \beta^{s-1} \left(\Pi_{j=1}^{s} \phi_{t+j-1}^{j-1} \right) \mathbb{E}_t \left[u(c_{t+s-1}^{s}, 1 - l_{t+s-1}^{s}) + v(g_{t+s-1}) \right], \tag{1}$$

OLG Model: Households



- with instantaneous utility $u(c, 1-l)$:

$$u(c, 1-l) = \frac{\left(c^{\gamma}(1-l)^{1-\gamma}\right)^{1-\eta}}{1-\eta}, \tag{2}$$

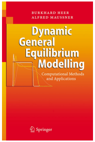
β — discount factor
$1/\eta$ — intertemporal elasticity of substitution (IES)
γ — share of consumption in utility
$v(g_t)$ — additive utility from government consumption

- Net labor income:

$$y_t^s = (1 - \tau_t^l - \tau_t^p)\epsilon(s, \theta, e)A_t w_t l_t^s$$

A_t — aggregate productivity with growth rate g_A

OLG Model: Households



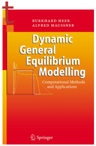
- idiosyncratic productivity:

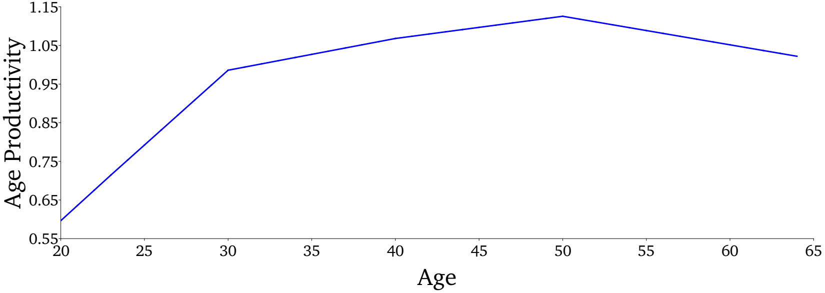
$$\epsilon(s, \theta, e) = \theta e \bar{y}^s$$

$\bar{y}^s$ — age component of wage
$e \in \{0.57, 1.43\}$ — permanent productivity type (high school/college)
θ — stochastic component: AR(1) process

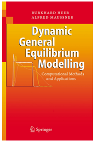
$$\ln \theta' = \rho \ln \theta + \xi, \quad \xi \sim N(0, \sigma_\xi) \tag{3}$$

OLG Model: Age-specific productivity component $\bar{y}^s$





OLG Model: Households

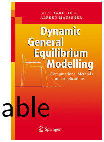


- Retirees receive lump-sum pension $y_t^s = pen_t$
- Budget constraint of the household at age s in period t:

$$k_{t+1}^{s+1} + b_{t+1}^{s+1} + (1+\tau_t^c)c_t^s = y_t^s + tr_t + \left[1 + (1-\tau_t^k)(r_t - \delta)\right] k_t^s + (1+r^b)b_t^s \tag{4}$$

$a_t^s = k_t^s + b_t^s$ — assets (wealth) of the s-year old in period t
b_t^s, k_t^s — capital, government bonds
r_t — rate of return on capital
r_t^b — real interest rate on government bonds
δ — depreciation rate
τ_t^k — capital income tax rate
τ_t^l — labor income tax rate
τ_t^p — pension contribution rate
τ_t^c — consumption tax rate
tr_t — government transfers

OLG Model: Households



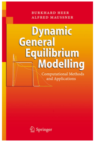
- Value function of the household with individual state variable $z_s = (\theta, e, a)$ at age s:

$$V_t(z_s) = \max_{c,l,a'} \left\{ u(c, 1-l) + v(g) + \beta \phi_t^s \sum_{\theta'} prob(\theta'|\theta)\; V_{t+1}(z_{s+1}) \right\} \tag{5}$$

subject to the budget constraint (4) and the credit constraint

$$a \geq 0.$$

OLG Model: Firms



3. Firms

- Perfect competition in goods and factor markets
- Cobb-Douglas production function:

$$Y_t = F(K_t, L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha}. \tag{6}$$

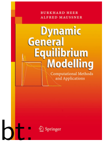
- Profits:

$$\Pi_t = Y_t - w_t A_t L_t - r_t K_t.$$

- Factors are rewarded with their marginal product:

$$\begin{aligned} w_t &= (1-\alpha)K_t^{\alpha}(A_t L_t)^{-\alpha}, && (7)\\ r_t &= \alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha}. && (8) \end{aligned}$$

OLG Model: Government



4. Government and Social Security

- Government budget is financed by issuing government debt:

$$Tr_t + G_t + r_t^b B_t = Tax_t + Beq_t + B_{t+1} - B_t, \qquad (9)$$

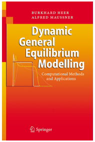
with taxes Tax_t:

$$Tax_t = \tau_t^l A_t L_t w_t + \tau_t^k (r_t - \delta) K_t + \tau_t^c C_t, \qquad (10)$$

C_t — aggregate consumption
G_t — government consumption
B_t — government bonds with rate of return r^b

- Government collects accidental bequests Beq_t.

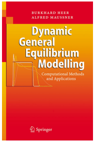
OLG Model: Government



- Balanced social security budget:

$$Pen_t = \tau_t^p A_t L_t w_t. \tag{11}$$

OLG Model: Equilibrium



5. Equilibrium

1. Capital market equilibrium:

$$\Omega_t = K_t + B_t$$

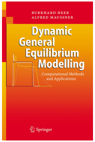
   Ω_t — aggregate wealth

2. Equal after-tax return on bonds on capital:

$$(1 - \tau_t^k)(r_t - \delta) = r_t^b$$

3. Aggregate consistency conditions: labor L_t, wealth Ω_t, consumption C_t

OLG Model: Equilibrium

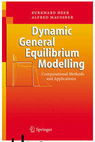


$\rightarrow$ sum of individual variables = aggregate variable

4. Goods markets equilibrium:

$$Y_t = C_t + G_t + K_{t+1} - (1 - \delta)K_t \tag{12}$$

OLG Model: Stationary Equilibrium



6. Stationary Equilibrium

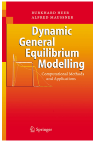
- Stationary individual variables $\tilde{x}_t^s$ for $x \in \{c, y, k, b\}$ (with the exception of labor supply l_t):

$$\tilde{x}_t^s \equiv \frac{x_t^s}{A_t}$$

- Stationary aggregate variables $\tilde{X}_t$ (with the exception of aggregate efficient labor L_t) for $X \in \{Pen, Tr, G, B, Beq, Tax, Y, K, C, \Omega\}$:

$$\tilde{X}_t \equiv \frac{X_t}{A_t N_t}.$$

OLG Model: Stationary Equilibrium



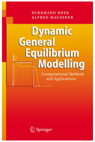
- Aggregate stationary labor $\tilde{L}_t = L_t/N_t$.
- Moreover, individual and aggregate government transfers are identical:

$$\widetilde{Tr}_t = \tilde{tr}_t.$$

- The budget constraint of the household in stationary variables is given by

$$(1+\tau_t^c)\tilde{c} = \tilde{y} + \left[1 + (1-\tau_t^k)(r_t - \delta)\right]\tilde{a} + \tilde{tr} - (1+g_A)\tilde{a}' \quad (13)$$

OLG Model: Stationary Equilibrium



with

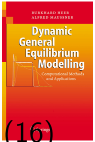
$$\tilde{y} = \begin{cases} (1-\tau_t^l-\tau_t^p)\epsilon(s,\theta,e)\, lw_t & s=1,\dots,T^W, \\ \widetilde{pen} & s=T^W+1,\dots,T. \end{cases} \tag{14}$$

- Stationary Bellman equation:

$$V_t(\tilde{z}_s) = \max_{\tilde{c},l,\tilde{a}'} \left\{ u(\tilde{c},1-l) + v(\tilde{g}) + (1+g_A)^{\gamma(1-\eta)} \beta\phi_t^s \sum_{\theta'} prob(\theta'|\theta)\, V_{t+1}(\tilde{z}_{s+1}) \right\}, \tag{15}$$

with the terminal condition $V_t(\tilde{z}_{T+1}) = 0$ and:

OLG Model: Stationary Equilibrium

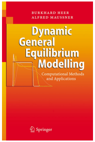


$$\tilde{z}_s = \begin{cases} (\theta, e, \tilde{a}) & \text{case 1} \\ (\theta, e, \tilde{a}, \tilde{x}) & \text{case 2,} \end{cases} \tag{16}$$

$$u(\tilde{c}, 1-l) = \frac{\left(\tilde{c}^{\gamma}(1-l)^{1-\gamma}\right)^{1-\eta}}{1-\eta}, \tag{17}$$

$$= \frac{u(c, 1-l)}{A_t^{\gamma(1-\eta)}} \tag{18}$$

OLG Model: Stationary Equilibrium



Stationary Equilibrium

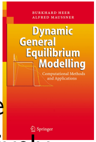
1. Total population N_t is equal to the sum of all cohorts:

$$N_t = \sum_{s=1}^{T} N_t(s)$$

with associated constant shares of the s-year-old cohorts

$$\mu^s = \frac{N_t(s)}{N_t}.$$

OLG Model: Stationary Equilibrium

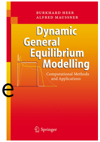


2. Population N_t and the youngest cohort $N_t(1)$ grow at the same rates $g_{N,t} = \frac{N_{t+1}}{N_t} - 1$ and $n_t = \frac{N_{t+1}(1)}{N_t(1)} - 1$, respectively, implying:

$$\frac{N_{t+1} - N_t}{N_t} = n.$$

3. Households maximize their lifetime utility subject to their budget constraint (34) and the non-negative constraint on wealth, $\tilde{a} \geq 0$, as described by the solution to the Bellman equation (15) implying the optimal policy functions $\tilde{a}'(\tilde{z})$, $\tilde{c}(\tilde{z})$ and $l(\tilde{z})$ for next-period wealth, consumption and labor supply.

OLG Model: Stationary Equilibrium



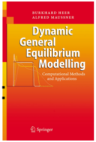
4. Aggregate effective labor supply is equal to the sum of the individual effective labor supplies:

$$\tilde{L}_t = \sum_{s=1}^{T^w} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_a=1}^{n_a} \epsilon(s, \theta_{i_\theta}, e_j)\, l(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a})\, f(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a}). \tag{19}$$

5. Aggregate wealth $\tilde{\Omega}$ is equal to the sum of the individual wealth levels:

$$\tilde{\Omega} = \sum_{s=1}^{T} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_a=1}^{n_a} \tilde{a}_{i_a}\, f(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a}). \tag{20}$$

OLG Model: Stationary Equilibrium



6. Firms maximize profits implying the factor prices w and r:

$$w = (1-\alpha)\tilde{K}^{\alpha}\tilde{L}^{-\alpha}, \tag{21a}$$

$$r = \alpha\tilde{K}^{\alpha-1}\tilde{L}^{1-\alpha}. \tag{21b}$$

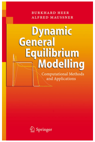
7. The after-tax returns on the two assets $\tilde{K}$ and $\tilde{B}$ are equal:

$$r^b = (1-\tau_r^k)(r-\delta). \tag{22}$$

8. In capital market equilibrium,

$$\tilde{\Omega} = \tilde{B} + \tilde{K}. \tag{23}$$

OLG Model: Stationary Equilibrium

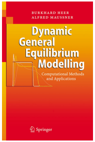


9. At the beginning of period $t+1$, the government collects accidental bequests from the s-year old households who do not survive from period t until period $t+1$:

$$\frac{\widetilde{Beq}'}{(1+n)(1+g_A)} = \sum_{s=2}^{T} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_a=1}^{n_a} (1-\phi^s) \times \left[1+(1-\tau^k)(r-\delta)\right] \tilde{a}'(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a}) \, f(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a}),$$

where $\widetilde{Beq}'$ denotes next-period accidental bequests.

OLG Model: Stationary Equilibrium



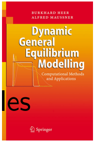
10. The goods markets clear:

$$\tilde{Y} = \tilde{C} + \tilde{G} + (1 + g_A)(1 + n)K' - (1 - \delta)K, \tag{24}$$

where aggregate consumption $\tilde{C}$ is the sum of individual consumptions:

$$C = \sum_{s=1}^{T} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_a=1}^{n_a} \tilde{c}(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a}) \, f(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a}). \tag{25}$$

OLG Model: Stationary Equilibrium

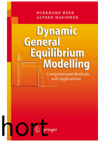


11. The densitiy function $f(s, \theta, e, \tilde{a})$ (and the associated distribution function $F(s, \theta, e, \tilde{a})$) of the per-capita variables (detrended by aggregate productivity A_t) are constant, $f(.) = f'(.)$ $(F'(.) = F(.))$. The dynamics of the distribution function $F(s, \theta, e, \tilde{a})$ evolves according to

$$F'(s+1, \theta', e, \tilde{a}') = \sum_{\theta'} \sum_{\tilde{a}=\tilde{a}'^{-1}(s,\theta,e,\tilde{a})} Prob(\theta'|\theta) \frac{\phi^s}{1+n} F(s, \theta, e, \tilde{a}), \tag{26}$$

where, on the right-hand side of the equation, we sum over all the productivity types θ' in period $t+1$ in the outer sum and the maximum wealth levels $\tilde{a}$ in period t that imply a next-period level $\tilde{a}'$ for given (s, θ, e) in the inner sum.

OLG Model: Stationary Equilibrium

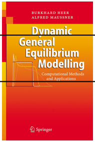


The distribution of $z = (s, \theta, e, \tilde{a})$ among the newborn cohort is constant and is presented by:

$$F(1, \theta, e, \tilde{a}) = \begin{cases} \mu^1 \times \nu(\theta) \times \pi(e) & \text{if } \tilde{a} = 0 \\ 0 & \text{else,} \end{cases}$$

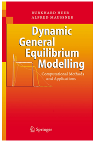
where $\nu(\theta)$ and $\pi(e)$ denote the shares of the θ and e productivity types in the cohorts (assumed to be constant over age s).

Calibration



Parameter	Value	Description
α	0.35	production elasticity of capital
δ	8.3%	depreciation rate of capital
g_A	2.0%	growth rate of output
$1/\eta$	1/2	intertemporal elasticity of substitution
γ	0.33	preference parameter for utility weight of consumption
β	1.011	discount factor
n	0.754%	population growth rate
$\tau^l + \tau^p$	28%	tax on labor income
τ^k	36%	tax on capital income
τ^c	5%	tax on consumption
G/Y	18%	share of government spending in steady-state production
B/Y	63%	debt-output ratio
$repl$	35.2%	gross pension replacement rate
$\{e_1, e_2\}$	$\{0.57, 1.43\}$	permanent productivity types

Calibration



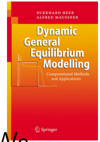
Calibration of the labor efficiency of the s-year old household, $\epsilon(s,\theta,e) = \theta e \bar{y}^s$

- Permanent efficiency types: $\{e_1, e_2\} = \{0.57, 1.43\}$ with $\pi(e_j) = 1/2$, $j = 1,2$

$\Rightarrow$ wages of college graduates are about 150% higher than that of the high school graduates

- Age-efficiency $\bar{y}^s$ as estimated by Hansen (1993)
- Stochastic component θ
  - $n_\theta = 5$ grid points
  - log-normal distribution of wages for the 21-year old with $\sigma_{y_1} = 0.38$

Calibration



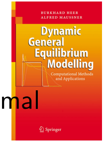
- θ_{i_θ} are equally spaced and range from $-m\sigma_{y_1}$ to $m\sigma_{y_1}$. We choose $m = 1.0$ so that the Gini coefficient of hourly wages amounts to 0.374 implying:

$$\Theta = (0.4688, 0.6847, 1.0000, 1.4605, 2.1332)$$

with corresponding logarithmic values

$$\ln \Theta = (-0.7576, -0.3788, 0.0000, 0.3788, 0.7576).$$

Calibration



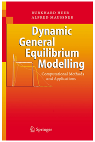
- Probability of having productivity θ_{i_θ}: area under the normal distribution implying the initial distribution among the 21-year-old agents for each permanent productivity type e_i, $i = 1, 2$:

$$\nu(\theta) = \begin{pmatrix} 0.1783 \\ 0.2010 \\ 0.2413 \\ 0.2010 \\ 0.1783 \end{pmatrix}$$

- AR(1) process for $\ln\theta$: $\ln\theta' = \rho\ln\theta + \nu\mu\xi$ with $\xi \sim N(0, \sigma_\xi)$

$\Rightarrow$ $\rho = 0.96$ and $\sigma_\xi^2 = 0.045$ as in Huggett (1996)

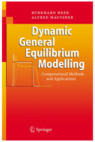
Calibration



$\Rightarrow$ The transition probabilities are computed using Tauchen's method as described in Algorithm 12.2.1 implying the finite Markov-chain transition matrix:

$$Prob(\theta'|\theta) = \begin{pmatrix} 0.7734 & 0.2210 & 0.0056 & 0.0000 & 0.0000 \\ 0.1675 & 0.6268 & 0.2011 & 0.0046 & 0.0000 \\ 0.0037 & 0.1823 & 0.6281 & 0.1823 & 0.0033 \\ 0.0000 & 0.0046 & 0.2011 & 0.6268 & 0.1675 \\ 0.0000 & 0.0000 & 0.0056 & 0.2210 & 0.7734 \end{pmatrix}. \tag{27}$$

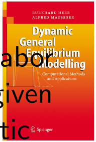
Computation of the Stationary Equilibrium



Algorithm

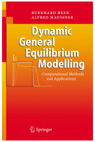
1. Parameterize the model and choose asset grids for the individual state space.
2. Make initial guesses of the steady state values of the aggregate capital stock $\tilde{K}$, labor $\tilde{L}$, mean working hours $\bar{l}$, labor income taxes τ^l, the social security contribution rate τ^p and government transfers $\widetilde{tr}$.
3. Compute the values w and r which solve the firm's first-order conditions and compute $\widetilde{pen}$.
4. Compute the household's decision functions by backward induction using value function iteration.

Computation of the Stationary Equilibrium



5. Compute the optimal path for consumption, savings and labor supply for the new-born generation by forward induction given the initial asset level $\tilde{a}^1 = 0$ and distribution of idiosyncratic productivities e and θ.
6. Compute the aggregate savings $\tilde{\Omega}$, labor supply $\tilde{L}$, mean working hours $\bar{l}$, aggregate taxes $\widetilde{Tax}$ and transfers $\widetilde{tr}$.
7. Update the aggregate variables and return to step 3 until convergence.
8. Update the asset grid of the individual state space if necessary and return to step 3 until convergence.

Computation of the Stationary Equilibrium



Program `AK70_stochastic_income`: PYTHON, JULIA or GAUSS

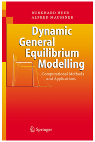
- PYTHON, JULIA and GAUSS CODES (with slides):

  https://www.uni-augsburg.de/de/fakultaet/wiwi/prof/vwl/maussner/dgebook/.

- PYTHON CODE explained in detail:

  PYTHON program tutorial

Computation of the Stationary Equilibrium



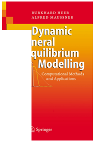
Step 1: Parameterize the model and choose asset grids for the individual state space.

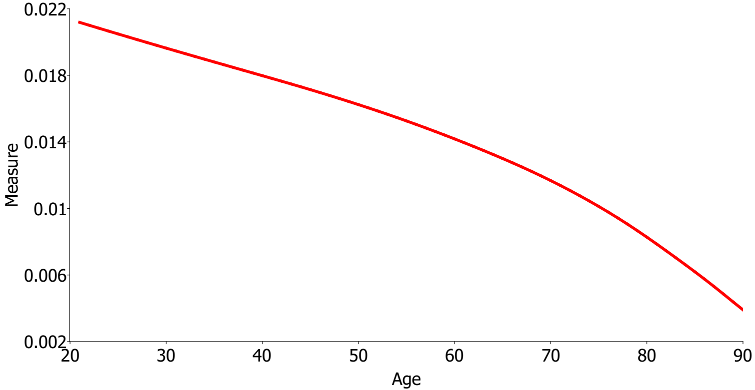
- Asset grid $\mathcal{A}$ on asset space:
  - Policy function: $n_a = 500$
  - Distribution function: $n_{ag} = 1,000$
- Computation of the stationary cohort shares μ^s:
  - Set $\mu^1 = 1.0$
  - Iterate over $s = 2, \ldots, 70$:

$$\mu^{s+1} = \frac{\phi^s}{1+n}\mu^s \tag{28}$$

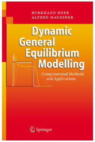
  - Normalize the sum of the measures to one: Divide the μ^s by $\sum_s \mu^s$

Computation of the Stationary Equilibrium





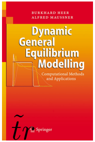
Computation of the Stationary Equilibrium



Step 2: Make initial guesses of the steady state values

- $\bar{l} = 0.3$
- $\tilde{L} = 0.30$ (workforce share in population: 78%)
- $r = 3\% \Rightarrow \tilde{K} = 1.708$
- $\tilde{\Omega} \approx 1.26\tilde{K}$ (follows from $B/Y = 0.63$ in model and $K/Y \approx 3.0$ in the US)
- $\widetilde{tr} = 0.01$

Computation of the Stationary Equilibrium



Step 3: Compute the values w, r and $\widetilde{pen}$

$\rightarrow$ initial step in the outer loop over aggregate variables $\tilde{K}$, $\tilde{L}$, $\tilde{tr}$

$\rightarrow$ outer loop iterates over Step 3-7

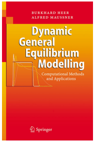
- Computation of factor prices w and r:

$$w_t = (1-\alpha)\tilde{K}_t^{\alpha}\left(\tilde{L}_t\right)^{-\alpha}, \tag{29a}$$

$$r_t = \alpha\tilde{K}_t^{\alpha-1}\left(\tilde{L}_t\right)^{1-\alpha}. \tag{29b}$$

- Computation of $\widetilde{pen} = repl \times w\,\bar{l}$

Computation of the Stationary Equilibrium



Step 4: Compute the household's policy functions with value function iteration

$$V_t(\tilde{z}_s) = \max_{\tilde{c}, l, \tilde{a}'} \left\{ u(\tilde{c}, 1-l) + v(\tilde{g}) + (1+g_A)^{\gamma(1-\eta)} \beta \phi_t^s \sum_{\theta'} prob(\theta'|\theta)\, V_{t+1}(\tilde{z}_{s+1}) \right\}, \quad (30)$$

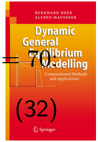
- Retiree: One-dimensional optimization w.r.t. $\tilde{a}'$ using GOLDEN SECTION SEARCH (GSS)
- Worker: Two-dimensional optimization w.r.t. $\tilde{a}'$ and l

$\Rightarrow$ nested optimization

- Outer function (value function): $\tilde{a}'$ using GSS
- Inner function (foc labor): l

$$l = \gamma - \frac{1-\gamma}{(1-\tau^l-\tau^p)\epsilon(s,\theta,e)w} \left(\left[1 + (1-\tau^k)(r-\delta)\right] \tilde{a} + \widetilde{tr} - (1+g_A)\tilde{a}' \right) \quad (31)$$

Computation of the Stationary Equilibrium



- Solving the value function backwards starting in last period of life $s = 70$:

$$V^T(\tilde{a}^T) = u(\tilde{c}^T, 1) \tag{32}$$

with

$$\tilde{c}_t^T = \frac{\widetilde{pen} + [1 + (1-\tau^k)(r-\delta)]\tilde{a}^T + \widetilde{tr}}{1+\tau^c}. \tag{33}$$

- Ages $s = 69 = T - 1$:

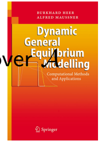
$$\begin{aligned} V^{T-1}(\tilde{a}^{T-1}) \quad = \quad & \max_{\tilde{a}^T} \left\{ u\left(\frac{\widetilde{pen} + [1 + (1-\tau^k)(r-\delta)]\tilde{a}^{T-1} + \widetilde{tr} - (1+g_A)\tilde{a}^T}{1+\tau^c}, 1 \right) \right. \\ & \left. + (1+g_A)^{\gamma(1-\eta)} \beta \phi^{T-1} \, V^T(\tilde{a}^T) \right\}. \end{aligned}$$

  - $\tilde{a}^T$ may not be a grid point

  $\Rightarrow$ interpolation: linear, cubic

- Ages $s = 68, \ldots, 46$ as above

Computation of the Stationary Equilibrium



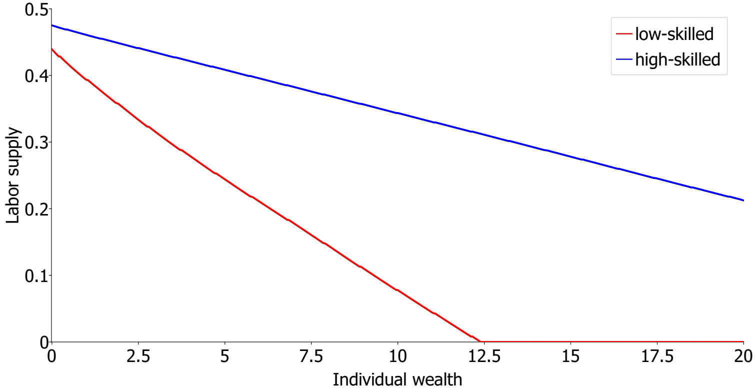
- Worker's value function at ages $s = 45, \ldots, 1$ at all n_a grid points over $\mathcal{A}$ and for all productivity types $\{e_1, e_2\}$ and $\{\theta_1, \ldots, \theta_5\}$:
  - Maximize right-hand side of the Bellman equation
  - Each time when we compute the term $u(c, 1-l)$, we
    - compute l from the first-order condition w.r.t. labor
    - and c with the help of the budget constraint:

$$(1+\tau^c)\tilde{c} = (1-\tau^l-\tau^p)\epsilon(s,\theta,e)\,lw + \left[1+(1-\tau^k)(r-\delta)\right]\tilde{a} + \tilde{tr} - (1+g_A)\tilde{a}' \tag{34}$$

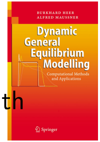
  - We store the value function $V(.)$ and the optimal policy functions $\tilde{a}'(.)$, $\tilde{c}(.)$ and $l(.)$ at all grid points and for all productivity types.

$\Rightarrow$ Time-consuming step

Computation of the Stationary Equilibrium



Computation of the Stationary Equilibrium



- Assessing the accuracy of the optimization:
  - Euler equation residua for the s-year old worker with wealth level $\tilde{a}$ and productivity type $\epsilon(s,\theta,e)$,:

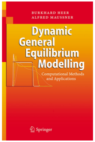
$$R(\tilde{a}) = 1 - \frac{\tilde{u}_c(\tilde{c}, 1-l)}{\beta(1+r^b)(1+g_A)^{\gamma(1-\eta)-1}\phi^s \mathbb{E}\left\{\tilde{u}_c(\tilde{c}', 1-l')\right\}} \tag{35}$$

derived from:

$$\frac{\partial \tilde{u}(\tilde{c}, 1-l)}{\partial \tilde{c}} = (1+g_A)^{\gamma(1-\eta)-1}\beta(1+r^b)\phi^s \mathbb{E}\left\{\frac{\partial \tilde{u}(\tilde{c}', 1-l)}{\partial \tilde{c}'}\right\}.$$

  - We compute the mean of all grid points (alternatively: weighted by measures of the households)

Computation of the Stationary Equilibrium



Step 5: Compute the distribution

- Endogenous wealth distribution $f(s, e, \theta, \tilde{a})$ over equispaced grid on $[\tilde{a}^{min}, \tilde{a}^{max}]$ with $n_{ag} = 1,000$ points
- Total grid points of $f(.)$:

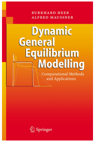
$$n_{ag} \times n_e \times n_\theta \times T^W + n_{ag} \times T^R = 1000 \times 2 \times 5 \times 45 + 1000 \times 25 = 475,000. \tag{36}$$

$\rightarrow$ stored in the variables 'gkw[$\tilde{a}$,θ,e,s]' and 'gkr[$\tilde{a}$,s]' in the program *AK70_stochastic_income.py*

- Distribution at age $s = 1$:

$$gkw[\tilde{a}, \theta, e, 1] = \begin{cases} \frac{1}{2}\mu^1 v(\theta) & \text{if } \tilde{a} = 0 \\ 0 & \text{else.} \end{cases}$$

Computation of the Stationary Equilibrium



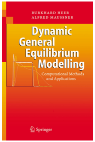
- Computation of the distribution at age $s = 2$:
  - We iterate over all grid points $\tilde{a}_{i_a}$, $i_a = 1, \ldots, n_{ag}$ at age $s = 1$ (and e_j and θ_{i_θ}) and compute the distribution of the 2-year-old workers in period $t + 1$
  - As an example, consider low-skilled worker, $e = e_1$, with the idiosyncratic productivity $\theta_4 = 1.4605$ and zero wealth $\tilde{a} = 0.0$ at age $s = 1$

$\rightarrow$ with measure 'gkw[0,e_1,θ_4,1]=0.00213' and next-period wealth a' given by 'awopt[0,e_1,θ_4,1]=0.008365'.

  - Measure of those agents with $(0, e_1, \theta_4, 1)$ who survive until next period:

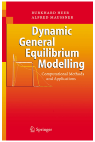
$$0.002129 \frac{\phi^1}{1+n} = 0.002111.$$

Computation of the Stationary Equilibrium



- the share $(1-\phi^s)$ dies
- population grows with factor $(1+n)$

Computation of the Stationary Equilibrium



- $\tilde{a}' = 0.008365$ is not a grid point:

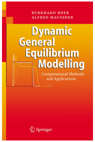
$$\tilde{a}_1 = 0 < 0.0008365 < \tilde{a}_2 = 0.02002.$$

- As in Chapter 7.2, we allocate the following shares to next-period wealth $\tilde{a}_1$ and $\tilde{a}_2$:
    - $(a_2 - \tilde{a}')/(\tilde{a}_2 - \tilde{a}_1) = 0.6$ at point $\tilde{a}_1$
    - $(\tilde{a}' - \tilde{a}_1)/(\tilde{a}_2 - \tilde{a}_1) = 0.4$ at point $\tilde{a}_2$
- We have to consider the transition dynamics of the productivity type θ:

$\rightarrow$ E.g, the measure at point $(\tilde{a}_1, e_1, \theta_3, 2)$ increases by

$$0.6 \cdot \underbrace{prob(\theta_3|\theta_4)}_{0.2011} \cdot \underbrace{gkw[0, e_1, \theta_4, 1]}_{0.00211} \cdot \underbrace{\frac{\phi^1}{1+n}}_{0.9917} = 0.0002525$$

Computation of the Stationary Equilibrium

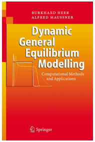


- Computational time of Step 5:
  - PYTHON: 33 minutes
  - JULIA: 11 seconds
  - GAUSS: 8 seconds

$\Rightarrow$ PYTHON is extremely slow in iterations

  - We used exactly the same number of operations and stored the numbers with the same accuracy

Computation of the Stationary Equilibrium



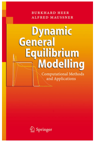
Step 6: Compute $\tilde{\Omega}$, $\tilde{L}$, $\bar{l}$, $\widetilde{Tax}$ and $\tilde{tr}$

- For given $\tilde{K}$ and $\tilde{L}$, we can compute $\tilde{Y}$ and, hence, $\tilde{B} = 0.63\tilde{Y}$
- Aggregate wealth $\tilde{\Omega}$: sum of $\tilde{a}$ weighted by $f(s, e, \theta, \tilde{a})$
- Aggregate labor supply $\tilde{L}$: sum of $(\bar{y}^s e\theta)l$ weighted by measure $f(s, e, \theta, \tilde{a})$
- Capital market equilibrium:

$$\tilde{K} = \tilde{\Omega} - \tilde{B}$$

- Pension contribution rate τ^p: computed with the help of the social security budget.
- Computation of government transfers:
    - government consumption $\tilde{G} = 0.18\tilde{Y}$

Computation of the Stationary Equilibrium



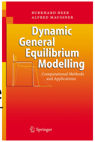
- accidental bequests $\widetilde{Beq}$
- aggregate consumption $\tilde{C}$
- total taxes:

$$\widetilde{Tax} = \tau^l w\tilde{L} + \tau^k (r-\delta)\tilde{K} + \tau^c \tilde{C}$$

- government transfers: residual from government budget constant debt:

$$\widetilde{Tr} = \widetilde{Tax} + \left[(1+g_A)(1+n) - (1+r^b)\right]\tilde{B} - \tilde{G}$$

Computation of the Stationary Equilibrium



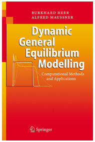
Step 7: Update aggregate variables and return to step 3 until convergence

- update the aggregate values of $\tilde{K}$, $\tilde{L}$, τ^p, τ^l, $\widetilde{Tr}$, $\bar{l}$
- we use a simple linear updating scheme: 80% old value plus 20% new value

Step 8: Update the asset grid of the individual state space if necessary

- Study distribution if $\tilde{a}^{max}$ is a reasonable upper boundary for the asset grid
- Study the Euler residual: accuracy satisfactory?

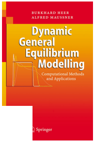
Computation: PYTHON, JULIA or GAUSS?

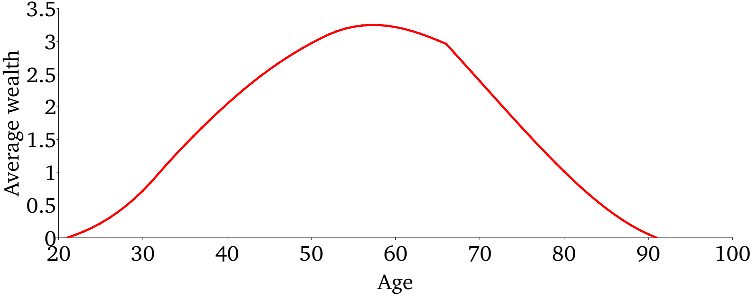


Computational Methods

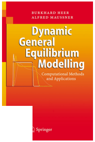
Interpolation	linear	cubic	cubic
Grid points			
n_a	500	500	300
n_{ag}	1,000	1,000	1,000
Accuracy			
Young	0.00085	0.00018	0.00032
Old	0.00231	0.00052	0.00140
Run time			
Julia	1h:29m:56s	1h:32m:43s	45m:37s
Gauss	27m:38s	1h:16m:34s	51m:56s
Python	32h:49m:37s	55h:30m:33s	48h:17m:04s

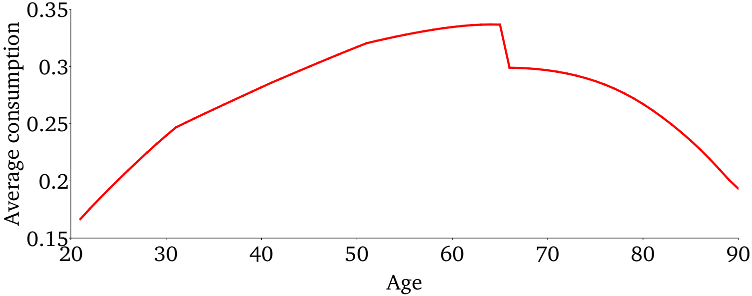
Results: Wealth-Age Profile



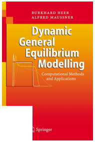


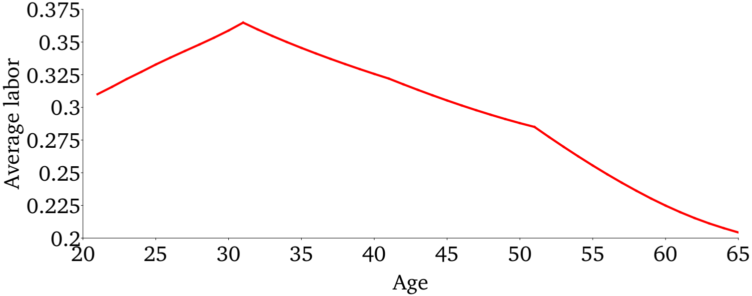
Results: Consumption-Age Profile



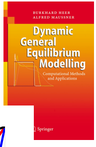


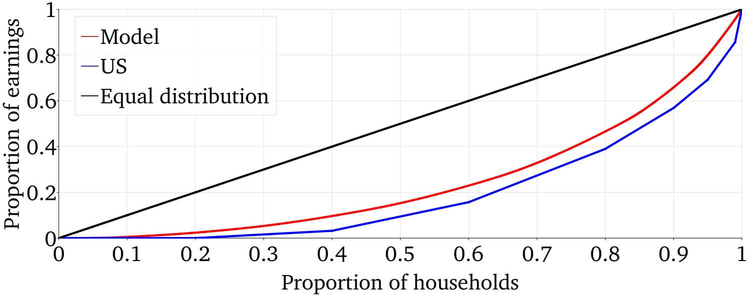
Results: Labor-Supply-Age Profile



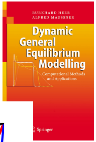


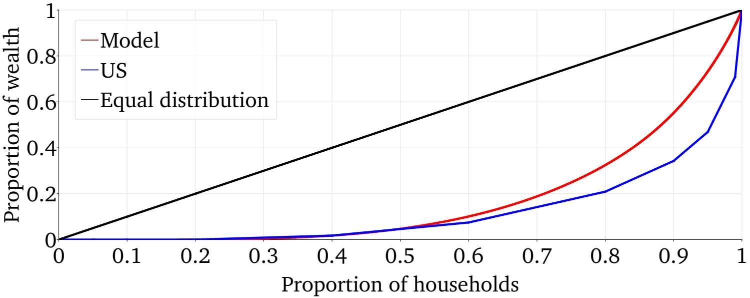
Results: Lorenz Curves Earnings



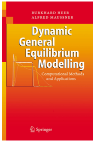


Results: Lorenz Curves Wealth





Results

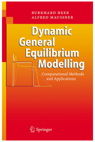


	Gini Coefficient US	Gini Coefficient OLG Model
wages	0.375	0.375
earnings	0.43-0.66	0.505
wealth	0.80	0.66

Missing elements to replicate wealth heterogeneity:

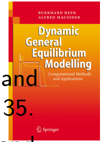
1. Bequests: De Nardi and Yang (2016)
2. Unemployment: Heer (2003)
3. Asset-based means tests of social security: Hubbard et al. (1995)

Results



4. Entrepreneurship (Bill Gates): Quadrini (2000), Cagetti and de Nardi (2009)
5. Stochastic health: Jung and Tran (2016)
6. Family heterogeneity, e.g. number of children, marital status: Holter et al. (2019)

References



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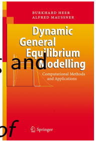

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