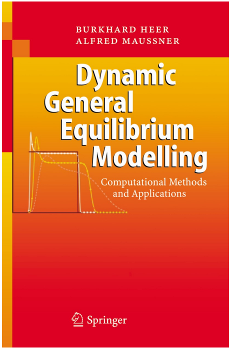
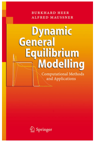
Chapter 10.2.2: KRUSELL-SMITH Algorithm and Overlapping Generations Model



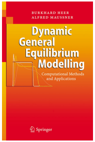
Burkhard Heer and Alfred Maußner
University of Augsburg, Germany
February 28, 2022

Krusell-Smith Algorithm: Applications



- Cyclical behavior of income distribution: Castañeda, Díaz-Giménez, and Ríos-Rull (1998)
- Great Recession 2007-09:
  - Wealth and income dynamics: Krueger, Mitman, and Perri (2016)
  - Housing: Kaplan, Mitman, and Violante (2020)

Contents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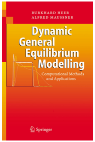
Motivation



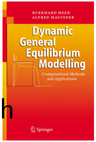
Research questions:

1. Cyclical income and wealth distribution: How can we model it?

$\Rightarrow$ Stochastic Overlapping Generations (OLG) model with income uncertainty/heterogeneity

2. How can we compute it?
   1. Steady state perturbation: no idiosyncratic uncertainty $\rightarrow$ Chapters 2 & 10.2.1 in Heer/Maußner
   2. Krusell-Smith Algorithm $\rightarrow$ this presentation

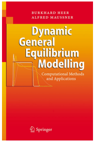
Motivation



Overlapping generations model with endogenous income/wealth distribution:

1. life-cycle savings
2. uncertain lifetime
3. uncertain earnings
4. pay-as-you-go pensions
5. endogenous labor supply

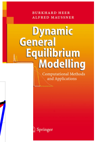
Motivation

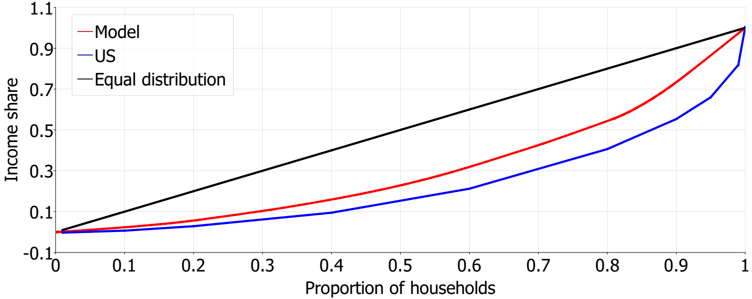


	Gini Coefficients	
	US	OLG Model
wages	0.375	0.302
gross income	0.553	0.386
wealth	0.78-0.80	0.558

Note: US Data is taken from Budría Rodríguez, Díaz-Giménez, Quadrini, and Ríos-Rull (2002) and Krueger, Mitman, and Perri (2016).

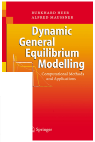
Lorenz Curve: Income

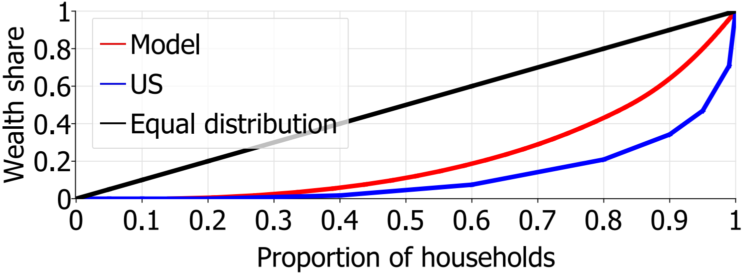




Note: US Data is taken from Budría Rodríguez, Díaz-Giménez, Quadrini, and Ríos-Rull (2002).

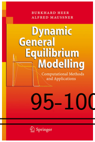
Lorenz Curve: Wealth





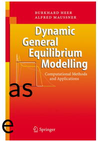
Note: US Data is taken from Budría Rodríguez, Díaz-Giménez, Quadrini, and Ríos-Rull (2002).

Correlation of income shares with GDP



	0-20%	20-40%	40-60%	60-80%	80-95%	95-100%
US	0.53	0.49	0.31	-0.29	-0.64	0.00
Castañeda et al. (1998)	0.95	0.92	0.73	-0.56	-0.90	-0.84
our model						
i) $pen_t(\epsilon) = \epsilon \cdot \bar{pen}_t$	0.85	0.54	0.60	-0.30	0.23	-0.44
ii) $pen_t(\epsilon) = \bar{pen}_t$	0.75	-0.48	0.41	0.08	0.06	-0.29

Notes: Entries in rows 1 and 2 are reproduced from Table 4 in Castañeda et al. (1998). Annual logarithmic output has been detrended using the Hodrick-Prescott filter with smoothing parameter $\lambda = 100$.



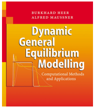
Topic for PhD thesis: Cyclical behavior of (gross) income distribution

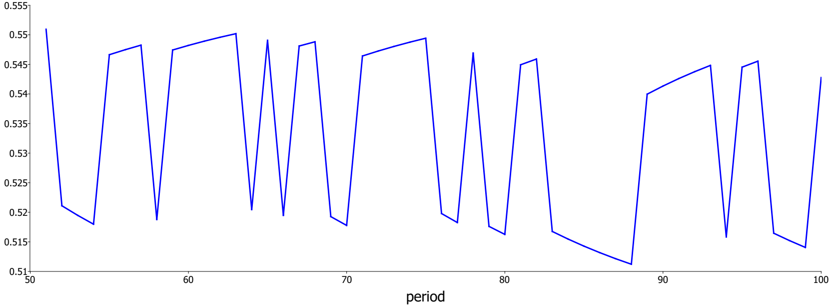
- Labor income: a combination of unemployment dynamics as in Castañeda, Díaz-Giménez, and Ríos-Rull (1998) and the OLG model seems promising
- Add demand shock and sticky wages
- Use more than two points to approximate technology $Z_t \Rightarrow$ behavior of Y_t jumps up and down otherwise!
- Capital income: problematic

$\rightarrow$ recessions are not alike:

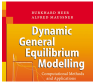
2008 Great recession, SP 500 return approx. -38%
2019 Covid-19 crisis, SP 500 return approx. +28%

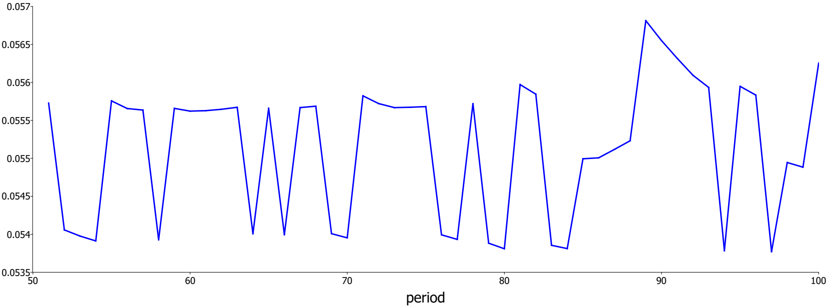
Jump behavior of aggregate output $\tilde{Y}_t$ during simulation



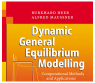


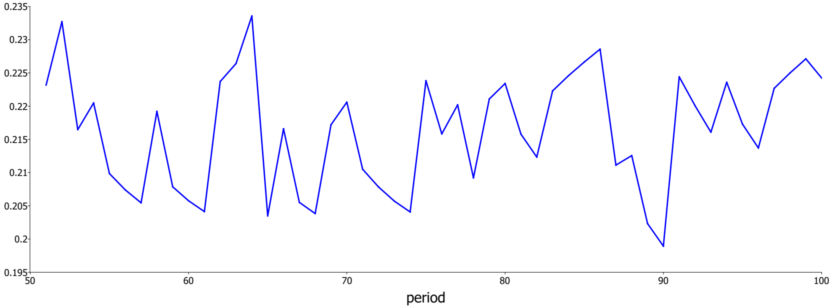
Dynamic behavior of factor share: bottom quintile



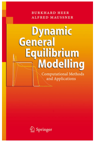


Dynamic behavior of factor share: 60-80% quintile





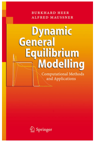
Computation: PYTHON or GAUSS?



RUN TIME: Krusell-Smith Algorithm

	PYTHON	GAUSS
Non-stochastic steady state	11m:53s	2m:19s
Policy functions	39m:03	14m:18s
Simulation (200 periods)	1h:13m:52s	3m:52s
Total	1d:21h:40m:18s	5h:18m:40s

DOWNLOADS: PYTHON or GAUSS code



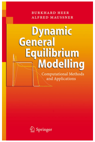
- PYTHON and GAUSS CODES (with slides):

  https://www.uni-augsburg.de/de/fakultaet/wiwi/prof/vwl/maussner/dgebook/.

- Or send an email: Burkhard.Heer@wiwi.uni-augsburg.de

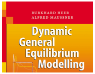
OLG Model: Demographics

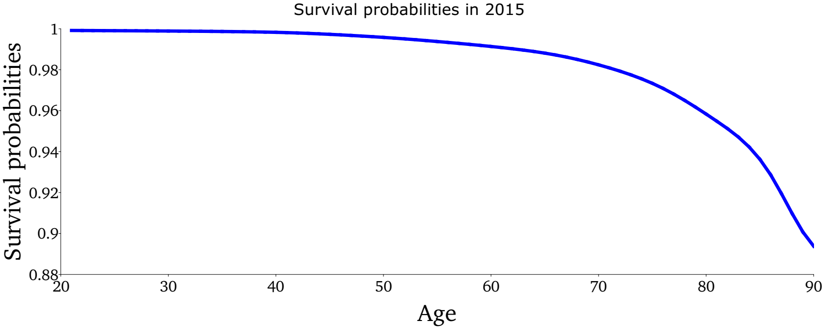
1. Demographics



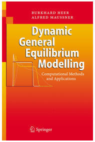
- Every year, a new generation of equal measure is born at real-life age 21 corresponding to age $s = 1$ in the model.
- Households live a maximum of $T = 70$ years (corresponding to real-life age 90).
- Survival from age s to age $s + 1$ is stochastic with probability ϕ_t^s.
- During their first $T^W = 45$ years as workers, agents supply labor l_t^s at age s in period t enjoying leisure $1 - l_t^s$.
- After T^W years, retirement is mandatory ($l_t^s = 0$ for $s > T^W$).
- The maximum number of retirement periods amounts to T^R.

OLG Model: Demographics





OLG Model: Demographics



- $N_t(s)$ — number of households of age s at t.
- N_t — total population
- Population growth rate $g_{n,t}$:

$$N_{t+1} = (1 + g_{n,t})N_t$$

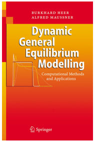
- Newborn cohort growth rate n_t:

$$N_{t+1}(1) = (1 + n_t)N_t(1)$$

- **In the stationary equilibrium**:
  - ϕ^s constant
  - $n = g_n = 0.754\%$

OLG Model: Households

2. Households



- Households maximize expected intertemporal utility:

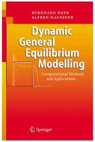
$$\max \sum_{s=1}^{T} \beta^{s-1} \left(\Pi_{j=1}^{s} \phi_{t+j-1}^{j-1} \right) \mathbb{E}_t \left[u(c_{t+s-1}^{s}, 1 - l_{t+s-1}^{s}) \right], \qquad (1)$$

- with instantaneous utility $u(c, 1-l)$:

$$u(c, 1-l) = \frac{\left(c^{\gamma} (1-l)^{1-\gamma} \right)^{1-\eta}}{1-\eta}, \qquad (2)$$

β — discount factor
$1/\eta$ — intertemporal elasticity of substitution (IES)
γ — share of consumption in utility

OLG Model: Households



- Net labor income:

$$y_t^s = (1 - \tau_t^l - \tau_t^p)\epsilon(s, e, \theta)A_t w_t l_t^s$$

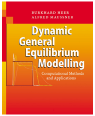
  A_t — aggregate productivity with growth rate g_A

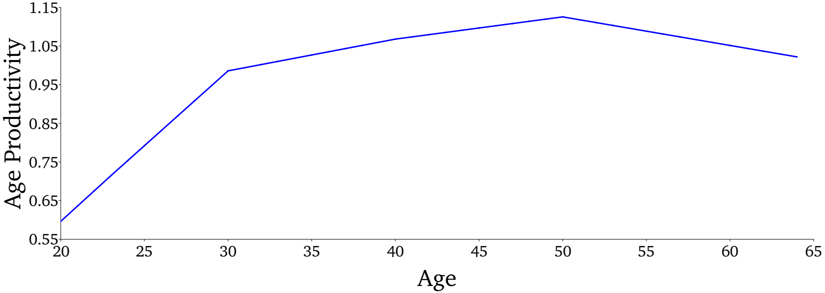
- idiosyncratic productivity: $\epsilon(s, e, \theta) = e\theta\bar{y}^s$

  $\bar{y}^s$ — age component of wage, Hansen (1993)
  $e \in \{0.57, 1.43\}$ — permanent productivity type (high school/college)
  $\theta \in \{0.727, 1.233\}$ — stochastic component: Markov process from Storesletten, Telmer, and Yaron (2007)

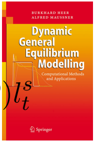
$$\pi(\theta'|\theta) = \begin{pmatrix} \pi_{11}^\theta & \pi_{12}^\theta \\ \pi_{21}^\theta & \pi_{22}^\theta \end{pmatrix} = \begin{pmatrix} 0.98 & 0.02 \\ 0.02 & 0.98 \end{pmatrix} \qquad (3)$$

OLG Model: Age-specific productivity component $\bar{y}^s$





OLG Model: Households

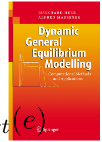


- Budget constraint of the worker at age s in period t:

$$k_{t+1}^{s+1} = \Big(1 + (1-\tau^k) r_t\Big) k_t^s + (1 - \tau^l - \tau_t^p) w_t A_t \epsilon(s,e,\theta) l_t^s \\ + tr_t - (1+\tau^c) c_t^s, \quad s = 1, \ldots, 45, \tag{4}$$

k_t^s — capital (wealth) of the s-year old in period t
r_t — rate of return on capital
δ — depreciation rate
τ_t^k — capital income tax rate
τ_t^l — labor income tax rate
τ_t^p — pension contribution rate
τ_t^c — consumption tax rate
tr_t — government transfers

OLG Model: Households



- Retirees receive productivity-dependent pension $y_t^s = pen_t(e)$
- Budget constraint of the retiree at age $s = 46, \ldots, 70$:

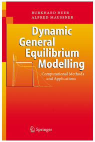
$$k_{t+1}^{s+1} = (1 + (1 - \tau^k) r_t) k_t^s + pen_t(e) + tr_t - (1 + \tau^c) c_t^s, \quad (5)$$

  with $k_t^{71} \equiv 0$ and $l_t^{46} = l_t^{47} = \ldots = l_t^{70} \equiv 0$.

- Credit constraint: $k_t^s \geq 0$

OLG Model: Firms

3. Firms



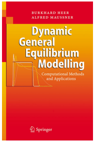
- Perfect competition in goods and factor markets
- Cobb-Douglas production function:

$$Y_t = Z_t F(K_t, A_t L_t) = Z_t K_t^{\alpha} (A_t L_t)^{1-\alpha}. \tag{6}$$

- Stochastic productivity Z_t: 2-state Markov process
  - Annual standard deviation of output is approximately equal to 2% implying $\{Z_1, Z_2\} = \{0.98, 1.02\}$.
  - Transition matrix: average duration of cycle $=$ 6 years

$$Prob(Z'|Z) = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}. \tag{7}$$

OLG Model: Firms



- Profits:

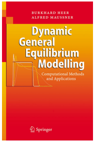
$$\Pi_t = Y_t - w_t A_t L_t - r_t K_t - \delta K_t.$$

- Factors are rewarded with their marginal product:

$$\begin{aligned} w_t &= (1-\alpha) Z_t K_t^{\alpha} (A_t L_t)^{-\alpha}, &(8)\\ r_t &= \alpha Z_t K_t^{\alpha-1} (A_t L_t)^{1-\alpha} - \delta. &(9)\end{aligned}$$

OLG Model: Government

4. Government and Social Security



- Government budget is financed by taxes and accidental bequests:

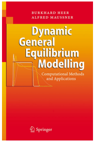
$$Tr_t + G_t = \tau_t^l A_t L_t w_t + \tau_t^k r_t K_t + \tau_t^c C_t + Beq_t, \qquad (10)$$

C_t — aggregate consumption
G_t — government consumption

- Government collects accidental bequests Beq_t.
- Balanced social security budget:

$$Pen_t = \tau_t^p A_t L_t w_t. \qquad (11)$$

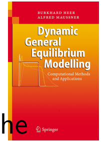
OLG Model: Government



- Two types of pensions:
  1. Lump-sum pensions: $pen_t = repl \cdot (1 - \tau^l - \tau^p) A_t w_t \bar{l}$
  2. Pensions depending on permanent productivity type: $pen_t = repl \cdot (1 - \tau^l - \tau^p) A_t w_t e \bar{l}$

  $\bar{l}$ — steady-state average working hours
  $repl$ — net replacement rate of pensions with respect to wages

OLG Model: Stationary Variables



5. Stationary Variables

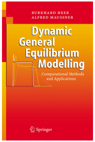
- Stationary individual variables $\tilde{x}_t^s$ for $x \in \{c, y, k\}$ (with the exception of labor supply l_t):

$$\tilde{x}_t^s \equiv \frac{x_t^s}{A_t}$$

- Stationary aggregate variables $\tilde{X}_t$ (with the exception of aggregate efficient labor L_t) for $X \in \{Pen, Tr, G, Beq, Y, K, C\}$:

$$\tilde{X}_t \equiv \frac{X_t}{A_t N_t}.$$

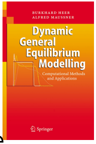
OLG Model: Stationary Variables



- Aggregate stationary labor $\tilde{L}_t = L_t/N_t$.
- Moreover, individual and aggregate government transfers are identical:

$$\widetilde{Tr}_t = \widetilde{tr}_t.$$

OLG Model: Equilibrium



6. Equilibrium

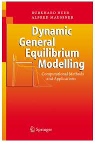
1. Population N_t and the youngest cohort $N_t(1)$ grow at the same rates $g_{N,t} = \frac{N_{t+1}}{N_t} - 1$ and $n_t = \frac{N_{t+1}(1)}{N_t(1)} - 1$, respectively, implying:

$$\frac{N_{t+1} - N_t}{N_t} = n.$$

2. Total population N_t is equal to the sum of all cohorts:

$$N_t = \sum_{s=1}^{T} N_t(s)$$

OLG Model: Equilibrium



with associated constant shares of the s-year-old cohorts

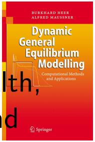
$$\mu^s = \frac{N_t(s)}{N_t}.$$

3. Firms maximize profits implying the factor prices w and r:

$$w = (1-\alpha) Z \tilde{K}^{\alpha} \tilde{L}^{-\alpha}, \tag{12a}$$

$$r = \alpha Z \tilde{K}^{\alpha-1} \tilde{L}^{1-\alpha} - \delta. \tag{12b}$$

OLG Model: Equilibrium



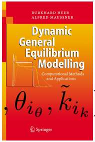
4. Households maximize their lifetime utility subject to their budget constraint and the non-negative constraint on wealth, $\tilde{k} \geq 0$ implying the optimal policy functions $\tilde{k}'(\tilde{z})$, $\tilde{c}(\tilde{z})$ and $l(\tilde{z})$ for next-period wealth, consumption and labor supply with stationary state variable $\tilde{z}$:

$$\tilde{z} = (s, e, \theta, \tilde{k}, Z, \tilde{K})$$

5. Distribution function $f_t(s, e, \theta, \tilde{k})$ of the individual state variable $(s, e, \theta, \tilde{k})$ in period t $\Rightarrow$ discrete approximation
6. Aggregate consistency conditions: labor L_t, wealth K_t, consumption C_t, bequests Beq_t

$\rightarrow$ sum of individual variables = aggregate variable

OLG Model: Equilibrium



- For example, labor supply:

$$\tilde{L}_t = \sum_{s=1}^{T^w} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_k=1}^{n_k} \epsilon(s, e_j, \theta_{i_\theta})\, l_t(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k})\, f_t(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k}).$$

7. The fiscal budget is balanced:

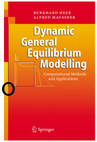
$$\tilde{T}r_t + \tilde{G}_t = \tau_t^l \tilde{L}_t w_t + \tau_t^k r_t \tilde{K}_t + \tau_t^c \tilde{C}_t + \widetilde{Beq}_t,$$

where aggregate consumption $\tilde{C}$ is the sum of individual consumptions:

$$\tilde{C}_t = \sum_{s=1}^{T} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_k=1}^{n_k} \tilde{c}_t(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k})\, f_t(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k}). \quad (13)$$

and $\tilde{G}_t = \tilde{G}$ constant.

OLG Model: Equilibrium



8. At the beginning of period $t+1$, the government collects accidental bequests from the s-year old households who do not survive from period t until period $t+1$:

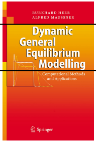
$$\frac{\widetilde{Beq}_t}{(1+g_A)} = \sum_{s=2}^{T} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_k=1}^{n_k} (1-\phi_t^s)\ \tilde{k}_t'(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k})\ f_t(s, e_j, \theta_{i_\theta}, \tilde{k}_{i_k}),$$

9. The goods markets clear:

$$\tilde{Y}_t = \tilde{C}_t + \tilde{G}_t + (1+g_A)(1+n)K_{t+1} - (1-\delta)K_t. \tag{14}$$

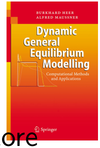
10. Dynamics of the distribution function: $\rightarrow$ described in the next section

Calibration



Parameter	Value	Description
α	0.35	production elasticity of capital
δ	8.3%	depreciation rate of capital
g_A	2.0%	growth rate of output
$1/\eta$	1/2	intertemporal elasticity of substitution
γ	0.29	preference parameter for utility weight of consumption
β	1.011	discount factor
n	0.754%	population growth rate
$\tau^l + \tau^p$	28%	tax on labor income
τ^k	36%	tax on capital income
τ^c	5%	tax on consumption
G/Y	18%	share of government spending in steady-state production
$repl$	49.4%	net pension replacement rate
$\{e_1, e_2\}$	$\{0.57, 1.43\}$	permanent productivity types

Krusell-Smith Algorithm



Krusell-Smith Algorithm: Krusell and Smith (1998)

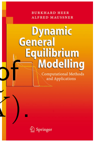
1. Compute the non-stochastic steady state with $Z_t \equiv 1$. Store the policy functions and the steady-state distribution of $(s, \epsilon, \theta, \tilde{k})$.
2. Choose an initial parameterized functional form for the law of motion for the aggregate next-period capital stock and present period employment and transfers:

$$\tilde{K}_{t+1} = g^K(Z_t, \tilde{K}_t) \tag{15a}$$

$$\tilde{L}_t = g^L(Z_t, \tilde{K}_t) \tag{15b}$$

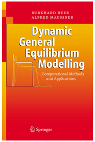
$$\widetilde{Tr}_t = g^{Tr}(Z_t, \tilde{K}_t). \tag{15c}$$

Krusell-Smith Algorithm



3. Solve the consumer's optimization problem as a function of the individual and aggregate state variables, $(s, e, \theta, \tilde{k}; Z\tilde{K})$.
4. Simulate the dynamics of the distribution function.
5. Use the time path for the distribution to estimate the law of motion for $\tilde{K}_t$, $\tilde{L}_t$ and $\widetilde{Tr}_t$.
6. Iterate until the parameters of the functions g^K, g^L and g^{Tr} converge.
7. Test the goodness of fit for the functional form using, for example, R^2. If the fit is satisfactory, stop, otherwise choose a different functional form for $g(.) = (g^K, g^L, g^{Tr})'$.

Krusell-Smith Algorithm



Step 1: Computation of the non-stochastic steady state

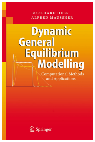
Two Loops:

1. Inner loop: Computation of the inidividual policy functions given $(\tilde{K}, \tilde{L}, \widetilde{Tr}, \widetilde{pen})$
2. Outer loop: Computation of $(\tilde{K}, \tilde{L}, \widetilde{Tr}, \widetilde{pen})$

Two options for the computation of the individual optimal policies:

1. Value function iteration: numerical optimization
2. Euler equations: non-linear eqs. solver ✓
    - We start in the last period of life where next-period capital stock is equal to zero: $\tilde{k}'(s, e, \theta, \tilde{k}) \equiv 0$

Krusell-Smith Algorithm



- Iterating backwards over ages $s = 69, 68, \ldots, 1$
- Euler eq.:

$$rf(\tilde{z}) = u_c(\tilde{c}(\tilde{z}, 1-l(\tilde{z}))) - \beta \mathbb{E}_t u_c(c'(\tilde{z}', 1-l(\tilde{z}')))(1+(1-\tau^k)r).$$

with

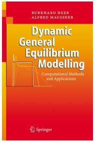
$$l = \gamma - (1-\gamma)\frac{\widetilde{tr} + (1+(1-\tau^k))r\tilde{k} - (1+g_A)\tilde{k}'}{(1-\tau^l-\tau^p)e\theta\bar{y}^s w} \tag{16a}$$

$$\tilde{c} = \frac{(1-\tau^l-\tau^p)e\theta\bar{y}^s wl + \widetilde{tr} + (1+(1-\tau^k))r\tilde{k} - (1+g_A)\tilde{k}'}{1+\tau^c} \tag{16b}$$

if $l \geq 0$, otherwise $l = 0$.

- much faster than value function iteration

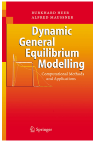
Krusell-Smith Algorithm



- test Euler equation for $\tilde{k}' = 0$: $rf > 0$?
- Problem: initial values at $(s, e, \theta, \tilde{k}_{i_k})$:
  1. Solution for $\tilde{k}_{i_k - 1}$
  2. Solution at age $s+1$
  3. $\tilde{k}' = \tilde{k}$
  4. $\tilde{k}' = 0$
  5. Computation of maximum possible value $\tilde{k}'$ for small $\tilde{c} \approx 0.001$, using half of this value

  $\Rightarrow$ Pick value with lowest absolute value for the residual function rf!
- Next-period capital stock $\tilde{k}'$ not a grid point: linear interpolation of $\tilde{c}'(.)$ and $\tilde{l}'(.)$

Krusell-Smith Algorithm

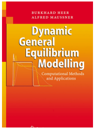


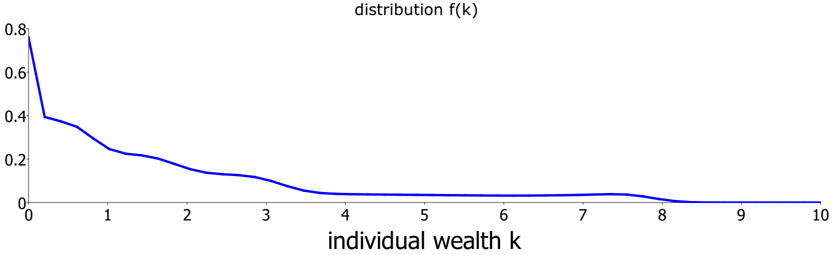
- Choice of asset grid over $\tilde{k}$: $[0, 10.0]$ with 50 grid points

$\rightarrow$ upper boundary non-binding

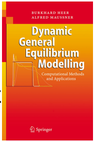
- Choice of initial values for $\tilde{K}$, $\tilde{L}$: $\bar{l} = 0.30$, $\tilde{L} = 0.20$, $r = 2.0\%$
- Update scheme of aggregate variables: linear adjustment of old value by 30% of the difference
- Computational time of Step 1:
  - GAUSS: $\approx$ 2 minutes
  - PYTHON: $\approx$ 12 minutes

OLG Model: Non-stochastic steady state distribution of $\tilde{k}$





Krusell-Smith Algorithm



Step 2:

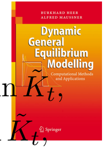
Choose an initial parameterized functional form for the law of motion for the aggregate next-period capital stock and present period employment and transfers:

$$\tilde{K}_{t+1} = g^K(Z_t, \tilde{K}_t) \tag{17a}$$

$$\tilde{L}_t = g^L(Z_t, \tilde{K}_t) \tag{17b}$$

$$\widetilde{Tr}_t = g^{Tr}(Z_t, \tilde{K}_t). \tag{17c}$$

Krusell-Smith Algorithm



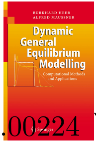
Specification of $g = (g^K, g^L, g^{Tr})'$:

$$
\begin{aligned}
\ln K_{t+1} &= \omega_{K,0} + \omega_{K,1} \ln \tilde{K}_t + \omega_{K,2} \mathbf{1}_{Z_t = Z_1} + \omega_{K,3} \mathbf{1}_{Z_t = Z_1} \ln \tilde{K}_t, \\
\ln \tilde{L}_t &= \omega_{L,0} + \omega_{L,1} \ln \tilde{K}_t + \omega_{L,2} \mathbf{1}_{Z_t = Z_1} + \omega_{L,3} \mathbf{1}_{Z_t = Z_1} \ln \tilde{K}_t, \\
\widetilde{Tr}_t &= \omega_{Tr,0} + \omega_{Tr,1} \tilde{K}_t + \omega_{Tr,2} \mathbf{1}_{Z_t = Z_1} + \omega_{Tr,3} \mathbf{1}_{Z_t = Z_1} \tilde{K}_t,
\end{aligned}
$$

Notice: $\widetilde{Tr}_t$ can be negative $\Rightarrow$ not a logarithmic function

Initial values: ω_0 such that the values are equal to their non-stochastic steady state counterparts.

Krusell-Smith Algorithm



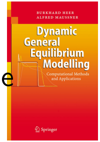
Solution:

$$\omega_K = \begin{pmatrix} 0.0386 \\ 0.0176 \\ 0.9257 \\ -0.0075 \end{pmatrix}, \quad \omega_L = \begin{pmatrix} -1.2663 \\ 0.0227 \\ 0.0748 \\ -0.0046 \end{pmatrix}, \quad \omega_{Tr} = \begin{pmatrix} 0.00224 \\ 0.00555 \\ -0.00173 \\ 0.00149 \end{pmatrix}$$

Regression of simulated $\tilde{K}_{t+1}$ and $\tilde{L}_t$ on $(1, \mathbf{1}_{Z_t=Z_1}, \ln \tilde{K}_t, \mathbf{1}_{Z_t=Z_1} \times \ln \tilde{K}_t)'$ and $\widetilde{Tr}_t$ on $(1, \mathbf{1}_{Z_t=Z_1}, \tilde{K}_t, \mathbf{1}_{Z_t=Z_1} \times \tilde{K}_t)'$:

$\Rightarrow$ all three regressions with $R^2 = 1.000$.

Krusell-Smith Algorithm



Step 3:

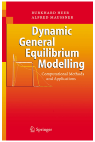
Solve the consumer's optimization problem as a function of the individual and aggregate state variables, $(s, e, \theta, \tilde{k}; Z\tilde{K})$.

- Again, solution of the Euler equation with a non-linear eq. solver
- We start in the last period of life where next-period capital stock is equal to zero: $\tilde{k}'(\tilde{z}) \equiv 0$
- Factor prices depend on $(Z_t, \tilde{K}_t)$ in period t and $t+1$

$$rf(\tilde{z}) = u_c(\tilde{c}(\tilde{z}, 1-l(\tilde{z}))) - \beta \mathbb{E}_t u_c(c'(\tilde{z}', 1-l(\tilde{z}')))(1+(1-\tau^k)r_{t+1}).$$

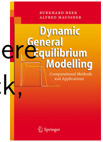
- State vector in period t: $\tilde{z}_t = (s, e, \theta, \tilde{k}_t; Z_t, \tilde{K}_t)$

Krusell-Smith Algorithm



- Computation of next-period utility in the aggregate state $Z_{t+1} = Z'$ and individual state $\theta_{t+1} = \theta'$:
  - Probability: $Prob(Z'|Z) \times Prob(\theta'|\theta)$
  - computation of $\tilde{K}_{t+1}$ with the help of $g^K(.)$
  - Given Z_{t+1}, $\tilde{K}_{t+1}$, we can compute $\tilde{L}_{t+1}$ using $g^L(.)$
  - Given Z_{t+1}, $\tilde{K}_{t+1}$ and $\tilde{L}_{t+1}$, we can compute r_{t+1}
  - To compute $\tilde{c}'$ and l' at age $s+1$ with productivity θ' and aggregates Z_{t+1} and $\tilde{K}_{t+1}$, we interpolate $\tilde{c}(.)$ and $l()$ bi-linearly at $(\tilde{k}_{t+1}, \tilde{K}_{t+1})$

Krusell-Smith Algorithm



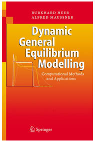
- Grid over aggregate capital stock: $\tilde{K} \in [0.8\bar{K}, 1.2\bar{K}]$, where $\bar{K} = 1.818$ is the non-stochastic steady state capital stock, using $n_K = 7$ grid points
  $\rightarrow$ little curvature of policy function in $\tilde{K}$
  $\rightarrow$ we need to check if the series for the simulated capital stock remains in this interval ✓
- Total number of grid points for policy function:

$$T \times n_\theta \times n_e \times n_k \times n_K \times n_Z = 70 \times 2 \times 2 \times 50 \times 7 \times 2 = 28,000$$

$\rightarrow$ time consuming part:

- GAUSS: 14 minutes
- PYTHON: 39 minutes

Krusell-Smith Algorithm



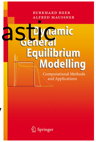
Step 4:

Simulate the dynamics of the distribution function.

Two options:

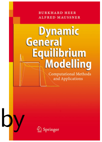
1. Monte-Carlo simulation for e.g. 1,000 households in each cohort $\Rightarrow$ too time-consuming
2. Use discrete approximation of distribution at grid points and study dynamics for the approximation $\Rightarrow$ faster with high accuracy

Krusell-Smith Algorithm



- Initial distribution $f_0(s, e, \theta, \tilde{k})$ in period $t = 0$: non-stochastic steady state
- Distributions f_0, f_1, $\ldots$ are normalized to one (divided by population size N_t)
- Aggregate capital stock in the initial period is equal to $\tilde{K}_0 = \bar{K}$
- Period $t = 0$: Random number generator, with probability 1/2 each the technology state Z_0 in the initial period is either Z_1 or Z_2
- Compute optimal policy functions for next-period wealth $\tilde{k}_1 = \tilde{k}'$ for each value of $(s, e, \theta, \tilde{k})$ on the grid points
- Add the mass at this grid point to the distribution $g_1(s, e, \theta', \tilde{k}')$ with probability $Prob(\theta'|\theta)$

Krusell-Smith Algorithm

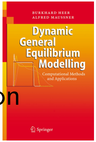


- If $\tilde{k}'$ is not a grid point: split the mass between the neighboring two grid points

  → see Section 7.2 in Heer/Maußner
- Weigh the mass by the survival probability ϕ^s and divide by $(1+n)$ to correct for growing population

- Use random number generator to compute Z_1 given Z_0 and $Prob(Z'|Z)$
- Continue iteration for $t = 2, \ldots, 200$
- Computational time for the simulation of 200 periods:
  - GAUSS: 3 minutes
  - PYHTON: 1 hour:14 minutes

Krusell-Smith Algorithm

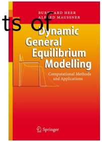


- Distribution function $f_1(s,e,\theta,\tilde{k})$ in period $t+1$ given distribution function $f_0(s,e,\theta,\tilde{k})$ in period t with predetermined aggregate capital stock $\tilde{K}_{t+1}$ and realization $Z_{t+1} \in \{Z_1, Z_2\}$:
  - At age $s=1$: newborn cohort.

$$g_1(1,e,\theta,0) = \frac{\mu^1}{4}$$

μ^1 — share of 1-year old in total population
$1/4$ — share of each individual productivity type (e,θ) among workers

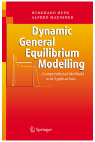
Krusell-Smith Algorithm



- Update at ages $s=2,\ldots,70$: summing over all grid points of $f_0(s,.,.,.)$ at ages $s=1,\ldots,69$:
  1. At grid point $(s,e_j,\theta_{i_\theta},\tilde{k}_{i_k})$, next-period wealth $\tilde{k}'(s,e_j,\theta_{i_\theta},\tilde{k}_{i_k})=k_i$ is a grid point:
  $$f_1(s+1,e_j,\theta',\tilde{k}_i)=Prob(\theta'|\theta_{i_\theta})\times f_0(s,e_j,\theta_{i_\theta},\tilde{k}_{i_k})\times\frac{\phi^s}{1+n},$$
  $$\text{for } \theta'\in\{\theta_1,\theta_2\}$$
  2. Next-period wealth $\tilde{k}'(s,e_j,\theta_{i_\theta},\tilde{k}_{i_k})$ lies between k_i and k_{i+1}
  $$f_1(s+1,e_j,\theta',\tilde{k}_i)=\frac{\tilde{k}_{i+1}-\tilde{k}'}{\tilde{k}_{i+1}-\tilde{k}_i}Prob(\theta'|\theta_{i_\theta})\times f_0(s,e_j,\theta_{i_\theta},\tilde{k}_{i_k})\times\frac{\phi^s}{1+n},$$
  $$f_1(s+1,e_j,\theta',\tilde{k}_{j+1})=\frac{\tilde{k}'-k_i}{k_{i+1}-k_i}Prob(\theta'|\theta_{i_\theta})\times f_0(s,e_j,\theta_{i_\theta},\tilde{k}_{i_k})\times\frac{\phi^s}{1+n},$$
  $$\text{for } \theta'\in\{\theta_1,\theta_2\}$$

$\Rightarrow$ code can easily be parallelized in the loop over ages $s=1,\ldots,69$.

Krusell-Smith Algorithm

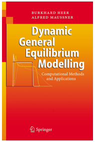


Step 5:

Use the time path for the distribution to estimate the law of motion for $\tilde{K}_t$, $\tilde{L}_t$ and $\widetilde{Tr}_t$.

1. Compute $\tilde{K}$, $\tilde{L}$, $\tilde{C}$, $\widetilde{Beq}_t$: summing up individual variables using mass $g_t(.)$
2. Compute τ^p that balances the social security budget
3. Compute aggregate taxes
4. Compute transfers $\widetilde{Tr}_t$ from the fiscal budget constraint
5. Use OLS regression of $\tilde{K}_{t+1}$, $\tilde{L}_t$ and $\widetilde{Tr}_t$ on $\ln \tilde{K}_t$ and $\tilde{K}_t$ to derive the parameters ω_K, ω_L and ω_{Tr}

Krusell-Smith Algorithm



Step 6:

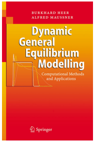
Iterate until the parameters of the functions g^K, g^L and g^{Tr} converge.
$\Rightarrow$ linear update by 30% of the difference

number of iterations in outer loop: 23

total computational time:

- GAUSS: $\approx$ 5 hours
- PYTHON: $\approx$ 2 days

Krusell-Smith Algorithm

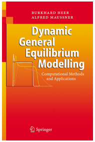


Step 7:

Test the goodness of fit for the functional form using, for example, R^2. If the fit is satisfactory, stop, otherwise choose a different functional form for $g(.) = (g^K, g^L, g^{Tr})'$.

$\Rightarrow R^2 = 1.000$ in all regressions
$\rightarrow$ see discussion in Chapter 8.3 in Heer/Maußner for a discussion of R^2 as a measure of accuracy

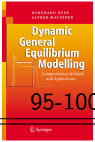
Computation: PYTHON or GAUSS?



RUN TIME: Krusell-Smith Algorithm

	PYTHON	GAUSS
Non-stochastic steady state	11m:53s	2m:19s
Policy functions	39m:03	14m:18s
Simulation (200 periods)	1h:13m:52s	3m:52s
Total	1d:21h:40m:18s	5h:18m:40s

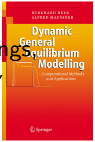
Correlation of income shares with GDP



	0-20%	20-40%	40-60%	60-80%	80-95%	95-100%
US	0.53	0.49	0.31	-0.29	-0.64	0.00
Castañeda et al. (1998)	0.95	0.92	0.73	-0.56	-0.90	-0.84
our model						
i) $pen_t(\epsilon) = \epsilon \cdot \bar{pen}_t$	0.85	0.54	0.60	-0.30	0.23	-0.44
ii) $pen_t(\epsilon) = \bar{pen}_t$	0.75	-0.48	0.41	0.08	0.06	-0.29

Notes: Entries in rows 1 and 2 are reproduced from Table 4 in Castañeda et al. (1998). Annual logarithmic output has been detrended using the Hodrick-Prescott filter with smoothing parameter $\lambda = 100$.

References



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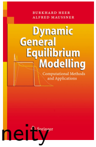

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