Sandwich Caregivers, Elderly Care, and Employment in an OLG Model with Endogenous Fertility *

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Abstract

"Sandwich caregivers" are defined as adults who must care for their dependent parents and children simultaneously. In this paper, we consider an overlapping generations model to study the consequences of such a double burden on their decisions on fertility and career (type of employment). In a fertility-declining environment, subsequent generations face a heavier care burden since they have fewer siblings to share. We show that if the pressure of elderly care on a young worker's time is sufficiently large, more workers will resort to taking nonregular jobs. Furthermore, a sufficiently high upskilling time cost (due to increased studying and training requirements) can prevent the economy from achieving a full regular employment ratio in the long run, regardless of the initial value of the capital stock. We then extend the model to analyze the case when a government elderly care support program is in place. A numerical simulation shows that such a policy can have a positive welfare effect and improve the full-time employment capability of the economy.

Keywords: sandwich care, elderly care, occupational choice, fertility decline, overlapping generations

JEL Classification: E13, J13, J14, J22, J24, O11

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1 Introduction

In many developed countries, the increase in the elderly population is linked to longer life expectancy and lower birth rates, leading to rising demand for long-term care. According to WTO (2015), the number of people aged 60 years or more who require daily care can increase by 42 percent by 2050. The reduction in family size can also put more burden on informal caregivers such as children and spouses of the elderly (Degiuli, 2010).

Despite some governments offering publicly funded care facilities, there are still significant challenges. A report by the OECD in 2023 [1] highlighted that the average cost of institutional care for severe needs is more than twice the median income of people in retirement age, which renders inpatient or hospitalization unaffordable. As a result, a large portion of the elderly still rely on family members for care and nursing at home. Another dimension is the capacity of LTC facilities [2], which can be measured by the number of LTC beds per 1000 population aged 65 years and older. While some countries, such as Germany, France, and Korea, can provide more than the OCED average, many countries, such as Japan, the US, and Italy, have relatively less capacity. The same report shows that for OECD countries in 2021, on average, about 11.5% of people aged 65 and over received LTC, either at home or in LTC facilities. The number is larger than 20% in only four countries (Germany, Switzerland, Israel, Lithuania), while it is less than 4% in many countries, including Japan, Portugal, Canada, the US, etc. Aside from the high cost and limited capacity, norms and cultural preferences must not be ignored. Compared to formal LTC, informal care provided by spouses or children creates a familiar physical and social environment, as well as more reliable and empathetic feelings for the dependents (Lehnert et al., 2019). Therefore, informal care will still play a significant role as a substitute for formal care in the near future (Bonsang, 2009).

When a large part of LTC responsibility is being put on the children, who are often middle-aged adults, the effect on the market labor supply can be expected. Since these adults would have to take care of their dependent parents and children simultaneously, they are often called "sandwich caregivers". This can lead to displacement or discouragement from traditional jobs with demanding schedules (Bolin et al., 2008; Van Houtven et al., 2013). As a result, many people, especially women, may opt for nonregular employment with flexible hours to better support their parents and children. Several empirical papers have addressed the adverse relationship between informal care time pressure and labor supply, including market participation (Leigh, 2010), employment status (Bauer and Sousa-Poza, 2015), and even the ability of workers to accumulate human capital (Skira, 2015).

In this paper, we focus on the long-term effects of such an increasing burden of elderly care on middle-aged adults and see how it affects fertility, career choices, and growth. To that end, we utilize an overlapping generations framework. The dynamic interactions between fertility and career choices are the main ingredients in our analysis. In contrast to previous literature, we consider elderly care time-costly and fertility-sensitive for the family members providing the care. When fertility is high, each individual has more siblings on average, so the burden of elderly care can be shared and alleviated. However, when fertility declines, the reverse happens. Elderly care now takes up a larger portion of the available time endowment. Since the elderly care work directly competes with the time that can be spent on childrearing, it is likely that fewer children will be born in the next generation. As a result, fertility decline has a self-reinforcing mechanism. Simultaneously, the time available to do other activities also reduces. Regular employment often requires more effort and commitment, such as studying, training, or unpaid overtime work. When individuals cannot afford enough time to commit to such activities, they may resort to nonregular (part-time) work, which often does not require as much effort. In that case, individual wealth decreases because the pay from nonregular jobs is lower than regular ones, which affects the ability to accumulate capital.

There is a growing literature on elderly care and sandwich caregivers using an overlapping generations framework. Regarding children's taking care of dependent parents, Tabata (2005) shows its effect on economic growth by introducing parents' health care as a good cost. Our approach, on the other hand, is more similar to Mizushima (2009) where elderly care is time intensive. This perspective is useful when we want to consider its influence on career choice, as regular jobs often require more time commitment than nonregular jobs. Nevertheless, these studies have not considered one important aspect - fertility. Although elderly care can pose a significant burden on children, if the siblings are large enough, then such a burden can be alleviated.

Furthermore, when a society experiences declining birth rates, the burden might increase throughout generations. To explore these dynamics, our paper is heavily influenced by Yakita (2023). Their paper has two main features: the ability to share the elderly care burden among siblings and a state-provided long-term care policy. Based on that approach, we extend the model further by including an endogenous employment decision. Since this fact has been verified by many empirical studies, this particular element can be crucial to understanding the inter-generational consequences of declining fertility in the job market.

For the analysis of this paper, we use a Diamond (1965)-type overlapping generations model with endogenous fertility and career choice. Since full-time jobs often require higher educational levels and skills, we adopt the skill-complimentary capital framework from Krusell et al. (2000). Similar to Yakita (2023) and Pestieau and Sato (2008), we assume that children are obliged to care for their old parents. There is no bargaining, so children must provide all the necessary care their dependent parents require. It can be due to moral obligation, altruistic nature or filial piety. The fertility is endogenized and only demands childcare time (Barro and Becker, 1989). We abstract ourselves from modeling education or upskilling requirements explicitly (de la Croix and Doepke, 2003). Hence, childcare time, elderly care time, and upskilling requirements to become a regular worker are all presented as opportunity costs. As a result, we can consider the employment choice based on the framework of Galor and Zang (1997); Kimura and Yasui (2007); Chen (2010).

By utilizing this model, the paper brings forth three notable contributions to the literature. First, we have analyzed another dimension that can explain declining fertility: elderly care. Existing literature has suggested several mechanisms behind the choice of the low fertility rate, such as the child quality-quantity tradeoff (Becker and Lewis, 1973), children versus self-education tradeoff (Kimura and Yasui, 2007), or increasing life longevity (Chen, 2010; Futagami and Konishi, 2019). Belonging to this class of model, we incorporate the elderly care time as another constraint to each agent's time (Yakita, 2023) and find that it can also explain the decline in fertility trend.

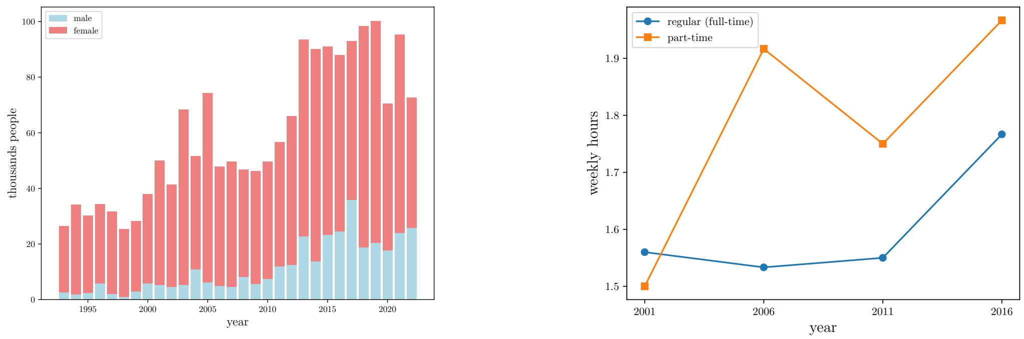
The second contribution concerns the dynamics of regular workers – those equipped with skills to work with capital. In this paper, elderly care is linked to the fertility decisions made by previous generations. When there are many siblings, the elderly care burden is sufficiently low. Individuals reduce the number of children so they can afford enough time to upskill and become regular workers. On the other hand, when there are too few siblings, per-capita elderly care opportunity cost is high. As a result, the time available for childcare and upskilling is reduced, which lowers fertility and makes it harder for an individual to become a regular worker.

Thus, the model proposes two steady-state equilibrium regimes: a full-time equilibrium where everyone works as a regular worker and a part-time equilibrium where only a fraction of the population can afford regular work. The threshold to separate these equilibria depends on a key parameter: the upskilling time requirements, such as tertiary education and personal development training and other unpaid overtime work activities. Intuitively, if the upskilling time is sufficiently low, the economy can converge to an equilibrium where everyone can afford the time to work regular jobs.

Finally, by extending the model to incorporate a policy for elderly care support, we demonstrate that a policy intervention can help improve the situation. Since elderly care demands a higher opportunity cost from regular workers, alleviating this burden frees up time for individuals to upskill and secure full-time jobs. One side effect is that a tax increase may further reduce the average fertility rate, especially for care workers. Nevertheless, the improvement in welfare thanks to a higher level of goods consumption can make up for the loss in altruistic utility, which makes the policy desirable. This result confirms some empirical evidence on policy effectiveness, such as Sugawara and Nakamura (2014); Fu et al. (2017); Løken et al. (2017); Korfhage and Fischer-Weckemann (2024).

The paper is organized as follows. Section 2 presents some stylized facts on the Japanese labor market as a motivation for this paper. Section 3 introduces the structure of the model with middle-aged representative adults who have to take care of their dependent parents and children simultaneously. In Section 4, we characterize the equilibrium. Section 5 analyzes the stability of the equilibrium dynamics and derives the threshold value of the upskilling cost that separates the two regimes. In section 6, we extend the model to capture an elderly care support policy. We then perform numerical exercises to illustrate the dynamics with and without policy intervention in section 7. Finally, section 8 concludes, discusses the main findings of the model, and suggests some future direction for further research.

2 Stylized Facts



(a) Annual job leavers due to nursing or caring for family members in Japan in absolute number.

(b) Time spent on nursing and caring for non-child family members by forms of employment.

Figure 1: Some demographic data related to caring for elder family members in Japan. Source: Japan Labour Force Survey and Survey on Time Use and Leisure Activities.

Since Japan has been at the forefront of super-aging societies with low birth rates and a high age dependency ratio, we take it as an example to motivate the paper and support the theoretical results. As depicted in Fig.1(a), the number of people leaving jobs to care for family members has increased significantly over the past three decades. The phenomenon used to be thought of as being exclusive to women has now taken a toll on both genders since more men also have to leave their jobs to take care of the ill elderly. Every year, around 0.3% of male workers and 1.4% of female workers leave their jobs for nursing. Taking care of the elderly also takes away time that could be spent working. Using 5-year surveys on time usage for Japanese adults, Figure 1(b) reveals that part-time workers spend more time providing informal care for family members than full-time workers. The number in both cohorts has been increasing over time. The data imply that leaving a time-demanding job (full-time employment) to switch to a more flexible job (part-time) seems to be an optimal choice for many people who have to spend time taking care of dependent family members.

The increasing trend in the number of people leaving work to care for the elderly correlates with the rise in the ratio of nonregular workers in Japan (Kitagawa et al., 2018). On the labor demand side, such an upward trend in nonregular workers can be attributed to the change in policy. After the bubble economy burst in 1990, lifetime employment became obsolete. Japanese firms rapidly increased part-time workers to become more flexible and responsive to changes in the global economy (Kato, 2001; Ono, 2010). On the supply side, there is also evidence of time constraints on individuals' ability to commit to full-time jobs and overall labor market participation, especially females (Shimizutani et al., 2008; Yamada and Shimizutani, 2015). Niimi

(2021) also shows that allowing a longer paid leave reduces workers' probability of leaving their jobs within one year to provide parental care, which directly confirms the adverse effect of elderly care responsibility on labor supply. Overall, what we can learn from the case of Japan is that the elderly care burden has an unintelligible contribution to the increasing trend of nonregular workers.

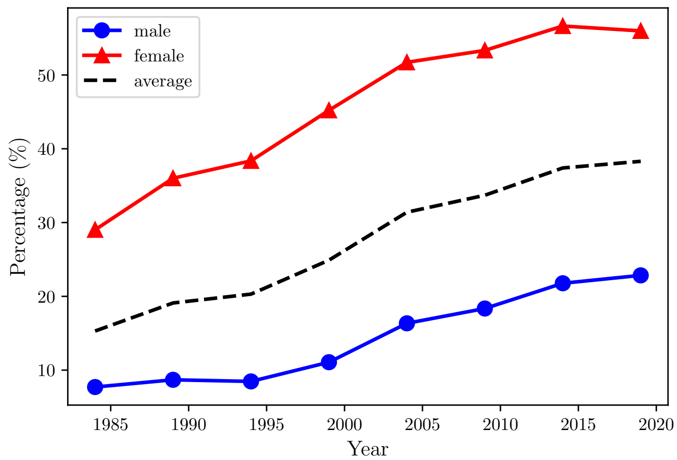
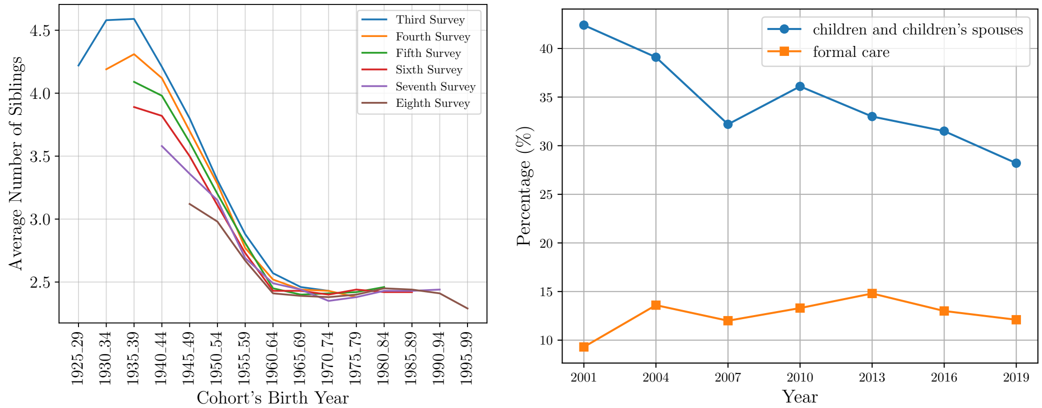


Figure 2: Nonregular Worker Ratio in Japan from 1984 to 2019. Source: Gender Equality Bureau Cabinet Office and Esteban-Pretel and Fujimoto (2020).

Although there is a market for elderly care, in Japan (Inamori, 2017; Kolpashnikova and Kan, 2021) or Germany (Fischer et al., 2022), children are still the main providers of informal care. The increasing burden of elderly care can also be attributed to the decreasing trends in birth rates and the number of siblings per person. Figure 3(a) shows the average number of siblings based on six waves of Japan's National Survey on Household Changes. The average number of siblings dropped significantly from 2.41 for the cohort born between 1960 and 1964 to 2.29 for the generation born between 1995 and 1999. This is the direct result of decreasing birth rates. As a result, the burden of elderly care put on each person will be more severe after each generation.

Furthermore, as shown in Fig.3(b), more than 30% of principal caregivers are the elderly's children and their spouses. Even though recent public and marketized LTC care developments have alleviated the burden, as seen by a rise in formal care coverage, fewer average siblings mean the caregiving responsibilities can still be significant. We can see such an effect in Figures 1a and 1b. The Comprehensive Survey of Living Conditions by the Ministry of Health, Labor and Welfare also reveals that in 2019, around 40% of the caregivers whose care hours are "almost all day" were the children (or their spouses) of the persons who require the care. That is to say, parental care takes a significant amount of time from the children. When they have to spend more time taking care of ill parents, the rate of unemployment or job displacement (changing from regular to nonregular, or increasing use of paid/ unpaid consecutive leaves) increases (Ikeda, 2017; Fu et al., 2017).



(a) Average Siblings/Person (b) Principal Caregivers' Distribution

Figure 3: Average siblings per person and percentage distribution of principal caregivers by their relationship to persons requiring long-term care for Japan. Source: National Survey on Household Changes and Comprehensive Survey of Living Conditions.

3 The Model

Time is discrete, starting from 0 to infinity. The population is a continuum of individuals who live for three periods: childhood, adulthood (middle-aged), and old age. All decisions are made in adulthood. To simplify the analysis, we assume that when an individual becomes old, they require a specific amount of care that takes a fraction h of an adult's total time endowment. Since the elderly care in this paper is not considered a good cost (such as Hashimoto and Tabata (2010)), savings carried to old age are spent only for good consumption. As time cannot be saved, this setup is natural when we want to consider the opportunity cost of elderly care for the next generation.

3.1 Middle-aged Adults

People are born with the same innate abilities. After childhood, they become middle-aged adults, enter the labor market and make decisions on the type of employment they want to pursue. Hence, there are two types of middle-aged adults after the occupation choice is made: type f who have full-time (or regular) employment and type p who work part-time (or nonregular) jobs. In the context of this paper, "full-time" (part-time) is used interchangeably with "regular" (nonregular). A middle-aged adult $i \in \{f, p\}$ makes decisions on consumption when young c_t^i, consumption when old d_{t+1}^i, and the number of children n_{t+1}^i [3].

The lifetime utility function has the form as follows.

$$u_t^i = \ln c_t^i + \gamma \ln n_{t+1}^i + \beta \ln d_{t+1}^i, \tag{1}$$

where $\gamma, \beta \in (0,1)$ are positive parameters capturing altruistic weight and future subjective discount. We assume that each individual is endowed with 1 unit of time. The portion of time to be spent on taking care of dependent parents is $x_t \in (0,1)\ \forall t$, while the time to be spent on each child is assumed to be $z \in (0,1)$ and is constant over time. In order to participate in regular work, an individual has to spend a portion of time $\sigma \in (0,1)$ for upskilling, which can take the form of tertiary education, personal development, etc. This is because regular jobs tend to require a higher level of education[4]. On the other hand, there is no such requirement for nonregular or part-time jobs since there is no skill entry bar.

Denote by w_t^f and w_t^p the wage for regular and nonregular jobs, respectively. We can write the budget constraints for an individual of each employment group as

$$c_t^f + s_t^f = (1 - \sigma - z n_{t+1}^f - x_t) w_t^f, \tag{2}$$
$$c_t^p + s_t^p = (1 - z n_{t+1}^p - x_t) w_t^p. \tag{3}$$

When people retire, they consume the savings accumulated by working when young, so the budget constraint in this period is

$$d_{t+1}^i = R_{t+1} s_t^i, \tag{4}$$

for $i = \{f, p\}$. The optimization problem for a middle-aged adult is to maximize (1) with respect to (2), (3), (4). The optimal choices are

$$(c_t^f) : c_t^f = \frac{1}{1+\gamma+\beta}(1-\sigma-x_t)w_t^f,$$
$$(s_t^f) : s_t^f = \frac{\beta}{1+\gamma+\beta}(1-\sigma-x_t)w_t^f,$$
$$(n_{t+1}^f) : n_{t+1}^f = \frac{\gamma}{z(1+\gamma+\beta)}(1-\sigma-x_t),$$

and

$$(c_t^p) : c_t^p = \frac{1}{1+\gamma+\beta}(1-x_t)w_t^p,$$
$$(s_t^p) : s_t^p = \frac{\beta}{1+\gamma+\beta}(1-x_t)w_t^p,$$
$$(n_{t+1}^p) : n_{t+1}^p = \frac{\gamma}{z(1+\gamma+\beta)}(1-x_t).$$

In this setup, with the absence of childrearing good cost, regular workers tend to have fewer children than nonregular workers because they have to spend more time at work. There is evidence in the Japanese labor market where "married females are less likely to find and sustain a regular job and are more likely to exit the labor market than single females" (Esteban-Pretel and Fujimoto, 2020). Thus, it is reasonable to assume that the fertility rates of nonregular workers are higher than regular ones.

From the optimal choices, it is clear that the fertility motive is influenced by upskilling time σ and commitment to elderly care x. When x_t increases, the pressure of caring for the elderly forces people to have fewer children. In turn, even when there is no change in the demand for elderly care, the next generation might still have to spend more time caring for their parents due to having fewer siblings to share the burden.

3.2 Elderly Care

This section discusses the formulation of the elderly care time cost x_t. For now, we assume that there is no government-supported LTC program, so children provide all the necessary care. This assumption is relaxed in Section 6. Each elderly individual demands a fixed amount of care time $h \in (0, 1)$ on their children. Since children are the only providers of care, the family care production is

$$\delta x_t N_t = h N_{t-1}, \tag{5}$$

where x_t is the amount of time a child spends providing LTC for parents and $\delta > 0$ is home care labor productivity. This setup is similar to Yakita (2023). The LHS represents the total supply, while the RHS is the total demand for elderly care. Dividing both sides by n_t, acknowledging that $N_t = N_{t-1} n_t$ where n_t is the number of children from the middle-aged adults at time $t - 1$, we have

$$\delta x_t n_t = h.$$

The burden of elderly care is shared among siblings. Thus, on average, an individual's time spent on LTC is

$$x_t = \begin{cases} \dfrac{h}{\delta n_t} & \text{if } n_t > \underline{n}, \\ \bar{x} < 1 - \sigma & \text{otherwise,} \end{cases} \tag{6}$$

where $\underline{n}$ is the minimum attainable fertility such that $x(\underline{n}) = 1 - \sigma$. The last term in Eq.(6) restricts the bound of fertility, so the time constraint imposed on each individual is not violated, i.e., ensuring the term $(1 - \sigma - x_t)$ always remains positive [5].

Equation (6) implies that the burden of elderly care is increasing in the care demand h and decreasing in the number of siblings n. Thus, when the economy experiences a fertility decline, more burdens will be placed on each working adult because they will have fewer siblings to share the responsibility. As seen from Eq.(6), we do not consider the case of heterogeneous elderly care among different types of agents. Since regular and nonregular workers can be considered siblings in the model, it is fair to assume that they share the burden of caring for their parents equally.

3.3 Final Good Production

Let the full-time and part-time populations be the N_t^f and N_t^p, respectively. We can denote by $\phi_t \in [0,1]$ the fraction of workers doing regular jobs as

$$\phi_t = \frac{N_t^f}{N_t}. \tag{7}$$

Then, the labor allocation in regular and nonregular jobs is

$$\begin{aligned} L_t^f &= (1-\sigma - zn_{t+1}^f - x_t)\phi_t N_t, \\ L_t^p &= (1 - zn_{t+1}^p - x_t)(1-\phi_t)N_t. \end{aligned} \tag{8}$$

The production of a final good (Y) takes three factors as inputs: physical capital K, regular labor L^f, and nonregular labor L^p. We assume that capital is more complementary to regular jobs than nonregular ones, as they often require a higher set of skills from worker [6]. With these features in mind, we use the following aggregate production function

$$Y_t = A\left[K_t^\alpha (L_t^f)^{1-\alpha} + bL_t^p\right]. \tag{9}$$

with $A > 0, b > 0$, and the capital share $\alpha \in (0,1)$. Let the capital per worker be

$$k_t = K_t/N_t.$$

In a perfectly competitive market, the input factor prices are

$$R_t = \frac{\partial Y_t}{\partial K_t} = A\alpha\left[\frac{k_t}{(1-\sigma - x_t - zn_{t+1}^f)\phi_t}\right]^{\alpha-1}, \tag{10}$$

$$w_t^f = \frac{\partial Y_t}{\partial L_t^f} = A(1-\alpha)\left[\frac{k_t}{(1-\sigma - x_t - zn_{t+1}^f)\phi_t}\right]^{\alpha}, \tag{11}$$

$$w_t^p = \frac{\partial Y_t}{\partial L_t^p} = Ab. \tag{12}$$

From here, we can derive the wage ratio of a regular to a nonregular worker as

$$\omega_t = \frac{w_t^f}{w_t^p} = \frac{1-\alpha}{b}\left[\frac{k_t}{(1 - x_t - \sigma - zn_{t+1}^f)\phi_t}\right]^{\alpha}. \tag{13}$$

4 Equilibrium

4.1 Temporal Equilibrium

Workers can freely move from regular to nonregular jobs or vice versa. Therefore, in equilibrium, they should be indifferent between working in either type of employment, which implies the following condition

$$u_t^f = u_t^p. \tag{14}$$

By plugging the optimal choices obtained in the individual's first-order conditions, the above condition is equivalent to

$$\frac{w_t^f}{w_t^p} = \left(\frac{1-x_t}{1-x_t-\sigma}\right)^{\eta}, \tag{15}$$

where $\eta = \dfrac{1+\gamma+\beta}{1+\beta} > 1$ (See A.8 for derivation). Substituting n_{t+1}^f from the household's solution into Eq. (13), we can write the right-hand side of Eq. (15) as

$$\frac{w_t^f}{w_t^p} = \frac{1-\alpha}{b} k_t^{\alpha} \phi_t^{-\alpha} (1-x_t-\sigma)^{-\alpha} \eta^{\alpha}. \tag{16}$$

Combining (15) and (16), the labor allocation ratio can be expressed as

$$\phi_t(x_t, k_t) = \left(\frac{1-\alpha}{b}\right)^{1/\alpha} \eta k_t \cdot \theta(x_t), \tag{17}$$

where

$$\theta(x_t) = \frac{(1-x_t-\sigma)^{\frac{\eta}{\alpha}-1}}{(1-x_t)^{\eta/\alpha}}. \tag{18}$$

Note that ϕ_t is unambiguously an increasing function of k_t. Since capital is complimentary to regular jobs, when per-capita capital increases, it increases the earnings from regular employment, incentivizing individuals to upskill and obtain a regular job. However, ϕ_t may decline with x_t since having fewer children implies a heavier elderly care burden on the next generation. Its partial effect on ϕ_t can be stated as follows.

Proposition 1. The regular job ratio ϕ_t is decreasing in the elderly burden, i.e., $\phi_t'(x_t) < 0$ if such a burden is sufficiently large, specifically

$$x_t \geq 1 - \frac{\eta\sigma}{\alpha}.$$

Proof: A.1.

Proposition 1 merits some discussion. When the burden of elderly care is sufficiently low ($x < 1 - \eta\sigma/\alpha$), the optimal decision for an individual is to reduce the number of children in order to have enough time to acquire skills and become a regular worker. As a result, the regular worker ratio increases with the decrease in fertility (which implies an increase in elderly care time). However, this is only possible if the increase in elderly care does not impede an individual's ability to spend time upskilling. Once the elderly care gets sufficiently large, the time available for childrearing and upskilling is significantly reduced. The smaller time endowment implies a lower fertility and regular worker ratio. Thus, a U-shaped relationship between x and ϕ can be obtained as depicted in Fig.4.

Since we are interested in the case when an increasing elderly care burden affects regular workers' ability to maintain their jobs, let us assume a lower bound for σ as follows.

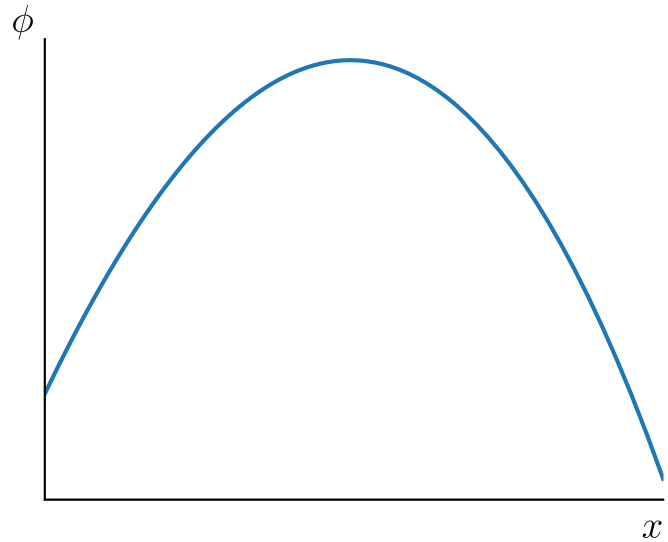


Figure 4: $\phi(x, k)$ when x change, holding k constant.

A higher x implies a lower level of fertility n and vice versa.

Assumption 1.

$$\sigma \geq \frac{\alpha}{\eta}.$$

The dependency of ϕ on k and x can be visualized in Figure 6. Each indifferent curve represents a possible value of ϕ. The closer the curve is to k and farther from x, the higher its value. When we compare point B and point C, we note that they are at the same level of elderly care burden x. However, ϕ_2 is bigger than ϕ_1 because it has a higher level of capital-per-worker. Since capital complements full-time workers, higher income incentivizes individuals to take up regular employment. Let us now compare points A and B. Although they are at the same level of capital per worker, the regular worker ratio at A is higher than B ($\phi_3 > \phi_2$) because of a lower elderly care burden. As each worker has more time endowment, more people can invest σ time proportion on upskilling and become regular workers.

4.2 Intertemporal Equilibrium

To derive the dynamics and characterize the steady-state equilibrium of capital per worker and the average fertility. We first study the dynamics of the average fertility, which can be calculated as

$$n_{t+1} = \phi_t n^f_{t+1} + (1 - \phi_t) n^p_{t+1}. \tag{19}$$

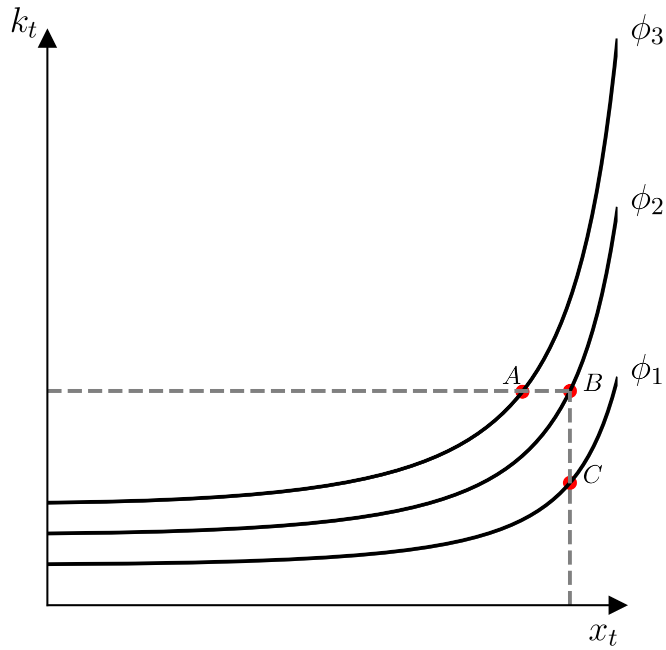


Figure 6: The possibility frontier of $\phi(k_t, x_t)$ with $\phi_1 < \phi_2 < \phi_3$.

Note that the upper bound of ϕ_t is 1 when everyone becomes a regular worker. Let us define $\hat{k}_t$ such that $\phi(x_t, \hat{k}_t) = 1$. Equating (17) to 1 yields

$$\hat{k}_t(x_t) = \left(\frac{b}{1-\alpha}\right)^{1/\alpha} \frac{1}{\eta}\theta^{-1}(x_t). \tag{20}$$

That is, for each value of x_t, we can find a corresponding value of $\hat{k}(x_t)$ such that at the point $(x_t, \hat{k}(x_t))$, $\phi_t = 1$. Since $\phi(x_t, k_t)$ is increasing in k, and the upper bound of ϕ is 1, we can state that

$$\phi_t = \begin{cases} \phi(x_t, k_t) < 1 & \text{if } k_t < \hat{k}_t, \\ 1 & \text{if } k_t \geq \hat{k}_t. \end{cases} \tag{21}$$

Accordingly, the average fertility can be expressed as

$$n_{t+1} = \begin{cases} \dfrac{\gamma}{z(1+\gamma+\beta)}(1 - x_t - \sigma\phi_t) & \text{if } k_t < \hat{k}_t, \\ \dfrac{\gamma}{z(1+\gamma+\beta)}(1 - x_t - \sigma) & \text{if } k_t \geq \hat{k}_t. \end{cases} \tag{22}$$

In the physical capital market, the market clearing condition reads

$$K_{t+1} = [\phi_t s_t^f + (1-\phi_t)s_t^p]N_t. \tag{23}$$

It is more convenient to analyze the fertility dynamics based on x_t instead of n_t. From (6), we can rewrite (22) as

$$x_{t+1} = \Psi(x_t, k_t) = \begin{cases} \dfrac{z(1+\gamma+\beta)hq}{\delta\gamma(1-x_t-\sigma\phi_t)} & \text{if } k_t < \hat{k}_t, \\ \dfrac{z(1+\gamma+\beta)hq}{\delta\gamma(1-x_t-\sigma)} & \text{if } k_t \geq \hat{k}_t. \end{cases} \tag{24}$$

Dividing both sides of (23) by N_{t+1}, we obtain

$$k_{t+1} = \frac{\phi_t s_t^f + (1-\phi_t)s_t^p}{n_{t+1}} = \frac{\beta}{1+\gamma+\beta} \cdot \frac{[(1-x_t-\sigma)\phi_t w_t^f + (1-x_t)(1-\phi_t)w_t^p]}{n_{t+1}}.$$

Using (11) and (12), we can simplify it to

$$k_{t+1} = \frac{A\beta z}{\gamma(1-x_t-\sigma\phi_t)} \left[(1-\alpha)(1-x_t-\sigma)^{1-\alpha}\eta^\alpha\phi_t^{1-\alpha}k_t^\alpha + (1-x_t)(1-\phi_t)b\right].$$

Substituting ϕ_t by equation (21), we can write the dynamics of k as

$$k_{t+1} = \Phi(x_t, k_t) = \begin{cases} \dfrac{A\beta z\left[(1-\alpha)(1-x_t-\sigma)^{1-\alpha}\eta^\alpha\phi_t^{1-\alpha}k_t^\alpha + (1-x_t)(1-\phi_t)b\right]}{\gamma(1-x_t-\sigma\phi_t)} & \text{if } k_t < \hat{k}_t, \\ \dfrac{A\beta z(1-\alpha)\eta^\alpha}{\gamma(1-x_t-\sigma)^\alpha}k_t^\alpha \text{ if } k_t \geq \hat{k}_t. \end{cases}$$

By substituting the formula of ϕ_t in Equation (17), we can further simplify the law of motion for physical capital per worker to

$$k_{t+1} = \Phi(x_t, k_t) = \begin{cases} \dfrac{A\beta zb}{\gamma}\left[\dfrac{(1-x_t)^\eta(1-x_t-\sigma)^{1-\eta}\phi_t + (1-x_t)(1-\phi_t)}{(1-x_t-\sigma\phi_t)}\right] & \text{if } k_t < \hat{k}_t, \\ \dfrac{A\beta z(1-\alpha)\eta^\alpha}{\gamma(1-x_t-\sigma)^\alpha}k_t^\alpha & \text{if } k_t \geq \hat{k}_t. \end{cases} \tag{25}$$

The steady-state equilibrium is characterized by two dynamical equations (24), (25). We can define the intertemporal equilibrium dynamics as follows.

Definition 1 (Intertemporal Equilibrium)**.** Given the initial capital per worker k_0, fertility n_0 and elderly care burden x_0, an equilibrium consists of a sequence of factor prices $\{R_t, w_t^f, w_t^p\}_{t=0}^\infty$, middle-aged agents' allocation $\{c_t^i, s_t^i, d_{t+1}^i, n_{t+1}^i\}_{t=0}^\infty$ for $i \in \{f, p\}$, regular worker ratio $\{\phi_t\}_{t=0}^\infty$, and a sequence of $\{K_t, N_t, x_t\}_{t=0}^\infty$ such that

1. Given initial conditions, the regular worker ratio ϕ_t is determined by Eq.(21).

2. Factor prices $\{R_t, w_t^f, w_t^p\}_{t=1}^\infty$ are determined by Eqs.(10),(11),(12). Given these prices, households make decisions on consumption, saving, and fertility by maximizing (1) subject to constraints (2), (3), (4).

3. The labor market clears following Eq.(8).
4. The capital market clears following Eq.(23).
5. The dynamics of the working population, average elderly care burden, and capital per worker follow Eqs.(22),(24), and (25).

5 Equilibrium Path

There are two potential equilibrium regimes. Let us call the regime with full regular employment ratio ($\phi^* = 1$) as full-time equilibrium and the other with partial regular employment ratio ($\phi^* < 1$) as part-time equilibrium. In what follows (section 5.1), we will prove that there exists a critical value of σ called $\hat{\sigma}$ that separates the two regimes. In particular, if σ is sufficiently high, the economy converges to the part-time equilibrium regardless of initial conditions (x_0, k_0).

In previous literature, Kimura and Yasui (2007) and Chen (2010) show that an economy with a sufficiently high initial capital always converges to the full regular employment regime. In their models, a low fertility choice leaves no consequences for the next generation, so the economy can achieve higher capital per capita by having fewer children. As a result, the dynamics in these models are reduced to be dependent solely on capital. However, in our model, fertility choice has intergenerational effects. To be precise, a low fertility decision of the previous generation can put downward pressure on the time available for their children to spend on upskilling. Hence, the employment choice is also affected by fertility dynamics, not just capital. Since there are two state variables, we use a phase diagram to analyze the model's dynamics under these two regimes.

5.1 Critical Value of σ

Before analyzing the stability of each regime, we have enough ingredients to investigate under which condition an economy will converge to this equilibrium but not the other. The key parameter is the upskilling time cost. Its importance can be stated as follows.

Proposition 2. Let $\hat{\sigma}_1$ be

$$\hat{\sigma}_1 = 1 - \sqrt{4Q},$$

where

$$Q = \frac{(1+\gamma+\beta)}{\gamma} \cdot \frac{h}{\delta} z.$$

Let $\hat{\sigma}_2$ be the unique solution to

$$\frac{\left[\frac{A\beta z(1-\alpha)\eta^\alpha}{\gamma}\right]^{1/(1-\alpha)}}{\left(\frac{b}{1-\alpha}\right)^{1/\alpha}\frac{1}{\eta}} \cdot \frac{(1-x^*-\hat{\sigma}_2)^{\frac{\eta}{\alpha}-1-\frac{\alpha}{1-\alpha}}}{(1-x^*)^{\eta/\alpha}} = 1.$$

where x^* satisfies

$$x^* = \frac{(1-\hat{\sigma}_2) - \sqrt{(1-\hat{\sigma}_2)^2 - 4Q}}{2}.$$

Then, if $\sigma \leq \min(\hat{\sigma}_1, \hat{\sigma}_2) \equiv \hat{\sigma}$, the steady state equilibrium ϕ^* is 1. Otherwise, $\phi^* < 1$.
Proof: A.2.

When $\sigma > \hat{\sigma}$, the model converges to the other equilibrium where only $\phi < 1$ can yield a solution. Intuitively, a higher upskilling cost implies that it is more difficult for agents to balance elderly care and a regular job. On a technical side, for $\hat{\sigma}$ to have economic meaning and avoid ambiguity, we impose the following two restrictions on the parameters.

Assumption 2.

$$0 < Q = \frac{(1+\gamma+\beta)}{\gamma} \cdot \frac{h}{\delta} z < 1/4.$$

Assumption 3.

$$1 + \eta/\alpha > 2/\hat{\sigma}.$$

Assumption 2 has h/δ as the effective time demanded by elderly care and z as the time demanded by child care. For the equilibrium to exist, they have to be sufficiently small. Otherwise, the combination of time cost for child care and elderly care will exceed an individual's time endowment. On the other hand, Assumption 3 is necessary to support the results of Lemma 1 (which will be shown shortly). This restriction on parameter choices does not hamper the main message of the paper while preserving its intuitive results. Therefore, we assume that Assumption 2 and 3 hold for the remainder of the analysis.

Let us explain the intuition behind Proposition 2. In this economy, the burden of elderly care is shared evenly among individuals. As a result, what they can do to mitigate such constraints is to control the number of children to raise. Although becoming a regular worker grants more income that can be used for consumption, an individual has to sacrifice by having fewer children, hence lower utility from altruism. When the upskilling time required to become a regular worker is sufficiently low, they can afford the time to commit to maintaining a regular job while simultaneously caring for children and parents. However, when the time required is sufficiently large, the marginal gain from regular employment is not enough to offset the marginal loss from altruism. As a result, many individuals may prefer to settle for nonregular jobs to afford enough time to take care of children and their dependent parents.

Figure 7 presents a graphical representation of Proposition 2. We can see that when σ lies below $\hat{\sigma}$, the economy can achieve full regular employment in the long run. The regular employment ratio gradually decreases as σ increases further from the critical $\hat{\sigma}$. We shall study the properties of the two regimes in the next two sections.

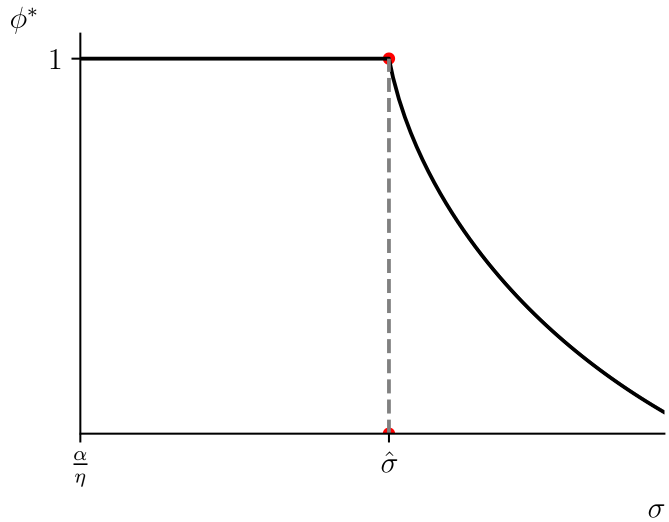


Figure 7: Equilibrium points of ϕ^* under different values of σ.

5.2 Part-time Equilibrium

In this regime, the steady state is $\phi^* < 1$. Following Proposition 2, $\sigma > \hat{\sigma}$ is assumed. The economy is characterized by the following system:

$$k_{t+1} = \Phi(x_t, k_t) = \frac{A\beta zb}{\gamma}\left[\frac{(1-x_t)^\eta(1-x_t-\sigma)^{1-\eta}\phi_t + (1-x_t)(1-\phi_t)}{(1-x_t-\sigma\phi_t)}\right], \tag{26}$$

$$x_{t+1} = \Psi(x_t, k_t) = \frac{Q}{1-x_t-\sigma\phi_t}, \tag{27}$$

with Q is defined in Proposition 2 and

$$\phi_t(x_t, k_t) = \epsilon k_t \theta(x_t) \text{ where } \epsilon = \eta\left(\frac{1-\alpha}{b}\right)^{1/\alpha} > 0. \tag{28}$$

First, we characterize the set of points (x_t, k_t) when there is no change in k_t. Solving $\Phi(x_t, k_t) = k_t$ for x_t is equivalent to solving the following

$$F_{kk}(x_t, k_t) = \Phi(x_t, k_t) - k_t = 0. \tag{29}$$

Although we cannot obtain an explicit solution, the shape of which can be investigated. Let $x_\Phi(x_t)$ be the function of x_t given k_t that satisfies (29). By the implicit function theorem, we have

$$x'_\Phi(k_t) = -\frac{\partial F_{kk}/\partial k_t}{\partial F_{kk}/\partial x_t} = -\frac{\partial\Phi(x_t, k_t)/\partial k_t - 1}{\partial\Phi(x_t, k_t)/\partial x_t}. \tag{30}$$

To see the sign direction of change in k_t, we differentiate $\Phi(x_t, k_t)$ with respect to x_t.

Lemma 1. Given $x_t \in (0, 1-\sigma)$ and Assumption 3 holds, we have

$$\frac{\partial \Phi(x_t, k_t)}{\partial x_t} < 0.$$

Proof: A.3.

As a result, points above the F_{kk} curve will have $k_{t+1} < k_t$, while points below will have $k_{t+1} > k_t$. Notice that $\Phi(x_t, k_t)$ is a convex function of k_t because

$$\frac{\partial \Phi(x_t, k_t)}{\partial k_t} = \frac{\frac{A\beta z b}{\gamma}(1-x_t)(1-x_t-\sigma)\left[\left(\frac{1-x_t}{1-x_t-\sigma}\right)^{\eta} - 1\right]}{(1-x_t-\sigma\phi_t)^2} \cdot \frac{\partial \phi_t}{\partial k_t} > 0$$

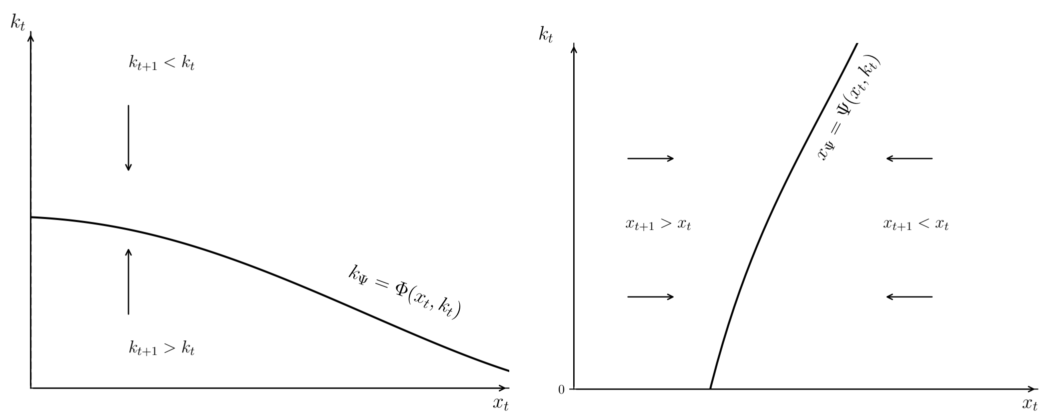
since

$$\left(\frac{1-x_t}{1-x_t-\sigma}\right)^{\eta} > 1 \text{ for all } x_t \in (0, 1-\sigma),\ \eta > 1\ ,$$
$$\frac{\partial \phi_t}{\partial k_t} = \epsilon\theta(x_t) > 0\ \forall x_t \in (0, 1-\sigma),$$

and the second-order derivative is also positive as

$$\frac{\partial^2 \Phi(x_t, k_t)}{\partial k_t^2} = 2\sigma\frac{A\beta z b}{\gamma}(1-x_t)(1-x_t-\sigma)\left[\left(\frac{1-x_t}{1-x_t-\sigma}\right)^{\eta} - 1\right](1-x_t-\sigma\phi_t)^{-3}\left(\frac{\partial \phi_t}{\partial k_t}\right)^2 > 0.$$

Together with the result from Lemma 1, we have enough information to realize the shape of the $k_t = \Phi(x_t, k_t)$ nullcline. Let k^* be the steady state value, then Eq.(30) is increasing in k_t for $k < k^*$ and is decreasing otherwise. The shape of the k-nullcline and its direction of motion are demonstrated on the left panel of Figure 8.



(a) (b)

Figure 8: The two nullclines when $\phi^* < 1$.

Now, we characterize the set of points (x_t, k_t) when there is no change in x_t. This is done by letting $x_t = \Psi(x_t, k_t)$ and solving for k_Ψ such that

$$x_t = \frac{Q}{1 - x_t - \sigma\epsilon\theta(x_t)k_t},$$

For each solution x_t, we can express the corresponding k_Ψ as

$$k_\Psi(x_t) = \frac{-x_t^2 + x_t - Q}{\sigma\epsilon x_t \theta(x_t)}.$$

Lemma 2. As long as Assumption 2 holds, the value of $k_\Psi(x_t)$ is always positive.
Proof: **A.4**

Lemma 2 is a direct consequence of Assumption 2, where we have restricted Q so that the model produces meaningful results. Based on this result, we can pin down the upper bound of x_t on the phase plane. Furthermore, differentiating $k_\Psi(x_t)$ with respect to x_t yields

$$k'_\Psi(x_t) = \frac{1}{\sigma\epsilon}\left[\frac{\theta(x_t)(Q - x_t^2) + \theta'(x_t)(x^2 - x_t + Q)}{x_t^2\theta^2(x_t)}\right]. \tag{31}$$

Lemma 3. $k'_\Psi(x_t) > 0$ for all x_t in $(0, 1-\sigma)$.
Proof: **A.5**.

Lemma 3 provides the information of the shape of the $x_t = \Psi(x_t, k_t)$ nullclines. To describe the direction of change in x_t for this nullcline, we differentiate $\Psi(x_t, k_t)$ to k_t and obtain

$$\Psi'(k_t) = \frac{Q\sigma\epsilon\theta(x_t)}{(1 - x_t - \sigma\epsilon\theta(x_t)k_t)^2} > 0.$$

Hence, $x_{t+1} > x_t$ above and $x_{t+1} < x_t$ below the curve. By restricting the field to contain only positive k_t, the $x_t = \Psi(x_t, k_t)$ curve is demonstrated in the right panel of Figure 8.

At this stage, we have enough information to plot all possible time paths for the dynamics in a single diagram, as shown in Figure 9. The intersection of the two phase lines represents the steady state. Given any initial combination of positive k and x, the time path converges unanimously to a single intersection. Thus, this steady state is a sink. The stability analysis presented here confirms that when σ is sufficiently high, the economy converges to the equilibrium where nonregular and regular workers co-exist, regardless of the initial values $\{k_0, x_0\}$.

5.3 Full-time Equilibrium

In contrast to the previous case, every individual can afford regular jobs in a full-time equilibrium. The steady state of ϕ^* equals 1. The following system characterizes the

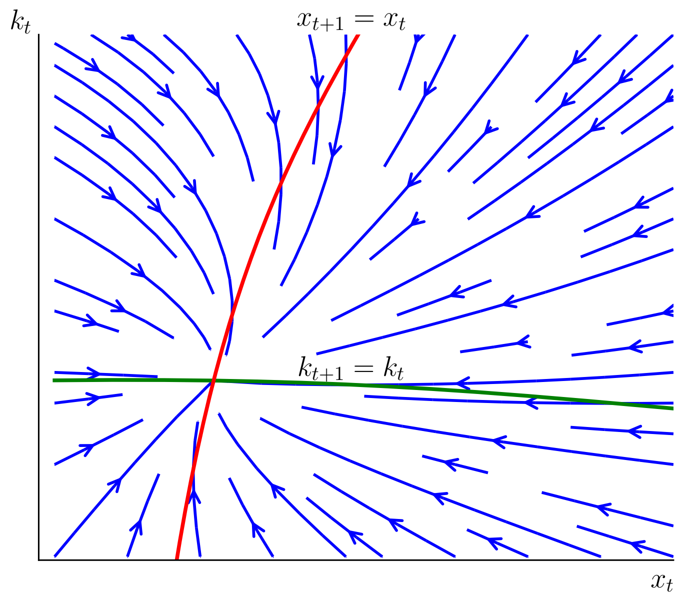


Figure 9: Phase portrait for $\phi^* < 1$ equilibrium.

dynamics of this economy

$$k_{t+1} = \Phi(x_t, k_t) = \frac{A\beta z(1-\alpha)\eta^\alpha}{\gamma(1-x_t-\sigma)^\alpha} k_t^\alpha,$$

$$x_{t+1} = \Psi(x_t) = \frac{Q}{\gamma(1-x_t-\sigma)}.$$

To study the stability of these dynamics, we construct a phase plane. In this case, the steady state is easy to find.

Lemma 4. The dynamical equation $x_{t+1} = \Psi(x_t)$ has at most two steady states

$$x_1^* = \frac{(1-\sigma) - \sqrt{(1-\sigma)^2 - 4Q}}{2}, \qquad x_2^* = \frac{(1-\sigma) + \sqrt{(1-\sigma)^2 - 4Q}}{2}.$$

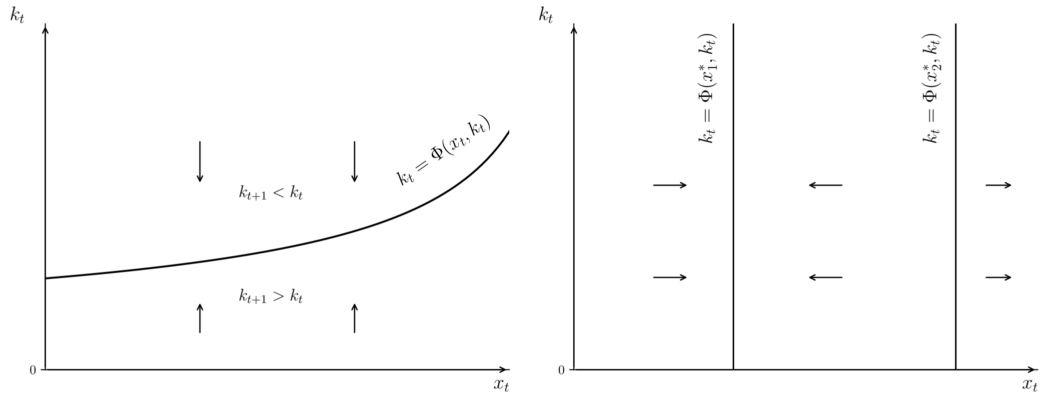
But only x_1^* is locally stable.
Proof: A.6.

The locally stable steady state k_1^* can be solved by letting $k_1^* = \Phi(x_1^*, k_1^*)$:

$$k_1^* = \left[\frac{A\beta z(1-\alpha)\eta^\alpha}{\gamma(1-x_1^*-\sigma)^\alpha}\right]^{\frac{1}{1-\alpha}}. \tag{32}$$

To show the stability of this equilibrium, we construct a phase plane similar to the previous case. First, we characterize the set of points where (k_t, x_t) where there is no change in k_t. Let $k_t = \Phi(x_t, k_t)$ and solve for x_t, we obtain

$$x_t = (1-\sigma) - \left[\frac{A\beta z(1-\alpha)\eta^\alpha}{\gamma k_t^{1-\alpha}}\right]^{1/\alpha} = x_\Phi(k_t).$$



(a) (b)

Figure 10: The two nullclines when $\phi^* = 1$.

Differentiating $x'_\Phi(k_t)$ leads to

$$x'_\Phi(k_t) = \frac{1}{\alpha}\left[\frac{A\beta z(1-\alpha)\eta^\alpha}{\gamma k_t^{1-\alpha}}\right]^{1/\alpha - 1}(1-\alpha)k_t^{-\alpha} > 0.$$

Hence, the function $x_\Phi(k_t)$ is increasing in k_t. Furthermore, since $\Phi(x_t, k_t)$ increases unambiguously with x_t, we have $k_{t+1} > k_t$ on the right (below) of the curve and $k_{t+1} < k_t$ on the left (above), as illustrated in Fig. 10(a).

On the other hand, the second dynamics $\Psi(x_t)$ does not depend on k_t. Since two steady states of x exist, we represent the $x_t = \Psi(x_t)$ by two vertical lines. The direction of motion for this system is plotted in Fig. 10(b). Note that only the lower steady state x_1^* is stable. Using all the information we have gathered so far, we can visualize all time paths of this equilibrium in Fig. 11. In this case, there are two steady states, but only the one with the lower elderly care burden (x_1^*, k_1^*) is stable.

Starting from any initial values of k and $x < x_2^*$ [7], the economy converges to the full-time equilibrium. The important implication drawn from Figure 9 and 11 is that the initial values of capital per worker k_0 and fertility n_0 (which implies elderly care x_0) on their respective feasible sets do not matter. To which equilibrium regime the dynamics converge to depends on the value of upskilling cost σ relative to its threshold $\hat{\sigma}$, as shown in Proposition 2.

5.4 A Rise in Upskilling Time Cost σ

One of the most important factors when a company employs a regular worker is the skill level, reflected by parameter σ in this model. Although not explicitly modeled, upskilling time often correlates with tertiary education and training. Since the length of education attainment has increased substantially (Barro and Lee, 2013), it is important to analyze the model dynamics when σ increases.

First, we consider the society where both regular and nonregular workers exist. The effect of a rise in σ can be summarized in the following proposition

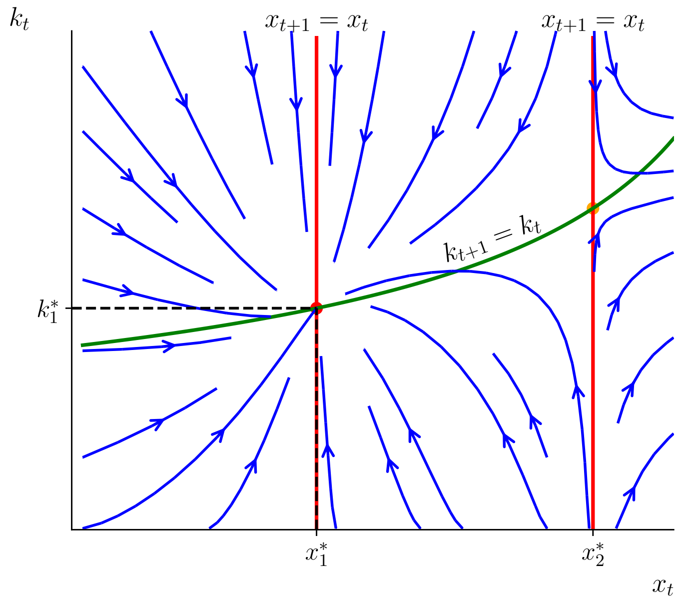


Figure 11: Phase portrait for $\phi^* = 1$ equilibrium.

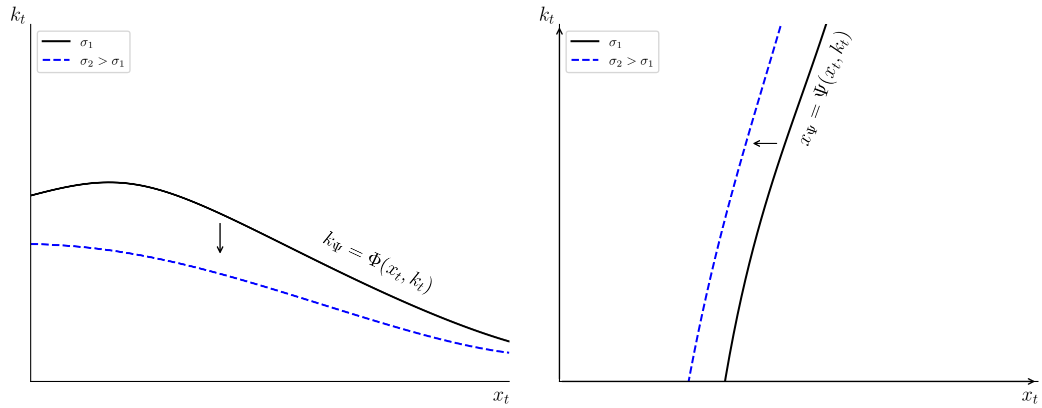
Proposition 3. Longer upskilling time requirements for regular jobs (a higher σ) will change the equilibrium states such that $\Phi'(\sigma) < 0$ and $\Psi'(\sigma) > 0$. Consequently, the equilibrium ϕ will be lowered.
Proof: A.7.

To understand proposition 3 more clearly, suppose that the economy is currently in the part-time equilibrium state when $\sigma = \sigma_1$. Assume that σ rises to a higher value at σ_2, then $\Phi'(\sigma) < 0$ implies that the $k_t = \Phi(k_t, x_t)$ nullcline will shift down while $\Psi(\sigma) > 0$ implies that the $x_t = \Psi(k_t, x_t)$ nullcline will shift up (to the left). We can visualize such movements in Fig. 12. The combination of these new changes in dynamics is visualized in Figure 13. Suppose the economy is already at a part-time equilibrium state where $\phi < 1$, then an increase in σ will lower ϕ even further.

Let us explain the mechanism behind these changes. When the upskilling time increases from σ_1 to σ_2, it will put pressure on the time allocation of the workers. Although workers can choose the number of children to have, they still have the responsibility to fulfill elderly care responsibility. Since more people now cannot afford σ_2, a larger share of workers will choose nonregular jobs compared to the previous generation. When the economy has fewer workers to work with capital, the capital accumulation is reduced, resulting in a downward shift in the left panel of Fig.12.

Consequently, since a non-regular worker has higher fertility than a regular worker, the average fertility improves, which reduces the elderly care burden x_t. As a result, we have a leftward shift of equilibrium x on the right panel of Fig.12. Note that with fewer full-time workers to work with capital, while fertility has risen due to an increase in nonregular workers, the capital shallowing effect dominates.

In what follows, we consider the effect of a rise in σ when the economy is currently



(a) (b)

Figure 12: The changes in the nullclines following a rise in σ for $\phi^* < 1$.

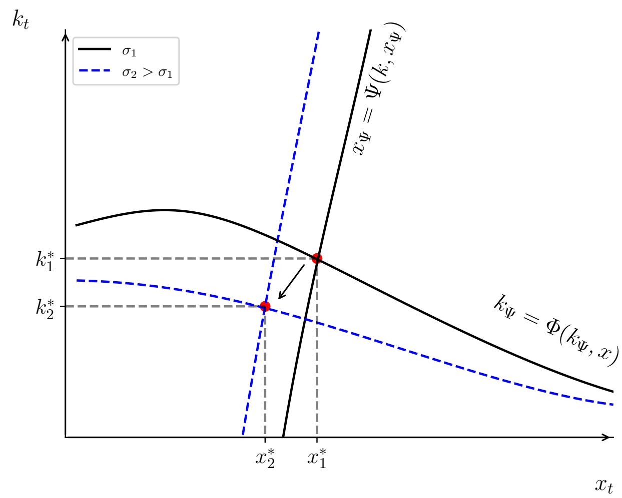


Figure 13: Changes in equilibrium when σ increases ($\phi^* < 1$).

settled with the $\phi^* = 1$ equilibrium. Two scenarios can happen. If σ increases to a new value smaller than $\hat{\sigma}$ as defined in Proposition 2, then there will be no change to the equilibrium ϕ^*. Moreover, based on the phase portrait we analyzed earlier, it is straightforward to derive the new dynamics. The nullcline $k_t = \Phi(x_t, k_t)$ shall shift upward, and the nullcline $x_t = \Psi(x_t, k_t)$ of the stable (lower) equilibrium shifts to the right while the other unstable nullcline shifts to the left. The new dynamics can be seen in Fig.14 (for clarity, we have restricted the x-axis to contain only the stable equilibrium x).

The economy converges to a new equilibrium with a higher capital per worker $k_2^* > k_1^*$ and a larger elderly care burden $x_2^* > x_1^*$. This is because a rise in σ forces individuals to spend more time upskilling rather than child-rearing. Thus, we have a lower fertility rate, which translates to a higher capital per worker due to capital deepening, and more time needs to be spent on elderly care due to fewer siblings.

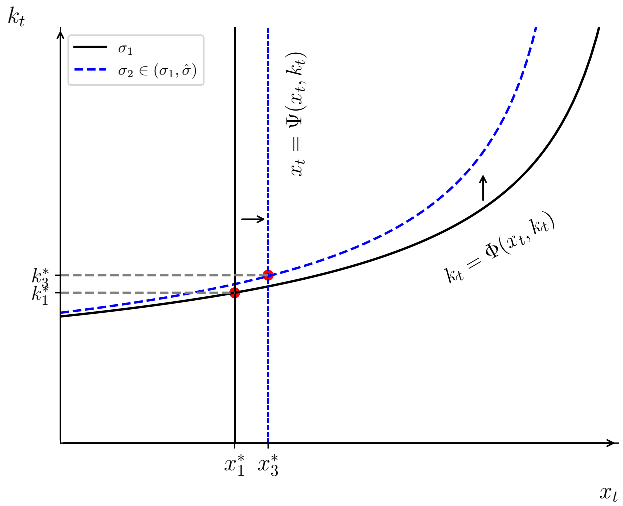


Figure 14: Changes in equilibrium when σ increases ($\phi^* = 1$).

The second scenario is when a rise in σ is substantial enough such that $\sigma_2 > \hat{\sigma}$. Under the new time constraints and work requirements, individuals cannot afford sufficient time for double caring. Some of them now choose to do part-time jobs. The economy has switched from the full-time regime to the other regime where both regular and nonregular workers exist. In this case, the phase diagram would look analogous to Fig.9.

6 Extension: Elderly Care Support Policy

The analysis thus far has not considered any government interventions. Indeed, this can be the case in developing countries where social security programs for elderly care are limited (Brodsky et al., 2003). In reality, however, many advanced countries have implemented some elderly care support policies to alleviate the problem, such as Japan (Ito and Sako, 2023). In this section, we extend the model by incorporating a government who collects taxes to fund a social security policy that provides a universal elderly care program [8]. For simplicity, we assume that there are only two elderly care providers: the state and the elderly's children.

6.1 The Environment

Every period, the government imposes a uniform lump-sum tax τ_t on every citizen to fund the Elderly Care Support Program. It employs labor on the market to perform care work under the price p_t. This price cannot be higher than the full-time wage w_t^f. Otherwise, no full-time labor will be allocated to the final good production as agents prefer to do care work. Then, production cannot be carried out as there are no full-time labor inputs to work with capital. As a result, part-time labor will be the main provider of elderly care.

Part-time labor employed in either final good production or care work does not use capital, so we assume they share the same level of productivity. In equilibrium, we must have the following relationship

$$p_t = w_t^p.$$

so that part-time workers are indifferent between working for the care sector and the final good production. Since $w_t^f > w_t^p$ for all t, full-time workers will provide no elderly care and pay the service at the price p_t. The new budget constraints for both types of agents can be written as follows

$$c_t^f + s_t^f = (1 - \sigma - zn_{t+1}^f)w_t^f - \tau_t, \tag{33}$$

$$c_t^p + s_t^p = (1 - zn_{t+1}^p - x_t^p)w_t^p + x_t^p p_t - \tau_t, \tag{34}$$

where x_t^p is the care provision provided by part-time workers. On the RHS of Eq.(34), the first term indicates earnings from final good production, and the second term indicates the earnings from working for the elderly care program. The demand and supply of elderly care in Eq.(5) are changed to

$$\delta x_t^p(1 - \phi_t)N_t = hN_{t-1}.$$

In per part-time worker terms, the elderly care time is given by

$$x_t^p = \frac{h}{\delta(1 - \phi_t)n_t}, \tag{35}$$

with n_t is still the average fertility rate. The government balances the budget every period such that

$$\tau_t N_t = p_t x_t^p (1 - \phi_t)N_t. \tag{36}$$

where the LHS represents tax revenue, and the RHS shows the total cost of the elderly care support policy. Substituting Eq.(35) into (36) allows us to determine the tax rate as

$$\tau_t = \frac{p_t h}{\delta n_t} = \frac{w_t^p h}{\delta n_t}. \tag{37}$$

Let

$$\chi_t(n_t) = \frac{h}{\delta n_t}, \tag{38}$$

we can simply express the tax amount as $\tau_t = w_t^p \chi_t$.

The production function of final goods remains unchanged from Equation (9). However, because this policy frees up the time constraint for workers, we have the new input prices as follows.

$$
\begin{aligned}
R_t &= A\alpha \left[\frac{k_t}{(1-\sigma - z n^f_{t+1})\phi_t} \right]^{\alpha-1}, \\
w^f_t &= A(1-\alpha) \left[\frac{k_t}{(1-\sigma - z n^f_{t+1})\phi_t} \right]^{\alpha}. \\
w^p_t &= Ab.
\end{aligned}
\tag{39}
$$

A middle-aged adult maximizes (1) with respect to the new constraints (33) and (34), taking the tax as given. The optimal choices for a full-time middle-aged adult are

$$
\begin{aligned}
c^f_t &= \frac{1}{1+\beta+\gamma}((1-\sigma)w^f_t - \tau_t), \\
s^f_t &= \frac{\beta}{1+\beta+\gamma}((1-\sigma)w^f_t - \tau_t), \\
n^f_{t+1} &= \frac{\gamma}{z(1+\beta+\gamma)} \left[(1-\sigma) - \frac{\tau_t}{w^f_t} \right],
\end{aligned}
$$

and for a part-time middle-aged adult (where we have used $p_t = w^p_t$ in equilibrium), the choices are

$$
\begin{aligned}
c^p_t &= \frac{1}{1+\beta+\gamma}(w^p_t - \tau_t), \\
s^p_t &= \frac{\beta}{1+\beta+\gamma}(w^p_t - \tau_t), \\
n^p_{t+1} &= \frac{\gamma}{z(1+\beta+\gamma)} \left(1 - \frac{\tau_t}{w^p_t} \right).
\end{aligned}
$$

As a result, we can see that tax policy will have a negative impact on saving and fertility.

6.2 Equilibrium Dynamics

In equilibrium, a middle-aged agent is indifferent between becoming a full-time or part-time worker. The equilibrium condition (14) must hold at all t. Substituting the new FOCs into (14), using the tax rate (37) and Equation (38), we obtain the equilibrium condition

$$
\chi_t = \frac{\omega_t^{1-\frac{1}{\eta}} - (1-\sigma)\omega_t}{\omega_t^{1-\frac{1}{\eta}} - 1}. \tag{40}
$$

where ω_t is the wage premium w^f_t / w^p_t (See B.1 for derivation). Since χ_t indicates the elderly care parameter that would determine policy, we can solve for ω_t given χ_t in equilibrium with Eq.(40).

Lemma 5. The increase in state-funded elderly care provision χ_t will reduce the wage premium.

$$\omega_t'(\chi_t) < 0.$$

Proof: B.2.

Using the wage formulations, we can obtain

$$\omega_t = \frac{w_t^f}{w_t^p} = \left(\frac{1-\alpha}{b}\right) k_t^{\alpha}\phi_t^{-\alpha}\left(\frac{1-\sigma+(\eta-1)\chi_t/\omega_t}{\eta}\right)^{-\alpha}.$$

Isolating ϕ_t on one side, we have

$$\phi_t = \left(\frac{1-\alpha}{b}\right)^{\frac{1}{\alpha}} \eta k_t \lambda(\chi_t). \tag{41}$$

where

$$\lambda(\chi_t) = \frac{\omega(\chi_t)^{-\frac{1}{\alpha}}}{1-\sigma+(\eta-1)\frac{\chi_t}{\omega(\chi_t)}}.$$

Similar to the previous analysis, ϕ_t is still an increasing function of capital-worker ratio k_t. We proceed to verify the effect of χ_t on ϕ_t. Differentiating ϕ_t with respect to χ_t gives the following result.

Proposition 4. The increase in state-funded elderly care provision χ_t can improve the full-time worker ratio since

$$\frac{\partial \phi_t}{\partial \chi_t} > 0.$$

Proof: B.3.

Contrary to the economy without any policy intervention, expanding public elderly care policy programs makes it easier for individuals to engage in regular jobs. Intuitively, since it is time-intensive, elderly care asserts a higher opportunity cost for a regular worker than its nonregular counterpart. With the elderly care support policy, that cost is relieved, which increases the marginal gain by taking a regular job. In exchange, however, savings and fertility will be affected due to the need to increase individual taxes. Note that the economy cannot function without caregivers. As a result, we must rule out the case $\phi_t = 1$ to reserve the fraction $(1-\phi_t)$ of the population as caregivers.

We now proceed to characterize the intertemporal equilibrium of this economy. The labor market clearing condition is

$$\begin{aligned} L_t^f &= (1-\sigma-zn_{t+1}^f)\phi_t N_t, \\ L_t^p &= (1-zn_{t+1}^p-x_t^p)(1-\phi_t)N_t. \end{aligned} \tag{42}$$

The dynamics of average fertility can be expressed as

$$n_{t+1} = \phi_t n_{t+1}^f + (1-\phi_t) n_{t+1}^p = \frac{\gamma}{z(1+\beta+\gamma)} \left[\left(\chi_t - \frac{1}{\omega_t}\chi_t - \sigma \right) \phi_t + (1-\chi_t) \right]. \tag{43}$$

The market clearing condition for the physical market is still

$$K_{t+1} = [\phi_t s_t^f + (1-\phi_t) s_t^p] N_t.$$

Hence, the capital per worker accumulates according to

$$\begin{aligned} k_{t+1} &= \frac{\phi_t s_t^f + (1-\phi_t) s_t^p}{n_{t+1}} = \frac{\beta z}{\gamma} \frac{\left(1-\sigma-\frac{\chi_t}{\omega_t}\right)\phi_t w_t^f + (1-\chi_t)(1-\phi_t) w_t^p}{\left(\chi_t - \frac{1}{\omega_t}\chi_t - \sigma\right)\phi_t + (1-\chi_t)} \\ &= \frac{\beta z}{\gamma} \cdot \frac{(1-\sigma-\omega_t^{-1})\phi_t + (1-\chi_t)\omega_t^{-1}}{(\chi_t - \chi_t\omega_t^{-1} - \sigma)\phi_t + (1-\chi_t)} \cdot w_t^f. \end{aligned}$$

From the wage equation of a full-time worker and the fertility choice, we can simplify the dynamics of k_t to the following equation

$$k_{t+1} = \frac{\beta z}{\gamma} \cdot \frac{(1-\sigma-\omega_t^{-1})\phi_t + (1-\chi_t)\omega_t^{-1}}{(\chi_t - \chi_t\omega_t^{-1} - \sigma)\phi_t + (1-\chi_t)} \cdot A(1-\alpha) \cdot \left(\frac{\eta k_t}{\phi_t[1-\sigma+(\eta-1)\chi_t/\omega_t]} \right)^{\alpha}. \tag{44}$$

We can then define the equilibrium dynamics of the economy. Given the initial capital per worker k_0, and fertility n_0, an equilibrium consists of a sequence of tax policy $\{\tau_t\}_{t=0}^{\infty}$, factor prices $\{R_{t+1}, w_t^f, w_t^p\}_{t=0}^{\infty}$, middle-aged households' decisions rules $(c_t^j, s_t^j, n_{t+1}^j)_{t=0}^{\infty}$ for $j \in \{f, p\}$, regular worker ratio $\{\phi_t\}_{t=0}^{\infty}$, and a sequence of $\{K_t, N_t, x_t^p\}_{t=0}^{\infty}$ such that:

1. Given the fertility, the government imposes a lump-sum tax to fund elderly care service $\{\tau_t\}_{t=0}^{\infty}$ by solving Eq.(38) and (37).

2. The wage premium $\{\omega_t\}_{t=0}^{\infty}$ is solved according to Eq.(40).

3. Based on the wage premium $\{\omega_t\}_{t=0}^{\infty}$, individuals choose suitable employment such that the utility from working full-time is equal to the utility from working part-time. The regular worker ratio $\{\phi_t\}_{t=0}^{\infty}$ is solved according to Eq.(41).

4. Factor prices $\{R_{t+1}, w_t^f, w_t^u\}_{t=0}^{\infty}$ are determined by a set of equations described in (39). Given these prices and tax $\{\tau_t\}_{t=0}^{\infty}$, $(c_t^f, s_t^f, n_{t+1}^f)_{t=0}^{\infty}$ and $(c_t^p, s_t^p, n_{t+1}^p)_{t=0}^{\infty}$ solve the middle-aged adults' maximization problem.

5. The labor market clears according to Eq.(42).

6. The capital market clears according to Eq.(23)

7. The dynamics of the working population and capital per worker follow Eqs.(43),(44).

The main result of an elderly care support policy lies in Proposition 4, where we show that public provision of elderly care will improve the ratio of workers engaged in full-time employment. With the rise of full-time employment under this policy, there are two immediate effects: (1) fertility may fall further because a larger fraction of the economy has become full-time workers, and (2) the wage premium is reduced. At this point, the model cannot be solved analytically. In the next section, we evaluate the effectiveness of this policy by numerical simulations.

7 Numerical Examples

7.1 Benchmark Case

In this section, we use numerical examples to illustrate the dynamics of the model. Although calibration is not considered, some parameters are based on observations of Japanese data.

One period (one generation) is equivalent to 30 years in real life. Let us assume that the typical working hours are 8. After accounting for sleep and personal activities, the rest of the time can be devoted to elderly care. A reasonable assumption is around 1/3 of the time endowment per day. According to the Comprehensive Survey of Living Conditions 2019, among the elderly, around 80% of males and 90% of females who require LTC are those aged 75 and older, so we can consider this age group as the main demand for care (about 30%). Furthermore, 40% of the “almost all day” caregivers are the children of the person requiring LTC. From these pieces of information, we can calibrate the average elderly care time cost per person $h = 0.3 \times 0.3 \times 0.4 = 0.04$, which translates to roughly 0.72 hours per day of elderly care. According to Urwin et al. (2023), the informal care time in the UK (averaging for weekdays and weekends) is 0.75 fraction of a 24-hour day with the average number of adults in the household of 2.52. Using this number, we set $\delta = 0.3$, implying that the care from one person can meet around 1/3 the total LTC demand of the elderly. As a result, the elderly care per household is $h/\delta = 0.13$, or equivalently 3.2 hours a day.

For other parameters, we follow the literature. Subjective discount is calibrated as $\beta = 0.99^{30}$, while the capital share α is set to 0.3. The altruism weight is set at 0.4 to ensure nondecreasing population growth. Following de la Croix and Doepke (2003), we set the time cost of childrearing at 0.08. Based on these parameters, we can calculate a threshold value $\hat{\sigma}$ of 0.42. This implies that if the unpaid time for upskilling, such as studying, self-development and training, takes up at least 0.42 fraction of working hours, the economy will have a higher ratio of nonregular workers. We now simulate two economies, one with a high value of $\sigma > \hat{\sigma}$ and the other with a low σ. Their respective dynamics are presented in Fig.15.

Imagine two developing economies with the same initial high fertility rates and low capital per worker. In Fig.15, country A, represented by solid lines, has $\sigma = 0.35$ and

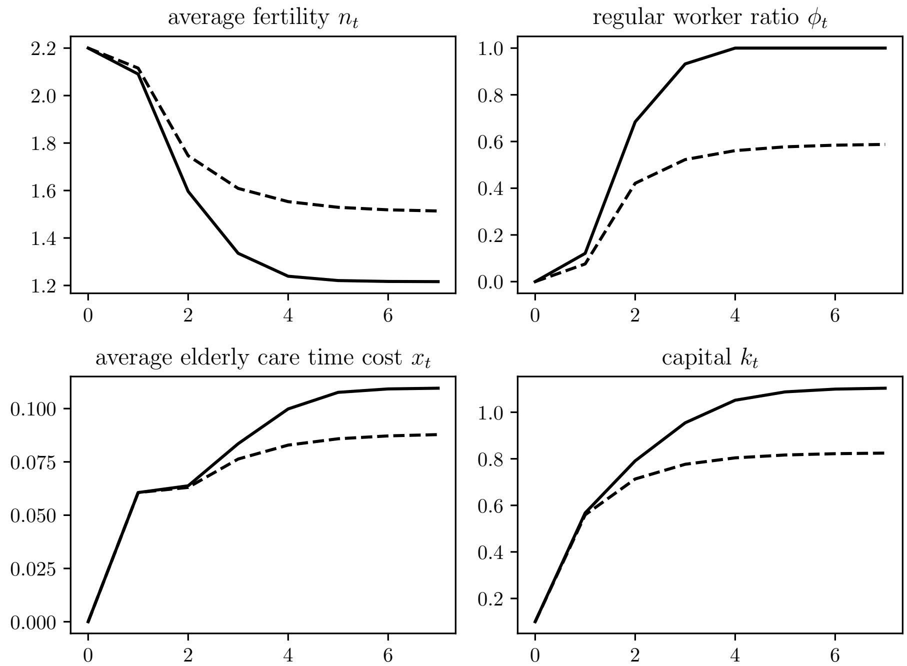


Figure 15: Simulation dynamics for two economies with the same initial states, with the common threshold $\hat{\sigma} = 0.45$. The solid line is for the economy A with $\sigma = 0.35$, and the dashed line represents the economy B with $\sigma = 0.5$. All other parameters are the same: $h = 0.04, b = 0.45, \delta = 0.3, \beta = 0.99^{30}, \gamma = 0.4, \alpha = 0.3, z = 0.08, A = 8$.

country B, represented by dashed lines, has $\sigma = 0.45$. A higher education time cost in country B implies that the productivity of education is lowered, i.e., it takes longer for an agent to finish upskilling. Both countries go through demographic changes. As the economy grows, the overall fertility rates decrease since more and more people spend time upskilling to secure regular jobs. With reduced family size, the burden of elderly care time costs increases, further accelerating fertility decline. Although we observe the same trends in the two simulated economies, their respective levels of wealth differ. Due to the high education opportunity cost, the regular worker ratio in country B converges to its steady state of 60%. In contrast, country A can achieve the 100% regular worker ratio after four periods.

However, country B's average fertility rate is higher than that of country A's, mainly because it has more part-time workers. Although the levels of capital per worker increase in both countries, the lower education efficiency in country B also implies a lower overall level of capital and a higher wage gap. In the steady state, those who acquire education and upskill can earn around 2.3 times more than part-time workers. Meanwhile, inequality does not exist in country A since everybody is employed as a regular worker.

Our model can also generate some interesting dynamics depending on the economy's initial state. In the previous example, we examined the case of two developing

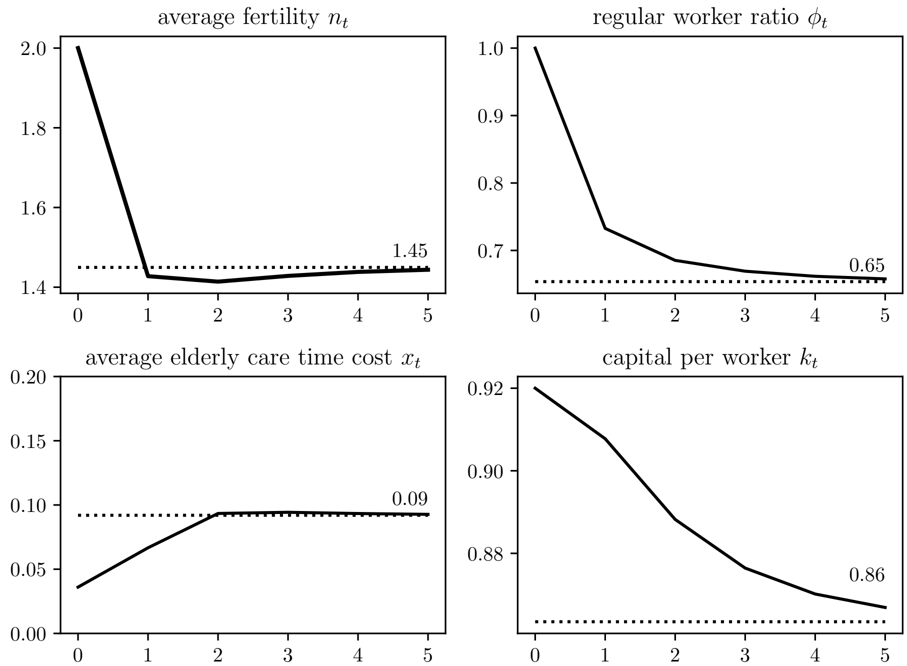


Figure 16: Simulation dynamics for an economy with initially high regular worker ratio. Parameters: $\sigma = 0.49, h = 0.04, b = 0.45, \delta = 0.3, \beta = 0.99^{30}, \gamma = 0.4, \alpha = 0.36, z = 0.08, A = 8$.

countries with low initial fertility and capital per worker. Only country A can achieve a high full-time employment rate. In the next example, we simulate the case of a moderately high fertility rate and a high capital per worker, such that it is initially at the full regular employment rate. The parameters are the same as those of country A. We then simulate a shock to the upskilling time cost where the new value of $\sigma = 0.45 > \hat{\sigma}$.

The dynamics are plotted in Fig.16. After the first period, average fertility falls from the initial level. Since the cost of education and elderly care opportunities puts pressure on individuals' ability to afford sufficient upskilling time, the regular-worker ratio fails to maintain its initial high level. It dips to 0.7. Without full-time workers to work with capital, the capital per worker decreases. After the first period, fertility rebounds. Since non-regular workers tend to have higher fertility rates, fertility improves if their ratio is sufficiently large. However, a higher population implies capital shallowing, which makes it less attractive for a worker to upskill. These dynamics jointly reduce the capital-per-worker and regular-worker ratio from its initial level.

7.2 Elderly Care Policy Consideration

In this section, we consider the case of a government intervention to build up a long-term care service. For example, Japan has started such a program since 2000 to alleviate the burden on informal caregivers. In this particular case, we will use the model simulated in Fig.16 as a starting point and adopt an Elderly care support program as described in section 6. The steady-state values of variables of interest are shown in Table 1.

Variable	Pre Policy	After Policy	Meaning
τ	0	0.38	lump-sum tax
τ/w^p	0	0.10	tax/nonregular wage ratio
τ/w^f	0	0.05	tax/regular wage
ϕ	0.59	0.88	regular worker's ratio
x^p	0.09	0.87	a nonregular worker's elderly care time
k	0.83	0.97	capital per worker
n	1.51	1.28	average fertility
n^p	2.13	2.09	nonregular worker's fertility
n^f	1.08	1.17	regular worker's fertility
ω	2.31	2.05	wage gap
u	0.79	0.98	life-time utility
$\ln(c^p) + \beta \ln(d^p)$	0.49	0.68	consumption utility for a nonregular worker
$\ln(c^f) + \beta \ln(d^f)$	0.76	0.92	consumption utility for a regular worker

Table 1: Steady state comparison of an elderly care subsidy policy.

In this case, the lump-sum tax would take a fraction of 10% and 5% of the non-regular and regular worker's wage rate. As a uniform lump-sum tax, it is relatively cheaper for a regular worker to spend on the elderly care service for their parents. The simulation results raise three interesting points.

First, we find that such a policy does significantly improve the regular worker ratio from 59% to 88% of the labor force. However, the burden of elderly care has shifted. In the previous case, since the state-funded program was not available, the average elderly care time costs a fraction of 0.09 from the time endowment of a nonregular worker. However, with the availability of the program, the nonregular labor force becomes the sole provider of elderly care for the whole society. In fact, in the non-regular job sector, on average, 87% of the time will be devoted to elderly care.

Second, the average fertility rate actually falls further to 28, lower than the case without the elderly care program. Even though the policy promotes higher fertility for regular workers (a rise from 1.08 to 1.17), due to a tighter time constraint on nonregular workers and the fact that a larger portion of workers have obtained regular jobs, the negative effect dominates, resulting in lower birth rates. Consequently, the further decline in fertility should put higher pressure on care workers.

Finally, we pay attention to the welfare effect. Notice that the interest rate is a decreasing function of capital per regular worker. Following the result from Proposition 5, an elderly care subsidy policy reduces the wage premium (since there are more regular workers on the market), which leads to the shallowing of capital per regular worker. As a result, the steady-state interest rates are higher in the case of policy intervention. Following an increase in regular workers, the overall savings of the economy also improved. Since fertility has decreased further, capital deepening leads to an increase in capital per worker. There is a strong income effect on an individual's utility, and it is more influential for nonregular workers. Regarding the utility of having children, it decreases slightly for nonregular workers but increases significantly for regular workers.

In conclusion, since lifetime welfare is improved following a policy intervention, this paper supports the implementation and wider coverage of an elderly care support program. By the same token, policies that reduce elderly care coverage may have unwanted effects on individuals' ability to commit to their regular employment.

8 Concluding Remarks

This research presents a general equilibrium model in an overlapping generation setting to study the influence of elderly care time demand on individuals' ability to afford regular work. The model provides a consequence of low fertility when the ability to share the burden of care is reduced: a lower participation rate in the regular employment market. Although higher capital earnings motivate people to upskill and become full-time workers, the constraints on time caused by the elderly care burden may prevent them from doing so. Furthermore, due to reduced time availability, individuals optimize by having fewer children, which further reinforces the dynamics of low fertility.

We also show that a threshold exists for the cost of upskilling time. Upskilling includes tertiary education time, self-improvement, or unpaid working hours required by firms. If the upskilling time cost is too high, the economy may observe a substantial fraction of workers resorting to unregular jobs to balance childcare and elderly care time. Our policy analysis shows that if the government can finance an elderly care support program, it can improve individuals' ability to engage in regular jobs. Although a full regular worker equilibrium might be hard to achieve as some portion of labor must be allocated to perform care work, the economic outcome is still better than an alternative scenario without the policy. Therefore, in developing countries with inadequate elderly care support programs, constructing such a program can be beneficial to avoid the long-lasting effect of the sandwich burden. On the contrary, policies that reduce elderly care support provision can have detrimental effects on the ability of individuals to sustain full-time employment.

This model imposes some simplifications that can be improved in future research. First, we have abstracted from family optimization. Although individuals are presented as siblings, they do not make collective decisions on elderly care and child care. Therefore, a prominent extension is exploring the effects of double caring under household units, such as a couple (Galor and Weil, 1996). Thus, if the preference for elderly care is more female demanding, then the burden put on female care workers (either formal or informal) can be heavier than the male counterparts. As a result, elderly care can affect the full-time work labor force participation rates differently between the two genders (Yakita, 2020). Another extension is to include life longevity. In this model, to focus on the results related to elderly care, we did not include life longevity. However, it is possible to include life span into the model either exogenously (Chen, 2010) or endogenously (Chakraborty, 2004; Futagami and Konishi, 2019). This extension can bring forth richer dynamics for the model, which can be important for calibration. Finally, we can extend the model by considering bequests as compensations paid to children in exchange for old-age support. In our model, the young generation is forced to spend time taking care of the elderly due to moral obligations. However, this is not always the case. Indeed, de la Croix and Dottori (2008) and recently Yakita (2024) have considered a bargaining process when the elderly leave a bequest in exchange for children's care provision. Although in this paper, we can choose the intensity of moral obligation by altering the exogenously given parameter h, a model with strategic bequest can provide a more satisfactory micro foundation.

Notes

1. OECD. 2023. ”Long-term care spending and unit costs” in ”Health at a Glance 2023: OECD Indicators”. Retrieved April 10, 2024, from https://www.oecd-ilibrary.org/sites/7a7afb35-en/1/3/10/index.html?itemId=/content/publication/7a7afb35-en&_csp_=6cf33e24b6584414b81774026d82a571&itemIGO=oecd&itemContentType=book

2. OECD. 2023. ”Access to long-term care” in ”Health at a Glance 2023: OECD Indicators”. Retrieved April 10, 2024 from https://www.oecd-ilibrary.org/sites/7a7afb35-en/1/3/10/index.html?itemId=/content/publication/7a7afb35-en&_csp_=6cf33e24b6584414b81774026d82a571&itemIGO=oecd&itemContentType=book

3. Conventionally, the number of children are often denoted by n_t as they are born at time t and become adults at time $t+1$. To conveniently keep track of the adult population, n_{t+1} is used instead. With child mortality implicitly considered, this notation can be interpreted as the net surviving children to adulthood (Eckstein et al., 1988).

4. Esteban-Pretel and Fujimoto (2020) show that in Japan, both male and female college graduates have a greater chance of obtaining and keeping regular jobs. Companies also prefer to offer those with adequate education full-time employment rather than those without.

5. Since the fertility rates in most countries start with a high level and slowly converge to low fertility during demographic changes, we can set appropriate parameters such that the steady-state fertility values are at least at the reproduceable level and higher than $\underline{n}$.

6. In Galor and Weil (1996), capital-complemented labor is called mental labor while the other is called physical labor. In Kimura and Yasui (2007); Chen (2010), they are called skilled and unskilled labor, respectively. Since regular jobs often demand a higher skill set, they are considered comparable to skilled labor.

7. A sufficiently low initial fertility rate implies a very high initial elderly care burden, which can consume all the time endowment that could be spent on market work. For this reason, we limit the feasible initial values of x. Since x_2^* is parameterized, we can add another condition to Eq.(6) by letting $\bar{x} < x_2^*$.

8. Note that the policy should be implemented when the economy is currently in the part-time equilibrium. Otherwise, if all agents can afford full-time employment while still being able to provide care for their parents, then government intervention is unnecessary.

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A Proofs (Main Model)

A.1 Proof of Proposition 1

Proof. The derivative of ϕ_t with respect to x_t is

$$\frac{\partial \phi_t}{\partial x_t} = \left(\frac{1-\alpha}{b}\right)^{1/\alpha} \eta k_t \theta'(x_t).$$

Since other terms are positive, the sign of $\partial\phi_t/\partial x_t$ is the sign of $\theta'(x_t)$. Taking the derivative of $\theta(x_t)$ with respect to x_t yields

$$\theta'(x_t) = \frac{(1-x_t-\sigma)^{\frac{\eta}{\alpha}-2}}{(1-x_t)^{\frac{\eta}{\alpha}+1}}\left(1-x_t-\frac{\eta\sigma}{\alpha}\right) = \frac{\theta(x_t)}{(1-x_t)(1-x_t-\sigma)}\left(1-x_t-\frac{\eta\sigma}{\alpha}\right). \tag{45}$$

The sign of $\theta'(x_t)$ depends on the sign of $1-x_t-\frac{\eta\sigma}{\alpha}$ since all other terms are positive. Thus, following the definition of x_t from Eq.(6), we have

$$\theta'(x_t) \leq 0 \text{ if } x_t \in \left[1-\frac{\eta\sigma}{\alpha}, 1-\sigma\right),$$

otherwise

$$\theta'(x_t) > 0 \text{ if } x_t \in \left(0, 1-\frac{\eta\sigma}{\alpha}\right).$$

■

A.2 Proof of Proposition 2

Proof. To find the critical value of σ, we use the steady state value of x^* in the case of $\phi^* = 1$. There are two conditions:

(i) At the steady state $\phi^* = 1$, a positive real solution for x^* must exist.

(ii) The following condition $k^* \geq \hat{k}(x^*)$ holds at the steady state.

For point (i), let $\phi^* = 1$ and solve for x when $x^* = \Psi(x^*)$, then its solution must satisfy

$$\gamma(1-x^*-\sigma)x^* = \frac{h}{\delta}z(1+\gamma+\beta),$$

Thus, x^* is a solution of the following quadratic equation

$$\gamma(x^*)^2 - \gamma(1-\sigma)x^* + \frac{h}{\delta}z(1+\gamma+\beta) = 0.$$

Following the quadratic formula, it has a solution if its discriminant is positive, equivalently

$$\gamma^2(1-\sigma)^2 - 4\gamma\frac{h}{\delta}z(1+\gamma+\beta) \geq 0,$$

which is true if and only if

$$\sigma \leq 1 - \sqrt{\frac{4\frac{h}{\delta}z(1+\gamma+\beta)}{\gamma}} = 1 - \sqrt{4Q} \equiv \hat{\sigma}_1. \tag{46}$$

For point (ii), we use the result from Lemma 4 and Eq.(32). At the steady state, the values of x^* and k^* can be expressed as functions of σ:

$$x^*(\sigma) = \frac{(1-\sigma) - \sqrt{(1-\sigma)^2 - 4Q}}{2},$$
$$k^*(x^*(\sigma)) = \left[\frac{A\beta z(1-\alpha)\eta^\alpha}{\gamma}\right]^{1/(1-\alpha)} (1-x^*-\sigma)^{-\alpha/(1-\alpha)}.$$

The threshold value for k_t at the steady state follows Eq.(20):

$$\hat{k}^*(x^*) = \left(\frac{b}{1-\alpha}\right)^{1/\alpha} \frac{1}{\eta} \frac{(1-x^*)^{\eta/\alpha}}{(1-x^*-\sigma)^{\frac{\eta}{\alpha}-1}}.$$

Condition (ii) states that $k^*(x^*) \geq \hat{k}^*(x^*)$. This condition is equivalent to

$$\frac{k^*}{\hat{k}^*} \geq 1.$$

By plugging the above derivations, the condition is equivalent to

$$\frac{\left[\frac{A\beta z(1-\alpha)\eta^\alpha}{\gamma}\right]^{1/(1-\alpha)}}{\left(\frac{b}{1-\alpha}\right)^{1/\alpha}\frac{1}{\eta}} \cdot \frac{(1-x^*(\sigma)-\sigma)^{\frac{\eta}{\alpha}-1-\frac{\alpha}{1-\alpha}}}{(1-x^*(\sigma))^{\eta/\alpha}} \geq 1. \tag{47}$$

In what follows, we will prove that the LHS is a decreasing function of σ. Denote the LHS of Eq.(47) as $\Xi(\sigma)$, using $x^*(\sigma)$, we can write it as

$$\Xi(\sigma) = D \cdot \frac{\left(1-\sigma+\sqrt{(1-\sigma)^2-4Q}\right)^{\frac{\eta}{\alpha}-1-\frac{\alpha}{1-\alpha}}}{\left(1+\sigma+\sqrt{(1-\sigma)^2-4Q}\right)^{\eta/\alpha}},$$

where

$$D = \frac{\left[\frac{A\beta z(1-\alpha)\eta^\alpha}{\gamma}\right]^{1/(1-\alpha)}}{\left(\frac{b}{1-\alpha}\right)^{1/\alpha}\frac{1}{\eta}} \times \left(\frac{1}{2}\right)^{-1-\frac{\alpha}{1-\alpha}}.$$

Let $\nu = 1 - \sigma + \sqrt{(1-\sigma)^2 - 4Q} > 0$ and $\varsigma = \frac{\eta}{\alpha} - 1 - \frac{\alpha}{1-\alpha} = \frac{\eta}{\alpha} - \frac{1}{1-\alpha} > 0$. It is obvious that

$$\nu' = \nu'(\sigma) = -1 - \frac{1}{\sqrt{(1-\sigma)^2 - 4Q}} < 0.$$

We can rewrite $\Xi(\sigma)$ succinctly as

$$\Xi(\sigma) = D \frac{\nu^{\varsigma}}{(\nu + 2\sigma)^{\varsigma + \frac{1}{1-\alpha}}}.$$

Differentiating Ξ with respect to σ yields

$$\begin{aligned}
\Xi'(\sigma) &= \frac{\varsigma \nu^{\varsigma-1} \nu' (\nu + 2\sigma)^{\varsigma + \frac{1}{1-\alpha}} - \left(\varsigma + \frac{1}{1-\alpha}\right) (\nu + 2)^{\varsigma + \frac{1}{1-\alpha} - 1} \nu^{\varsigma}}{\left[(\nu + 2\sigma)^{\varsigma + \frac{1}{1-\alpha}}\right]^2} \\
&= \frac{\nu^{\varsigma} \nu' \left[\frac{\varsigma (\nu + 2\sigma)^{\varsigma + \frac{1}{1-\alpha}}}{\nu} - \left(\varsigma + \frac{1}{1-\alpha}\right) (\nu')^{\varsigma + \frac{1}{1-\alpha} - 2}\right] - \left(\varsigma + \frac{1}{1-\alpha}\right) \cdot 2^{\varsigma + \frac{1}{1-\alpha} - 1} \nu^{\varsigma}}{\left[(\nu + 2\sigma)^{\varsigma + \frac{1}{1-\alpha}}\right]^2} < 0
\end{aligned}$$

since $\nu' < 0$. This verifies that the $\Xi(\sigma)$ is decreasing in σ. Let $\hat{\sigma}_2$ be the solution such that $\Xi(\hat{\sigma}_2) = 1$, then the condition (47) holds if and only if $\sigma < \hat{\sigma}_2$. Combining with condition (46), we can state that the steady state $\phi^* = 1$ is attainable if and only if $\sigma \leq \hat{\sigma}$ where $\hat{\sigma} = \min(\hat{\sigma}_1, \hat{\sigma}_2)$. ■

A.3 Proof of Lemma 1

Proof. For Eq.(26), grouping all the terms containing ϕ_t in the numerator yields

$$\begin{aligned}
\Phi(x_t, k_t) &= \frac{A\beta z b}{\gamma} \cdot \frac{\phi_t[(1-x_t)^{\eta}(1-x_t-\sigma)^{1-\eta} - 1] + (1-x_t)}{1 - x_t - \sigma\phi_t} \\
&= \frac{A\beta z b}{\gamma} \cdot \frac{\phi_t (1-x_t) \left[\left(\frac{1-x_t}{1-x_t-\sigma}\right)^{\eta-1} - 1\right] + (1-x_t)}{1 - x_t - \sigma\phi_t}.
\end{aligned}$$

Factoring by $(1-x_t)$, we have

$$\Phi(x_t, k_t) = \frac{A\beta z b}{\gamma} \cdot \frac{(1-x_t) \cdot v(x_t)}{1 - x_t - \sigma\phi(x_t)}.$$

where

$$v(x_t) = \phi(x_t) \left[\left(\frac{1-x_t}{1-x_t-\sigma}\right)^{\eta-1} - 1\right] + 1 > 0.$$

Let

$$g(x_t) = \left(\frac{1-x_t}{1-x_t-\sigma}\right)^{\eta-1},$$

It is obvious that $g(x_t) > 1\ \forall x_t \in (0, 1-\sigma)$ so that $g(x_t) - 1 > 0$. Furthermore, we have

$$g'(x_t) = -\sigma(\eta-1)\frac{\left(\frac{1-x_t}{1-x_t-\sigma}\right)^{\eta-2}}{(1-x_t-\sigma)^2} < 0 \text{ since } \eta > 1.$$

Hence,

$$v'(x_t) = \phi'(x_t)[g(x_t)-1] + g'(x_t)\phi(x_t) < 0,$$

since $\phi'(x_t) < 0$ by Proposition 1 under Assumption 1, $g'(x_t) < 0$ and $\phi(x_t) > 0$. Now, differentiate $\Phi(x_t, k_t)$ with respect to x_t [9] yields

$$\begin{aligned}
\frac{\partial \Phi(x_t,k_t)}{\partial x_t} &= \frac{[v_t' - v_t - x_t v_t'](1-x_t-\sigma\phi_t) + (1+\sigma\phi_t')(1-x_t)v_t}{(1-x_t-\sigma\phi_t)^2} \\
&= \frac{-v_t(1-x_t-\sigma\phi_t) + \sigma\phi_t' v_t(1-x_t) + v_t'(1-x_t)(1-x_t-\sigma\phi_t) + (1-x_t)v_t}{(1-x_t-\sigma\phi_t)^2} \\
&= \frac{v_t'(1-x_t)(1-x_t-\sigma\phi_t) + v_t\cdot[(1-x_t) - (1-x_t-\sigma\phi_t) + \sigma\phi_t'(1-x_t)]}{(1-x_t-\sigma\phi_t)^2} \\
&= \frac{v_t'(1-x_t)(1-x_t-\sigma\phi_t) + \sigma v_t\cdot[(1-x_t)\phi_t' + \phi_t]}{(1-x_t-\sigma\phi_t)^2}.
\end{aligned}$$

Since $v_t' < 0$, the first term in the numerator is negative. We want to find the sign of the other term. From Eq.(28) and (45), we have

$$\phi_t' = \phi_t \cdot \frac{1-x_t-\frac{\eta\sigma}{\alpha}}{(1-x_t)(1-x_t-\sigma)}.$$

Hence,

$$(1-x_t)\phi_t' = \phi_t \cdot q(x_t),$$

where

$$q(x_t) = \frac{1-x_t-\frac{\eta\sigma}{\alpha}}{1-x_t-\sigma} = 1 - \frac{\sigma\left(\frac{\eta}{\alpha}-1\right)}{1-x_t-\sigma}.$$

The limit for very small x_t is

$$\lim_{x\to 0}\frac{\sigma\left(\frac{\eta}{\alpha}-1\right)}{1-x_t-\sigma} = \frac{\sigma\left(\frac{\eta}{\alpha}-1\right)}{1-\sigma} > \frac{\hat{\sigma}\left(\frac{\eta}{\alpha}-1\right)}{1-\hat{\sigma}} > 2 \text{ since } 1+\eta/\alpha > 2/\hat{\sigma} \text{ (Assumption 3)},$$

which implies that

$$\lim_{x \to 0} q(x_t) < -1.$$

Due to Assumption 1, $q(x_t) < 0 \;\forall x_t \in (0, 1-\sigma)$. It is obvious that $q(x_t)$ is decreasing in x_t since $q'(x_t) < 0$. Hence, we can state that

$$q(x_t) < -1 \quad \forall x_t \in (0, 1-\sigma).$$

As a result, we have

$$(1 - x_t)\phi_t' + \phi_t = \phi_t(q(x_t) + 1) < 0 \quad \forall x_t \in (0, 1-\sigma).$$

Therefore, $\partial\Phi(x_t, k_t)/\partial x_t < 0 \quad \forall x_t \in (0, 1-\sigma)$. ■

A.4 Proof of Lemma 2

Proof. Since the denominator is always positive in

$$k_\Psi(x_t) = \frac{-x_t^2 + x_t - Q}{\sigma\epsilon x_t \theta(x_t)},$$

the positive condition is equivalent to

$$\begin{aligned} & x_t^2 - x_t + Q < 0 \\ \Leftrightarrow & \left(x_t - \frac{1}{2}\right)^2 < \left(\frac{1}{4} - Q\right). \end{aligned}$$

Both sides are positive since $Q < 1/4$ by Assumption 2. Taking square roots of both sides yields

$$\begin{aligned} & x_t - 1/2 < \sqrt{\frac{1}{4} - Q}, \\ \Leftrightarrow & \; 0 < \frac{1}{2} - \sqrt{\frac{1}{4} - Q} < x_t < \frac{1}{2} + \sqrt{\frac{1}{4} - Q}, \end{aligned}$$

which is true for all $x \in (0, 1-\sigma)$. ■

A.5 Proof of Lemma 3

Proof. If $0 < x_t \le \sqrt{Q}$, then we have $Q - x^2 > 0$. Since $\theta'(x_t) < 0$ by Assumption 1, and $x^2 - x_t + Q < 0$ due to Assumption 2 and Lemma 2, we can conclude that $k_\Psi' > 0$.

Let us investigate the case $1 - \sigma > x_t > \sqrt{Q}$. First, using equation (45), we can write the numerator of (31) as

$$\theta(x_t)\left[(Q - x^2) + \frac{(1 - x_t - \frac{\eta\sigma}{\alpha})(x^2 - x_t + Q)}{(1 - x_t)(1 - x_t - \sigma)}\right].$$

Let

$$q_1(x_t) = Q - x^2,$$
$$q_2(x_t) = \frac{(1 - x_t - \frac{\eta\sigma}{\alpha})(x_t^2 - x_t + Q)}{(1 - x_t)(1 - x_t - \sigma)}.$$

Since $x_t > \sqrt{Q}$, we have $q_1(x_t) < 0$. Since $\sigma > \alpha/\eta$ and $x^2 - x_t + Q < 0$, we have $q_2(x_t) > 0$. Hence, if $|q_2(x_t)| > |q_1(x_t)|$, then the sign of (31) will be positive. From the previous results, because (31) is positive when $x_t = \sqrt{Q}$, so $|q_2(\sqrt{Q})| > |q_1(\sqrt{Q})|$. Differentiate $q_1(x)$ yields

$$q_1'(x_t) = -2x_t < 0,$$

so $|q_1(x_t)|$ is decreasing in x_t. We can rewrite $q_2(x_t)$ in all positive terms as

$$q_2(x_t) = q_3(x_t)(-x_t^2 + x_t - Q),$$

where $q_3(x_t) = \dfrac{(x_t + \frac{\eta\sigma}{\alpha} - 1)}{(1 - x_t)(1 - x_t - \sigma)}$. Since $x_t > \sqrt{Q}$, we have $-x_t^2 + x_t - Q > \sqrt{Q} - 2Q = \sqrt{Q}(1 - 2\sqrt{Q}) > 0$ because $Q < 1/4$. Furthermore, differentiating $q_3(x_t)$ with respect to x_t yields

$$q_3'(x_t) = \frac{(1 - x_t)^2 + \eta\sigma[2(1 - x_t) - \sigma]}{\alpha(1 - x_t)^2(1 - x_t - \sigma)^2} > 0 \text{ since } 1 - x_t - \sigma > 0\ .$$

Thus, $q_3(x_t)$ is increasing in x_t, which implies that $|q_3(x_t)\sqrt{Q}(1 - 2\sqrt{Q})|$ is also increasing in x_t. That is, as x_t increases further from $\sqrt{Q}$, $|q_1(x_t)|$ decreases while $|q_3(x_t)|$ increases. Since $|q_2(x_t)| > |q_3(x_t)\sqrt{Q}(1 - 2\sqrt{Q})|$ for all $x_t > \sqrt{Q}$, we can conclude that $|q_2(x_t)| > |q_1(x_t)|$ for all $x_t > \sqrt{Q}$. Thus, (31) is positive for all x. ■

A.6 Proof of Lemma 4

Proof. First, we solve for the steady state of x_t by letting $\Psi(x_t) = x_t$. The dynamics of x_t is

$$x_{t+1} = \frac{Q}{1 - x_t - \sigma}.$$

To find its steady state, we solve for x^* such that

$$(x^*)^2 - (1 - \sigma)x^* + Q = 0.$$

Its discriminant is

$$\triangle = (1 - \sigma)^2 - 4Q.$$

Since $\sigma > \hat{\sigma}, \triangle > 0$, we have 2 solutions as follows

$$x_1^* = \frac{(1-\sigma) - \sqrt{\triangle}}{2}, \qquad x_2^* = \frac{(1-\sigma) + \sqrt{\triangle}}{2}.$$

The dynamical stability condition is

$$|\Psi'(x^*)| = \frac{Q}{((1-\sigma) - x^*)^2} < 1.$$

This is true only for $x_1^* = ((1-\sigma) - \sqrt{\triangle})/2$. We also show a graphical illustration of the solution in Figure 17. Since

$$\Psi'(x_t) = \frac{Q}{(1-\sigma-x_t)^2} > 0, \qquad \Psi''(x_t) = \frac{2Q}{(1-\sigma-x_t)^3} > 0.$$

Thus, $\Psi(x_t)$ is a convex function of x_t as shown in Fig.(17). As a result, there are two steady-state candidates, but only the lower one is stable. ∎

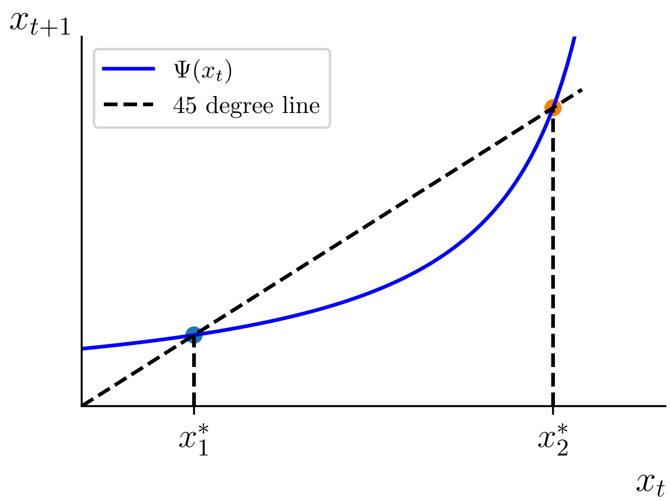

Figure 17: Dynamics of $x_{t+1} = \Psi(x_t)$. Only x_1^* is stable.

A.7 Proof of Proposition 3

Proof. In this equilibrium, we can represent the dynamics as functions of σ as follows

$$k_{t+1} = \Phi(\sigma) = C(1-x_t)\frac{\Lambda(\sigma)}{\Upsilon(\sigma)},$$
$$x_{t+1} = \Psi(\sigma) = \frac{Q}{\Upsilon(\sigma)},$$

where $\theta_t(\sigma)$ is defined in Eq.(18) and

$$\begin{aligned}\chi_t(\sigma) &= \frac{1-x_t}{1-x_t-\sigma} > 1,\\ \Lambda(\sigma) &= [\chi_t(\sigma)^{\eta-1}-1]\epsilon k_t\theta_t(\sigma)+1 > 0,\\ \Upsilon(\sigma) &= 1-x_t-\epsilon k_t\sigma\theta_t(\sigma) > 0.\end{aligned}$$

The derivative of Φ and Ψ with respect to σ is

$$\Phi'(\sigma) = C(1-x_t)\frac{\Lambda'(\sigma)\Upsilon(\sigma)-\Upsilon'(\sigma)\Lambda(\sigma)}{\Upsilon^2(\sigma)}, \tag{48}$$

$$\Psi'(\sigma) = -\frac{Q}{\Upsilon^2(\sigma)}\Upsilon'(\sigma). \tag{49}$$

First, notice that

$$\begin{aligned}\chi_t'(\sigma) &= \frac{1-x_t}{(1-x_t-\sigma)^2} > 0,\\ \theta_t'(\sigma) &= -\frac{\frac{\eta}{\alpha}-1}{1-x_t-\sigma}\theta_t(\sigma) < 0.\end{aligned}$$

Differentiating Λ with respect to σ yields

$$\begin{aligned}\Lambda'(\sigma) &= \epsilon k_t\left[(\eta-1)\chi_t(\sigma)^{\eta-2}\chi_t'(\sigma)\theta_t(\sigma)+\chi_t(\sigma)^{\eta-1}\theta_t'(\sigma)-\theta_t'(\sigma)\right]\\ &= \epsilon k_t\theta_t'(\sigma)\left[-\frac{(\eta-1)(1-x_t)}{\frac{\eta}{\alpha}-1}\chi_t(\sigma)^{\eta-2}+\chi_t(\sigma)^{\eta-1}-1\right]\\ &= \epsilon k_t\theta_t'(\sigma)\left[\chi_t^{\eta-1}\left(1-\frac{(\eta-1)(1-x_t-\sigma)}{\frac{\eta}{\alpha}-1}\right)-1\right].\end{aligned}$$

Given $k_t > 0$, the sign of $\Lambda'(\sigma)$ depends on the sign of the last term, which we will denote by $\lambda(x_t,\sigma)$. Differentiating λ with respect to x_t yields

$$\lambda'(x_t) = (\eta-1)\chi_t^{\eta-2}\chi_t'\left(1-\frac{(\eta-1)(1-x_t-\sigma)}{\frac{\eta}{\alpha}-1}\right)+\chi_t^{\eta-1}\left(\frac{\eta-1}{\frac{\eta}{\alpha}-1}\right) > 0.$$

Thus, λ is an increasing function of x_t. Furthermore

$$\lim_{x_t\to 0}\lambda(x_t) = \frac{1}{1-\sigma}-\frac{(\eta-1)\alpha}{\eta-\alpha}-1.$$

By Assumption 1, we have

$$\begin{aligned}&\sigma > \frac{\alpha}{\eta}\\ \Rightarrow &1-\sigma < 1-\frac{\alpha}{\eta}\\ \Rightarrow &\frac{1}{1-\sigma} > \frac{1}{1-\frac{\alpha}{\eta}} = \frac{\eta}{\eta-\alpha} > \frac{(\eta-1)\alpha}{\eta-\alpha}.\end{aligned}$$

Hence,

$$\frac{1}{1-\sigma} - \frac{(\eta-1)\alpha}{\eta-\alpha} - 1 > \frac{\eta}{\eta-\alpha} - \frac{(\eta-1)\alpha}{\eta-\alpha} - 1 = \frac{\alpha(2-\eta)}{\eta-\alpha} > 0 \text{ since } 1 < \eta < 2\ .$$

Therefore, $\lambda(x_t, \sigma) > 0$ for all $x > 0$ and $\sigma > \frac{\alpha}{\eta}$. Since $\theta_t'(\sigma) < 0$, we conclude that $\Lambda'(\sigma) < 0$.

Now, we differentiate Υ with respect to σ:

$$\begin{aligned}\Upsilon'(\sigma) &= -\epsilon k_t[\theta(\sigma) + \sigma\theta'(\sigma)] \\ &= -\epsilon k_t \theta'(\sigma)\left(\sigma - \frac{1-x_t-\sigma}{\frac{\eta}{\alpha}-1}\right) \\ &= -\epsilon k_t \theta'(\sigma)\left[\frac{\frac{\sigma\eta}{\alpha} - (1-x_t)}{\frac{\eta}{\alpha}-1}\right] > 0.\end{aligned}$$

because $\sigma > \alpha/\eta$ by Assumption 1 and $\theta'(\sigma) < 0$. Since $\Lambda'(\sigma) < 0$ and $\Upsilon'(\sigma) > 0$, Eq.(48) indicates that $\Phi'(\sigma) < 0$, which implies $k_t'(\sigma) < 0$; and Eq.(49) indicates that $\Psi'(\sigma) > 0$, implying that $x_t'(\sigma) < 0$ (see Lemma 3)

Differentiating Eq.(17) with respect to σ yields

$$\begin{aligned}\phi'(\sigma) &= \left(\frac{1-\alpha}{b}\right)^{1/\alpha} \eta\left[k_t'(\sigma)\theta_t(\sigma) + k_t(\sigma)\theta_t'(\sigma)\right] \\ &= \left(\frac{1-\alpha}{b}\right)^{1/\alpha} \eta\theta_t'(\sigma)\left[k_t(\sigma) - k_t'(\sigma)\frac{\frac{\eta}{\alpha}-1}{1-x_t-\sigma}\right] < 0.\end{aligned}$$

since $\theta_t'(\sigma) < 0$, $k_t'(\sigma) < 0$. ■

A.8 Derivation of Equation (15)

Using the FOCs, the utility function of any individual $i \in \{f, p\}$ can be written as

$$\begin{aligned}u_t^i &= \ln(c_t^i) + \gamma \ln\left(\frac{\gamma c_t^i}{z w_t^i}\right) + \beta \ln\left(R_{t+1} \cdot \beta c_t^i\right) \\ &= \ln(c_t^i) + \gamma\ln(c_t^i) + \beta\ln(c_t^i) - \gamma\ln(w_t^i) + \gamma\ln(\gamma/z) + \beta\ln(R_{t+1}\beta) \\ &= \ln\left(\frac{(c_t^i)^{1+\beta+\gamma}}{(w_t^i)^\gamma}\right) + \gamma\ln(\gamma/z) + \beta\ln(R_{t+1}\beta).\end{aligned}$$

Since the last two terms are common between the two types, the condition $u_t^f = u_t^p$ is equivalent to

$$\frac{(c_t^f)^{1+\beta+\gamma}}{(w_t^f)^\gamma} = \frac{(c_t^p)^{1+\beta+\gamma}}{(w_t^p)^\gamma}.$$

Cross multiplying yields

$$\left(\frac{c_t^f}{c_t^p}\right)^{1+\beta+\gamma} = \left(\frac{w_t^f}{w_t^p}\right)^{\gamma}. \tag{50}$$

Using the FOC conditions, we obtain

$$\left(\frac{1-x_t}{1-\sigma-x_t}\right)^{\frac{1+\beta+\gamma}{1+\beta}} = \left(\frac{w_t^f}{w_t^p}\right),$$

where the exponential $\frac{1+\beta+\gamma}{1+\beta}$ can be denoted as η.

B Proofs (Policy)

B.1 Derivation of Eq.(40)

This condition implies that

$$\begin{aligned}
(1+\beta)\ln((1-\sigma)w_t^f-\tau_t)+\gamma\ln\left(1-\sigma-\frac{\tau_t}{w_t^f}\right) &= (1+\beta)\ln\left(w_t^p-\tau_t\right) = \gamma\ln\left(1-\frac{\tau_t}{w_t^p}\right) \\
\Leftrightarrow (1+\beta+\gamma)\ln\left(\frac{(1-\sigma)w_t^f-\tau_t}{w_t^p-\tau_t}\right) &= \gamma\ln\left(\frac{w_t^f}{w_t^p}\right) \\
\Leftrightarrow (1+\beta+\gamma)\ln\left(\frac{(1-\sigma)\omega_t-\chi_t}{1-\chi_t}\right) &= \gamma\ln\omega_t.
\end{aligned}$$

where $\omega_t \equiv w_t^f/w_t^p$, and $\tau_t = w_t^p\chi_t$. This implies that

$$\begin{aligned}
&\frac{(1-\sigma)\omega_t-\chi_t}{1-\chi_t} = \omega_t^{\frac{\gamma}{1+\beta+\gamma}} = \omega_t^{1-\frac{1}{\eta}} \\
\Leftrightarrow&(1-\sigma)\omega_t-(1-\chi_t)\omega_t^{1-\frac{1}{\eta}}-\chi_t = 0 \\
\Leftrightarrow&\chi_t = \frac{\omega_t^{1-\frac{1}{\eta}}-(1-\sigma)\omega_t}{\omega_t^{1-\frac{1}{\eta}}-1} > 0 \ \text{ for all } \omega_t > 1, \eta > 1.
\end{aligned}$$

The labor time of a full-time worker is $1-\sigma-zn_{t+1}^f$, which can be expressed as

$$\begin{aligned}
&1-\sigma-\frac{\gamma}{1+\beta+\gamma}\left(1-\sigma-\frac{w_t^p\chi_t}{w_t^f}\right) \\
&= \frac{1+\beta}{1+\beta+\gamma}(1-\sigma)+\frac{\gamma}{1+\beta+\gamma}\frac{\chi_t}{\omega_t} \\
&= \frac{(1-\sigma)+(\eta-1)\chi_t/\omega_t}{\eta}.
\end{aligned}$$

B.2 Proof of Lemma 5

Proof. Denote Eq.(40) as a function of χ_t and ω_t such that

$$\Theta(\chi_t, \omega_t) = \frac{\omega_t^{1-\frac{1}{\eta}} - (1-\sigma)\omega_t}{\omega_t^{1-\frac{1}{\eta}} - 1} - \chi_t.$$

Differentiate $\Theta(\chi_t, \omega_t)$ with respect to χ_t yields

$$\frac{\partial\Theta(\chi_t, \omega_t)}{\partial\omega_t} = \frac{(1-\sigma) - \omega_t^{-1/\eta}\left[\left(1-\frac{1}{\eta}\right) + \frac{1}{\eta}(1-\sigma)\omega_t\right]}{\left(\omega_t^{1-\frac{1}{\eta}} - 1\right)^2}.$$

Let

$$\Omega(\omega_t) = \left(1 - \frac{1}{\eta}\right) + \frac{1}{\eta}(1-\sigma)\omega_t.$$

Ω_t is increasing in ω_t. Since $\omega_t > 1$, we have

$$\lim_{\omega_t \to 1} \Omega(\omega_t) = 1 - \frac{\sigma}{\eta} > 1 - \sigma \text{ since } \eta > 1,$$

which implies that $\omega_t^{-1/\eta}\Omega(\omega_t) > 1 - \sigma$ for all $\omega_t \in (1, \infty)$. Hence, $\frac{\partial\Theta(\chi_t,\omega_t)}{\partial\omega_t} < 0$. Furthermore

$$\frac{\partial\Theta(\chi_t, \omega_t)}{\partial\chi_t} = -1 < 0.$$

By the implicit function theorem

$$\frac{\partial\omega_t}{\partial\chi_t} = -\frac{\frac{\partial\Theta(\chi_t,\omega_t)}{\partial\chi_t}}{\frac{\partial\Theta(\chi_t,\omega_t)}{\partial\omega_t}} < 0.$$

■

B.3 Proof of Proposition 4

Proof. The sign of $\phi_t'(\chi_t)$ is the sign of $\lambda'(\chi_t)$. Differentiating $\lambda'(\chi_t)$ yields

$$\lambda'(\chi_t) = \frac{-\frac{1}{\alpha}\omega(\chi_t)^{-\frac{1}{\alpha}-1}\omega'(\chi_t)\left[1-\sigma+(\eta-1)\frac{\chi_t}{\omega(\chi_t)}\right] + (\eta-1)\left[\frac{\omega(\chi_t)-\omega'(\chi_t)\chi_t}{\omega^2(\chi_t)}\right]\omega(\chi_t)^{-\frac{1}{\alpha}}}{(1-\sigma+(\eta-1)\frac{\chi_t}{\omega(\chi_t)})^2}$$

$$= \frac{\omega(\chi_t)^{-1/\alpha}\left[(\eta-1)\omega(\chi_t) - \left(\frac{1-\sigma}{\alpha}\right)\frac{1}{\omega(\chi_t)\omega'(\chi_t)} - \left(1+\frac{1}{\alpha}\right)\frac{(\eta-1)\chi_t\omega'(\chi_t)}{\omega^2(\chi_t)}\right]}{(1-\sigma+(\eta-1)\frac{\chi_t}{\omega(\chi_t)})^2}.$$

Since $\omega(\chi_t) > 0, \omega'(\chi_t) < 0$, the numerator is positive.

Hence, $\lambda'(\chi_t) > 0 \Rightarrow \phi_t'(\chi_t) > 0$. ■