Syllabus

1. Difference Equations
   - One Variable
   - Eigenvalues and Eigenvectors
   - System of 2
2. Static Optimization
   - Unconstrained case
   - Constraint case
   - Kuhn-Tucker Theorem
3. Differential Equations
   - One Variable
   - A system of 2
   - Phase Diagrams
4. Dynamic Optimization
   - Dynamic Programming with Bellman
   - Optimal Control with Hamiltonian

One Variable Difference Equations

Linear

- General Solution
  - Linear first-order difference equation: $x_{t+1} = ax_t$
  - General solution: $x_t = x_0 a^t$
  - Including a constant b: $x_{t+1} = ax_t + b$
- Stability and dynamics
  - Equilibrium solution: $\bar{x} = \frac{b}{1-a}$
  - For $|a| < 1$, solution converges to $\bar{x}$
  - Illustrations of stable, oscillatory, and unstable behavior

Nonlinear

- General Solution
  - Autonomous first-order difference equation: $x_t = f(x_{t-1})$
  - Fixed point: $x^* = f(x^*)$
  - Linear approximation: $x_t = f(x^*) + f'(x^*)(x_{t-1} - x^*)$.
- Stability
  - If $|f'(x^*)| < 1$, then x^* is **locally asymptotically stable**
  - If $|f'(x^*)| > 1$, then x^* is **unstable**
  - If $|f'(x^*)| = 1$, the situation is inconclusive.

Eigenvalues and Eigenvectors

- Eigenvalues and eigenvectors are important concepts for square matrices.
- Eigenvalues λ and eigenvectors $\mathbf{v}$ satisfy: $\mathbf{Av} = \lambda\mathbf{v}$.
- Steps to find eigenvalues and eigenvectors:
  1. Set up the characteristic equation and solve for eigenvalues.
  2. Solve the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ to find eigenvectors.
  3. Checks
     - (trace) The sum of all the eigenvalues will be the sum of the diagonal of $\mathbf{A}$.
     - (determinants) The product of all the eigenvalues is the determinant.
- Eigenvalues determine stability: real parts affect convergence behavior.
- Stability: Let λ_1, λ_2 be the eigenvalues of $\mathbf{A}$
  1. If $|\lambda_1| \leq |\lambda_2| < 1$, then equilibrium is stable (sink).
  2. If $|\lambda_2| \geq |\lambda_1| > 1$, then equilibrium is unstable (source).
  3. If $|\lambda_1| < 1 < |\lambda_2|$, unique direction converge to eqm. (saddle point)

System of Difference Equations

Linear

- Equations can be written as:

$$\begin{aligned} x_{t+1} &= ax_t + by_t \\ y_{t+1} &= cx_t + dy_t \end{aligned} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\mathbf{J}} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

- The only equilibrium point is $(\bar{x}, \bar{y}) = (0, 0)$.

Nonlinear

- The equilibrium point is $\bar{x}, \bar{y}$.
- Linearization around the equilibrium

$$\begin{pmatrix} x_{t+1} - \bar{x} \\ y_{t+1} - \bar{y} \end{pmatrix} = \underbrace{\begin{pmatrix} f'_x(\bar{x}, \bar{y}) & f'_y(\bar{x}, \bar{y}) \\ g'_x(\bar{x}, \bar{y}) & g'_y(\bar{x}, \bar{y}) \end{pmatrix}}_{\mathbf{J}} \begin{pmatrix} x_t - \bar{x} \\ y_t - \bar{y} \end{pmatrix}$$

Stability: Let $\mathbf{D}$ be the det($\mathbf{J}$) and $\mathbf{D}$ be $tr(\mathbf{J})$.

1. If $|1 + \mathbf{D}| < |\mathbf{T}|$, the steady state is a saddle.
2. If $|1 + \mathbf{D}| > |\mathbf{T}|$ and $|\mathbf{D}| < 1$, the steady state is a sink.
3. If $|1 + \mathbf{D}| > |\mathbf{T}|$ and $|\mathbf{D}| > 1$, the steady state is a source.

Unconstrained Optimization

To find solutions of n choice variables $\mathbf{x} = (x_1, \ldots, x_n)$ that maximize $F(\mathbf{x})$.

Necessary Conditions

For a local max or min $\mathbf{x}^*$ of F:

$$\frac{\partial F}{\partial x_i}(\mathbf{x}^*) = 0 \text{ for } i = 1, \ldots, n$$

Sufficient Conditions

Using the Hessian matrix $D^2F(\mathbf{x}^*)$:

- If $D^2F(\mathbf{x}^*)$ is negative definite, $\mathbf{x}^*$ is strict local max. n leading principal minors of $D^2F(\mathbf{x}^*)$ alternate in sign.
- If $D^2F(\mathbf{x}^*)$ is positive definite, $\mathbf{x}^*$ is strict local min. All principal minors are positive.
- If $D^2F(\mathbf{x}^*)$ is indefinite, $\mathbf{x}^*$ is neither max nor min.

Using Eigenvalues

- All the real parts of eigenvalues are negative, $D^2F(\mathbf{x}^*)$ is negative definite.
- All the real parts of eigenvalues are positive, $D^2F(\mathbf{x}^*)$ is positive definite.

Constraint Optimization

We want to

$$\max f(x_1, x_2)$$
$$s.t.\ h(x_1, x_2) = c$$

The Lagrangian

$$\mathcal{L}(x_1, x_2, \lambda) \equiv f(x_1, x_2) - \lambda[h(x_1, x_2) - c].$$

Necessary Conditions

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0.$$

Sufficient Conditions The bordered Hessian matrix is

$$H = \begin{pmatrix} 0 & \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial^2 \mathcal{L}}{\partial x^2} & \frac{\partial^2 \mathcal{L}}{\partial x \partial y} \\ \frac{\partial h}{\partial y} & \frac{\partial^2 \mathcal{L}}{\partial y \partial x} & \frac{\partial^2 \mathcal{L}}{\partial y^2} \end{pmatrix}$$

1. if $\det(H) > 0$ at (x^*, y^*), then (x^*, y^*) is the local MAX of f on C_h.
2. if $\det(H) < 0$ at (x^*, y^*), then (x^*, y^*) is the local MIN of f on C_h.

Inequality Optimization

We want to

$$\max f(x, y)$$
$$s.t.\ g(x, y) \leq c$$

The Lagrangian

$$\mathcal{L}(x, y) = f(x, y) - \lambda(g(x, y) - c)$$

KKT Necessary Conditions

$$\mathcal{L}'_x = f'_x - \lambda g'_x = 0,$$
$$\mathcal{L}'_y = f'_y - \lambda g'_y = 0,$$
$$\lambda \cdot (g(x, y) - c) = 0,$$
$$\lambda \geq 0, \quad g(x, y) \leq c$$

Complimentary slackness condition

$$\lambda > 0, \text{ the constraint binds so that } g(x, y) = c$$
$$\lambda = 0, \text{ the constraint does not bind so that } g(x, y) < c$$

For a minimum problem, the FOCs are the same, except that $\lambda \leq 0$.

One-variable: Linear Case

Autonomous

- Simplest case: $\dot{x}(t) = \lambda x(t)$.
  Solution: $x(t) = x(0)e^{\lambda t}$.
- Constant Growth plus a Constant: $\dot{x}(t) = \lambda x(t) + b$.
  Solution: $x(t) = -\frac{b}{\lambda} + ke^{\lambda t}$.

Theorem

Stability condition:

- *If λ is negative, $x(t)$ decays to 0 (asymptotic stability).*
- *If λ is positive, $x(t)$ grows without bound (instability).*

Nonautonomous (self-study)

- Simple case: $\dot{x}(t) = \lambda x(t) + b(t)$.
  Solution: $x(t) = e^{\lambda t}\left(k + \int e^{-\lambda t} b(t) dt\right)$.
- General case $\dot{x}(t) = \lambda(t)x(t) + b(t)$.
  Solution: $x(t) = e^{\int \lambda(s)ds}\left(k + \int e^{-\int \lambda(s)ds} b(t) dt\right)$.

System of 2 Differential Equations

Linear Homogeneous System

$$\dot{x}_1 = ax_1 + bx_2,$$
$$\dot{x}_2 = cx_1 + dx_2$$

To find solutions, transformed to matrix form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

Solution (λ_1, λ_2 are eigenvalues and $\mathbf{u}, \mathbf{v}$ are eigenvectors of λ_1, λ_2)

$$\dot{\mathbf{x}} = k_1 e^{\lambda_1 t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + k_2 e^{\lambda_2 t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Steady state $(0, 0)$.
Stability

1. Stable: $tr(\mathbf{A}) < 0$ and $|\mathbf{A}| > 0$, i.e. both eigenvalues of $\mathbf{A}$ have **negative** real parts: $\lambda_1 < 0$ AND $\lambda_2 < 0$, $|\mathbf{A}| > 0$.
2. Unstable: $tr(\mathbf{A}) > 0$ and $|\mathbf{A}| > 0$, i.e. both eigenvalues of $\mathbf{A}$ have **positive** real parts: $\lambda_1 > 0$ AND $\lambda_2 > 0$, $|\mathbf{A}| > 0$,.
3. Saddle: If $|\mathbf{A}| < 0$, i.e. λ_1 AND λ_2 have opposite signs.

System of 2 Differential Equations

Nonlinear Homogeneous System

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y)$$

Transformed to matrix form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

with the Jacobian

$$\mathbf{A} = \begin{pmatrix} f'_x & f'_y \\ g'_x & g'_y \end{pmatrix}$$

$(\bar{x}, \bar{y})$ is the steady (equilibrium) state for the system.

1. If $tr(\mathbf{A}) < 0$ and $|\mathbf{A}| > 0$, both eigenvalues of $\mathbf{A}$ have **negative** real parts, then $(\bar{x}, \bar{y})$ is locally asymptotically stable.
2. If $tr(\mathbf{A}) > 0$ and $|\mathbf{A}| > 0$, both eigenvalues of $\mathbf{A}$ have **positive** real parts, then $(\bar{x}, \bar{y})$ is unstable.
3. If $|\mathbf{A}| < 0$, the eigenvalues are nonzero real numbers of OPPOSITE signs, $(\bar{x}, \bar{y})$ is a saddle.

Phase Diagrams

Systems of 2 linear differential equations

$$\dot{x} = ax + by + \kappa_1,$$
$$\dot{y} = cx + dy + \kappa_2. \tag{1}$$

Steps:

1. (A. Nullclines) Plot the nullclines, which are the loci $\dot{x} = 0$ and $\dot{y} = 0$.
2. (B. Steady State) The steady state is the intersection of the two nullclines.
3. (C. Directional Arrows) Determine the trajectories by analyzing the signs of

$$\frac{d\dot{x}}{dx} \qquad \frac{d\dot{x}}{dy} \qquad \frac{d\dot{y}}{dy} \qquad \frac{d\dot{y}}{dx}$$

4. (D. Trajectories) Using the information above, draw trajectories

The same process can be applied to Nonlinear system.

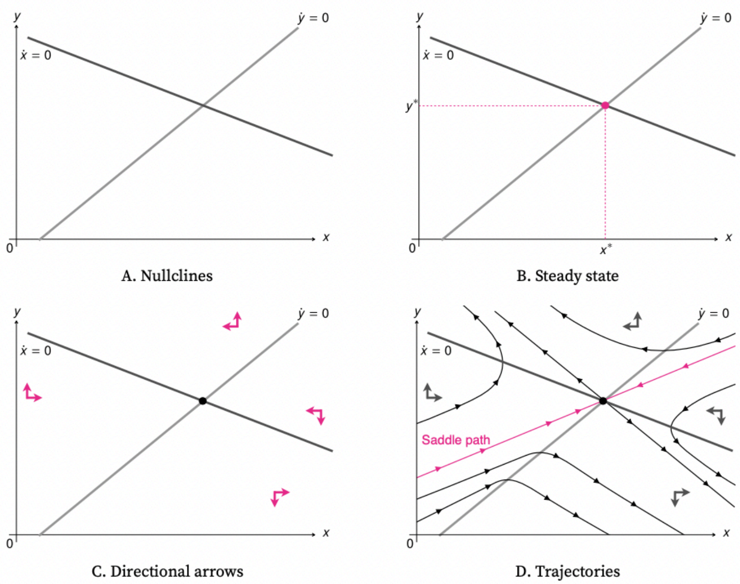

Figure: Phase diagram of the dynamical system (1) (Michaillat, 2023) .

Example of a Nonlinear case: Optimal Growth

Given the system

$$\dot{k} = f(k) - c - \delta k, \\ \dot{c} = [f'(k) - (\delta + \rho)]c \tag{2}$$

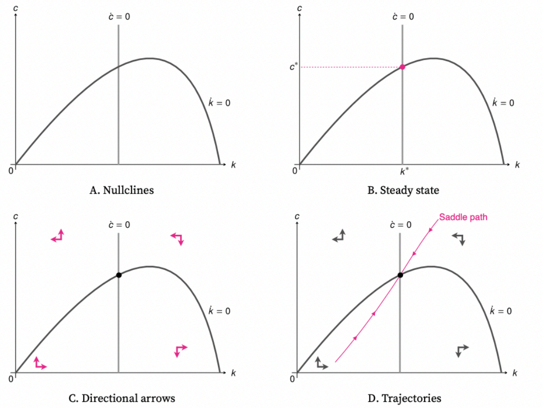

Figure: Phase Diagram of system (2) (Michaillat, 2023).

Dynamic Programming

$$\max_{c_t} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} &(\textbf{transition equation}) && k_{t+1} = f(k_t) + (1-\delta)k_t - c_t, \\ &(\textbf{initial condition}) && k_0 > 0 \\ &(\textbf{transversality condition}) && \lim_{t\to 0} \beta^t u'(c_t) k_{t+1} = 0 \end{aligned}$$

1. Write the Bellman equation
$$V(k_t) = [u(c_t) + \beta V(k_{t+1})] \tag{3}$$
2. Solve for policy function by maximizing V with respect to control variable
$$\frac{\partial V(k_t)}{\partial k_{t+1}} = 0 \Leftrightarrow \frac{\partial u(k_{t+1})}{\partial k_{t+1}} + \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}} = 0$$
3. Use Benveniste-Scheinkman Equation for $\frac{\partial V(k_t)}{\partial k_t}$ then forward to $t+1$.
4. Obtain Euler: $\frac{u'(c_t)}{u'(c_{t+1})} = \beta(f'(k_{t+1}) + (1-\delta))$. Policy function maps k_t to c_t.

Dynamic Programming

Further issues

1. How to obtain the closed-form Value function and Policy function?
2. Value function iteration algorithm.
3. Steady state
4. Stability of the steady state

Optimal Control

We want to $\max_{\{c_t\}_{t=0}^{T}} e^{-(\rho-n)t} u(c_t) dt$

$$\begin{aligned} &(\textbf{transition equation}) && \dot{k}_t = f(k_t) - \delta k_t - c_t, \\ &(\textbf{initial condition}) && k_0 > 0 \\ &(\textbf{transversality condition}) && \lim_{t\to\infty} \lambda_t k_t = 0 \end{aligned}$$

The control variable is c_t, and the state variable is k_t.

1. Write the present-value Hamiltonian
$$H_t = u(c_t)e^{-\rho t} + \lambda_t(\dot{k}_t)$$
2. Take FOC wrt to control variable
$$\frac{\partial H_t}{\partial c_t} = 0 \Leftrightarrow e^{-\rho t} u'(c_t) = \lambda_t.$$
3. Take FOCs wrt to the state and co-state variable
$$\dot{k}_t = \frac{\partial H_t}{\partial \lambda_t} = f(k_t) - c_t - \delta k_t, \qquad \dot{\lambda}_t = -\frac{\partial H_t}{\partial k_t} = -\lambda_t(f'(k_t) - \delta)$$
4. Derive the Euler equation by diff. control FOC wrt time. $\frac{\dot{c}_t}{c_t} = f'(k_t) - \delta - \rho$.