INSEIKAI Tohoku BootCamp 2024

Mathematics III

Special Topics in Micro-founded Macroeconomics

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Contents

"Economists use their economic models to explain or to understand the facts of the world by telling stories about how those facts might have arisen."

— Mary S. Morgan

Fact and Fiction in Economics (Mäki, 2002)

Chapter 1

Child Quality-Quantity Tradeoff

1.1 Reality

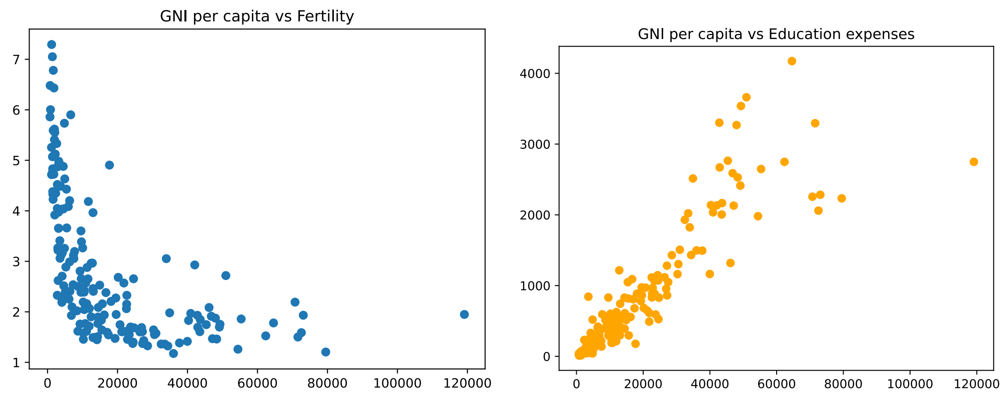


Fig. 1.1. World Development Indicators. Average: 2009–2019.

Rich countries have lower fertility rates and higher educational expenses, while poorer countries have more children and spend less on education. We illustrate this model based on de la Croix and Doepke (2003) and de la Croix (2013, Chapter 1).

1.2 The Model

We use an overlapping generations structure. Preferences are represented by the following utility function

$$\ln(c) + \gamma \ln(n \cdot h(e)),$$

where $c, n, h(e)$ are consumption, fertility rate, and children's human capital, which depends on educational expenses. Human capital follows

$$h = \mu(\theta + e)^{\eta},$$

where θ is a child's innate ability, μ is the efficiency parameter, and $\eta \in (0, 1)$ is the elasticity of human capital to total education input (or returns to educational expenses). Now, we need a budget constraint

$$c + ne = w(1 - \phi n),$$

where ϕ is the time cost of childrearing. How is wage determined? For simplicity, assume the production function follows

$$Y = vL,$$

where v is labor productivity, L is total labor input. Hence, we have the wage per person equals product per person equals

$$w = Y'(L) = v.$$

Construct the problem of maximizing individual preferences with respect to the budget constraint and prove the following

Proposition 1. If $w > \theta/(\eta\phi)$ then

$$e = \frac{\eta\phi w - \theta}{1-\eta},$$
$$n = \frac{(1-\eta)\gamma w}{(\phi w - \theta)(1+\gamma)}.$$

Otherwise,

$$e = 0,$$
$$n = \frac{\gamma}{\phi(1+\gamma)}.$$

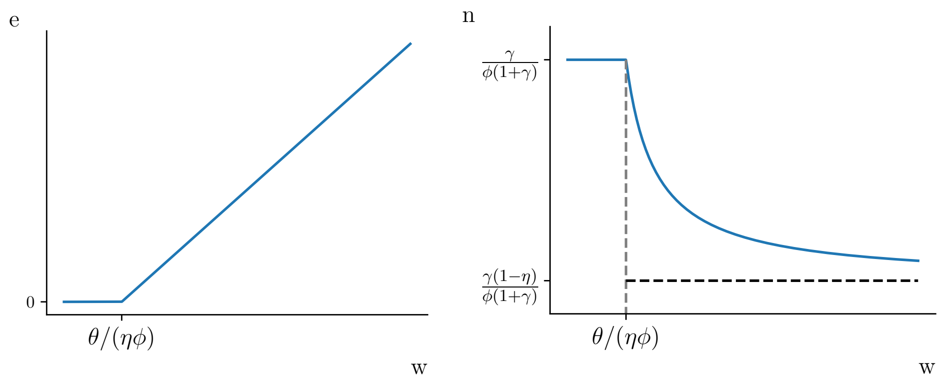


Fig. 1.2. Illustration for Proposition 1

1. For rich parents, differentiate e, n with respect to w to see how decisions change given a change in income w? What can you say about this relationship?
2. Why do the fertility and education choices differ between rich and poor parents? (Hint: the opportunity cost of raising children. Note that e and n are substitutes by construction.)
3. Find the minimum and maximum fertility by solving

$$n_{min} = \lim_{w \to 0} n \qquad n_{max} = \lim_{w \to \infty} n$$

Model Variable	Description	Real-world Equivalence
n	Fertility (net)	Fertility per woman × (1 - infant mortality rate/1000)
y	Productivity	GNI per person
$e+\theta$	Education expenses	Adjusted savings: education expenditure (% of GNI) × y
v	Wage	implied from $y = v(1-\phi n)$

Table 1.1: Estimation Strategy.

1.3 Calibration

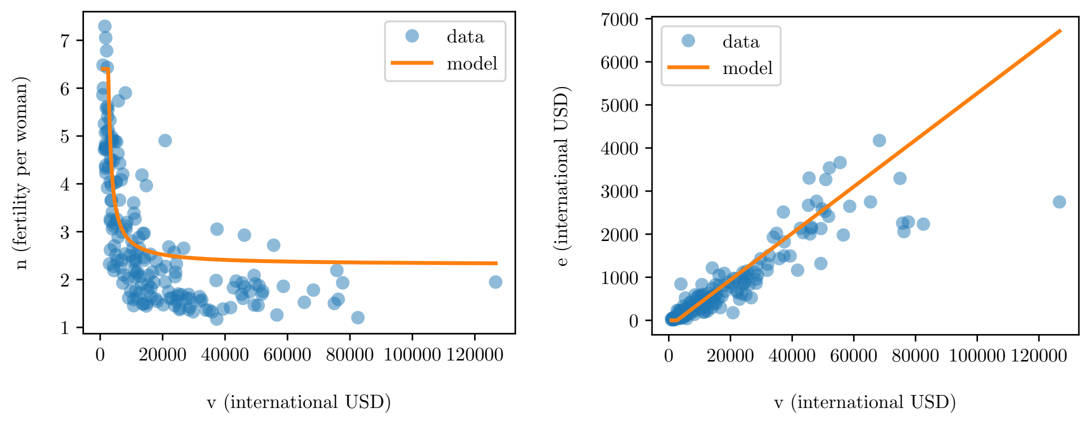
Many people may say this claim is not true. Thus, we show a calibration of the model to see how well the theoretical model matches the real-world data.

For this exercise, I use the average values from 2009 to 2018 (10 years) for all countries in the World Development Indicators.

To calibrate the data, we can use Maximum Likelihood Estimation. You can do this in Python. To perform a good calibration, you need a good guess. A good guess often comes from previous research. For this, we rely on de la Croix (2013), who provided parameters for the 1998-2002 average data. Using them as an initial guess, we updated the parameters as follows.

Parameter	Guess (1998-2002 average)	Calibrated Value
η	0.572	0.639
ϕ	0.039	0.031
θ	51.61	51.80
γ	0.103	0.243

Table 1.2: Calibrated Parameters.



(a) Fertility vs Productivity. Fitness: $R^2 = 0.657$, t-stats = 18.654, p-val = 0.00

(b) Education vs Productivity. Fitness: $R^2 = 0.796$, t-stats = 26.687, p-val = 0.00

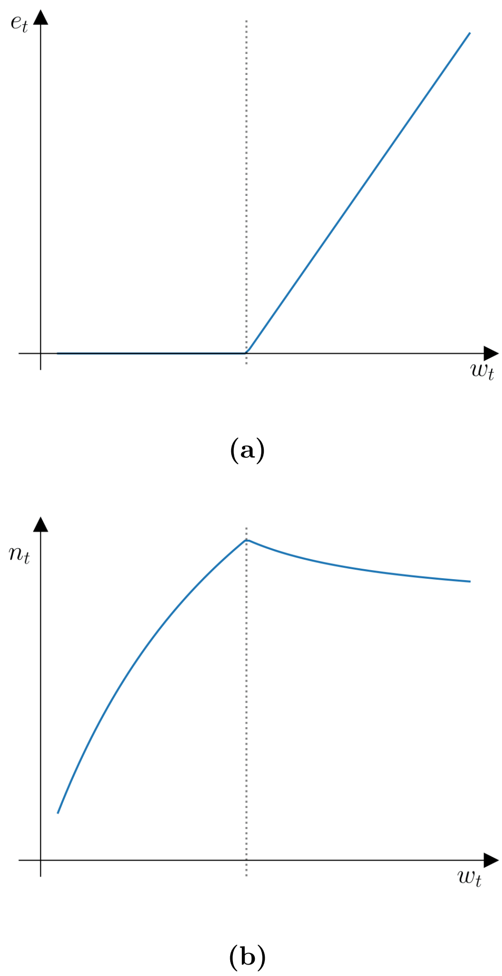
Fig. 1.3. Model performance vs real-world data

As you can see, economics theory is not a fairy tale. This particular theory about child quality-quantity tradeoff was first proposed by Barro and Becker (1989) and has dominated fertility economics ever since.

1.4 Application

Ex. 1. Add a good cost of childrearing to the budget constraint, say ψn.

1. Restate the problem
2. How does the problem change? Pay attention to fertility choice when $e = 0$.
3. Justify your argument with this graph. What is the value of the threshold?



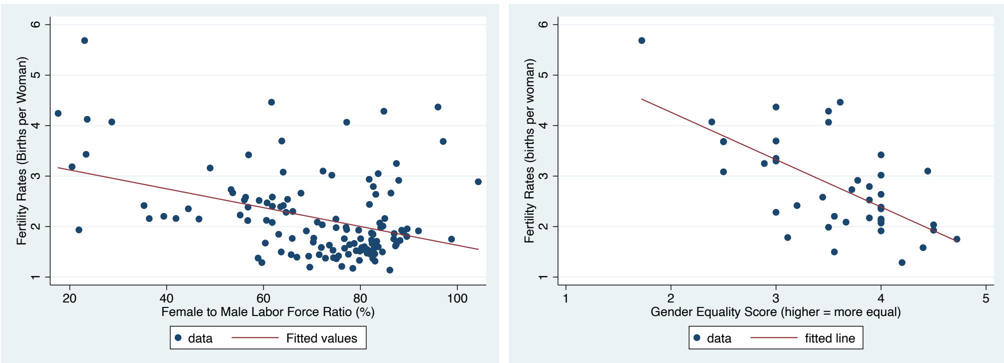
(a) **(b)**

Chapter 2

Gender Equality

2.1 Reality

The second observation is gender equality. More females tend to have fewer children when they join the labor force.



(a) Female to Male Labor Participation **(b)** CPIA gender equality rating

Fig. 2.1. Average WDI data from 2008 to 2017. Excluding Gulf and African countries.

We study a toy model demonstrated in Doepke et al. (2023, p.181).

2.2 Model

In this household, only the wives take care of the children, so their time to work in the market is

$$l_f = 1 - \phi n.$$

Men do not spend time taking care of children; they only do so at work. A couple's budget constraint is

$$c = w_m + l_f w_f.$$

A couple's utility

$$\ln(c) + \delta \ln(n).$$

Prove the following

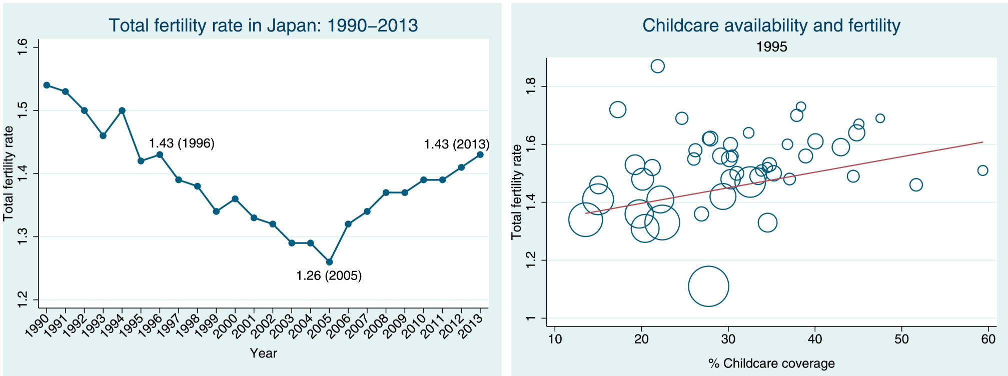
Proposition 2. The optimal fertility choice for the couple is

$$n = \frac{\delta}{1+\delta}\frac{1}{\phi}\left(1 + \frac{w_m}{w_f}\right).$$

Show that when the gender pay gap closes, the fertility declines.

2.3 Marketization

Female participation in the labor market does not necessarily lead to declining birth rates. A paper by Fukai (2017) shows that the availability of childcare availability can support women in balancing childcare and work.



(a) TFR in Japan. **(b)** Childcare availability & fertility: 1995.

Fig. 2.2. Empirical Evidence from Fukai (2017).

Suppose now a childcare market has been developed. Instead of spending time on childcare directly, a couple can buy on the childcare market under the price $p_s > 0$. We denote by $s \in [0, \bar{s}]$ the share of childcare they can buy with $\bar{s}$ is the maximum amount of childcare that can be outsourced. A couple's choice problem is

$$\max_{c,n,s} u := \ln(c) + \delta \ln(n)$$
$$\text{s.t. } c + \psi n + s p_s n \phi = w_m + w_f[1 - (1-s)n\phi].$$

Show that the optimal fertility choice is

$$n = \frac{\delta}{1+\delta} \cdot \frac{w_m + w_f}{\psi + [s p_s + (1-s)w_f]\phi}.$$

Show that the utility u can be expressed as a function of s as follows

$$u(s) = \ln\left(\delta^{\delta}\left[\frac{w_m + w_f}{1+\delta}\right]^{1+\delta}\right) - \delta \ln(\psi + [s p_s + (1-s)w_f]\phi).$$

1. Differentiate $u'(s)$.
2. Under what condition will the couple decide not to buy s? i.e., $u'(s) \le 0$.
3. Under what condition will this couple buy s? How much will they buy?

Prove the following proposition by differentiating n w.r.t w_f.

Proposition 3. Assume that the good cost of childrearing is smaller than the time cost $\psi < w_m\phi$. Let

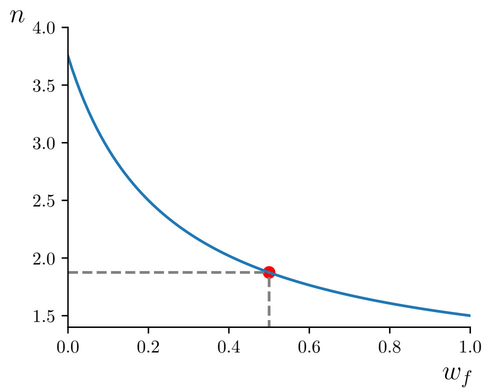
$$\hat{s} = \frac{w_m\phi - \psi}{(p_s + w_m)\phi},$$

then we have

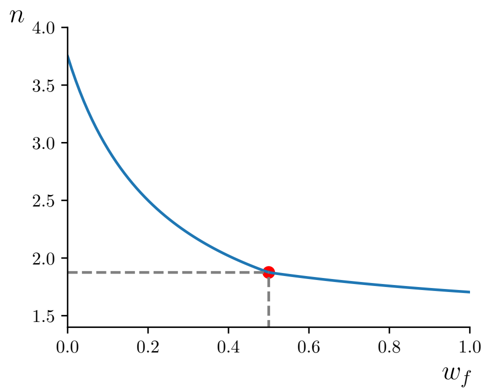
$$\frac{\partial n}{\partial w_f} \begin{cases} < 0 & \text{if } s < \hat{s}, \\ = 0 & \text{if } s = \hat{s}, \\ > 0 & \text{if } s > \hat{s}. \end{cases}$$

Explain what happens if

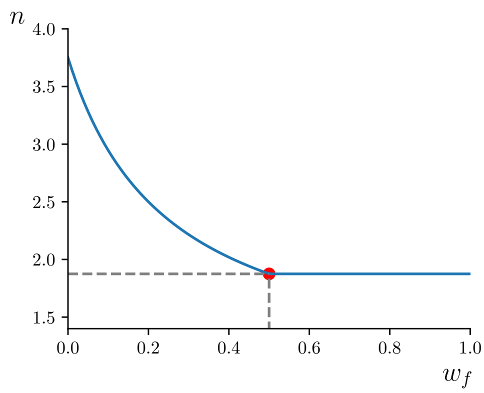
1. $\bar{s} = 0$?
2. $\bar{s} = \hat{s} < 1$?
3. $\bar{s} = \hat{s} = 1$?



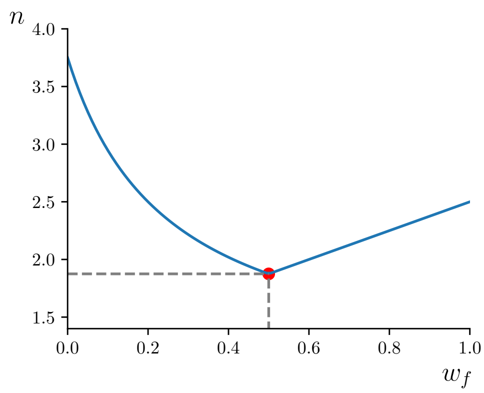
(a) $\bar{s} = 0$



(b) $\bar{s} < \hat{s} < 1$



(c) $1 > \bar{s} = \hat{s}$



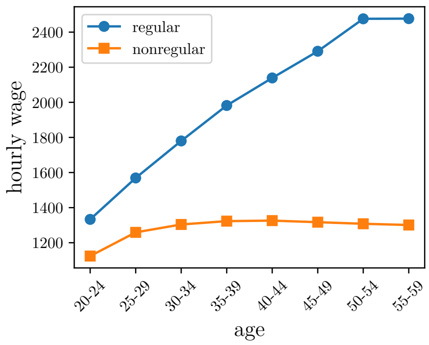
(d) $\bar{s} = \hat{s} = 1$

Fig. 2.3. Illustration of Proposition 3. Parameters: $\delta = 0.6, \psi = 0.1, \phi = 0.4, w_m = 1, p_s = 0.5$.

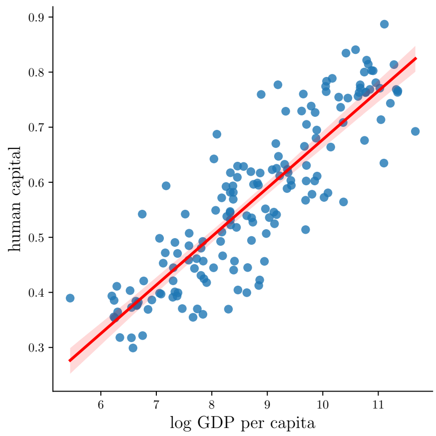
Chapter 3

Human Capital

3.1 Reality



(a) Full-time vs Part-time Wage (Japan, 2019, Source: https://www.mhlw.go.jp/stf/wp/hakusyo/kousei/19/backdata/01-01-03-23.html)



(b) log GDP per capita and human capital in 2018. Source: WDI

Fig. 3.1. Some Indication of Human Capital.

3.2 Model

So far, education has not contributed anything to productivity. Let's change this assumption by adopting an idea from Ceroni (2001). For simplicity, let us assume that fertility is exogenous and is normalized to 1. Preferences are

$$u_t = \ln(c_t) + \delta h_{t+1},$$

subject to

$$h_{t+1} = f(e_{t+1}) = \ln(\gamma e_{t+1} + v),$$
$$c_t = w_t - e_{t+1}.$$

Labor productivity is now h_t instead of ν, hence:

$$w_t = h_t(e_t).$$

Now, prove the following proposition by doing optimization.

Proposition 4. Let $\bar{h} = v/\delta\gamma$, then

$$e_{t+1} = \begin{cases} 0 \text{ if } h_t < \bar{h}, \\ \dfrac{\delta}{1+\delta} h_t - \dfrac{v}{\gamma(1+\delta)} \text{ if } h_t \geq \bar{h}. \end{cases}$$

Show the law of motion of human capital accumulation

$$h_{t+1} = \phi(h_t) = \begin{cases} \ln(v) \text{ if } h_t < \bar{h}, \\ \ln\left[\dfrac{\delta(\gamma h_t + v)}{1+\delta}\right] \text{ if } h_t \geq \bar{h}. \end{cases}$$

3.3 Poverty Trap

Describe the dynamics of $\phi(h_t)$. Find conditions for poverty traps to emerge.

1. Condition 1 (concavity): $\phi'(\bar{h}) > 1$ for any $h_t > \bar{h}$. (why?)
2. Condition 2 (lowest steady state): $\phi(h_t < \bar{h}) < \bar{h}$. (why?)

Your result should be

$$\frac{v}{\delta} \frac{1}{\ln(v)} > \gamma > \frac{v}{\delta}(1+\delta).$$

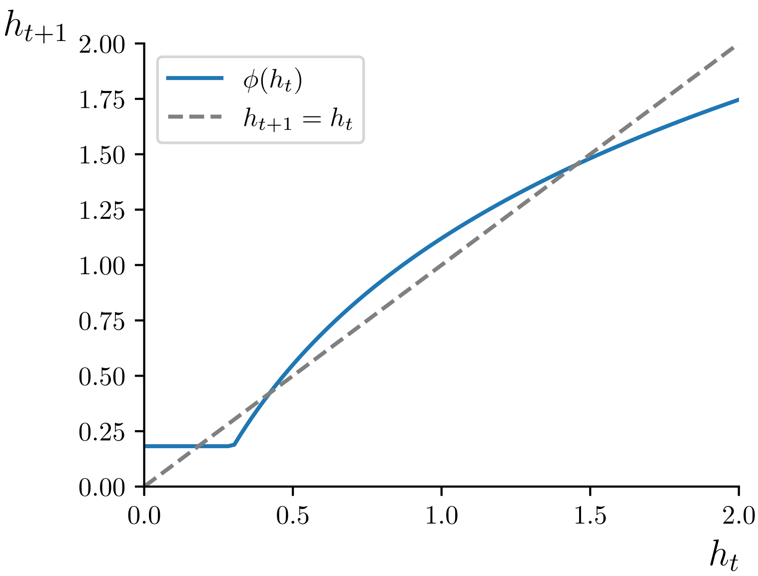


Fig. 3.2. Dynamics of $\phi(h_t)$ with $\gamma = 8, \delta = 0.5, v = 1.2$.

Describe what the poverty trap is. Why is it called a trap? Is there a way to escape this trap? Let us assume that everyone in this economy receives a one-time aid to boost the "crude" productivity (productivity without education) from v to $\tilde{v} = xv$ where $x > 1$.

What is the condition of x so the economy can escape the poverty trap?

3.4 Counterfactual

1. What happens if $\gamma > \frac{v}{\delta}\frac{1}{\ln(v)}$? How do you know?
2. What happens if $\gamma < \frac{v}{\delta}(1+\delta)$? How do you know?

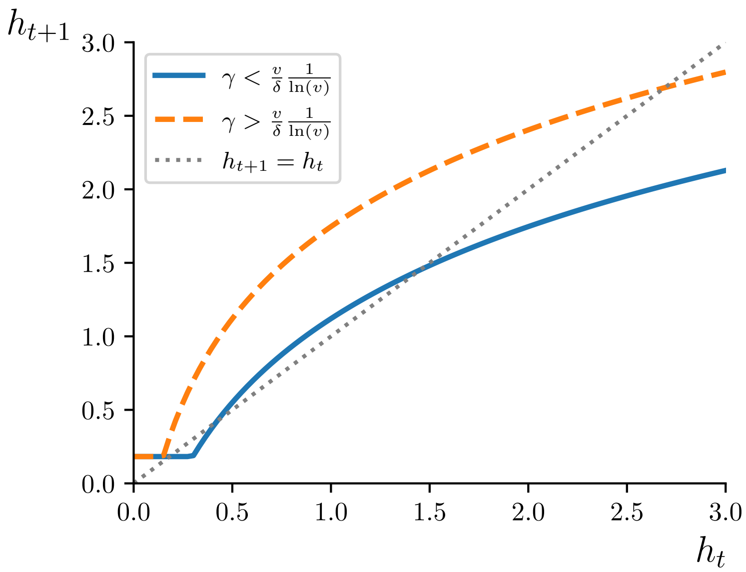


Fig. 3.3. A rise in γ

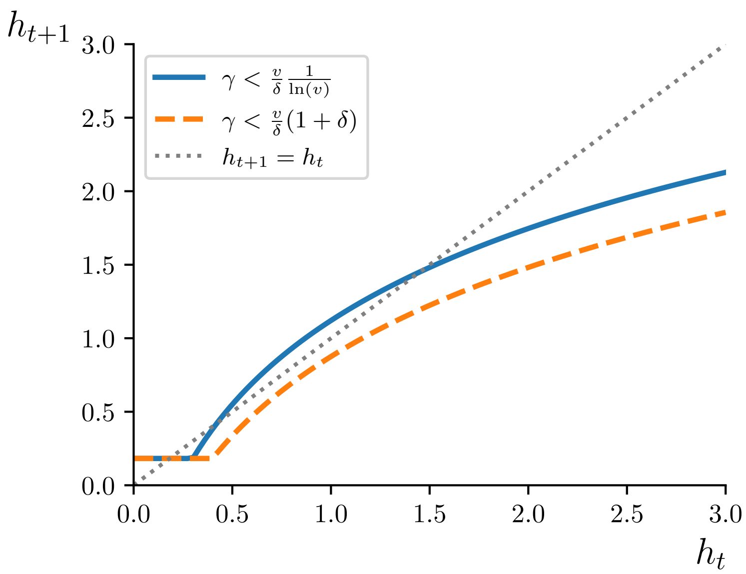


Fig. 3.4. A decrease in γ

Chapter 4

Career Choice

4.1 Reality

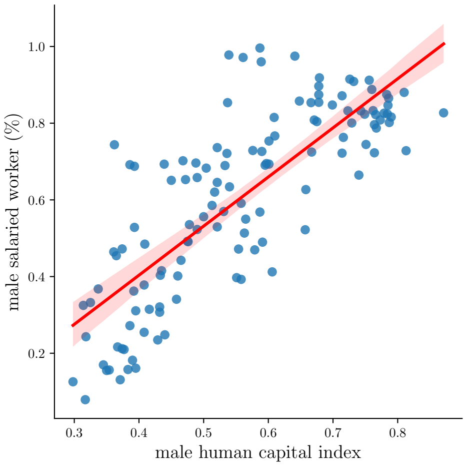


Fig. 4.1. Human Capital Index and Salaried Workers Ratio (% of employment) for Male. Global data from WDI in 2017.

We will use the models by Kimura and Yasui (2007) and Galor and Weil (1996).

4.2 Model

Production includes physical capital K, skilled labor H, and unskilled labor L such that

$$Y_t = A[K_t^{\alpha} H_t^{1-\alpha} + bL_t],$$

where $A > 0, b > 0, \alpha \in (0, 1)$. Capital is skill-complimentary.

To become a skilled worker, an individual needs to spend τ portion of time to acquire education. To study how education might affect fertility decisions, we assume that τ competes with the time required to care for children, which is z times the number of children n. The time that can be used in the market for each type of worker is

$$(H) : 1 - \tau - zn_t^H,$$
$$(L) : 1 - zn_t^L.$$

Let ϕ be the portion of skilled workers over the whole population. Hence

$$\phi_t = \frac{N_t^H}{N_t^L + N_t^H} = \frac{N_t^H}{N_t}.$$

where N_t is the population. The labor supply is

$$(H): H = (1 - \tau - z n_t^H)\phi_t N_t,$$
$$(L): L = (1 - z n_t^L)(1 - \phi_t)N_t.$$

Input factor prices under a competitive market are

$$w_t^H = \frac{\partial Y_t}{\partial H_t} = A(1-\alpha)\left[\frac{k_t}{(1-\tau-zn_t^H)\phi_t}\right]^{\alpha}, \tag{4.1}$$
$$w_t^L = \frac{\partial Y_t}{\partial L_t} = Ab, \tag{4.2}$$

where $k_t = K_t/N_t$ is the capital per capita. Regardless of type, an agent has the preference

$$u_t = \gamma \ln(n_t) + (1-\gamma)\ln(c_{t+1}).$$

The budget constraint for a skilled worker is

$$n_t^H + s_t^H = w_t^H(1 - \tau - z n_t^H),$$
$$c_{t+1}^H = (1 + r_{t+1})s_t,$$

and for an unskilled worker is

$$n_t^L + s_t^L = w_t^L(1 - z n_t^L),$$
$$c_{t+1}^L = (1 + r_{t+1})s_t^L.$$

Utility maximization shows the FOCs such that

$$n_t^H = \frac{\gamma(1-\tau)}{z}, \tag{4.3}$$
$$s_t^H = w_t^H(1-\tau)(1-\gamma), \tag{4.4}$$

and

$$n_t^L = \frac{\gamma}{z}, \tag{4.5}$$
$$s_t^L = w_t^L(1-\gamma). \tag{4.6}$$

4.3 Equilibrium Dynamics

In equilibrium, agents are indifferent between becoming skilled or unskilled. Thus, the following condition must hold

$$u_t^H = u_t^R.$$

By substituting the FOC conditions for the utility function, we have

$$\frac{w_t^L}{w_t^H} = (1-\tau)^{1/(1-\gamma)}. \tag{4.7}$$

From (4.1), (4.2), (4.3), (4.5), one can derive

$$\phi(k_t) = \frac{(1-\tau)^{\frac{1}{\alpha(1-\gamma)}-1}}{1-\gamma}\left(\frac{1-\alpha}{b}\right)^{\frac{1}{\alpha}} k_t \equiv \theta k_t. \tag{4.8}$$

Since ϕ cannot be larger than 1, define $\bar{k}$ such that $\phi(\bar{k}) = 1$, then

$$\bar{k} = \theta^{-1}.$$

We have

$$\phi(k_t) = \begin{cases} \theta k_t \text{ if } k_t < \bar{k}, \\ 1 \text{ if } k_t \geq \bar{k}. \end{cases}$$

Let the total fertility rate at time t be m_t, then from (4.3) and (4.5)

$$m_t = \phi_t n_t^H + (1-\phi_t)n_t^L = (1-\tau\phi_t)\frac{\gamma}{z}.$$

Note that $N_{t+1} = m_t N_t$. Using (4.8), it can be written as a function of k_t

$$m(k_t) = \begin{cases} (1-\tau\theta k_t)\frac{\gamma}{z} \text{ if } k_t < \bar{k}, \\ (1-\tau)\frac{\gamma}{z} \text{ if } k_t \geq \bar{k}. \end{cases}$$

The dynamics of the model can now be pinned down solely on the dynamics of k. The capital accumulates according to

$$K_{t+1} = [\phi_t s_t^H + (1-\phi_t)s_t^L]N_t.$$

Dividing both sides by N_{t+1} and use the saving function from (4.4), (4.6), we have

$$k_{t+1} = \frac{z(1-\gamma)}{\gamma}\frac{\phi_t w_t^H(1-\tau) + (1-\phi_t)w_t^L}{1-\tau\phi_t}.$$

Using the wage from (4.1), (4.2), the equilibrium dynamics is a sequence of k_t starting from $t = 0$ to ∞ where k_0 is given historically such that

$$k_{t+1} = \phi(k_t) = \begin{cases} \dfrac{Az(1-\gamma)}{\gamma}\dfrac{1}{1-\tau\theta k_t}\left[\dfrac{(1-\alpha)(1-\tau)^{1-\alpha}\theta^{1-\alpha}k_t}{(1-\gamma)^\alpha} + (1-\theta k_t)b\right] \text{ if } k_t < \bar{k}, \\ \dfrac{Az(1-\gamma)}{\gamma}\dfrac{(1-\alpha)}{(1-\tau)^\alpha(1-\gamma)^\alpha}k_t^\alpha \text{ if } k_t \geq \bar{k}. \end{cases} \tag{4.9}$$

Since capital is skill-complimentary, a rise in k encourages people to acquire education and become skilled to earn the premium by forgone fertility and time to study. Skilled agents earn more utility from consumption and less utility from altruism. Meanwhile, unskilled agents earn more utility from altruism and less from consumption. This trade-off makes sense only when k is sufficiently low. When k is high enough, fertility is independent of k, so agents can spend sufficient time cost τ without sacrificing fertility.

1. $\phi(0) > 0$. Why do we need to check this?

2. $\lim_{k_t\to\infty} \phi'(k_t) = 0$. Why do we need to check this?

3. $\phi'(k_t) > 0$ if $k_t < \bar{k}$ and $\phi'(k_t) > 0$ for $k_t \geq \bar{k}$.

4. $\phi''(k_t) > 0$ if $k_t < \bar{k}$ and $\phi''(k_t) < 0$ for $k_t \geq \bar{k}$.

5. What is the shape of $\phi(k_t)$ when $k_t < \bar{k}$?
6. What is the shape of $\phi(k_t)$ when $k_t \geq \bar{k}$?
7. (advanced) What is the condition of A for the model to have 3 equilibria? Pay attention to the function $\bar{k}$.

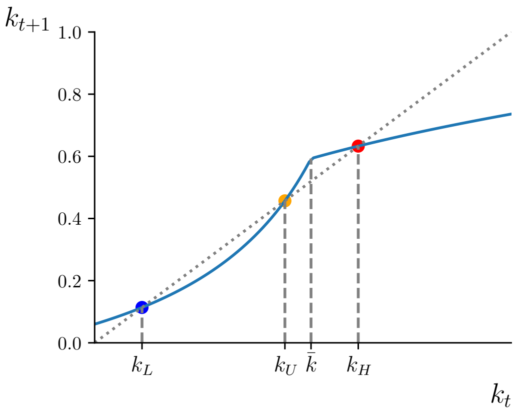
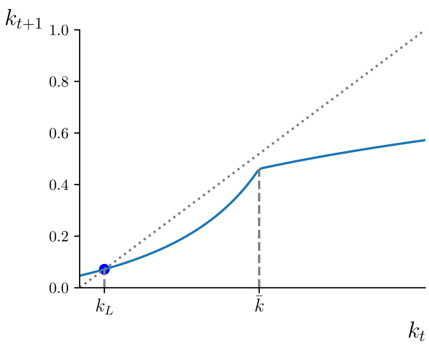
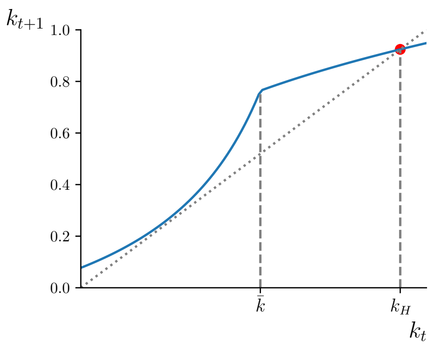


Fig. 4.2. Development trap case with three equilibria where k_L, k_H are stable and k_U is unstable. Parameters: $\alpha = 0.33, \tau = 0.6, \gamma = 0.6, z = 0.2, b = 0.1, A = 4.5$.

What happens when A increases or decreases?



(a) Same params except $A = 3.5$



(b) Same params except $A = 5.8$

A	k^*	m^*	$\phi*$
4.5	0.11	2.6	0.22
4.5	0.63	1.2	1
3.5	0.07	2.75	0.13
5.8	0.92	1.2	1

Table 4.1: Steady-state values

Chapter 5

Project: Life Expectancy

In a paper by Chen (2010), the author shows a model linking life expectancy and fertility with career choice. We will proceed with the structure previously demonstrated, from reality to model to simulation.

1. (Reality Check): In this paper, the theoretical model shows that an increase in life expectancy tends to increase savings, increase the labor force participation rate of educated workers, and reduce fertility. Use some variables from World Development Indicators to check if the claim is true.

Variable	Description
life expectancy	Life expectancy at birth, total (years)
savings	log of Gross domestic savings (current US$)
fertility	Fertility rate, total (births per woman)
educated labor (%)	Labor force with intermediate education
educated labor 2 (%)	Labor force with advanced education

Table 5.1: Suggested variables. You can use alternatives if you see fit.

2. (Model): Follow the structure of the model and try to derive the main results (basically, the propositions in the paper). If you don't have time, try to make it until the equilibrium dynamics.

3. (Graphical Presentation): Try to make Fig.1 or Fig.2 in the paper.

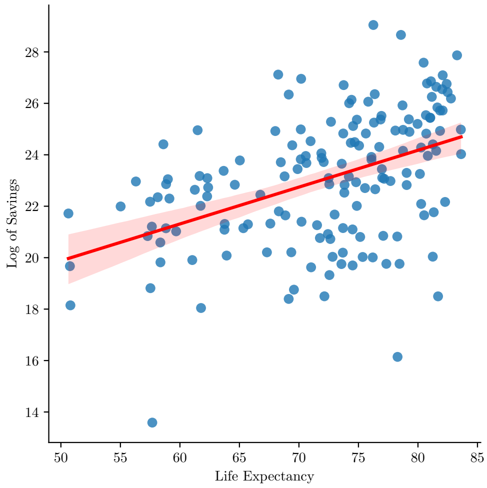
   Remark 1 (Multiple Equilibria Condition)**.** These are the conditions for multiple equilibria to exist.

   (a) There exists a $k_t < \bar{k}$ such that $\xi(k_t) < k_t$. You should find a condition of A that satisfies this condition (called the upper bound of A). This condition gives us the convex part seen in Fig.1 of the paper.

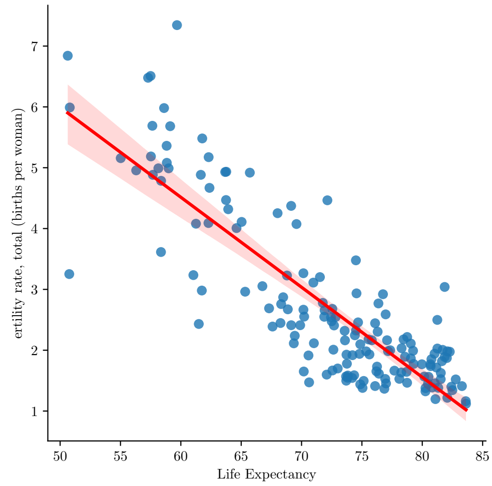
   (b) The dynamics when $k_t = \bar{k}$ satisfies $\xi(\bar{k}) > \bar{k}$. Or $\xi'(\bar{k}) > 1$. Finding the values of A satisfying this condition gives you the lower bound of A. This condition gives us a concave dynamic in Fig.1.

   (c) Combining these 2 conditions will give you a range of values of A (and other parameters) to have multiple equilibria.

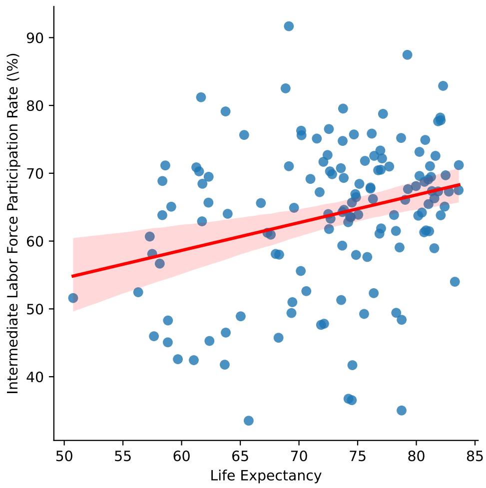
Link to the paper:
https://link.springer.com/article/10.1007/s00148-008-0202-y



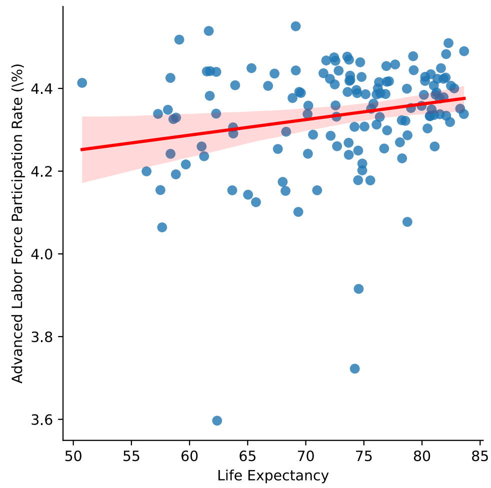
(a) Life Expectancy and Savings



(b) Life Expectancy and Fertility



(c) Life Expectancy and Intermediate Educated Worker Participation Rate



(d) Life Expectancy and Advanced Educated Worker Participation Rate

Fig. 5.1. WDI data averaging 2008–2017.

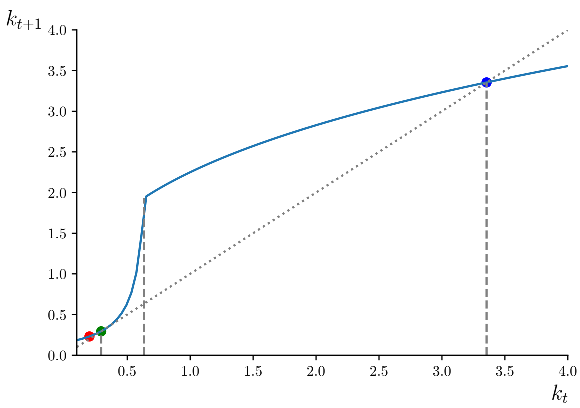


Fig. 5.2. An example dynamics. Parameters: $A = 6.14, b = 0.1, z = 0.1, \gamma = 0.142105, \sigma = 0.9, \alpha = 0.33, \beta = 0.99^{35}, x = 0.5$.

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