Inseikai Bootcamp Spring 2025
Computation Notes

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Contents

Chapter 1

Review of Optimization

1.1 Unconstrained optimisation

- The first-order condition for a point x^* to be a max or min of a function f is that $f'(x^*) = 0$ so that x^* is a critical point of f.
- For an n-dimensional point $\mathbf{x}^*$, the local max or min must satisfy

$$\frac{\partial F}{\partial x_i}(\mathbf{x}^*) = 0 \text{ for } i = 1, \dots, n$$

Example 1 (Profit-Maximisation)**.** Suppose the production function F is twice differentiable (e.g. Cobb-Douglas)

$$Q = F(K, L) = K^\alpha L^{1-\alpha},$$

where K, L are capital and labour. The input price is p, capital rental cost is r, wage rate is w such that $p, r, w > 0$. The firm chooses the optimal input factors (K, L) to maximise the following profit

$$\max_{K,L} \pi = pF(K, L) - rK - wL$$

The FOCs are:

$$\pi'_K = p\alpha K^{\alpha-1} L^{1-\alpha} - r = 0,$$
$$\pi'_L = pK^\alpha (1-\alpha) L^{-\alpha} - w = 0.$$

This is a system of 2 equations of 2 unknowns, so there exists a unique solution for K^*, L^*. We can also derive the capital-labour ratio by dividing the two equations:

$$\frac{p\alpha K^{\alpha-1} L^{1-\alpha}}{pK^\alpha (1-\alpha) L^{-\alpha}} = \frac{r}{w}$$

$$\frac{K}{L} = \frac{w}{r}\frac{\alpha}{1-\alpha}$$

Ex. 1. Solve the profit maximisation problem for a firm with the following production function

$$Q = F(K, L) = 12K^{1/2}L^{1/4}$$

with $p = 1, w = 0.6, r = 1.2$. [Answers: $K = L = 625$.]

1.2 Equality-constrained optimisation

Let $f(x_1, x_2)$ and $h(x_1, x_2)$ be differentiable functions of two variables. To maximize or minimize $f(x_1, x_2)$ subject to a constraint $h(x_1, x_2) = c$, we can write the Lagrangian

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda[h(x_1, x_2) - c]$$

The critical point $(x_1^*, x_2^*, \lambda^*)$ must satisfy

$$\frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial x_2} = \frac{\partial L}{\partial \lambda} = 0.$$

Example 2. The problem is

$$\max_{x,y} \ f(x,y) = x^{\frac{1}{3}}y^{\frac{2}{3}}$$
$$s.t. \ x + y = 60$$

The Lagrangian is

$$\mathscr{L} = x^{\frac{1}{3}}y^{\frac{2}{3}} - \lambda(x + y - 60)$$

The FOCs:

$$\mathscr{L}'_x = \frac{1}{3}x^{-\frac{2}{3}}y^{\frac{2}{3}} - \lambda = 0,$$
$$\mathscr{L}'_y = x^{\frac{1}{3}}\frac{2}{3}y^{-\frac{1}{3}} - \lambda = 0,$$
$$\mathscr{L}'_\lambda = x + y - 60 = 0,$$

From the FOCs, we obtain:

$$\lambda = \frac{1}{3}x^{-\frac{2}{3}}y^{\frac{2}{3}} = \frac{2}{3}x^{\frac{1}{3}}y^{-\frac{1}{3}}$$
$$\Rightarrow y = 2x$$

$$x + 2x = 60$$

which yields the solution $(x^*, y^*) = (20, 40)$.

In this example, we have a Cobb-Douglas form utility function with only two goods. In the section below, we will generalise it to a many-goods problem with Constant Elasticity of Substitution (CES).

1.3 Economic application

The above Lagrangian technique is very frequently used in economic modelling. We will look at the following consumer utility maximisation problem to see its application.

The preferences of a representative consumer are given by a CES utility function over a continuum of goods indexed by i:

$$U = \left[\sum_i q_i^\rho\right]^{\frac{1}{\rho}} \tag{1.1}$$

The parameter ρ controls the curvature of the utility function. In standard economic models, we require $\rho < 1$ to ensure the utility function is concave (i.e. exhibit diminishing marginal utility). When $\rho \to 0$, the CES form becomes Cobb-Douglas.

We will try to solve this consumer maximisation problem and derive the aggregate price index.

$$\max_{q_i} U = \left[\sum_i q_i^\rho\right]^{\frac{1}{\rho}} \tag{1.2}$$

In consumer problems, the budget constraint is normally implied by:

$$\sum_i p_i q_i = I \tag{1.3}$$

So now we have the familiar equality-constrained optimisation problem, and we can set up the Lagrangian:

$$\begin{aligned} L &\equiv U - \lambda \cdot \text{budget constraint} \\ &= \left[\sum_i q_i^\rho\right]^{\frac{1}{\rho}} - \lambda\left(\sum_i p_i q_i - I\right) \end{aligned} \tag{1.4}$$

Recall from (1.2) we are to find the optimal quantity demanded by consumers. The FOC is therefore differentiating with respect to q_i:

$$\frac{\partial L}{\partial q_i} = \frac{\partial U}{\partial q_i} - \lambda p_i = 0$$

$$
\begin{aligned}
&= \frac{1}{\rho}\left[\sum_i q_i^{\rho}\right]^{\frac{1}{\rho}-1}\frac{\partial[\ ]}{\partial q_i} - \lambda p_i \\
&= \frac{1}{\rho}\left[\sum_i q_i^{\rho}\right]^{\frac{1}{\rho}-1}\rho q_i^{\rho-1} - \lambda p_i \\
&= \left[\sum_i q_i^{\rho}\right]^{\frac{1}{\rho}-1} q_i^{\rho-1} = \lambda p_i
\end{aligned}
\tag{1.5}
$$

In standard optimisation, we just find the solution based on this FOC and the derivative wrt λ. However in this practice, we will introduce an intermediate variable to see the relative relation and derive the aggregate price index first. With this definition in hand, you can try the standard approach and see what happens to λ.

By analogy, the FOC for another consumer j is:

$$
\left[\sum_j q_j^{\rho}\right]^{\frac{1}{\rho}-1} q_j^{\rho-1} = \lambda p_j \tag{1.6}
$$

Now we can eliminate the λ term and obtain a relationship between q_i and q_j:

$$
\frac{\left[\sum_i q_i^{\rho}\right]^{\frac{1}{\rho}-1} q_i^{\rho-1}}{\left[\sum_j q_j^{\rho}\right]^{\frac{1}{\rho}-1} q_j^{\rho-1}} = \frac{p_i}{p_j} \tag{1.7}
$$

The term in the square bracket can be thought of as an aggregate utility (same for all consumers) so it can be eliminated as well.

$$
\begin{aligned}
\frac{q_i^{\rho-1}}{q_j^{\rho-1}} &= \frac{p_i}{p_j} \\
q_i &= \left(\frac{p_i}{p_j}\right)^{\frac{1}{\rho-1}} q_j
\end{aligned}
\tag{1.8}
$$

This is the relative relation between quantity of two goods, based on their relative price and substitution parameter. Now we substitute this into the utility function (1.1):

$$
\begin{aligned}
U &= \left[\sum_i \left(\frac{p_i}{p_j}\right)^{\frac{\rho}{\rho-1}} q_j^{\rho}\right]^{\frac{1}{\rho}} \\
U &= \frac{1}{p_j}^{\frac{1}{\rho-1}} q_j \left[\sum_i p_i^{\frac{\rho}{\rho-1}}\right]^{\frac{1}{\rho}}
\end{aligned}
$$

$$q_j = U p_j^{\frac{1}{\rho-1}} \left[\sum_i p_i^{\frac{\rho}{\rho-1}} \right]^{-\frac{1}{\rho}} \tag{1.9}$$

Now substitute into the budget constraint (1.3):

$$\begin{aligned}
\sum_i p_i q_i \equiv \sum_j p_j q_j &= I \\
\sum_j p_j U p_j^{\frac{1}{\rho-1}} \left[\sum_i p_i^{\frac{\rho}{\rho-1}} \right]^{-\frac{1}{\rho}} &= I \\
\left[\sum_j p_j^{\frac{\rho}{\rho-1}} \right] U \left[\sum_i p_i^{\frac{\rho}{\rho-1}} \right]^{-\frac{1}{\rho}} &= I \\
\left[\sum_i p_j^{\frac{\rho}{\rho-1}} \right]^{\frac{\rho-1}{\rho}} &= \frac{I}{U} \equiv P
\end{aligned} \tag{1.10}$$

Note here in the derivation, we used the fact that summing over i and j essentially gives the same value. The aggregate price index P is defined as total expenditure (same as income) divided by total utility, which is equivalent to cost per utility across all goods.

Many papers define $\sigma = \frac{\rho}{\rho-1}$ as the elasticity of substitution between goods. This measure is more economically meaningful and measurable. Rearranging gives $\rho = \frac{\sigma}{\sigma-1}$ so many papers use this instead of ρ (for example, see the function in Section 2.1.2).

Ex. 2. With the definition of aggregate price index P, do the optimisation again using the standard approach (without introducing j) and see what happens. **Hint**: You should obtain some nice and easy relation between λ and P. From there you can then obtain the demand function in terms of utility and price index:

$$q_i = U \left(\frac{P}{p_i} \right)^{\frac{1}{1-\rho}}$$

This demand function can also be obtained by combining (1.8), (1.9) and (1.10).

Ex. 3 (**Challenge**). We have derived the FOC and aggregate price index for a CES utility function with one good in each variety. It is possible to combine goods into one variety. Consider the representative consumer has two-tier preferences: Cobb-Douglas preferences over the two types of differentiated goods, M and A, and CES preferences over varieties of each differentiated good:

$$X_i = \left[\int_{\omega \in \Omega_i} q(\omega)^{1-\rho_i} x(\omega)^{\rho_i} d\omega \right]^{\frac{1}{\rho_i}}, \quad \rho_i = \frac{\sigma_i - 1}{\sigma_i}, \quad \sigma_i > 1$$

Confirm that the aggregate price index P_i and demand $x(\omega)$ are:

$$P_i = \left(\int_{\omega \in \Omega_i} q(\omega) p(\omega)^{1-\sigma_i} d\omega \right)^{\frac{1}{1-\sigma_i}}, \qquad x(\omega) = Y p(\omega)^{-\sigma_i} q(\omega) P_i^{\sigma_i - 1}$$

Chapter 2

OLG

In this section, we study the basic OLG model and then use Dynare to solve it.

2.1 Model

Typically, a life-cycle model earning profile looks like Figure 2.1. Our purpose is to build a model that can capture this shape.

Figure 2.1: Labor Income by Age in Japan.

2.1.1 Households

Imagine a household that can live through 6 periods. She spends the first 4 periods working and the last 2 periods retiring. In reality, you can imagine each period is equivalent to 10 years, and the person starts to work at the age of 20. Thus, she works from age 21 until age 60 and retires from age 61 to 80. When she works, she earns labor income and capital returns and pays social security tax. When she retires, she stops working, so the only source of income is capital returns and social security payments (pensions).

Let c, n, l be consumption, working time, and leisure. At the age of 1, she maximizes her lifetime utility

$$\max \sum_{s=1}^{6} \beta^{s-1} u(c^s_{t+s-1}, 1 - n^s_{t+s-1})$$

Note that s indicates age and t indicates time. The following utility function is used

$$u(c,l) = \frac{c^{1-\eta} \cdot (1-n)^{\gamma(1-\eta)} - 1}{1-\eta}.$$

When she works, the following constraint is used.

$$c^s_t + k^{s+1}_{t+1} = (1+r_t)k^s_t + (1-\tau_t)w_t n^s_t \text{ for } s = 1, 2, 3, 4$$

When she retires, the following constraint is used.

$$c^s_t + k^{s+1}_{t+1} = (1+r_t)k^s_t + b_t \text{ for } s = 5, 6$$

The first-order conditions of the working household are given by

$$\text{(consumption-leisure FOC)} \qquad (1-\tau_t)w_t = \gamma \frac{c^s_t}{1-n^s_t}, \tag{2.1}$$

$$\text{(consumption-saving Euler)} \qquad \frac{1}{\beta} = (1+r_{t+1}) \frac{(c^{s+1}_{t+1})^{-\eta}(1-n^{s+1}_{t+1})^{\gamma(1-\eta)}}{(c^s_t)^{-\eta}(1-n^s_t)^{\gamma(1-\eta)}} \tag{2.2}$$

2.1.2 Firms

Technology

$$Y_t = N_t^{1-\alpha} K_t^{\alpha} \tag{2.3}$$

In a competitive market, factors are paid their marginal product:

$$w_t = (1-\alpha) K_t^{\alpha} N_t^{-\alpha}, \tag{2.4}$$

$$r_t = \alpha K_t^{\alpha-1} N_t^{1-\alpha} - \delta. \tag{2.5}$$

where δ is the depreciation rate.

2.1.3 Government

The government uses the revenues from labor income tax to finance its spending on social security. Its budget is balanced every period:

$$\tau_t w_t N_t = \frac{1}{\overline{T}} T^R \times b_t, \tag{2.6}$$

where $\overline{T}$ is the total lifespan (in this case, it is 6), and T^R is the total retirement lifespan (in this case, it is 2). The factor $1/\overline{T}$ is the individual mass of each cohort. The LHS represents total tax revenue. The RHS represents the total pension payment. Since there are 6 generations alive every period, $T^R/\overline{T}$ represents the fraction of retirees.

2.1.4 Equilibrium

On the aggregate level, we have

$$N_t = \frac{1}{\overline{T}} \sum_{s=1}^{4} n_t^s, \tag{2.7}$$

$$K_t = \frac{1}{\overline{T}} \sum_{s=1}^{6} k_t^s, \tag{2.8}$$

$$C_t = \frac{1}{\overline{T}} \sum_{s=1}^{6} c_t^s \tag{2.9}$$

The aggregate labor supply N_t is equal to the sum of labor supplies from the working cohort, weighted by its mass. Similarly, the aggregate capital supply is equal to the sum of capital supplies of all cohorts (weighted by their mass).

2.2 Steady-state computation

In the steady state, the distribution of capital stock over generations will be constant, specifically $\{k_t^s\}_{s=1}^6 = \{k_{t+1}^s\}_{s=1}^6 = \{k^s\}_{s=1}^6$ – so we can remove the time indicator t. In that case, aggregate capital K and labor supply N will also be constants. As a result, factor prices w and r will also be a constant, and so are tax τ and pension b.

2.2.1 Calibration

First and foremost, we pay attention to the replacement ratio. It is the ratio of the pension benefit with respect to the average income of the agent during working periods. For the benchmark model, we assume that it is 0.3 (pension is about 1/3 the average income during working life).

Table 2.1: Model Parameters

Parameter	Definition	Value
α	Capital share	0.30
β	Subjective discount factor	0.90 $(= 0.99^{10})$
η	Preference parameter	2.0
δ	Depreciation rate	0.40 $(= 1 - 0.95^{10})$
$\bar{n}$	Average steady-state labor supply	0.35
γ	Leisure parameter	2.0
$\dfrac{b}{(1-\tau)w\bar{n}}$	Replacement ratio	0.3

2.2.2 Solving with Dynare

First, we need to declare the initial and terminal values of capital. Agents are assumed to be born without wealth. Furthermore, since they live only for 6 periods, all capital must be consumed at age 6, leaving no capital for age 7 or 8.

Second, the government first sets the replacement ratio. For simplicity, we can assume that the government also sets a tax rate based on the replacement ratio. Once the tax is decided, pension benefits can be calculated automatically.

Third, since agents start with no wealth, we can skip k^1. Retirees do not supply labor so $n^5 = n^6 = 0$. This means that we can reduce the number of choice variables from 18 to 15. Other variables include the aggregate variables C, K, L, factor prices r, w and pension b. For the total of 21 endogenous variables, we need 21 equations to solve the model. They include all the FOCs, the budget constraints (law of motion), and equilibrium conditions (which will be the aggregation).

If you input the code correctly, you will get the steady states as follows.

This allows us to plot in figure 2.2 the age-profile capital and labor.

The 6-period model is highly stylized. However, by the same method, we can easily extend the lifecycle to 60 periods (that is, work at the age 21, and die at the age 80). This practice will increase the number of equations to 222 that need to be solved simultaneously. But Dynare can solve this in less than 2 milliseconds (!). A 60-period lived age profile is illustrated in Figure 2.3

Variable	Steady state
k2	0.0372527
k3	0.0683529
k4	0.089901
k5	0.0971652
k6	0.0616097
c1	0.122728
c2	0.129427
c3	0.136491
c4	0.143941
c5	0.113666
c6	0.126412
n1	0.394588
n2	0.361545
n3	0.326698
n4	0.28995
C	0.128778
K	0.0590469
L	0.228797
w	0.466254
r	0.37428
b	0.0417433

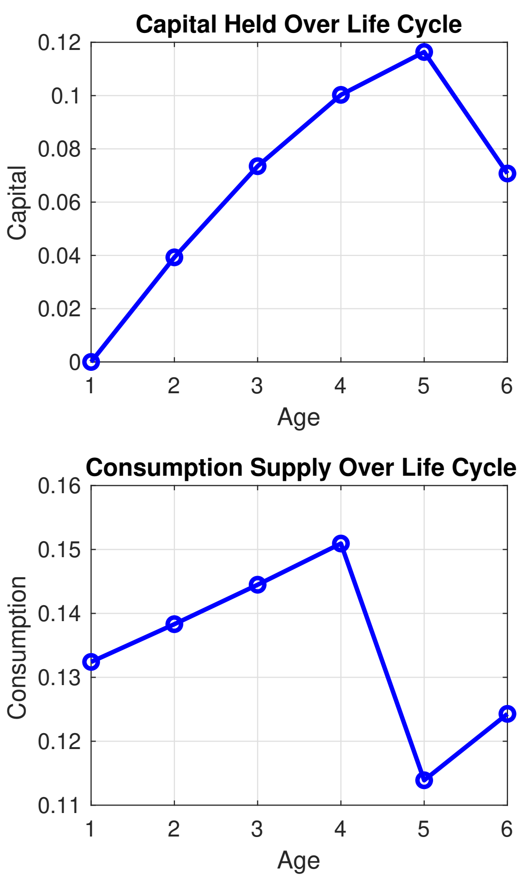

Figure 2.2: Steady state distribution of a 6-period OLG

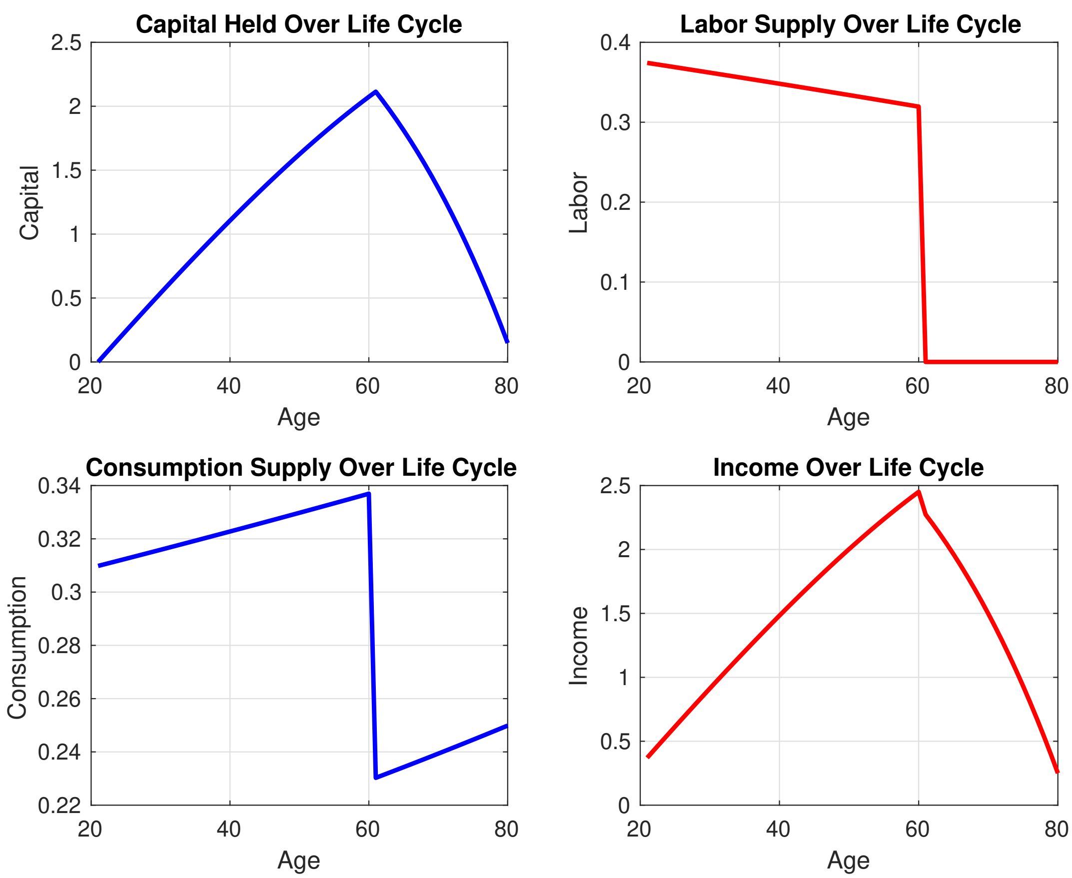

Figure 2.3: Steady state distribution of a 60-period OLG

2.2.3 Solving by hand

The following algorithm can be used to write a program that finds the steady state by hand.

1. Make initial guesses of the steady state values of K and N.
2. Compute w, r, τ, b that solve the firm's problem and the government's budget set.
3. Compute the optimal path for consumption, savings, and labor supply by backward iteration.
   (a) We know $k^1 = k^7 = 0$. Make a guess of the terminal capital holding k^6.
   (b) Retirees: With k^6, k^7 known, you can solve for k^5. Then, with k^5 and k^6 known, you can solve for k^4 and since $n^5 = 0$, you can solve for n^4. With k^5, k^4, n^4, you can solve for k^3 and n^3. Continue until you reach k^1, n^1.
   (c) Compute k^1:
      i. if $k^1 = 0$. Stop the loop and output the series of $k^1, \ldots, k^6$ and $n^1, \ldots, n^4$.
      ii. else, if $k^1 \neq 0$, update k^6 and go back to step 3(a).
4. Recompute the new aggregate K and N.
5. If they are the same as the initial guess. Stop. Otherwise, update a new K and N and go back to step 2 until convergence.

In step 3(b), you need to solve 2 equations (using the FOC (2.1) and Euler (2.2) to solve 2 unknowns. This is a simple multi-variable root finding.

In step 3(c).ii, we need to have a method to update the initial guess. You can use the secant method

$$k_i^6 = k_{i-1}^6 - \frac{k_{i-1}^6 - k_{i-2}^6}{k_{i-1}^1 - k_{i-2}^1} k_{i-1}^1.$$

where i indexes the i-th loop (or iteration).

In step 5. It is actually sufficient to make only a guess of N since K must scale up with it due to Cobb-Douglas technology. Since there is a target for the steady state N, we should not make a random guess. In the code, *nbar* is the aggregate N.

The update algorithm is simple:

$$N_j^{guess} = \phi N_{j-1}^{guess} + (1-\phi) N_{j-1}^{out}.$$

with ϕ is the learning rate, j is the j^{th} iteration. N^{out} is the outcome of an iteration taking N^{guess} as an input. The algorithm should update and converge at iteration $\hat{j}$ such that $N_{\hat{j}}^{guess} = N_{\hat{j}}^{out}$.

For the first 2 initial guesses, use $N_1 = 0.15$, $N_2 = 0.35$.

Chapter 3

The Real Business Cycle

3.1 Model

This section presents the most basic DSGE model with endogenous savings and labor. There is no government or money. The consumer is an infinitely-lived representative household. There is a representative firm that produces the final good.

3.1.1 Households

Each period, the consumer receives income from labor and capital ownership. The consumer's budget constraint is [1]

$$C_t + I_t = W_t L_t + R_t K_t. \tag{3.1}$$

The investment I_t (equivalent to savings S_t) is saved to form capital in the next period

$$K_{t+1} = (1-\delta)K_t + I_t. \tag{3.2}$$

with δ as the depreciation rate. The intertemporal maximization problem for households is

$$\max_{\{C_t, L_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t, L_t) \tag{3.3}$$

subject to (3.1), (3.2). We now specify a functional form for utility. Throughout the chapter, we will use the easy log-linear utility function. Assume that each individual is endowed with one unit of time. Let C_t, L_t be consumption and labor supply. The utility function is Cobb-Douglas such that

$$u(C_t, L_t) = \log[C_t^{\gamma}(1-L_t)^{1-\gamma}]$$

[1]Technically, the RHS should be $W_t L_t + R_t K_t + \Pi_t$ where Π_t is the profit transferred from the firms. But since there is no profit in production, we can ignore this term here.

which can be written as

$$u(C_t, L_t) = \gamma \log C_t + (1-\gamma)\log(1-L_t) \tag{3.4}$$

where $1 - L_t$ represents leisure [2].

Solving (3.3) with respect to (3.4), (3.1), and (3.2), the Lagrangian is [3]

$$\mathscr{L} = \sum_{t=0}^{\infty} \beta^t \Big\{ \gamma \log C_t + (1-\gamma)\log(1-L_t) + \lambda_t [W_t L_t + (1 + R_t - \delta)K_t - C_t - K_{t+1}] \Big\}$$

The FOC:

$$\begin{aligned}
(C_t):&\quad \beta^t \left(\frac{\gamma}{C_t} - \lambda_t \right) = 0,\\
(L_t):&\quad \beta^t \left(\frac{1-\gamma}{1-L_t} - \lambda_t W_t \right) = 0,\\
(K_{t+1}):&\quad \beta^{t+1}\lambda_{t+1}(1 + R_{t+1} - \delta) - \beta^t \lambda_t = 0,\\
(\lambda_t):&\quad C_t + K_{t+1} - (1 + R_t - \delta)K_t - W_t L_t = 0.
\end{aligned}$$

The consumption-leisure decision rule

$$\frac{1-\gamma}{\gamma}\frac{C_t}{1-L_t} = W_t$$

The Euler equation

$$1 = \beta \mathbb{E}_t \left[\frac{C_t}{C_{t+1}} (R_{t+1} + 1 - \delta) \right].$$

[2] Alternatively, we can have the CRRA utility often used in standard NK

$$u(C_t, L_t) = \frac{C^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}$$

or Cobb-Douglas nested within the CRRA, which is more commonly used in OLG

$$u(C_t, L_t) = \frac{(C_t^{\omega}(1-L_t)^{1-\omega})^{1-\sigma}}{1-\sigma}$$

[3] Although correct from a theoretical standpoint, the formulation of the Lagrangian is different in practice. In Dynare, investment must be transformed into capital stock in the same period since this amount cannot disappear from the economy and then magically return the next period. The problem can be avoided by setting the budget and capital accumulation differently

$$\begin{aligned}
C_t + I_t &= W_t L_t + R_t K_{t-1},\\
K_t &= (1-\delta)K_{t-1} + I_t
\end{aligned}$$

3.1.2 Firms

Production technology is Cobb-Douglas [4]

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}. \tag{3.5}$$

The profit is

$$\Pi_t = A_t K_t^{\alpha} L_t^{1-\alpha} - W_t L_t - R_t K_t$$

The FOCs are

$$\begin{aligned} R_t &= \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} = \alpha \frac{Y_t}{K_t}, \\ W_t &= (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} = (1-\alpha) \frac{Y_t}{L_t} \end{aligned}$$

Finally, productivity level (TFP) is subject to the following stochastic process

$$\log A_{t+1} = (1-\rho_A) \log \overline{A} + \rho_A \log A_{t-1} + \varepsilon_t \tag{3.6}$$

$$\varepsilon_t \sim N(0, \sigma^2) \tag{3.7}$$

The parameter $\rho_A \in (0,1)$ represents the persistence of the shocks.

3.1.3 Competitive Equilibrium

A sequence of $\{C_t, I_t, K_t, L_t, R_t, W_t, Y_t\}_{t=0}^{\infty}$ such that

$$\begin{aligned} \frac{1-\gamma}{\gamma} \frac{C_t}{1-L_t} &= W_t, \\ 1 &= \beta \mathbb{E}_t \left[\frac{C_t}{C_{t+1}} (R_{t+1} + 1 - \delta) \right], \\ R_t &= \alpha \frac{Y_t}{K_t}, \\ W_t &= (1-\alpha) \frac{Y_t}{L_t}, \\ Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha}, \\ K_{t+1} &= (1-\delta) K_t + I_t, \\ C_t + I_t &= Y_t. \end{aligned}$$

[4] Just like the Household's problem, in Dynare, we must specify the capital as follows.

$$Y_t = A_t {K_{t-1}}^{\alpha} L_t^{1-\alpha}$$

as capital must already exist before production starts. Savings can be transformed into capital stock in the same period but do not enter production until the next period.

To calculate the non-stochastic steady state, we let $C_{t+1} = C_t = \overline{C}$; $K_{t+1} = K_t = \overline{K}$; $A_{t+1} = A_t = \overline{A}$ and solve for other variables in the steady state as

$$\begin{aligned}
\frac{1-\gamma}{\gamma}\frac{\overline{C}}{1-\overline{L}} &= \overline{W}, \\
1 &= \beta[1+\overline{R}-\delta], \\
\overline{R} &= \alpha\frac{\overline{Y}}{\overline{K}}, \\
\overline{W} &= (1-\alpha)\frac{\overline{Y}}{\overline{L}}, \\
\overline{Y} &= \overline{A}\overline{K}^{\alpha}\overline{L}^{1-\alpha}, \\
\overline{I} &= \delta\overline{K}, \\
\overline{C}+\overline{I} &= \overline{Y}.
\end{aligned}$$

Given $\overline{A} = 1$, we solve explicitly

$$\begin{aligned}
\overline{R} &= \frac{1}{\beta}+\delta-1, \\
\overline{L} &= \frac{\gamma(1-\alpha)(1-\beta+\beta\delta)}{(1-\gamma)(1-\beta+(1-\alpha)\beta\delta)+\gamma(1-\alpha)(1-\beta+\beta\delta)}, \\
\overline{Y} &= \left(\frac{\alpha}{\overline{R}}\right)^{\alpha/(1-\alpha)}\overline{L}, \\
\overline{K} &= \frac{\alpha}{\overline{R}}\overline{Y}, \\
\overline{I} &= \delta\overline{K}, \\
\overline{C} &= \overline{Y}-\overline{I}.
\end{aligned}$$

Handling and solving non-linear models is generally very arduous. When the model is very simple, it is possible to find an approximation of the policy function by recursively solving the value function. On the other hand, linear models are often easier to solve. The problem is converting a non-linear model to a sufficiently adequate linear approximation so that its solution helps in understanding the underlying non-linear system's behavior.

3.2 Log-Linearization

We follow Uhlig's log-linearization method.

Define the log differences of a variable $\tilde{X}_t$ as

$$\tilde{X}_t = \ln X_t - \ln \overline{X}.$$

where X_t is the value at time t and $\overline{X}$ is the stationary value. Conveniently, when the difference is small, it is approximately *the percentage change compared to the steady state value* at time t. This definition of the log differences allows us to write the variable as

$$X_t = \overline{X}e^{\tilde{X}_t}.$$

since

$$\overline{X}e^{\tilde{X}_t} = \overline{X}e^{\ln X_t - \ln \overline{X}} = \overline{X}e^{\ln X_t/\overline{X}} = \overline{X}X_t/\overline{X} = X_t.$$

The following rules help us derive the first-order approximations when $\tilde{X}_t$ and $\tilde{Y}_t$ are small.

$$\begin{aligned} e^{\tilde{X}_t} &\approx 1 + \tilde{X}_t, \\ e^{\tilde{X}_t + a\tilde{Y}_t} &\approx 1 + \tilde{X}_t + a\tilde{Y}_t, \\ \tilde{X}_t\tilde{Y}_t &\approx 0, \\ E_t[ae^{\tilde{X}_{t+1}}] &\approx E_t[a\tilde{X}_{t+1}] + \text{a constant}. \end{aligned}$$

The log-linearized representation has an intuitive interpretation: percentage deviation from the steady state (provided that the log difference is small). Let $\log A_t = z_t$, our model can be reduced to the following six:

$$\begin{aligned} \text{(Euler)}: &\quad 1 = \beta\mathbb{E}_t\left[\frac{C_t}{C_{t+1}}(R_{t+1} + 1 - \delta)\right], \\ \text{(Consumption-Leisure FOC)}: &\quad \frac{1-\gamma}{\gamma}C_t = (1 - L_t)(1-\alpha)\frac{Y_t}{L_t}, \\ \text{(Law of motion for capital)}: &\quad C_t = Y_t + (1-\delta)K_t - K_{t+1}, \\ \text{(Production)}: &\quad Y_t = e^{z_t}K_t^{\alpha}L_t^{1-\alpha}, \\ \text{(Interest rate)}: &\quad R_t = \alpha\frac{Y_t}{K_t}, \\ \text{(Stochastic Proc.)}: &\quad z_{t+1} = \rho_A z_t + \varepsilon_t \end{aligned}$$

For each equation, use the log-linearized identity to replace all "sub t" variables by their log differences from the steady state, and use the rules and steady-state identities to simplify the expressions.

Detailed derivations are in Appendix A.1.

$$
\begin{aligned}
&\text{(Euler)}: && 0 \approx \tilde{C}_t - \mathbb{E}_t\tilde{C}_{t+1} + \beta\overline{R}\mathbb{E}_t\tilde{R}_{t+1}, && (3.8)\\
&\text{(Labor)}: && 0 \approx \tilde{Y}_t - \frac{\tilde{L}_t}{1-\overline{L}} - \tilde{C}_t, && (3.9)\\
&\text{(Resource)}: && 0 \approx \overline{Y}\tilde{Y}_t - \overline{C}\tilde{C}_t + \overline{K}[(1-\delta)\tilde{K}_t - \tilde{K}_{t+1}], && (3.10)\\
&\text{(Production)}: && 0 \approx z_t + \alpha\tilde{K}_t + (1-\alpha)\tilde{L}_t - \tilde{Y}_t, && (3.11)\\
&\text{(Interest rate)}: && 0 \approx \tilde{Y}_t - \tilde{K}_t - \tilde{R}_t, && (3.12)\\
&\text{(Stochastic Proc.)}: && z_{t+1} = \rho_A z_t + \varepsilon_t && (3.13)
\end{aligned}
$$

To solve this system, we can define a vector of endogenous state variables x_t and choice variables y_t. We can rewrite the system in a state-space representation and separate equations that include expectations from those that do not.

A detailed solution method is written in Appendix A.2. The final linearized system is

$$
\begin{aligned}
x_t &= Px_{t-1} + Qz_t,\\
y_t &= Rx_{t-1} + Sz_t
\end{aligned}
$$

with $x_t = [\tilde{K}_{t+1}]$ and $y_t = [\tilde{Y}_t \quad \tilde{C}_t \quad \tilde{L}_t \quad \tilde{R}_t]'$. Based on this specification, given a set of calibrated parameters, we can solve for the parameters P, Q, R, S and the model.

3.3 Computation

3.3.1 Calibration

The process of calibration is to choose a set of parameters that best describes the economy we are investigating. For common parameters, such as α, β, δ, we can follow the literature and infer the values from aggregate data. Quarterly data normally imply $\beta \in [0.96, 0.99], \delta \in [0.025, 0.06], \alpha \in [0.3, 0.4]$. After that, we select target variables, which are normally working hours and interest rates, and then use these values to calibrate the rest of the parameters based on how they are calculated.

Table 3.1: Model Parameters

Parameter	**Definition**	**Value**
α	Capital share	0.35
β	Subjective discount factor	0.97
γ	Preference parameter	0.40
δ	Depreciation rate	0.06
ρ_A	TFP autoregressive parameter	0.95
σ_A	TFP standard deviation	0.01

Using these numbers, we can calculate the steady state as follows.

Variable	Value	Ratio to $\overline{Y}$
$\overline{Y}$	0.74469	1.000
$\overline{C}$	0.57270	0.769
$\overline{I}$	0.17199	0.231
$\overline{K}$	2.86649	3.849
$\overline{L}$	0.36039	
$\overline{R}$	0.09092	
$\overline{W}$	1.34312	
$\overline{A}$	1.00000	

3.3.2 IRFs with Dynare

In practice, you can input the model into Dynare in levels or logs. The IRFs, however, will be shown in log difference from the steady state.

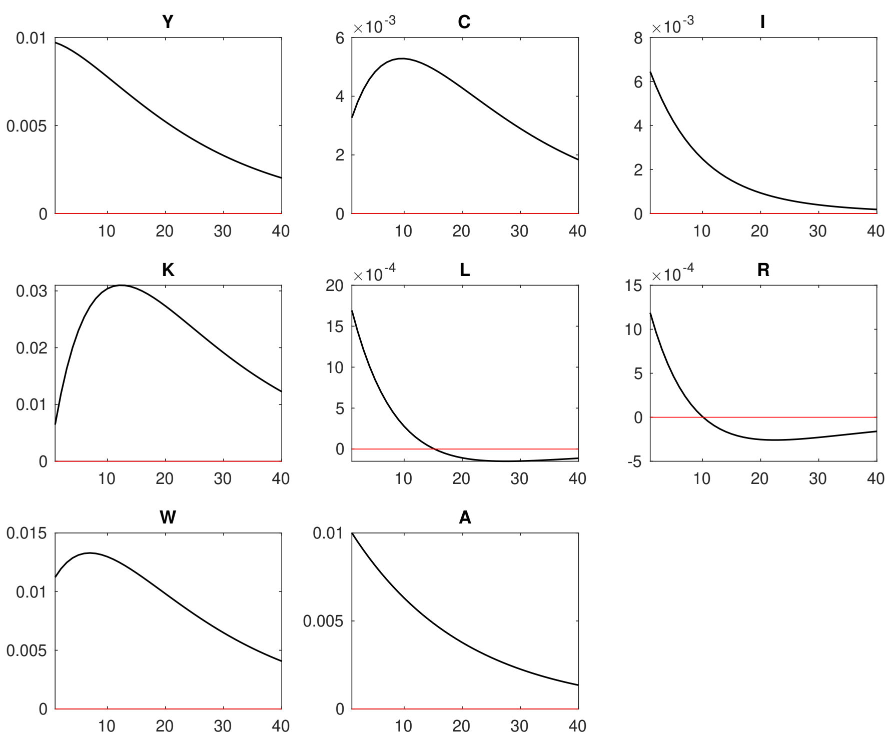

Figure 3.1: IRF to a positive productivity shock

3.3.3 IRFs by hand

One can also calculate IRFs by hand if you want to understand the process better. Applying the formula in Appendix A.2 and using the steady states, we obtain:

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & -2.8665 & 0 & 0 \end{bmatrix}' \\
B &= \begin{bmatrix} 0 & 2.6945 & 0.35 & -1 \end{bmatrix}' \\
C &= \begin{bmatrix} 1 & -1 & -1.5635 & 0 \\ 0.7447 & -0.5727 & 0 & 0 \\ -1 & 0 & 0.65 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \\
D &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}' \\
F &= [0],\ G = [0],\ H = [0], \\
J &= \begin{bmatrix} 0 & -1 & 0 & 0.0882 \end{bmatrix}, \\
K &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \\
L &= [0],\ M = [0], \\
N &= [0.95]
\end{aligned}
$$

Then, the matrix P must satisfy

$$0 = 2.7629 \cdot P^2 - 5.6622 \cdot P + 2.8483$$

There are two solutions $P = 1.1627$ and $P = 0.8866$. Only the second value results in a stable equilibrium (its modulus is less than one), so we are going to use that value. The next calculation gives us $Q = 0.2251$. Then, we can find

$$
\begin{aligned}
R &= \begin{bmatrix} 0.2124 & 0.5433 & -0.2116 & -0.7876 \end{bmatrix}' \\
S &= \begin{bmatrix} 1.3054 & 0.5709 & 0.4698 & 1.3054 \end{bmatrix}'
\end{aligned}
$$

The laws of motion around the steady state for the five equations can be written out as

$$
\begin{aligned}
\tilde{K}_{t+1} &= 0.8866\tilde{K}_t + 0.2251\tilde{A}_t, \\
\tilde{Y}_t &= 0.2124\tilde{K}_t + 1.3054\tilde{A}_t, \\
\tilde{C}_t &= 0.5433\tilde{K}_t + 0.5709\tilde{A}_t, \\
\tilde{L}_t &= -0.2116\tilde{K}_t + 0.4698\tilde{A}_t, \\
\tilde{R}_t &= -0.7876\tilde{K}_t + 1.3054\tilde{A}_t.
\end{aligned}
$$

We are now interested in observing how a model responds to an impulse applied to one of its error terms. The economy begins in a stationary state. All shocks to stochastic processes are set to zero. Since the model is calculated in log differences from the steady state, all variables are set to zero. Then, we apply a small, positive, one-period change to the shock of interest and calculate how the economy responds to this shock. If a technology shock is applied, we have

$$\tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_t.$$

where the shock is ε_t. If the shock is applied in period $t = t^*$, then $\varepsilon_t = 0 \; \forall t > t^*$. The time path the economy follows is defined by the law of motions of linear policy functions

$$\tilde{K}_{t+1} = P\tilde{K}_t + Q\tilde{A}_t$$

and with $y_t = \begin{bmatrix} \tilde{Y}_t & \tilde{C}_t & \tilde{L}_t & \tilde{R}_t \end{bmatrix}'$, the responses of other variables can be found from

$$y_t = R\tilde{K}_t + S\tilde{A}_t.$$

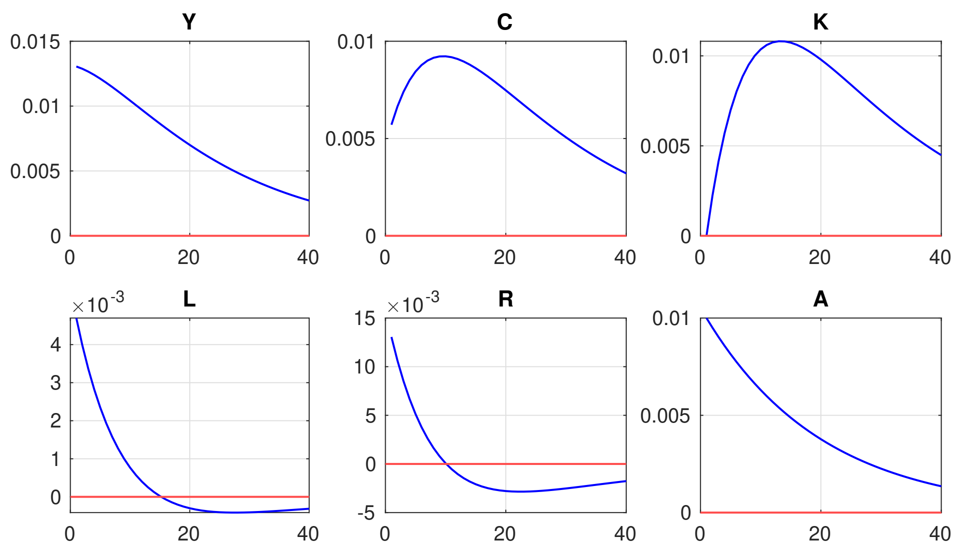

Figure 3.2: Caption

Chapter 4

New-Keynesian

Next we will examine New Keynesian (NK) models. They are micro-founded models which incorporate consumers and firms as we have done. The RBC model's general assumptions still hold but we introduce other assumptions that better reflect the real economy.

In the following example, we will introduce price stickiness and monopolistic competition. The consumer and firm problem derivations are omitted for your own exercise. We will mainly focus on Calvo pricing in this section.

4.1 Model

4.1.1 Consumers

The representative household optimises the following utility function:

$$\max_{C_{j,t},L_{j,t},K_{j,t+1}} E_t \sum_{t=0}^{\infty} \beta^t \left(\frac{C_{j,t}^{1-\sigma}}{1-\sigma} - \frac{L_{j,t}^{1+\varphi}}{1+\varphi} \right) \tag{4.1}$$

subject to the intertemporal budget constraint

$$P_t(C_{j,t} + I_{j,t}) = W_t L_{j,t} + R_t K_{j,t} + D_t \tag{4.2}$$

and capital accumulation

$$K_{j,t+1} = (1-\delta)K_{j,t} + I_{j,t} \tag{4.3}$$

The FOCs take the familiar forms:

$$C_{j,t}^{\sigma} L_{j,t}^{\varphi} = \frac{W_t}{P_t} \tag{4.4}$$

$$\left(\frac{E_t C_{j,t+1}}{C_{jt,}} \right)^{\sigma} = \beta \left[(1-\delta) + E_t \left(\frac{R_{t+1}}{P_{t+1}} \right) \right] \tag{4.5}$$

4.1.2 Firms

The economy's producing sector is divided into two parts: an intermediate goods sector (wholesale firms) and a final goods sector (retail firms). The intermediate goods sector consists of a large number of companies, each one producing differentiable goods.

These companies decide the quantity of factors of production to be used and the prices of their goods using a production function. In the final goods sector, there is a single firm that aggregates intermediate goods into one single good that will be consumed by economic agents.

The final good market is still perfectly competitive but the intermediate good market is now of monopolistic competition.

Final goods (Wholesale firms)

The production function takes the form:

$$Y_t = \left(\int_0^1 Y_{j,t}^{\frac{\psi-1}{\psi}} dj \right)^{\frac{\psi}{\psi-1}} \tag{4.6}$$

With P_t as the nominal price of a retail product and $P_{j,t}$ as the nominal price of wholesale good j, the price of each wholesale good is taken as a given by retail firms. Therefore, the problem of the representative retail firm is maximising its profit function:

$$\max_{Y_{j,t}} \pi_t \equiv P_t Y_t - \int_0^1 P_{j,t} Y_{j,t} dj \tag{4.7}$$

The demand function and aggregate price index can then be derived (exercise for you):

$$Y_{j,t} = Y_t \left(\frac{P_t}{P_{j,t}} \right)^{\psi} \tag{4.8}$$

$$P_t = \left(\int_0^1 P_{j,t}^{1-\psi} dj \right)^{\frac{1}{1-\psi}} \tag{4.9}$$

Intermediate goods (Retail firms)

The retail firm solves its problem in two stages. First, the firm takes the prices of the factors of production (return on capital and wages) and determines the amount of capital and labor that it will use to minimise its total production cost:

$$\min_{J_{j,t}, K_{j,t}} TC_{j,t} \equiv W_t L_{j,t} + R_t K_{j,t} \tag{4.10}$$

subject to the Cobb-Douglas production function:

$$Y_{j,t} = A_t K_{j,t}^{\alpha} L_{j,t}^{1-\alpha} \tag{4.11}$$

with law of motion of productivity:

$$\log A_t = (1-\rho_A)\log A_{ss} + \rho_A \log A_{t-1} + \varepsilon_t \tag{4.12}$$

Note that this law of motion of productivity can be changed according to your research.

Setting the Lagrangian and solving yields:

$$\alpha \frac{Y_{j,t}}{K_{j,t}} \mu_{j,t} = R_t \tag{4.13}$$

$$(1-\alpha)\frac{Y_{j,t}}{L_{j,t}} \mu_{j,t} = W_t \tag{4.14}$$

Note that it can be verified that the Lagrange multiplier is equivalent to the Marginal Cost (MC):

$$\begin{aligned}
MC_{j,t} &= \frac{\partial TC_{j,t}}{\partial Y_{j,t}} \\
&= \frac{\partial (W_t L_{j,t} + R_t K_{j,t})}{\partial Y_{j,t}} \\
&= \frac{\partial [(1-\alpha)\mu_{j,t} Y_{j,t} + \alpha \mu_{j,t} Y_{j,t}]}{\partial Y_{j,t}} \\
&= \frac{\partial (\mu_{j,t} Y_{j,t})}{\partial Y_{j,t}} \\
&= \mu_{j,t}
\end{aligned}$$

Dividing the two FOCs gives the capital-labour ratio:

$$\frac{1-\alpha}{\alpha}\frac{K_{j,t}}{L_{j,t}} = \frac{W_t}{R_t} \tag{4.15}$$

Rearranging and substitute into the production function:

$$K_{j,t} = \frac{W_t}{R_t}\frac{\alpha}{1-\alpha} L_{j,t} \tag{4.16}$$

$$Y_{j,t} = A_t \left(\frac{W_t}{R_t}\frac{\alpha}{1-\alpha} L_{j,t}\right)^{\alpha} L_{j,t}^{1-\alpha}$$

$$L_{j,t} = \frac{Y_{j,t}}{A_t}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\left(\frac{W_t}{R_t}\right)^{-\alpha} \tag{4.17}$$

Similarly,

$$K_{j,t} = \frac{Y_{j,t}}{A_t}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha-1}\left(\frac{W_t}{R_t}\right)^{1-\alpha} \tag{4.18}$$

Equations (2.17) and (2.18) give the optimal labour and capital respectively. Finally, the MC is:

$$MC_{j,t} = \frac{\partial TC_{j,t}}{\partial Y_{j,t}} = \frac{1}{A_{j,t}} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} \left(\frac{R_t}{\alpha} \right)^{\alpha} \tag{4.19}$$

Calvo pricing

The second stage of the problem of the wholesale firm is defining the price of its goods. This firm decides how much to produce in each period according to the Calvo rule.

Definition 1. Calvo's rule *Establishes that in each period t, a fraction $0 < \theta < 1$ of firms is randomly selected and allowed to define the prices of its goods for the period. The rest of the firms (the θ fraction) keeps the prices of its goods defined by a stickiness rule which, in the literature, may follow one of the three possibilities below:*

1. *Maintain previous period's price*
$$P_{j,t} = P_{j,t-1}$$
2. *Update the price using the steady state gross inflation rate (π)*
$$P_{j,t} = \pi P_{j,t-1}$$
3. *Update the price using the previous period's gross inflation rate (π_{t-1})*
$$P_{j,t} = \pi_{t-1} P_{j,t-1}$$

To keep things simple, we will use the first rule. You can try the second rule yourself as an exercise. For the third rule, you can refer to Jesús Fernández-Villaverde's paper for example.

Now the problem of the wholesale firm that is capable of readjusting the price of its good is:

$$\max_{P^*_{j,t}} E_t \sum_{i=0}^{\infty} (\beta\theta)^i (P^*_{j,t} Y_{j,t+i} - TC_{j,t+i}) \tag{4.20}$$

Substituting (2.8) in (2.20):

$$\max_{P^*_{j,t}} E_t \sum_{i=0}^{\infty} (\beta\theta)^i \left[P^*_{j,t} Y_{t+i} \left(\frac{P_{t+i}}{P^*_{j,t}} \right)^{\psi} - Y_{t+i} \left(\frac{P_{t+i}}{P^*_{j,t}} \right)^{\psi} MC_{j,t+i} \right] \tag{4.21}$$

The FOC is:

$$0 = E_t \sum_{i=0}^{\infty} (\beta\theta)^i \left[(1-\psi)(P^*_{j,t})^{-\psi} Y_{t+i} P^{\psi}_{t+i} + \psi (P^*_{j,t})^{-\psi-1} Y_{t+i} P^{\psi}_{t+i} MC_{j,t+i} \right]$$

$$0 = E_t \sum_{i=0}^{\infty} (\beta\theta)^i \left[(1-\psi) Y_{j,t+i} + \psi \frac{Y_{j,t+i}}{P^*_{j,t}} MC_{j,t+i} \right]$$

$$P^*_{j,t} = \frac{\psi}{\psi - 1} \frac{E_t \sum_{i=0}^{\infty} (\beta\theta)^i Y_{j,t+i} MC_{j,t+i}}{E_t \sum_{i=0}^{\infty} (\beta\theta)^i Y_{j,t+i}} \tag{4.22}$$

The aggregate price index from (2.9) can be the sum of the price of two groups:

$$\begin{aligned}
P_t^{1-\psi} &= \int_0^{\theta} P_{t-1}^{1-\psi} dj + \int_{\theta}^{1} P_{j,t}^{*1-\psi} dj \\
P_t^{1-\psi} &= \left[j P_{t-1}^{1-\psi} \right]_0^{\theta} + \left[j P_{j,t}^{*1-\psi} \right]_{\theta}^{1} \\
P_t^{1-\psi} &= \theta P_{t-1}^{1-\psi} + (1-\theta) P_t^{*1-\psi} \\
P_t &= \left[\theta P_{t-1}^{1-\psi} + (1-\theta) P_t^{*1-\psi} \right]^{\frac{1}{1-\psi}}
\end{aligned} \tag{4.23}$$

4.1.3 Market equilibrium

Now that each agent's behaviour has been described, the interaction between them must be studied in order to determine macroeconomic equilibrium. Households decide how much to consume (C), how much to invest (I) and how much labour to supply (L), with the aim of maximising utility, taking prices as given.

On the other hand, firms decide how much to produce (Y) using available technology and choosing the factors of production (capital and labour), taking these prices as given.

Therefore, the model's equilibrium consists of the following blocks:

1. a price system, W_t, R_t and P_t;

2. an endowment of values for goods and inputs Y_t, C_t, I_t, L_t and K_t; and

3. a production-possibility frontier described by the following equilibrium condition of the goods market (aggregate supply = aggregate demand).

$$Y_t = C_t + I_t \tag{4.24}$$

Competitive equilibrium consists of finding a sequence of endogenous variables in the model such that the conditions that define equilibrium are satisfied. In short, this economy's model consists of the following equations:

4.1.4 Steady states

After defining the economy's equilibrium, the steady state values must be defined. The model is steady in the sense that there exists a value for the variables that is maintained over time: an endogenous variable x_t is at the steady state in each t, if $E_t x_{t+1} = x_t = x_{t-1} = x_{ss}$.

Table 4.1: Model structure

Equation	Definition
$C_t^{\sigma} L_t^{\varphi} = \frac{W_t}{P_t}$	Labour supply
$\left(\frac{E_t C_{t+1}}{C_t}\right)^{\sigma} = \beta \left[(1-\delta) + E_t \left(\frac{R_{t+1}}{P_{t+1}}\right)\right]$	Euler Equation
$K_{t+1} = (1-\delta)K_t + I_t$	Law of motion of capital
$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$	Production function
$K_t = \alpha \frac{MC_t Y_t}{R_t}$	Demand for capital
$L_t = (1-\alpha)\frac{MC_t Y_t}{W_t}$	Demand for labor
$MC_t = \frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha}$	Marginal cost
$P_t^{*} = \frac{\psi}{\psi-1}\frac{E_t \sum_{i=0}^{\infty}(\beta\theta)^i Y_{j,t+i} MC_{j,t+i}}{E_t \sum_{i=0}^{\infty}(\beta\theta)^i Y_{j,t+i}}$	Optimal price level
$P_t = \left[\theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*\,1-\psi}\right]^{\frac{1}{1-\psi}}$	General price level
$\pi_t = \frac{P_t}{P_{t-1}}$	Gross inflation rate
$\mathbf{Y}_t = C_t + I_t$	Equilibrium condition
$\log A_t = (1-\rho_A)\log A_{ss} + \rho_A \log A_{t-1} + \epsilon_t$	Productivity shock

Households

$$C_{ss}^{\sigma} L_{ss}^{\phi} = \frac{W_{ss}}{P_{ss}} \tag{4.25}$$

$$1 = \beta\left(1 - \delta + \frac{R_{ss}}{P_{ss}}\right) \tag{4.26}$$

$$\delta K_{ss} = I_{ss} \tag{4.27}$$

Firms

$$L_{ss} = \frac{(1-\alpha)MC_{ss}Y_{ss}}{W_{ss}} \tag{4.28}$$

$$K_{ss} = \frac{\alpha MC_{ss}Y_{ss}}{R_{ss}} \tag{4.29}$$

$$Y_{ss} = K_{ss}^{\alpha} L_{ss}^{1-\alpha} \tag{4.30}$$

$$MC_{ss} = \left(\frac{W_{ss}}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_{ss}}{\alpha}\right)^{\alpha} \tag{4.31}$$

$$P_{ss} = \frac{\psi}{\psi - 1} MC_{ss} \tag{4.32}$$

where $\sum_{i=0}^{\infty} (\beta\theta)^i = \frac{1}{1-\beta\theta}$.

Equilibrium condition

$$Y_{ss} = C_{ss} + I_{ss} \tag{4.33}$$

The economy's general price level is often normalised ($P_{ss} = 1$). From (2.26) and (2.32):

$$R_{ss} = P_{ss} \left[\frac{1}{\beta} - (1 - \delta) \right] \tag{4.34}$$

$$MC_{ss} = \frac{\psi - 1}{\psi} P_{ss} \tag{4.35}$$

Applying these in (2.31) gives

$$W_{ss}^{1-\alpha} = \frac{MC_{ss}(1-\alpha)^{1-\alpha}\alpha^{\alpha}}{R_{ss}^{\alpha}}$$

$$W_{ss} = (1-\alpha) MC_{ss}^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{R_{ss}} \right)^{\frac{\alpha}{1-\alpha}} \tag{4.36}$$

Next, we try to obtain some relations before calculating the exact values. Substituting (2.20) in (2.27):

$$I_{ss} = \frac{\delta\alpha MC_{ss}}{R_{ss}} Y_{ss} \tag{4.37}$$

Substituting (2.28) in (2.25):

$$C_{ss}^{\sigma} \left[(1-\alpha) MC_{ss} \frac{Y_{ss}}{W_{ss}} \right]^{\varphi} = \frac{W_{ss}}{P_{ss}}$$

$$C_{ss} = \frac{1}{Y_{ss}^{\frac{\varphi}{\sigma}}} \left\{ \frac{W_{ss}}{P_{ss}} \left[\frac{W_{ss}}{(1-\alpha) MC_{ss}} \right]^{\varphi} \right\}^{\frac{1}{\sigma}} \tag{4.38}$$

Now we use (2.37) and (2.38) to meet (2.33).

$$Y_{ss} = \frac{\delta\alpha MC_{ss}}{R_{ss}} Y_{ss} + \frac{1}{Y_{ss}^{\frac{\varphi}{\sigma}}} \left\{ \frac{W_{ss}}{P_{ss}} \left[\frac{W_{ss}}{(1-\alpha) MC_{ss}} \right]^{\varphi} \right\}^{\frac{1}{\sigma}}$$

$$\left(1 - \frac{\delta\alpha MC_{ss}}{R_{ss}} \right) Y_{ss} = \frac{1}{Y_{ss}^{\frac{\varphi}{\sigma}}} \left\{ \frac{W_{ss}}{P_{ss}} \left[\frac{W_{ss}}{(1-\alpha) MC_{ss}} \right]^{\varphi} \right\}^{\frac{1}{\sigma}}$$

$$Y_{ss}^{1+\frac{\varphi}{\sigma}} = \left(\frac{R_{ss}}{R_{ss} - \delta\alpha MC_{ss}} \right) \left\{ \frac{W_{ss}}{P_{ss}} \left[\frac{W_{ss}}{(1-\alpha) MC_{ss}} \right]^{\varphi} \right\}^{\frac{1}{\sigma}}$$

$$Y_{ss} = \left(\frac{R_{ss}}{R_{ss} - \delta\alpha MC_{ss}} \right)^{\frac{\sigma}{\sigma+\varphi}} \left\{ \frac{W_{ss}}{P_{ss}} \left[\frac{W_{ss}}{(1-\alpha) MC_{ss}} \right]^{\varphi} \right\}^{\frac{1}{\sigma+\varphi}} \tag{4.39}$$

Last but not least, we set $A_{ss} = 1$.

Table 2.2 shows the calibrated values that will be used in the NK model's simulation. Table 2.3 shows the steady state values with the given parameters.

4.2 Computation

We are going to use Dynare to calculate the steady states and IRFs.

Table 4.2: Structural model's parameters

Parameter	Meaning of the parameter	Cal1brated
σ	Relative risk aversion coefficient	2
φ	Marginal disutility with respect to labor supply	1.5
α	Elasticity of output with respect to capital	0.35
β	Discount factor	0.985
δ	Depreciation	0.025
ρ_A	Autoregressive parameter of productivity	0.95
σ_A	Standard deviation of productivity	0.01
θ	Price stickiness parameter	0.75
ψ	Elasticity of substitution between intermediate goods	8

Table 4.3: Steady state

A	R	MC	W	Y	I	C	L	K
1	0.040	0.2286	0.2152	0.778	0.039	0.739	0.537	1.547

4.2.1 Log-linearization

Using the good old Uhlig's method, we transform the level model to a log-linearized model. This is useful for IRF simulations. Detailed derivations are given in Appendix A.3.

4.2.2 Productivity shock

Figure 4.1 shows the dynamics when there is a one-time positive shock in productivity.

The productivity shock in question caused the values of the marginal labor and capital products to rise. Consequently, firms increased their demand for inputs (labor and capital). The prices

Table 4.4: Structure of the log-linear model

Equation	**Definition**
$\sigma\tilde{C}_t + \varphi\tilde{L}_t = \tilde{W}_t - \tilde{P}_t$	Labor supply
$\frac{\sigma}{\beta}(\mathbb{E}_t\tilde{C}_{t+1} - \tilde{C}_t) = \frac{R_{ss}}{P_{ss}}\mathbb{E}_t(\tilde{R}_{t+1} - \tilde{P}_{t+1})$	Euler equation
$\tilde{K}_{t+1} = (1-\delta)\tilde{K}_t + \delta\tilde{I}_t$	Law motion of capital
$\tilde{Y}_t = \tilde{A}_t + \alpha\tilde{K}_t + (1-\alpha)\tilde{L}_t$	Production function
$\tilde{L}_t = \widetilde{MC}_t + \tilde{Y}_t - \tilde{W}_t$	Labor demand
$\tilde{K}_t = \widetilde{MC}_t + \tilde{Y}_t - \tilde{R}_t$	Demand for capital
$\widetilde{MC}_t = [(1-\alpha)\tilde{W}_t + \alpha\tilde{R}_t - \tilde{A}_t]$	(Marginal cost)
$\tilde{\pi}_t = \beta\mathbb{E}_t\tilde{\pi}_{t+1} + \dfrac{(1-\theta)(1-\beta\theta)}{\theta}(\widetilde{MC}_t - \tilde{P}_t)$	(Phillips equation)
$\tilde{\pi}_t = \tilde{P}_t - \tilde{P}_{t-1}$	(inflation)
$Y_{ss}\tilde{Y}_t = C_{ss}\tilde{C}_t + I_{ss}\tilde{I}_t$	Equilibrium condition
$\tilde{A}_t = \rho_A\tilde{A}_{t-1} + \varepsilon_t$	(productivity shock)

of these inputs thus responded positively to this greater level of demand. Bearing in mind that higher wages increase the income of households, if, on the one hand, this higher level of income increases the acquisition of goods (I and C), on the other, it increases the demand for another ”good”, leisure (income effect). This fall in labor supply explains the higher resistance of wages returning to the steady state, while returns on capital fell below even their initial level (steady state) in period 10. With the growth in aggregate supply, the elements that make up aggregate demand increase, most notably investments, whose result is 10 times greater than that of consumption goods. This higher capital supply (strong growth in investments) explains the returns on capital's swifter return to the steady state.

Higher productivity increased the spending variables (consumption and investment) and input prices, with wages showing greater persistence when returning to a steady state. With regard to the factors of production, capital widened, exhibiting a bell-shaped curve with an inflection point in period 20. However, labor supply decreased because of a strong predominance of the income effect.

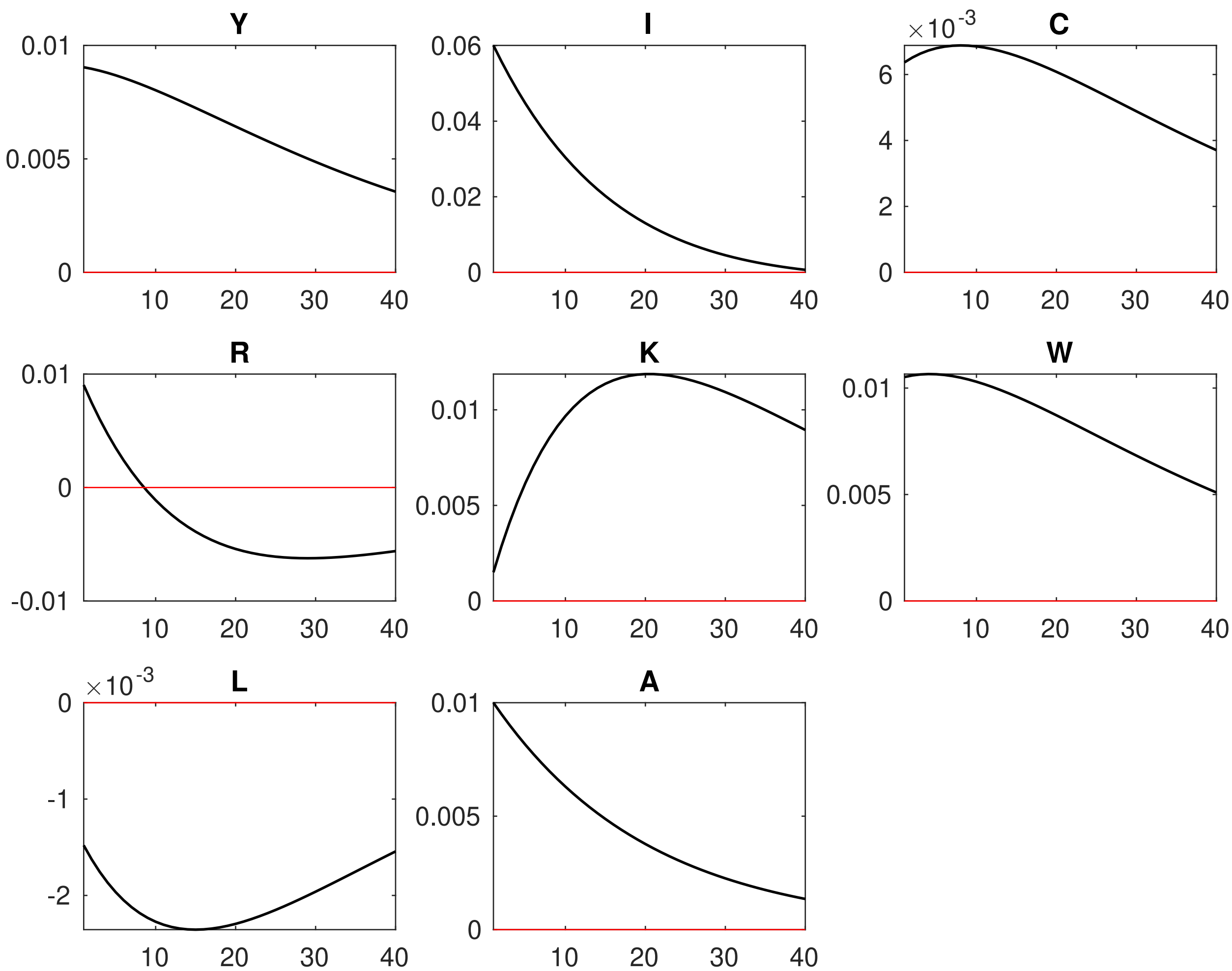

Figure 4.1: New-Keynesian's IRF when there is a positive productivity shock

Chapter 5

Projects

Choose 1 of the 2 projects below (or both if you like) to practice coding with Dynare.

Project 1. In this project, you will extend the original OLG model to a more realistic lifespan.

1. Expand the model from 6-period lived to 12-period lived agents with the first 8 periods spent on working, and the last 4 periods spent on retiring.

   Recalibrate the parameters as follows.

   Table 5.1: Model Parameters

Parameter	**Definition**	**Value**
α	Capital share	0.30
β	Subjective discount factor	0.99^5
η	Preference parameter	2.0
δ	Depreciation rate	$1-0.95^5$
$\bar{n}$	Average steady-state labor supply	0.35
γ	Leisure parameter	2.0
$\dfrac{b}{(1-\tau)w\bar{n}}$	Replacement ratio	0.3

2. What happens to the steady state when the replacement ratio is reduced to 0.2 (i.e, the government decides to reduce the pension benefits paid to the retirees)?

For reference, here is a sample plot of the 6-period case.

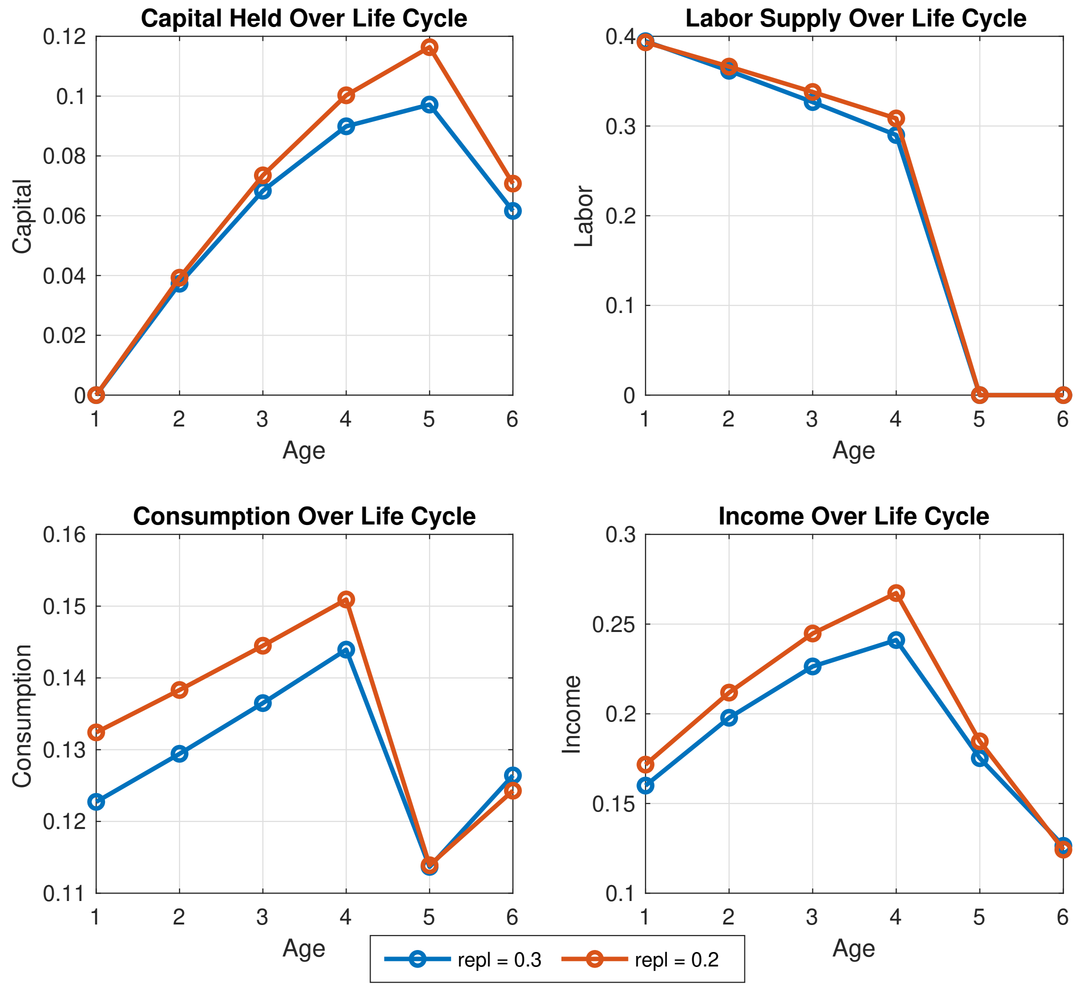

Figure 5.1: Steady state comparison of a change in replacement ratio.

Project 2. In this project, you will practice an RBC simulation when there is a preference shock.

1. Let us add one preference shock to the Household's problem

$$\max_{C_{j,t},L_{j,t},K_{j,t+1}} E_t \sum_{t=0}^{\infty} \beta^t \left(\frac{C_{j,t}^{1-\sigma}}{1-\sigma} - \exp(\tau_t) \frac{L_{j,t}^{1+\varphi}}{1+\varphi} \right) \tag{5.1}$$

   where the law of motion for the preference shock is

$$\tau_t = \rho_\tau \tau_{t-1} + \varepsilon_t^\tau. \text{ with } \varepsilon_t^\tau \sim N(0, \sigma_\tau^2) \tag{5.2}$$

   Everything else remains the same.

2. Derive the new FOCs for the household.

3. Use the following parameter: $\rho_\tau = 0.5, \sigma_\tau = 0.02$. Write a Dynare code and simulate IRFs given a negative preference shock.

4. What happens if you put the preference shock on consumption? Is there any difference from the other case?

Appendix A

Derivations in Chapter 2

A.1 Log-linearization in RBC

(Euler) + use: $\overline{R} = 1/\beta + 1 - \delta$

$$
\begin{aligned}
1 &\approx \beta \mathbb{E}_t \left[\frac{\overline{C} e^{\tilde{C}_t}}{\overline{C} e^{\tilde{C}_{t+1}}} (\overline{R} e^{\tilde{R}_{t+1}} + 1 - \delta) \right] \\
1 &= \beta \mathbb{E}_t \left[\overline{R} e^{\tilde{C}_t - \tilde{C}_{t+1} + \tilde{R}_{t+1}} + (1-\delta) e^{\tilde{C}_t - \tilde{C}_{t+1}} \right] \\
1 &\approx \beta \left[\overline{R} \mathbb{E}_t \left(1 + \tilde{C}_t - \tilde{C}_{t+1} + \tilde{R}_{t+1}\right) + (1-\delta) \mathbb{E}_t \left(1 + \tilde{C}_t - \tilde{C}_{t+1}\right) \right] \\
1 &= \beta [\overline{R} \tilde{R}_{t+1} + (\overline{R} + 1 - \delta) \mathbb{E}_t (1 + \tilde{C}_t - \tilde{C}_{t+1})] \\
1 &= \mathbb{E}_t \left[1 + \tilde{C}_t - \tilde{C}_{t+1} + \beta \overline{R} \tilde{R}_{t+1}\right] \\
0 &\approx \tilde{C}_t - \mathbb{E}_t \tilde{C}_{t+1} + \beta \overline{R} \mathbb{E}_t \tilde{R}_{t+1}
\end{aligned}
$$

(Labor) + use: $\frac{1-\gamma}{\gamma} \overline{C} = (1-\overline{L})(1-\alpha)\overline{Y}/\overline{L}$

$$
\begin{aligned}
\frac{1-\gamma}{\gamma} \overline{C} e^{\tilde{C}_t} &= (1-\alpha) \left(\frac{\overline{Y}}{\overline{L}} e^{\tilde{Y}_t - \tilde{L}_t} - \overline{Y} e^{\tilde{Y}_t} \right) \\
\frac{1-\gamma}{\gamma} \overline{C} (1 + \tilde{C}_t) &\approx (1-\alpha) \frac{\overline{Y}}{\overline{L}} (1 + \tilde{Y}_t - \tilde{L}_t) - (1-\alpha) \overline{Y} (1 + \tilde{Y}_t) \\
\frac{1-\gamma}{\gamma} \overline{C} \tilde{C}_t &= (1-\alpha) \frac{\overline{Y}}{\overline{L}} (1 - \overline{L}) \tilde{Y}_t - (1-\alpha) \frac{\overline{Y}}{\overline{L}} \tilde{L}_t \\
\tilde{C}_t &= \tilde{Y}_t - \frac{\tilde{L}_t}{1 - \overline{L}} \\
0 &\approx \tilde{Y}_t - \frac{\tilde{L}_t}{1 - \overline{L}} - \tilde{C}_t
\end{aligned}
$$

(Resource) + use $\overline{Y} = \overline{C} + \delta \overline{K}$

$$
\overline{C} e^{\tilde{C}_t} = \overline{Y} e^{\tilde{Y}_t} + (1-\delta) \overline{K} e^{\tilde{K}_t} - \overline{K} e^{\tilde{K}_{t+1}}
$$

$$
\begin{aligned}
\overline{C}(1+\tilde{C}_t) &\approx \overline{Y}(1+\tilde{Y}_t) + (1-\delta)\overline{K}(1+\tilde{K}_t) - \overline{K}(1+\tilde{K}_{t+1}) \\
\overline{C}\tilde{C}_t &= \overline{Y}\tilde{Y}_t + \overline{K}[(1-\delta)\tilde{K}_t - \tilde{K}_{t+1}] \\
0 &\approx \overline{Y}\tilde{Y}_t - \overline{C}\tilde{C}_t + \overline{K}[(1-\delta)\tilde{K}_t - \tilde{K}_{t+1}]
\end{aligned}
$$

(Production): simply take logs

$$
\begin{aligned}
\ln Y_t &= z_t + \alpha \ln K_t + (1-\alpha)\ln L_t, \\
\ln \overline{Y} &= \alpha \ln \overline{K} + (1-\alpha)\ln \overline{L}
\end{aligned}
$$

Subtracting the second equation from the first

$$
0 \approx z_t + \alpha\tilde{K}_t + (1-\alpha)\tilde{L}_t - \tilde{Y}_t
$$

(Interest rate) + use $\overline{R} = \alpha\overline{Y}/\overline{K}$

$$
\begin{aligned}
\overline{R}e^{\tilde{R}_t} &= \alpha\frac{\overline{Y}}{\overline{K}}e^{\tilde{Y}_t - \tilde{K}_t} \\
0 &\approx \tilde{Y}_t - \tilde{K}_t - \tilde{R}_t
\end{aligned}
$$

The stochastic process is already linear.

A.2 Uhlig's Method of Undetermined Coefficients

State variable vector

$$
x_t = [\tilde{K}_{t+1}]
$$

Endogenous variables vector

$$
y_t = [\tilde{Y}_t \quad \tilde{C}_t \quad \tilde{L}_t \quad \tilde{R}_t]'
$$

The linear version of the model can be written as

$$
\begin{aligned}
((3.8)):\ & 0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t, \\
((3.9)-(3.12)):\ & 0 = \mathbb{E}_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t], \\
((3.13)):\ & z_{t+1} = Nz_t + \varepsilon_{t+1},\ E(\varepsilon_{t+1}) = 0.
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & -\overline{K} & 0 & 0 \end{bmatrix}' \\
B &= \begin{bmatrix} 0 & (1-\delta)\overline{K} & \alpha & -1 \end{bmatrix}'
\end{aligned}
$$

$$
\begin{aligned}
C &= \begin{bmatrix} 1 & -1 & -1/(1-\overline{L}) & 0 \\ \overline{Y} & -\overline{C} & 0 & 0 \\ -1 & 0 & 1-\alpha & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \\
D &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}' \\
F &= [0],\ G = [0],\ H = [0], \\
J &= \begin{bmatrix} 0 & -1 & 0 & \beta\overline{R} \end{bmatrix}, \\
K &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \\
L &= [0],\ M = [0], \\
N &= [\rho]
\end{aligned}
$$

The solution is a set of matrices P, Q, R, S, that satisfy

$$
\begin{aligned}
x_t &= Px_{t-1} + Qz_t, \\
y_t &= Rx_{t-1} + Sz_t
\end{aligned}
$$

Note that in this model, the matrix C is full rank and has a well-defined inverse C^{-1}. If the equilibrium laws of motion exist, they must fulfill

$$
\begin{aligned}
0 &= (F - JC^{-1}A)P^2 - (JC^{-1}B - G + KC^{-1}A)P - KC^{-1}B + H, \\
R &= -C^{-1}(AP + B)
\end{aligned}
$$

Q satisfies

$$
\begin{aligned}
\Big(N' \otimes (F - JC^{-1}A) + I_k \otimes (JR + FP + G - KC^{-1}A)\Big)\mathrm{vec}(Q) &= \mathrm{vec}\Big((JC^{-1}D - L)N + KC^{-1}D - M\Big), \\
S &= -C^{-1}(AQ + D)
\end{aligned}
$$

where I_k is the identity matrix with $k \times k$ dimension with k is the number of columns in the matrix Q. The number k is based on the dimension of z_t. In this case, there is only one shock in productivity, so $k = 1$. The matrix P has one element, so finding its solution is just a standard quadratic equation. The rest is matrix algebra. Note that the value of P must have a modulus of less than 1.

A.3 Log-linearization in NK

This section derives the log-linearized NK model using Uhlig's method.

Labor Supply

Beginning with the labor supply function

$$C_t^{\sigma} L_t^{\varphi} = \frac{W_t}{P_t}$$

$X_t = X_{ss} e^{\tilde{X}_t}$ is replace in each variable equation

$$C_{ss}^{\sigma} L_{ss}^{\varphi} e^{\sigma \tilde{C}_t + \varphi \tilde{L}_t} = \frac{W_{ss}}{P_{ss}} e^{(\tilde{W}_t - \tilde{P}_t)}$$

Note that at the steady state $C_{ss}^{\sigma} L_{ss}^{\varphi} = \frac{W_{ss}}{P_{ss}}$, one gets

$$\sigma \tilde{C}_t + \varphi \tilde{L}_t = \tilde{W}_t - \tilde{P}_t \tag{A.1}$$

Euler equation for consumption

The Euler equation is

$$\frac{1}{\beta} \mathbb{E}_t \left(\frac{C_{t+1}}{C_t} \right)^{\sigma} = (1 - \delta) + \mathbb{E}_t \left(\frac{R_{t+1}}{P_{t+1}} \right)$$

Once again, $X_t = X_{ss} e^{\tilde{X}_t}$ is replace in each variable

$$\frac{1}{\beta} \left(\frac{C_{ss}^{\sigma}}{C_{ss}^{\sigma}} \right) e^{(\sigma \mathbb{E}_t \tilde{C}_{t+1} - \sigma \tilde{C}_t)} = (1 - \delta) + \frac{R_{ss}}{P_{ss}} e^{\left[\mathbb{E}_t (\tilde{R}_{t+1} - \tilde{P}_{t+1}\right]}$$

Using Uhlig rule and noting that at steady state $\frac{1}{\beta} = \frac{R_{ss}}{P_{ss}}{}_{(} 1 - \delta)$, we have

$$\frac{\sigma}{\beta} (\mathbb{E}_t \tilde{C}_{t+1} - \tilde{C}_t) = \frac{R_{ss}}{P_{ss}} \mathbb{E}_t (\tilde{R}_{t+1} - \tilde{P}_{t+1}) \tag{A.2}$$

To determine the New-Keynesian Phillips curve, we begin with the log-linearization of the equation that defines the optimal price level:

$$\tilde{P^*}_t = (1 - \beta\theta) \mathbb{E}_t \sum_{i=0}^{\infty} (\beta\theta)^i \widetilde{MC}_{t+i} \tag{A.3}$$

Taking the log-linear of the aggregate price yields

$$\tilde{P}_t = \theta \tilde{P}_{t-1} + (1 - \theta) \tilde{P}_t^* \tag{A.4}$$

By substituting in equation (A.3), we arrive at

$$\tilde{P}_t = \theta \tilde{P}_{t-1} + (1 - \theta)(1 - \beta\theta) \mathbb{E}_t \sum_{i=0}^{\infty} (\beta\theta)^i \widetilde{MC}_{t+i} \tag{A.5}$$

By computing the quasi-differencing $\tilde{P}_t - \beta\theta\mathbb{E}_t\tilde{P}_{t+1}$, one gets

$$\theta(\tilde{P}_t - \tilde{P}_{t-1}) = \beta\theta(\mathbb{E}_t\tilde{P}_{t+1} - \tilde{P}_t) + (1-\theta)(1-\beta\theta)(\widetilde{MC}_t - \tilde{P}_t)$$

With the gross inflation rates in t and $t+1$ being: $\tilde{\pi}_t = \tilde{P}_t - \tilde{P}_{t-1}$, we arrive at the New-Keynesian Phillips equation:

$$\tilde{\pi}_t = \beta\mathbb{E}_t\tilde{\pi}_{t+1} + \frac{(1-\theta)(1-\beta\theta)}{\theta}(\widetilde{MC}_t - \tilde{P}_t) \tag{A.6}$$