Appendix A

Cheatsheets

A.1 Differentiation

See Tran and Zhang (2023, Ch. 5).

Basic Rules

$$(cf)' = cf' \qquad (f+g)' = f' + g'$$

$$(fg)' = f'g + fg' \qquad (f/g)' = \frac{f'g - fg'}{g^2} \quad (g \neq 0)$$

$$(f \circ g)' = f'(g(x)) \cdot g'(x)$$

Common Derivatives

$$\frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Chain Rule

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

Product Rule

$$\frac{d}{dx}(uv) = u'v + uv'$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2} \quad (v \neq 0)$$

Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Exponential Rule

$$\frac{d}{dx}(e^{u(x)}) = u'(x)e^{u(x)}$$

A.2 Integration

Basic Integrals

$$\int k\,dx = kx + C \quad \text{(where } k \text{ is a constant)}$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad \text{(where } n \neq -1\text{)}$$

$$\int e^x\,dx = e^x + C, \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

Common Integrals

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int \ln x\,dx = x\ln x - x + C$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

For more detail, see Tran and Zhang (2023, Ch. 8).

A.3 Taylor Series Formula

See Tran and Zhang (2023, Ch. 8). The Taylor expansion of $f(x)$ at a point a is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Or, in sigma notation:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Order 1

The Taylor series formula of order 1 for a function $f(x)$ centered at a is given by:

$$f(x) = f(a) + f'(a)(x-a)$$

Order 2

The Taylor series formula of order 2 for a function $f(x)$ centered at a is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

Geometric Series

$$\sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1-r} \quad \text{given } |r| < 1.$$

A.4 Implicit Function Theorem

See Tran and Zhang (2023, Ch 10.6). Let $F(x, y)$ be a continuously differentiable function defined in a neighborhood of a point (a, b). If $F(a, b) = 0$ and $\frac{\partial F}{\partial y} \neq 0$ at (a, b), then there exists an open interval I containing a and an open interval J containing b, and a unique continuously differentiable function $f : I \to J$, such that for all x in I, the equation $F(x, f(x)) = 0$ holds.

Furthermore, the derivative $f'(x)$ where $y = f(x)$ is given by:

$$f'(x) = -\frac{\frac{\partial F}{\partial x}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))}$$

Sometimes, when explicit solutions are not attainable, you can perform comparative statics with this theorem.

Example A.1. Find $y'(x)$ when $xy = 5$.

Let $F(x, y) = xy$. Then $F'_x = y, F'_y = x$. For $x \neq 0$, the IFT says that

$$y' = -\frac{F'_x}{F'_y} = -\frac{y}{x}$$

A.5 Intermediate Value Theorem

See Tran and Zhang (2023, Ch 6.10). Let f be a continuous function on the closed interval $[a, b]$. If y is any number between $f(a)$ and $f(b)$, then there exists at least one number c in the open interval (a, b) such that $f(c) = y$.

Application: Mostly to prove the existence of a solution.

We can use the Intermediate Value Theorem (IVT) to show that certain equations have solutions, or that certain polynomials have roots. For instance, the polynomial $f(x) = x^4 + x - 3$ is complicated, and finding its roots is very complicated. However, it's easy to check that $f(-1) = -3$ and $f(2) = 15$. Since $-3 < 0 < 15$, there has to be a point c between -1 and 2 with $f(c) = 0$. In other words, $f(x)$ has a root somewhere between -1 and 2. We don't know where, but we know it exists.

In a more general concept, if you need to solve

$$f(x) = g(x)$$

Sometimes, solving it is difficult. Instead, you can use numerical methods (that is, let the computer do the hard part). However, you may still want to prove the existence of such an x^*. Then you can show that $f(x)$ is increasing from $[-\infty, \infty]$, while $g(x)$ is decreasing from $[-\infty, \infty]$. Thus, they must cross somewhere, and that somewhere is x^*. And if f, g are monotone, then this x^* is unique.

Example A.2. Prove that the equation

$$2x - 5e^{-x}(1 + x^2) = 0$$

has a unique solution, which lies in the interval (0, 2).

Solution:

Define $g(x) = 2x - 5e^{-x}(1 + x^2)$. Then $g(0) = -5$ and $g(2) = 4 - 25/e^2$. In fact $g(2) > 0$ because $e > 5/2$. According to the intermediate value theorem, therefore, the continuous function g must have at least one zero in $(0, 2)$. Moreover, note that $g'(x) = 2 + 5e^{-x}(1 + x^2) - 10xe^{-x} = 2 + 5e^{-x}(1 - 2x + x^2) = 2 + 5e^{-x}(x - 1)^2$.But then $g'(x) > 0$ for all x, so g is strictly increasing. It follows that g can have only one zero.

A.6 Matrix Algebra

See Tran and Zhang (2023, Ch. 9)

Matrix Operations

Addition and Subtraction

$$A + B = B + A \quad \text{(Commutative)}$$

$$(A + B) + C = A + (B + C) \quad \text{(Associative)}$$

Scalar Multiplication

$$c(A + B) = cA + cB$$

$$(c + d)A = cA + dA$$

Matrix Multiplication

$$A(BC) = (AB)C \quad \text{(Associative)}$$

$$A(B + C) = AB + AC \quad \text{(Distributive)}$$

$$(cA)B = A(cB) = c(AB)$$

Matrix Multiplication

Given matrices $A_{m\times p}$ and $B_{p\times n}$, their matrix product $C = AB$ is defined by the formula:

$$C_{ij} = \sum_{k=1}^{p} A_{ik} \cdot B_{kj}$$

where $1 \leq i \leq m$ and $1 \leq j \leq n$.

In other words, the entry in the i-th row and j-th column of C is obtained by multiplying the elements in the i-th row of A with the corresponding elements in the j-th column of B, and then summing up these products.

Transpose

A transpose $\mathbf{A}^T$ of a $k \times n$ matrix is the $n \times k$ matrix obtained by interchanging rows and columns of $\mathbf{A}$. Notation: $\mathbf{A}^T$ or $\mathbf{A}'$.

Definition A.1 (Rules for Transposition)**.**

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T,$$

$$(\mathbf{A} - \mathbf{B})^T = \mathbf{A}^T - \mathbf{B}^T,$$

$$(\mathbf{A}^T)^T = \mathbf{A},$$

$$(r\mathbf{A})^T = r\mathbf{A}^T,$$

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T.$$

A matrix is **symmetric** iff $\mathbf{A} = \mathbf{A}'$

Determinants

Order 2

The determinant of a 2×2 matrix is given by

$$|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Order 3

The determinant of a 3×3 matrix is given by:

$$\begin{aligned} |A| &= \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= a_{11}[a_{22}a_{33} - a_{32}a_{23}] - a_{12}[a_{21}a_{33} - a_{31}a_{23}] + a_{13}[a_{21}a_{32} - a_{31}a_{22}] \end{aligned}$$

For higher order, see Tran and Zhang (2023, Ch. 9.4.2).

Inverse

2×2 matrix

Given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The inverse matrix A^{-1} is given by:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided that $|A| \neq 0$.

3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The inverse matrix A^{-1} is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

Where $\det(A)$ is the determinant of matrix A, and the elements e_{ij} are given by:

$$\begin{aligned} e_{11} &= a_{22}a_{33} - a_{23}a_{32} \\ e_{12} &= a_{13}a_{32} - a_{12}a_{33} \\ e_{13} &= a_{12}a_{23} - a_{13}a_{22} \\ e_{21} &= a_{23}a_{31} - a_{21}a_{33} \\ e_{22} &= a_{11}a_{33} - a_{13}a_{31} \\ e_{23} &= a_{13}a_{21} - a_{11}a_{23} \\ e_{31} &= a_{21}a_{32} - a_{22}a_{31} \\ e_{32} &= a_{12}a_{31} - a_{11}a_{32} \\ e_{33} &= a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

The matrix's inverse exists if and only if the determinant $\det(A)$ is non-zero.

Cramer's Rule

For a system of equations:

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned}$$

The solutions x and y can be found using Cramer's Rule:

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Where: - A is the coefficient matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- B_x is the matrix formed by replacing the first column of A with the constants column:

$$B_x = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$$

- B_y is the matrix formed by replacing the second column of A with the constants column:

$$B_y = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$$

Cramer's Rule is valid when the determinant $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is non-zero.

A.7 Jacobians, Gradients, and Hessians

The Gradient ∇ is a vector that points in the direction of the steepest increase of a scalar-valued (1D) function of multiple variables. Its magnitude $|\nabla|$ is the rate of change in that direction. The gradient is a vector and has the same dimension as the number of variables in the function. For a function of n variables, the gradient is an n-dimensional vector.

The Jacobian is a matrix that represents the collection of all first-order partial derivatives of a vector-valued function with respect to multiple variables. It provides information about how each component of the vector function changes as the variables change. The Jacobian is a matrix whose size is determined by the number of components in the vector-valued function and the number of variables. For a function with m components (functions) and n variables, the Jacobian is an $m \times n$ matrix.

In summary, the gradient is a vector that describes the rate of change of a scalar-valued (1D) function, while the Jacobian is a matrix that describes the rate of change of a vector-valued (n-D) function.

The Hessian is just a matrix of second-order mixed partials of a scalar field (that is, gradients). The Hessian matrix is symmetric. This means that the element in row i and column j is the same as the element in row j and column i. The mixed partial derivatives in the Hessian matrix satisfy the equality of mixed partials $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. The eigenvalues of the Hessian matrix are indicators of the curvature of the function at a critical point. Positive eigenvalues suggest a local minimum, negative eigenvalues suggest a local maximum, and a mix of positive and negative eigenvalues suggest a saddle point. For more detail, see Tran and Zhang (2023, Ch 10).

1D (1 dimension)

Let $f(x)$ be a one-dimensional function. The Gradient ∇ is a vector of the first order derivative of f with respect to x mapping $\mathbb{R}^1 \to \mathbb{R}^1$ (a scalar field)

$$\nabla = \frac{df}{dx}$$

The Hessian is a matrix of second-order mixed partials of a scalar field.

$$\mathbf{H} = \frac{d^2 f}{dx^2}$$

The Jacobian is a matrix of gradients for components of a vector field (in this case, only 1 component), thus

$$\mathbf{J} = \frac{dF}{dx} = \frac{df}{dx}$$

So in 1D, Jacobian and Gradient are the same.

2D (2 dimensions)

Let $f(x, y)$ be a two-dimensional function. The Gradient is defined as:

$$\nabla = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

The Hessian is

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

Consider $f(x, y)$ and $g(x, y)$ be a system of two-dimensional functions. The Jacobian matrix $\mathbf{J}$ for this system is defined as:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

3D (3 dimensions)

Let $f(x, y, z)$ be a three-dimensional function. The Gradient is:

$$\nabla = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}$$

The Hessian is

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

Consider $f(x, y, z), g(x, y, z), h(x, y, z)$ a system of three-dimensional functions. The Jacobian matrix $\mathbf{J}$ for this system is defined as:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix}$$

In a sense, The Hessian is the Jacobian of the gradient of a function that maps from n-dimension to 1-Dimension.