Chapter 1

First-Order Difference Equations

In economic growth theory, in studies of the extraction of natural resources, in many models in environmental economics, and in several other areas of economics that have one key variable moves based on its past values, you will have to deal with dynamics. If we talk about dynamics, we talk about difference (in discrete time), or differential (in continuous time) equations. For the scope of this course, we are only concerned about the first-order difference equation, that is, tomorrow's value only depends on today, not including yesterday.

In preparation of the materials presented here, we reference Sydsæter and Hammond (2008); Sydsæter et al. (2008) and Chiang (1984, p.616). For differential equations, read Sydsæter et al. (2008, Chapter 5, 6), Simon and Blume (1994, Chapter 23,24,25)

1.1 One Variable Difference Equations

1.1.1 Linear case

A simple example of a linear first-order difference equation is

$$x_{t+1} = ax_t$$

for $t = 0.1, \ldots$ and a is a constant. Suppose x_0 is given, if we repeatedly applying the function for different t, we get

$$\begin{aligned}
x_1 &= ax_0, \\
x_2 &= ax_1 = a^2 x_0, \\
x_3 &= ax_2 = a^3 x_0, \ldots
\end{aligned}$$

which we can generalize it as

$$x_t = x_0 a^t$$

So that at any time t, given a_0 is known, we can calculate the current value of x_t. We can expand it by adding a constant b[1]. so the difference equation becomes

$$x_{t+1} = ax_t + b \tag{1.1}$$

[1] If $b = g(t)$, that is, the difference equation also depends on time t, then it is called non-autonomous. If the difference equation does not depend on t, then it is called autonomous

which gives us the SOLUTION as follows.

$$x_t = a^t \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a} \qquad (a \neq 1)$$

We now discuss the notion of **stationary point**. Consider the solution of (1.1). If

$$x_0 = \frac{b}{1-a}$$

then

$$x_t = \frac{b}{1-a} \ \forall t$$

This solution $\bar{x} = b/(1-a)$ is called an equilibrium, or stationary, or steady state of (1.1). To find such an equilibrium state $\bar{x}$, we need to find $\bar{x}$ such that

$$\bar{x} = a\bar{x} + b$$

Suppose that $|a| < 1$ then

$$\lim_{t \to \infty} a^t = 0$$

implying that

$$\lim_{t \to \infty} x_t = \frac{b}{1-a}$$

Hence, so long as $-1 < a < 1$, the solution converges to the equilibrium state as $t \to \infty$ and we say that the difference equation is globally asymptotically stable. But would happen otherwise?

Let us now discuss **stability** by the following illustration.

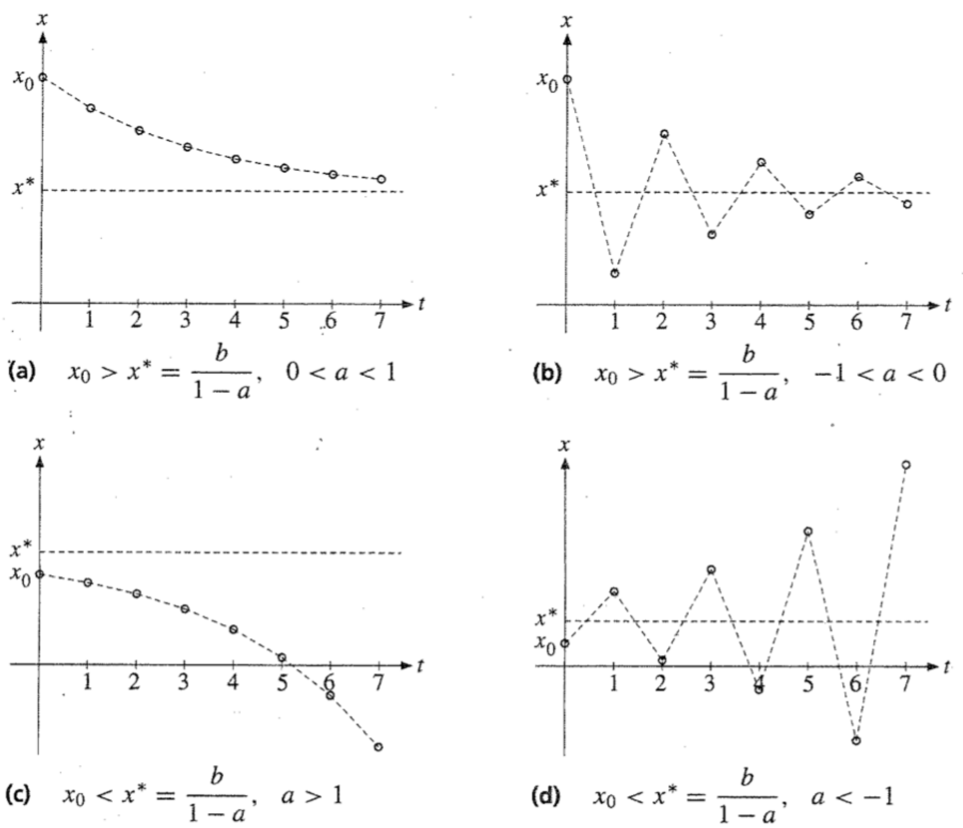


Fig. 1.1. Dynamics of stable and unstable equations (Sydsæter et al., 2008, p.393)

We summarize the cases in Fig.1.1 as follows.

(a) x_t decreases monotonically and converges to the equilibrium state x^*.

(b) x_t exhibits decreasing fluctuations, or **damped oscillations** around x^*.

(c) x_t tends to $-\infty$ monotonically (it never reaches x^*).

(d) x_t exhibits increasing fluctuations, or **explosive oscillations** around x^*. It does cross x^* at some point but never converges to it.

Thus, the condition that $|a| < 1$ is necessary to guarantee that the dynamics of x_t converge to the steady state x^*.

1.1.2 Nonlinear case

So far, we only consider linear difference equations. Although this case is easy, we almost never encounter them in economics because most dynamics in economics are nonlinear. Let us consider an autonomous first-order difference equation of one variable

$$x_t = f(x_{t-1}) \tag{1.2}$$

The procedure to find the stationary point is still the same. We still need to solve for x^* such that

$$x^* = f(x^*)$$

The solution to this is called a fixed point. How can we determine the stability of this fixed point?

It turns out, the idea behind the stability conditions stems from linear approximation. The Taylor expansion shows how the function behaves about one specific point. The behavior of x_t about this point is

$$f(x_{t-1}) = f(x^*) + f'(x^*)(x_{t-1} - x^*) + R_2(x_{t-1}, x^*)$$

R is the remainder. If at the initial point, x_0 is sufficiently close to x^*, then $R \approx 0$. Since we are estimating the point close to x^*, we can ignore R. The point x_t about x^* can be now expressed as

$$x_t = f(x^*) + f'(x^*)(x_{t-1} - x^*).$$

This is just similar to (1.1), so we just apply their conditions to ours, and we arrive at the following conclusion.

Theorem 1.1.1 (Stability condition)**.** *Let $\bar{x}$ be a stationary state for the difference equation $x_{t+1} = f(x_t)$, and suppose that f is differentiable in an open interval around x^*.*

1. *If $|f'(x^*)| < 1$, then x^* is **locally asymptotically stable***
2. *If $|f'(x^*)| > 1$, then x^* is **unstable***
3. *If $|f'(x^*)| = 1$, the situation is inconclusive (you will need to analyze the higher orders) – see Shone (2002, p. 89)*

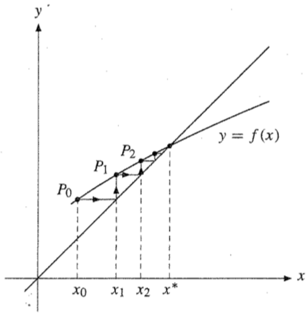


Figure 1 x^* stable, $f'(x^*) \in (0, 1)$.

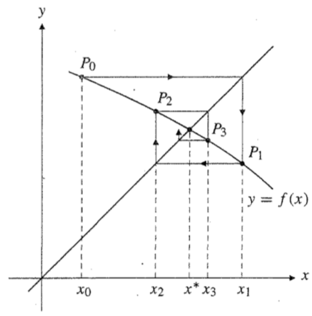


Figure 2 x^* stable, $f'(x^*) \in (-1, 0)$.

(a) Stable $f'(x_t) < 1$

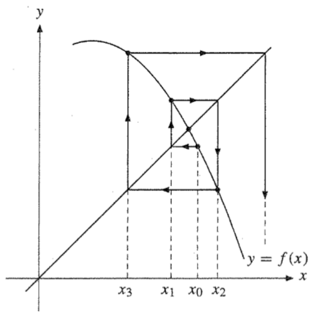


Figure 3 x^* unstable, $|f'(x^*)| > 1$.

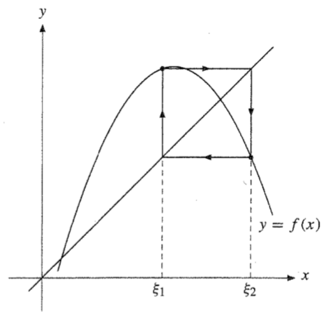


Figure 4 A cycle of period 2.

(b) Unstable $f'(x_t) > 1$

Fig. 1.2. Dynamics of some nonlinear equations (Sydsæter et al., 2008, p.420)

For formal proof, see Sydsæter et al. (2008, p.419)

Again, let us examine the stability illustrated in Fig.1.2.

1. Here, $0 < f'(x^*) < 1$ (derivative is positive) so the sequence monotonically converges to x^*.

2. In this case, $-1 < f'(x^*) < 0$ (derivative is negative) so x_t alternates between values above and below the equilibrium state x^*. Eventually, it converges to x^*.

3. This is the first case of unstable dynamics. Since $f'(x^*) > 1$, x is getting farther away from the equilibrium point.

4. This is a special case where solutions exhibit cyclic behavior (in this case, a cycle of period 2).

Let us explore the last case in more detail. A cycle of period 2 has the following property

$$\begin{aligned} &\text{first cycle} && x_0 = x_2 = x_4 = \dots, \\ &\text{second cycle} && x_1 = x_3 = x_5 = \dots, \\ &\text{the 2 cylces are different} && x_0 \neq x_1 \end{aligned}$$

This case happens if and only if there are 2 solutions for (1.2). Let us call them ξ_1, ξ_2. (pronounced /ksai/ or /zai/). Hence

$$\xi_1 = f(\xi_1),$$
$$\xi_2 = f(\xi_2).$$

If we let $F = f \circ f$, it is clear that ξ_1 and ξ_2 must be fixed points of F and hence the equilibria of the difference equation

$$y_{t+1} = F(y_t) = f(f(y_t)),$$

where $y_t = x_t x_{t+1}$. Now, we change the focus only to the stability of y_t. Applying Theorem 1.1.1, the dynamic is stable if and only if $F'(y_t) < 1$. By Chain rule

$$F'(x) = f'(f(x)) \cdot f'(x)$$

so that

$$F'(\xi_1) = f'(f(\xi_2))f'(\xi_1) = f'(\xi_2)f'(\xi_1) = F'(\xi_2).$$

Therefore, we can state

Theorem 1.1.2. *If (1.2) admits a cycle of period 2, alternating between values ξ_1 and ξ_2, then:*

1. *If $|f'(\xi_1)f'(\xi_2)| < 1$, the cycle is locally asymptotically stable.*
2. *If $|f'(\xi_1)f'(\xi_2)| > 1$, the cycle is unstable.*

1.1.3 Economic Applications

Ex. 1.1. Find the fixed point and determine their stability (Shone, 2002, p.97)

1. $y_{t+1} = 2y_t - y_t^2$,
2. $y_{t+1} = 3.2y_t - 0.8y_t^2$

Ex. 1.2 (Harrod-Domar growth)**.** Consider the following model

$$\begin{aligned} S_t &= sY_t, \\ I_t &= \nu(Y_t - Y_{t-1}), \\ S_t &= I_t \end{aligned}$$

1. Write the fundamental difference equation relating $Y_t = F(Y_{t-1})$.
2. When does the economy grow without bounds?

Ex. 1.3 (Solow Growth)**.** Consider the economy where

$$\begin{aligned} S_t &= sY_t, \\ I_t &= K_t - (1-\delta)K_{t-1}, \\ S_t &= I_t, \\ Y_t &= AK_t^{\alpha}L_t^{1-\alpha}, \\ L_t &= (1+n)L_{t-1} \end{aligned}$$

1. Define $k_t = K_t/L_t$, write the fundamental equation.
2. Derive the steady state k^*.
3. Let $A = 5, \alpha = 0.25, s = 0.1, n = 0.02, \delta = 0.05, k_0 = 20$, derive numerically.
4. Use linear approximation to investigate its stability.

1.2 System of 2 Difference Equations

Things get more complicated when there are 2 variables revolving around each other. Consider the following case system

$$\begin{aligned} x_{t+1} &= f(x_t, y_t) \\ y_{t+1} &= g(x_t, y_t) \end{aligned} \tag{1.3}$$

Functions ϕ, ψ can be linear or nonlinear, just like the case with one variable. Before diving into things, let us take a detour and do some exercises on finding the eigenvalues and eigenvectors of a matrix.

1.2.1 Eigenvalues and Eigenvectors

Definition: Given a square matrix $\mathbf{A}$, an eigenvalue of $\mathbf{A}$ is a scalar λ for which there exists a non-zero vector v such that the following equation holds:

$$\mathbf{A}v = \lambda v$$

Here, v is called an eigenvector corresponding to the eigenvalue λ. In other words, when the matrix $\mathbf{A}$ is applied to the vector v, the resulting vector is a scalar multiple of v (scaled by λ). [2]

Steps in finding the Eigenvectors and Eigenvalues

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \lambda\mathbf{x} \\ \Rightarrow (\mathbf{A} - \lambda)\mathbf{x} &= 0 \\ \Rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= 0 \\ \iff \det(\mathbf{A} - \lambda\mathbf{I}) &= 0 \end{aligned}$$

Solving this expression gives us the Eigenvalues λ and the eigenvector $\mathbf{x}$.

Checks when you found the eigenvalues

1. (trace) the sum of all the eigenvalues will be the sum of the diagonal of $\mathbf{A}$
2. (determinants) the product of all the eigenvalues is the determinant

The eigenvectors tell you the directions that do not change during some linear transformation, while the eigenvalues tell you the scaling vector of these eigenvectors.

Proof. Suppose $\mathbf{A}$ is a square 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then finding the eigenvalues is to solve

$$\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$$

And the eigenvalues λ are the solutions of the following **Characteristic Equation**:

$$\begin{aligned} (a - \lambda)(d - \lambda) - bc &= 0 \\ \Leftrightarrow \lambda^2 - \underbrace{(a + d)}_{\beta}\lambda + \underbrace{(ad - bc)}_{\alpha} &= 0 \end{aligned} \tag{1.4}$$

[2] To understand more intuitively, visit: https://www.youtube.com/watch?v=PFDu9oVAE-g&list=PL0-GT3co4r2y2YErbmuJw2L5tW4Ew205B&index=14

This is called the characteristic polynomial and the eigenvalues are the roots. You can find the solutions by using quadratic formula.

Let

$$\triangle = \beta^2 - 4\alpha$$

1. if $\triangle = 0$, there is one real root $\lambda = -\frac{\beta}{2}$
2. if $\triangle > 0$, there are 2 real roots $\lambda_{1,2} = \frac{-\beta \pm \sqrt{\triangle}}{2}$.
3. if $\triangle < 0$, there are 2 complex roots $\lambda_{1,2} = \frac{-\beta \pm i\sqrt{|\triangle|}}{2}$ where $i = \sqrt{-1}$.

The verification process actually is a corollary of the Vieta's formulas.

$$\begin{aligned}
\lambda_1 + \lambda_2 &= -\beta \equiv a + d = tr(\mathbf{A}), \\
\lambda_1 \lambda_2 &= \alpha \equiv (ad - bc) = \det(\mathbf{A})
\end{aligned}$$

For each eigenvalue λ, find the corresponding eigenvector by solving the system of equations $(A - \lambda I)\mathbf{v} = \mathbf{0}$:

$$(A - \lambda\mathbf{I})\mathbf{v} = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For λ_1:

$$\begin{pmatrix} a - \lambda_1 & b \\ c & d - \lambda_1 \end{pmatrix} \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For λ_1:

$$\begin{pmatrix} a - \lambda_2 & b \\ c & d - \lambda_2 \end{pmatrix} \begin{pmatrix} v_{2,1} \\ v_{2,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

■

Example 1.1. Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$$

Solution:

1. (eigenvalues) We need to solve

   $$\begin{vmatrix} 2 - \lambda & 3 \\ 5 & 4 - \lambda \end{vmatrix} = 0$$

   which yields $\lambda_1 = 7$ and $\lambda_2 = -1$. These are the eigenvalues. We can do a quick verification.

   (a) (trace) $\lambda_1 + \lambda_2 = 6 = 2 + 4$ (diagonal of $\mathbf{A}$)

   (b) (determinants) $\lambda_1 \times \lambda_2 = -7 = \det\mathbf{A}(= 2 \times 4 - 5 \times 3)$.

2. (eigenvector) Let the eigenvectors be $\mathbf{v} = (v_1, v_2), \mathbf{u} = (u_1, u_2)$

   Now, for $\lambda_1 = 7$, one need to solve

$$\begin{pmatrix} 2-7 & 3 \\ 5 & 4-7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   v_1 and v_2 are the solutions of

$$\begin{aligned} -5v_1 + 3v_2 &= 0 \\ 5v_1 - 3v_2 &= 0 \end{aligned}$$

   which gives the eigenvector $\mathbf{v} = \left(\frac{3}{5}, 1\right)$ for theeigenvalue $\lambda_1 = 7$.

   Similarly, we can find the other eigenvector $\mathbf{u} = (-1, 1)$ for the eigenvalue $\lambda_2 = -1$.

Ex. 1.4. Find the eigenvalues and verify them for the following matrices

$$\mathbf{A} = \begin{pmatrix} 3 & 8 \\ 4 & -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & 6 \\ -1 & 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix},$$
$$\mathbf{E} = \begin{pmatrix} -1 & 2 \\ 4 & -5 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \mathbf{G} = \begin{pmatrix} 4 & -2 & 1 \\ 1 & 5 & 2 \\ 0 & 1 & 6 \end{pmatrix}$$

1.2.2 System of 2 Linear Difference Equations

We consider the linear dynamics in $\mathbb{R}^2$ as follows

$$\begin{aligned} x_{t+1} &= ax_t + by_t \\ y_{t+1} &= cx_t + dy_t \end{aligned} \tag{1.5}$$

Since we want a system involving both x and y, we assume that $b, d \neq 0$. For example

$$\begin{aligned} x_{t+1} &= 0.9x_t - 0.2y_t \\ y_{t+1} &= 0.1x_t + 0.7y_t \end{aligned} \tag{1.6}$$

We can exploit the matrix notation and write it as

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\mathbf{J}} \begin{pmatrix} x_t \\ y_t \end{pmatrix} \tag{1.7}$$

Yes, $\mathbf{J}$ is the Jacobian matrix. To characterize the properties of the Jacobian, we need to involve the notion of eigenvalues. But first, let's find the steady state by solving

$$\begin{aligned} \bar{x} &= a\bar{x} + b\bar{y} \\ \bar{y} &= c\bar{x} + d\bar{y} \end{aligned}$$

which can be written as

$$\begin{aligned} \bar{x} &= \frac{b}{1-a}\bar{y}, \\ \bar{y} &= \frac{d}{1-c}\bar{x} \end{aligned}$$

implying that

$$\bar{x} = \frac{bd}{(1-a)(1-c)}\bar{x}$$

Now, since $b, d \neq 0$, $\frac{bd}{(1-a)(1-c)} \neq 0$. Assume further that $a, c \neq 1$, then the only solution to the above equation is

$$\bar{x} = 0.$$

Thus, the only equilibrium point is $(\bar{x}, \bar{y}) = (0, 0)$.

In studying the stability around the steady state, the eigenvalues of a Jacobian matrix provide valuable insights into the behavior of a dynamical system near its equilibrium points. They indicate how nearby trajectories behave over time. If the real parts of the eigenvalues are negative (i.e., the absolute value of the eigenvalues are less than 1), trajectories that start near the equilibrium point will converge towards it, indicating **stability**. If the real parts are positive, trajectories will diverge, leading to instability.

Theorem 1.2.1. *Let the eigenvalues of the Jacobian matrix be $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then:*

1. *If $|\lambda_1| \leq |\lambda_2| < 1$, then*

$$\text{for all } \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \lim_{t\to\infty} \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   *and the steady state $(0, 0)$ is **stable** in $\mathbb{R}^2$. Any value for (x_0, y_0) will lead the dynamics to the steady state. The steady state $(0, 0)$ is said to be a **sink**. Furthermore. assume that $|\lambda_2| > |\lambda_1|$.*

   - *if $\lambda_2 > 0$, the long run dynamics are monotonic*
   - *if $\lambda_2 > 0$, the long run dynamics are oscillating.*

2. *If $|\lambda_2| \geq |\lambda_1| > 1$, all trajectories starting from*

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   ***explode**. The steady state $(0, 0)$ is **unstable**. It is said to be a **source**.*

3. *If $|\lambda_1| < 1 < |\lambda_2|$, there exists a unique direction along which the dynamics converge to $(0, 0)$. This implies that for a given x_0, there is only one value of y_0 such that the trajectory converges to the steady state. The steady state $(0, 0)$ is said to be a saddle point.*

So far, we have assume that the 2 eigenvalues λ_1, λ_2 are real and not equal each other. The following analyzes such cases.

Repeated Eigenvalues

Write them as $\lambda_1 = \lambda_2 = \lambda$, then if

1. If $|\lambda < 1|$, all trajectories converge to $(0, 0)$, which is globally stable in $\mathbb{R}^2$.
2. If $|\lambda > 1|$, all trajectories starting from $(x_0, y_0) \neq (0, 0)$ explode and $(0, 0)$ is unstable.

Complex Eigenvalues
We can write them as

$$\lambda_1 = \alpha + i\beta,$$
$$\lambda_2 = \alpha - i\beta$$

There are then two possibilities:

1. If $\alpha^2 + \beta^2 = |\lambda_1|^2 = |\lambda_2|^2 < 1$, all trajectories converge to $(0, 0)$, which is globally stable in $\mathbb{R}^2$.

2. If $\alpha^2 + \beta^2 > 1$, all trajectories starting from $(x_0, y_0) \neq (0, 0)$ explode and $(0, 0)$ is unstable.

Example 1.2. Let's work through the stability analysis for the system (1.6).

Step 1: Equilibrium Points
To find the equilibrium points, we need to solve the equations:

$$x_{t+1} = x_t$$
$$y_{t+1} = y_t$$

For the given system, the equilibrium points are found by setting each equation to its corresponding variable:

$$0.9x - 0.2y = x$$
$$0.1x + 0.7y = y$$

Solving these equations simultaneously, the equilibrium point is $(x^*, y^*) = (0, 0)$.

Step 2: Jacobian Matrix
The Jacobian matrix is given by:

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

Where f and g are the functions defining the system.

For the given system, we have:

$$f(x, y) = 0.9x - 0.2y$$
$$g(x, y) = 0.1x + 0.7y$$

Calculating the partial derivatives, we get:

$$J = \begin{bmatrix} 0.9 & -0.2 \\ 0.1 & 0.7 \end{bmatrix}$$

Step 3: Eigenvalues and Stability
Evaluate the Jacobian matrix at the equilibrium point $(0, 0)$:

$$J^* = \begin{bmatrix} 0.9 & -0.2 \\ 0.1 & 0.7 \end{bmatrix}$$

Calculate the eigenvalues of J^*. The characteristic equation is given by:

$$\det(J^* - \lambda I) = 0$$

Solving this equation, we find that the eigenvalues are approximately 0.8 and 0.8.

Since both eigenvalues have negative real parts, the equilibrium point $(0, 0)$ is stable.

In conclusion, the equilibrium point $(0, 0)$ for the given system is stable due to the negative real parts of the eigenvalues of the Jacobian matrix.

1.2.3 System of 2 Nonlinear Difference Equations

We focus on the nonlinear dynamics, as they are the most common in economics.

Consider the linear dynamics in $\mathbb{R}^2 \to \mathbb{R}^2$ following (1.3). Given the initial state (x_0, y_0). Assume that

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{y}), \\ \bar{y} &= g(\bar{x}, \bar{y}) \end{aligned} \tag{1.8}$$

be the steady state $(\bar{x}, \bar{y})$ of the system (1.3). It is is locally stable if for any initial value (x_0, y_0) near enough to $(\bar{x}, \bar{y})$, the dynamics starting from (x_0, y_0) converge to $(\bar{x}, \bar{y})$.

Let us take a first-order Taylor expansion of $f(\cdot)$ around a steady state:

$$f(x, y) - f(\bar{x}, \bar{y}) \approx f'_x(\bar{x}, \bar{y})(x - \bar{x}) + f'_y(\bar{x}, \bar{y})(y - \bar{y})$$

Similarly for $g(\cdot)$:

$$g(x, y) - g(\bar{x}, \bar{y}) \approx g'_x(\bar{x}, \bar{y})(x - \bar{x}) + g'_y(\bar{x}, \bar{y})(y - \bar{y})$$

From (1.3),(1.8), we can write them in matrix form [3] as

$$\begin{pmatrix} x_{t+1} - \bar{x} \\ y_{t+1} - \bar{y} \end{pmatrix} = \underbrace{\begin{pmatrix} f'_x(\bar{x}, \bar{y}) & f'_y(\bar{x}, \bar{y}) \\ g'_x(\bar{x}, \bar{y}) & g'_y(\bar{x}, \bar{y}) \end{pmatrix}}_{\mathbf{J}} \begin{pmatrix} x_t - \bar{x} \\ y_t - \bar{y} \end{pmatrix} \tag{1.9}$$

where, as well all know by now, $\mathbf{J}$ is the Jacobian matrix. The system has been "linearized" and can be analyzed similarly to the linear case.

Theorem 1.2.2. *Let λ_1, λ_2 be the eigenvalues of the Jacobian matrix $\mathbf{J}$ evaluated at the steady state $(\bar{x}, \bar{y})$. Then*

1. *If $|\lambda_1| \leq |\lambda_2| < 1$, the steady state is locally stable. Any initial condition will lead the dynamics to the steady state. The steady state $(\bar{x}, \bar{y})$ is said to be a **sink**.*

2. *If $|\lambda_1| \geq |\lambda_2| > 1$, the steady state is unstable: for any initial condition different from the steady state, the trajectories are locally exploding. The steady state is said to be a **source**.*

3. *If $|\lambda_1| < 1 < |\lambda_2|$, the steady state is a saddle point. For a given initial condition on one variable, there is only one initial value of the other variable such that the trajectory converges to the steady state. Any other value for this variable would lead the trajectory to locally explode.*

When the eigenvalues are real and their moduli (absolute value) lie on the same side of 1, the steady state is also called a (stable or unstable) node.

From a practical point of view, it is often easier to use the **trace** (**T**) and the **determinant**(**D**) of the Jacobian matrix. The results are summarized as follows.

[3] This is called to "linearize" around the steady state

Theorem 1.2.3. *We have*

$$\mathbf{T} = tr(\mathbf{J}) = f'_x + g'_y$$

and

$$\mathbf{D} = \det(\mathbf{J}) = f'_x g'_y - f'_y g'_x$$

Then:

1. *If* $|1 + \mathbf{D}| < |\mathbf{T}|$, *the steady state is a saddle.*
2. *If* $|1 + \mathbf{D}| > |\mathbf{T}|$ *and* $|\mathbf{D}| < 1$, *the steady state is a sink.*
3. *If* $|1 + \mathbf{D}| > |\mathbf{T}|$ *and* $\mathbf{D}| > 1$, *the steady state is a source.*

This is an important and neat result. After the linearization around the steady state, the LHS of (1.9) becomes the "amount of change" between 2 periods. Naturally, if the change converges toward zero and stays there as stability is guaranteed, it means that the steady-state sustains forever.

1.3 Economic Applications

The following exercises are from Sydsæter et al. (2008, p.395–399).

Ex. 1.5. Find the solution and equilibrium point. The last is from Shone (2002, p.139)

$$\begin{aligned}
&(a)\ x_{t+1} = 2x_t + 4,\ x_0 = 1,\\
&(b)\ 3x_{t+1} = x_t + 2,\ x_0 = 2,\\
&(c)\ 2x_{t+1} + 3x_t + 2 = 0,\ x_0 = -1,\\
&(d)\ x_{t+1} - x_t + 3 = 0,\ x_0 = 3,\\
&(e)\ x_{t+1} = 3.84x_t(1 - x_t),\ x_0 = 0.1 \text{ (3 cycles, use Python or Excel)}
\end{aligned}$$

Ex. 1.6 (Cobweb Model (Kaldor, 1934) [(4)]). Assume the total cost of raising q pigs is

$$C(q) = \alpha q + \beta q^2$$

Suppose there are N identical pig farms. Let the demand function for pigs be

$$D(p) = \gamma - \delta p$$

where p is the price, $\alpha, \beta, \gamma, \delta$ are positive constants. Each farmer behaves competitively and takes price as given to maximize their profit according to

$$\pi(q) = pq - C(q)$$

1. Find the quantity q^* that maximizes profit.
2. Find the Aggregate Supply $S(p)$.

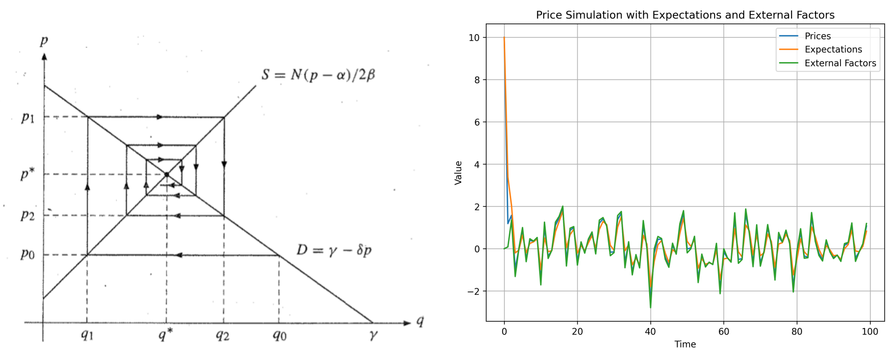
[(4)] There is no guy named "cobweb". Kaldor was the first to analyze the model and coined such a term because it looks like a web.

3. Now, suppose it takes 1 period to raise each pig. When choosing the number of pigs to raise for sale at time $t+1$, each farmer remembers the price p_t and expects p_{t+1} to be the same as p_t. Thus, aggregate supply at time $t+1$ is $S(p_t)$. Find the equilibrium price satisfying

$$S(p_t) = D(p_{t+1})$$

4. Write solution of p_t in terms of p_0 and a time path.

5. Find the equilibrium. Analyze its stability. When is it stable? When is it not?



(a) Stable cobweb (Sydsæter et al., 2008) **(b)** Inflation expectation

Fig. 1.3. The cobweb dynamics

Below are some variations of the cobweb model (Evans and Honkapohja, 2001). For numerical methods and simulation, you can visit https://python.quantecon.org/re_with_feedback.html.

Ex. 1.7 (Lucas (1973)'s model). Aggregate output

$$q_t = \bar{q} + \pi(p_t - p_t^e) + \zeta_t$$

where $\pi > 0$, p_t^e is the price expectation. Aggregate demand is

$$m_t + v_t = p_t + q_t$$

Money supply is random around a constant mean

$$m_t = \bar{m} + u_t$$

where all variables are in log form and u_t, v_t, ζ_t are white noise shocks. Can you achieve a reduced form of $p_t = F(p_t^e)$ (5).

Ex. 1.8 (Cagan (1956)'s model of hyperinflation). Demand for money depends linearly on expected inflation (the change in prices on the RHS).

$$m_t - p_t = -\psi(p_{t+1}^e - p_t), \quad \psi > 0$$

m_t, p_t, p_{t+1}^e are logs of money supply, price level and expectation of next-period price formed at time t. m_t, again, is i.i.d. around a constant mean $\bar{m}$.

Solve for p_t as a function of price expectation $F(p_{t+1}^e)$. (6)

(5) Ans: $p_t = (1+\pi)^{-1}(\bar{m} - \bar{q}) + \pi(1+\pi)^{-1}p_t^e + (1-\pi)^{-1}(u_t + v_t - \zeta_t)$
(6) Ans: $p_t = \alpha p_{t+1}^e + \beta m_t$ where $\alpha = \psi(1+\psi)^{-1}, \beta = (1+\psi)^{-1}$.

Chapter 2

Static Optimization and Economic Modeling

2.1 Unconstraint Optimization

Say we want to find the solutions of n choice variables ($\mathbf{x} = (x_1, \ldots, x_n)$)

$$\max_{\mathbf{x}} F(\mathbf{x})$$

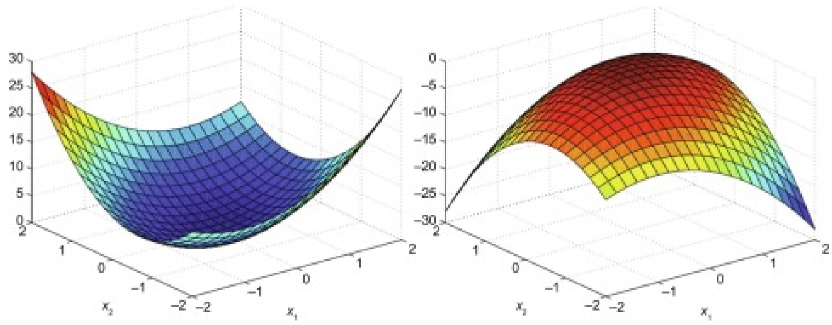


Fig. 2.1. Find the min (left) or max (right).

2.1.1 Necessary Conditions

This condition requires that the solution $\mathbf{x}^*$ must be a critical point of f, that is $f'(\mathbf{x}^*) = 0$. $\mathbf{x}^*$ will not be the endpoint of the interval under consideration, which means it lies in the INTERIOR of the domain of f.

Theorem 2.1.1. *Let $F : U \mapsto \mathbb{R}^1$ be a C^1 function defined on a subset U of $\mathbb{R}^n$. If $\mathbf{x}^*$ is a local max or min of F in U, and if $\mathbf{x}^*$ is an interior point of U, then*

$$\frac{\partial F}{\partial x_i}(\mathbf{x}^*) = 0 \text{ for } i = 1, \ldots, n$$

Basically, the FOCs to every choice variable must be 0. We can write the condition in the form of Jacobian

$$DF(\mathbf{x}^*) = \begin{pmatrix} \frac{\partial F}{\partial x_1}(\mathbf{x}^*) & \ldots & \frac{\partial F}{\partial x_n}(\mathbf{x}^*) \end{pmatrix} = \mathbf{0}$$

The solution $\mathbf{x}^* \neq 0$ for this problem is called a "non-trivial solution". Otherwise, if $\mathbf{x}^* = 0$, then it is called a "trivial" or corner solution, which is usually uninteresting in economics.

2.1.2 Sufficient Conditions

We need to use a condition on the second derivatives of F to determine whether the critical point is a max or a min. A C^2 function of n variables has n^2 second-order partial derivatives at each point in its domain. We combine them into a $n \times n$ matrix called the **Hessian** of F

$$D^2F(\mathbf{x}^*) = \begin{pmatrix} \frac{\partial^2 F}{\partial x_1^2}(\mathbf{x}^*) & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_n}(\mathbf{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial x_n \partial x_1}(\mathbf{x}^*) & \cdots & \frac{\partial^2 F}{\partial x_n^2}(\mathbf{x}^*) \end{pmatrix}$$

The Hessian is always a symmetric matrix. Whether the critical point is a min or max or neither depends on the definiteness of the Hessian matrix at that point.

Theorem 2.1.2. *Let $F : U \mapsto \mathbb{R}^1$ be a C^2 function whose domain is an open set U in $\mathbb{R}^n$. Suppose that $\mathbf{x}^*$ is a critical point of F, then*

1. *If the Hessian $D^2F(\mathbf{x}^*)$ is a NEGATIVE DEFINITE symmetric matrix, then $\mathbf{x}^*$ is a strict LOCAL MAX of F,*

2. *If the Hessian $D^2F(\mathbf{x}^*)$ is a POSITIVE DEFINITE symmetric matrix, then $\mathbf{x}^*$ is a strict LOCAL MIN of F,*

3. *If the Hessian $D^2F(\mathbf{x}^*)$ is INDEFINITE, then $\mathbf{x}^*$ is neither a local max nor a local min of F.*

In general, there are 2 methods to test for definiteness.

(1) The Signs of the Leading Minors

Theorem 2.1.3 (Sufficient Conditions for a MAX)**.** *Let $F : U \mapsto \mathbb{R}^1$ be a C^2 function whose domain is an open set U in $\mathbb{R}^1$. Suppose that*

$$\frac{\partial F}{\partial x_i}(\mathbf{x}^*) = 0 \textit{ for } i = 1, \ldots, n$$

and that the n leading principal minors of $D^2F(\mathbf{x}^*)$ alternate in sign

$$|F''_{x_1\ x_1}| < 0, \begin{vmatrix} F''_{x_1\ x_1} & F''_{x_2\ x_1} \\ F''_{x_1\ x_2} & F''_{x_2\ x_2} \end{vmatrix} > 0, \begin{vmatrix} F_{x_1\ x_1} & F_{x_2\ x_1} & F_{x_3\ x_1} \\ F_{x_1\ x_2} & F_{x_2\ x_2} & F_{x_3\ x_2} \\ F_{x_1\ x_3} & F_{x_2\ x_3} & F_{x_3\ x_3} \end{vmatrix} < 0, \ldots$$

at $\mathbf{x}^*$. Then $\mathbf{x}^*$ is a strict local max of F.

Theorem 2.1.4 (Sufficient Conditions for a MIN)**.** *Let $F : U \mapsto \mathbb{R}^1$ be a C^2 function whose domain is an open set U in $\mathbb{R}^1$. Suppose that*

$$\frac{\partial F}{\partial x_i}(\mathbf{x}^*) = 0 \textit{ for } i = 1, \ldots, n$$

and that the n leading principal minors of $D^2F(\mathbf{x}^*)$ <u>all positive</u>

$$|F_{x_1\ x_1}| > 0, \begin{vmatrix} F_{x_1\ x_1} & F_{x_2\ x_1} \\ F_{x_1\ x_2} & F_{x_2\ x_2} \end{vmatrix} > 0, \begin{vmatrix} F_{x_1\ x_1} & F_{x_2\ x_1} & F_{x_3\ x_1} \\ F_{x_1\ x_2} & F_{x_2\ x_2} & F_{x_3\ x_2} \\ F_{x_1\ x_3} & F_{x_2\ x_3} & F_{x_3\ x_3} \end{vmatrix} > 0, \ldots$$

at $\mathbf{x}^*$. Then $\mathbf{x}^*$ is a strict local min of F.

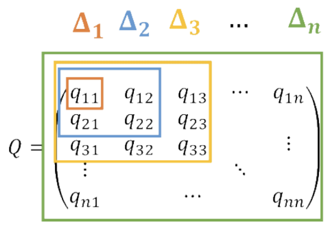


Fig. 2.2. Principal Minors

(2) The Signs of the Eigenvalues

Another way is to evaluate the Eigenvalues of the Hessian at critical points.

Theorem 2.1.5 (Eigenvalues Test for Sufficient Conditions)**.** *If the Hessian at a given point has* ***all positive eigenvalues****, it is said to be positive-definite, meaning the function is* ***concave up (convex)*** *at that point. If all the* ***eigenvalues are negative****, it is said to be a negative-definite, equivalent to* ***concave down****.*

To find the eigenvalues λ of a matrix $\mathbf{A}$, solve the following

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

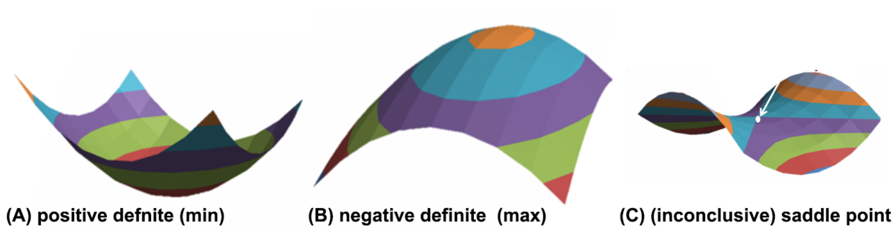


Fig. 2.3. The definiteness of the Hessian matrix

Intuitively, a negative definite Hessian matrix at the optimal point suggests that the objective function is concave in the vicinity of that point. This means that the function curves downward and resembles a bowl-like shape around the optimal point. In other words, if you move a little bit away from the optimal point in any direction, the function value will decrease. This curvature indicates that you are on the highest point in that particular region, and there is no other point nearby that can provide a higher value for the function. Mathematically, a matrix is negative definite if all its eigenvalues are negative. In the context of the Hessian matrix, the negative eigenvalues indicate that the curvature of the function in the corresponding directions is downward, which aligns with the idea of concavity.

2.1.3 Examples

Example 2.1 (Optimization)**.** Suppose

$$f(x,y) = x^4 + y^2 - xy$$

The critical point is found by

$$\begin{aligned}(x) : \frac{\partial f}{\partial x} &= 4x^3 - y = 0 \iff y = 4x^3, \\ (y) : \frac{\partial f}{\partial y} &= 2y - x = 0 \iff y = x/2.\end{aligned}$$

Solving for x yields the following critical points

$$(x^*, y^*) = (0,0),\ (\frac{1}{2\sqrt{2}}, \frac{1}{4\sqrt{2}}),\ (-\frac{1}{2\sqrt{2}}, -\frac{1}{4\sqrt{2}})$$

To verify the extremum, we evaluate the Hessian matrix at the critical points

$$H = \begin{pmatrix} 12x^2 & -1 \\ -1 & 2 \end{pmatrix}$$

The first principal minor is $12x^2$. The second principal minor is H itself where

$$|H| = 24x^2 - 1$$

Thus, we conclude that $(0,0)$ is a saddle point.

The other 2 critical points $(\frac{1}{2\sqrt{2}}, \frac{1}{4\sqrt{2}}),\ (-\frac{1}{2\sqrt{2}}, -\frac{1}{4\sqrt{2}})$ are minima.

Example 2.2. Consider the maximization problem

$$\max\ f(x) = -x^2 + 2ax + 4a^2$$

What is the effect of an increase in a on the maximal value of $f(x,a)$?

First, Find the critical point. Taking FOC yields $x^* = a$

1. (Direct Solution) Inserting to the objective function $f(a) = 5a^2$ and so the effect of a on f is $df/da = 10a$.

2. (Envelop Theorem)

$$f'(a) = \frac{\partial f}{\partial a} = 2x + 8a$$

evaluate at $x = a$ also yields $10a$.

2.1.4 Economic Applications

Ex. 2.1. Find the optimal solution for

$$\begin{aligned}
&(a)\ \min_x x^2 - 4x + 7,\\
&(b)\ \max_x \ln(x+1) \text{ for } x \geq 0\\
&(c)\ \max_x x^3 - 6x^2 + 9x \text{ for } x \in [-1, 4]\\
&(d)\ \max_x [800x - 2x^2] - (100 + 150x)
\end{aligned}$$

Ex. 2.2. Find the eigenvalues and evaluate them at given points, and determine whether the matrix is negative-definite, positive-definite, or indefinite.

$$\begin{aligned}
&(a) \begin{pmatrix} 12x^2 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{at } (3,1),\\
&(b) \begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix} \quad \text{at } (-1,2),\\
&(c) \begin{pmatrix} -2y^2 & -4xy \\ -4xy & -2x^2 \end{pmatrix} \quad \text{at } (1,-1) \quad \text{and} \quad (1,0)
\end{aligned}$$

Ex. 2.3. Find the optimal solutions for

$$\begin{aligned}
&(a)\ \max_{x,y} f(x,y) = -2x^2 - 3y^2 + 4xy\\
&(b)\ \max_{x,y} f(x,y) = -x^3 - 2y^3 + 3xy\\
&(c)\ \min_{x,y,z} f(x,y,z) = x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z.
\end{aligned}$$

Verify with SOC condition.

Ex. 2.4. Endogenous Fertility (de la Croix, 2012) p.12.

$$\max_{s_t, n_t} \ln(w(1 - \phi n_t) - s_t) + \beta \ln((1+r)s_t) + \gamma \ln(n_t)$$

where n is fertility decision, ϕ is the time-cost of raising children, s, w, r are savings, wage rate, and interest rate.

1. Find the optimal fertility decision.
2. Add a good cost of childcare per child (say $\theta > 0$), will it change the fertility decision?

Ex. 2.5. Endogenous Fertility with Bequest (de la Croix, 2012) p.189.

$$\max_{l_t, n_t, b_t} \ln((1 - l_t - \phi N_t^{\sigma} n_t)k_t - n_t b_t) + \varphi \ln(l_t) + \gamma \ln(n_t k_{t+1})$$

where l_t, n_t, b_t are leisure, fertility, and educational bequests. The idea is that population size asserts a negative externality on having children.

Productive assets accumulate according to

$$k_{t+1} = \mu b_t^{\eta} k_t^{\tau}$$

1. Write the first-order conditions.

2. Find the optimal values l^*, n^*, b^*.

Ex. 2.6. Principal-Agent problem (Varian, 1992), p.453
The Agent's problem is

$$\max_a \delta + \gamma a - \frac{\gamma^2 r}{2}\sigma^2 - c(a)$$

where a is the agent's effort. The Principal's problem is

$$\max_{\delta,\gamma,a}(1-\gamma)a - \delta$$

where $\delta + \gamma a - \frac{\gamma^2 r}{2}\sigma^2 - c(a) = 0$. Let $c(a)$ be a convex function.

1. Solve the Agent's problem to obtain a

2. For the Principal's problem, first, extract δ as a function of γ, a. Then, replace it back to the objective function, use the results from 1. and solve for a.

3. Assume $c(a) = 0.5a^2$. Derive the explicit solution.

Ex. 2.7 (Maximum Likelihood Estimation (MLE))**.** Application in Statistical Inference.

1. (1 parameter) Suppose a sample $x_1, \dots, x_n$ is modeled by a **Poisson** distribution with parameter denoted by λ, so that

$$f_X(x|\lambda) = \frac{\lambda^x}{x!}e^{-\lambda}$$

for some $\lambda > 0$. Estimate λ by MLE.

2. The **Gaussian** probability density function of a normally distributed i.i.d $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2\right]$$

In this problem, the probability density function is characterized by 2 parameters σ and μ. Use MLE to find them.

Hint: steps to estimate parameters θ using MLE.

1. Write the likelihood function

$$L_n(\theta) = \prod_{i=1}^n f(X_i|\theta)$$

2. Take logs to obtain log-likelihood function

$$\ell_n(\theta) = \log L_n(\theta) = \log(f(X_1|\theta)) + \cdots + \log(f(X_n|\theta)) = \sum_{i=1}^n \log f(X_i|\theta)$$

3. Find the estimator that maximizes this function

$$\hat{\theta} = \arg\max \ell_n(\theta)$$

2.2 Constraint Optimization

Let us consider an optimization problem for n variables with k constraints s.t.

$$\max_{\mathbf{x}}(\min) f\underbrace{(x_1,\ldots,x_n)}_{\mathbf{x}}$$
$$s.t.\ h_i(\mathbf{x}) = c_i \quad \text{for} \quad i = 1,\ldots,k.$$

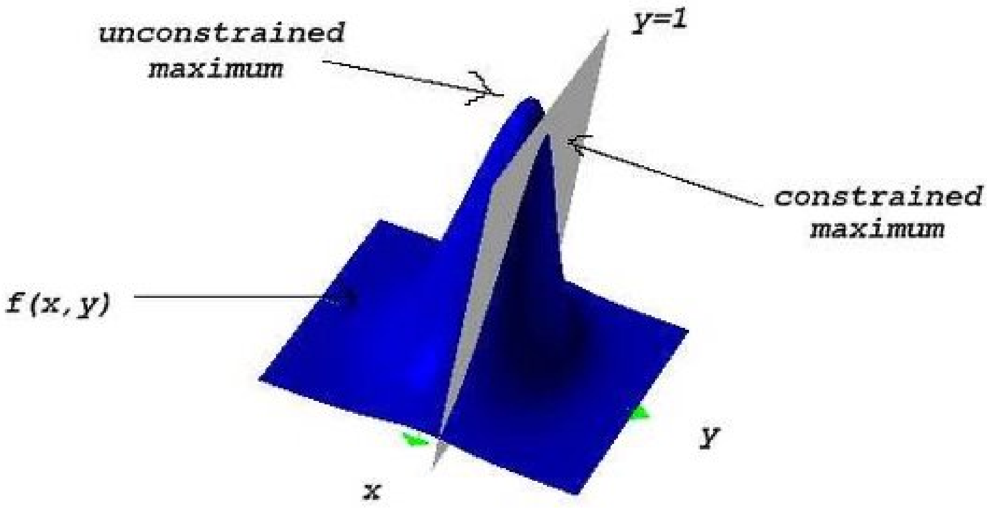


Fig. 2.4. An example of Constraint Optimization

2.2.1 Necessary First-order Conditions

Assume that NDQC is satisfied. There will be k Lagrangian multipliers λ_i for $i = 1,\ldots,k$. The Lagrangian is

$$\mathcal{L}(\mathbf{x},\lambda_i) = f(\mathbf{x}) - \sum_{i=1}^{k} \lambda_i(h_i(\mathbf{x}) - c_i)$$

As usual, the FOC is just

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \quad \text{for } i = 1,\ldots,n$$

Let us show what it means in a problem of 1 function of 2 variables.

Theorem 2.2.1. *Let f and h be C^1 functions of 2 variables. Suppose that $\mathbf{x}^* = (x_1^*, x_2^*)$ is a solution of the problem*

$$\max f(x_1,x_2)$$
$$s.t.\ h(x_1,x_2) = c$$

Suppose further that (x_1^*, x_2^*) is not a critical point of h. Then, there is a real number λ^* such that $(x_1^*, x_2^*, \lambda^*)$ is a critical point of the Lagrangian function

$$\mathcal{L}(x_1,x_2,\lambda) \equiv f(x_1,x_2) - \lambda[h(x_1,x_2) - c].$$

In other words, at $(x_1^*, x_2^*, \lambda^*)$ we can obtain the First-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0.$$

or

$$\nabla \mathcal{L}(x_1, x_2, \lambda) = 0$$

Example 2.3. The problem is

$$\max_{x,y} \ f(x,y) = xy$$
$$s.t. \ 2x + y = 100$$

Method 1: Lagrangian Method
The Lagrangian is

$$\mathcal{L} = xy - \lambda(2x + y - 100)$$

The FOCs:

$$\begin{aligned} \mathcal{L}'_x &= y - 2\lambda = 0, \\ \mathcal{L}'_y &= x - \lambda = 0, \\ \mathcal{L}'_\lambda &= 2x + y - 100 = 0, \end{aligned}$$

which yields the solution $(x^*, y^*) = (25, 50)$.

Method 2: Substitution Method or "Naive" Method
We can turn the constrained optimization problem into an unconstrained problem. From the constraint, we have $y = 100 - x$, the problem becomes

$$\max_x x(100 - 2x)$$

The FOC is

$$100 - 4x = 0,$$

which also gives $x^* = 25, y^* = 50$.

2.2.2 Sufficient Conditions

For sufficient conditions, we need to use the notion of Bordered Hessian Matrix. The construction of such a matrix is

$$H = \left(\begin{array}{ccc|ccc} 0 & \dots & 0 & B_{11} & \dots & B_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & B_{m1} & \dots & B_{mn} \\ \hline B_{11} & \dots & B_{m1} & a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{1n} & \dots & B_{mn} & a_{1n} & \dots & a_{nn} \end{array}\right)$$

in short, it looks like this

$$H = \begin{pmatrix} 0 & \mathbf{B} \\ \mathbf{B}^T & \text{Ł} \end{pmatrix}$$

where $\mathbf{B}$ is the matrix of derivatives of constraints h_i wrt to $\mathbf{x}$, and L is the matrix of second-order derivatives of the Lagrangian $\mathcal{L}$ wrt to $\mathbf{x}$.

In our case, it looks like this

$$H = \left(\begin{array}{ccc|ccc} 0 & \dots & 0 & \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{\partial h_k}{\partial x_1} & \dots & \frac{\partial h_k}{\partial x_n} \\ \hline \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_k}{\partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \dots & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial x_n} & \dots & \frac{\partial h_k}{\partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_n \partial x_1} & \dots & \frac{\partial^2 \mathcal{L}}{\partial x_n^2} \end{array}\right)$$

This $(k+n) \times (k+n)$ matrix has $k+n$ leading principal minors (the biggest one is H itself). The first m matrices $H_1, \dots, H_k$ are zero matrices. The next $k-1$ matrices $H_{k+1}, \dots, H_{2k-1}$ have zero determinant.

The determinant of the next minor H_{2k} is $\pm \det(H')^2$ where H' is the upper $k \times k$ minor of H after block of zeros, so $\det H_{2k}$ does not contain information about f.

And only the determinants of the last $n-k$ leading principal minors

$$H_{2k+1}, H_{2k+2}, \dots, H_{2k+(n-k)=k+n} \equiv H$$

carry information about both, the objective function f and the constraints h_i. Exactly these minors are essential for the following sufficient condition for constraint optimization.

Theorem 2.2.2 (Constraint SOC)**.** *Suppose* $\mathbf{x}^*$ *satisfies the FOCs.*

1. *For the bordered Hessian matrix H, the last $n-k$ leading principal minors*

$$H_{2k+1}, H_{2k+2}, \dots, H_{2k+(n-k)=k+n} \equiv H$$

   evaluated at the critical point ALTERNATE in signs where the last minor $H_{n+k} \equiv H$ has the sign as $(-1)^n$, then $\mathbf{x}^*$ is a LOCAL MAX.

2. *For the bordered Hessian matrix H, the last $n-k$ leading principal minors*

$$H_{2k+1}, H_{2k+2}, \dots, H_{2k+(n-k)=k+n} \equiv H$$

   evaluated at the critical point have the SAME sign where the last minor $H_{n+k} \equiv H$ has the sign as $(-1)^k$, then $\mathbf{x}^*$ is a LOCAL MIN.

which can be summarized as

	H_{2k+1}	H_{2k+2}	$\dots$	H_{k+n-1}	$H_{k+n} \equiv H$
max	$(-1)^{k+1}$	$(-1)^{k+2}$	$\dots$	$(-1)^{n-1}$	$(-1)^n$
min	$(-1)^k$	$(-1)^k$	$\dots$	$(-1)^k$	$(-1)^k$

We provide here only the sufficient conditions for a problem of **2 variables and 1 constraint**, which is the most common.

Theorem 2.2.3. *Let f, h be C^2 functions on $\mathbb{R}^2$. Consider the problem*

$$\max_{x,y} f(x,y) \quad \textbf{\textit{s.t.}} \;\; h(x,y) = c \;\; \text{for } c \in C_h \text{(constraint set)}$$

The Lagrangian is

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(h(x,y) - c)$$

Suppose that (x^*, y^*, λ^*) satisfies the following FOCs

$$\mathcal{L}'_x = \mathcal{L}'_y = \mathcal{L}'_\lambda = 0 \quad \text{at } (x^*, y^*, \lambda^*)$$

and the bordered Hessian matrix is

$$H = \begin{pmatrix} 0 & \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial^2 \mathcal{L}}{\partial x^2} & \frac{\partial^2 \mathcal{L}}{\partial x \partial y} \\ \frac{\partial h}{\partial y} & \frac{\partial^2 \mathcal{L}}{\partial y \partial x} & \frac{\partial^2 \mathcal{L}}{\partial y^2} \end{pmatrix}$$

1. *if* $\det(H) > 0$ *at* (x^*, y^*), *then* (x^*, y^*) *is the local MAX of* f *on* C_h.
2. *if* $\det(H) < 0$ *at* (x^*, y^*), *then* (x^*, y^*) *is the local MIN of* f *on* C_h.

2.2.3 Examples

Example 2.4 (1 objective function of 2 variables and 1 constraint)**.** Find the extremum of

$$\begin{aligned} F(x,y) &= xy \\ s.t. \; h(x,y) &= x + y = 6. \end{aligned}$$

The Lagrangian is

$$L(x,y) = xy - \lambda(x + y - 6)$$

The FOCs are

$$\begin{aligned} (x) &: \frac{\partial L}{\partial x} = y - \lambda = 0, \\ (y) &: \frac{\partial L}{\partial y} = x - \lambda = 0, \\ (\lambda) &: \frac{\partial L}{\partial \lambda} = x + y - 6 = 0, \end{aligned}$$

which gives

$$x^* = y^* = 3, \;\; \lambda = 3$$

To tell the nature of its extremum, we test the second-order conditions. The bordered Hessian is

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

We have $n = 2, k = 1$ so we have to check the $n - k = 1$ last leading principal minors, a.k.a, H itself. Calculation shows that $\det H = 2 > 0$ has the same sign with $(-1)^n = (-1)^2 > 0$ so our critical point is a MAX.

Example 2.5 (1 objective function of 2 variables and 2 constraint)**.** Find the extremum of

$$\begin{aligned} F(x,y,z) &= x^2+y^2+z^2 \\ s.t.\ h_1(x,y,z) &= 3x+y+z=5, \\ h_2(x,y,z) &= x+y+z=1 \end{aligned}$$

The Lagrangian is

$$L(x,y) = x^2+y^2+z^2 - \lambda_1(3x+y+z-5) - \lambda_2(x+y+z-1)$$

The FOCs are

$$\begin{aligned} (x) &: \frac{\partial L}{\partial x} = 2x - 3\lambda_1 - \lambda_2 = 0, \\ (y) &: \frac{\partial L}{\partial y} = 2y - \lambda_1 - \lambda_2 = 0, \\ (z) &: \frac{\partial L}{\partial z} = 2z - \lambda_1 - \lambda_2 = 0, \\ (\lambda_1) &: \frac{\partial L}{\partial \lambda_1} = 3x+y+z-5 = 0, \\ (\lambda_2) &: \frac{\partial L}{\partial \lambda_2} = x+y+z-1 = 0 \end{aligned}$$

which gives

$$x^* = 2, \quad y^* = -1/2, \quad z^* = -1/2, \quad \lambda_1 = 5/2, \quad \lambda_2 = -7/2$$

To tell the nature of its extremum, we test the second-order conditions. The bordered Hessian is

$$H = \left(\begin{array}{cc|ccc} 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ \hline 3 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{array}\right)$$

We have $n = 3, k = 2$ so we have to check the $n - k = 1$ last leading principal minors, a.k.a, H itself. Calculation shows that $\det H = 16 > 0$ has the same sign with $(-1)^k = (-1)^2 > 0$ so our critical point is a MIN.

2.2.4 Economic Applications

Ex. 2.8. Find the extremum, then verify it is either max or min.

$(a)\ f(x,y) = 7 - y - x^2$	s.t	$h(x,y) = x + y = 0;$
$(b)\ f(x,y) = x(y+4)$	s.t	$h(x,y) = x + y = 8;$
$(c)\ x_1^2 + x_2^2 + x_3^2$	s.t	$x_1 + x_2 + x_3 = 1;$
$(d)\ yz + xz$	s.t	$y^2 + z^2 = 1,\ xz = 3.$

Ex. 2.9 (Beckerian trade-off)**.** A parent's problem is

$$\begin{aligned} \max_{c_t,d_{t+1},n_t,e_t} \quad & \ln(c_t) + \beta\ln(d_{t+1}) + \gamma\ln(\Pi_t n_t) \\ \text{s.t. } & c_t + s_t + e_t n_t = (1-\phi n_t)w_t, \\ & d_{t+1} = (1+r_{t+1})s_t, \\ & \Pi_t = (\theta + e_t)^\eta \end{aligned}$$

where c, d, s, n, e are consumption when young, consumption when old, saving, number of children, and children's education. The parameters $\beta, \gamma, \phi, \theta, \eta \in (0,1)$. This problem is a simplified version of the model in de la Croix (2012), p.22.

1. Find the optimal solutions.
2. (if time allows) verify by constructing a Hessian (you should use the Naive method)
3. Is there a trade-off between children's quantity and quality?

Ex. 2.10 (Renewable Resources)**.** Section 9.2 of Farmer and Bednar-Friedl (2010) (p.120). A country has a stock of renewable resource R_t such that

$$R_{t+1} = R_t + g(R_t) - X_t$$

where $g(R_t)$ is the rate of regeneration, while X_t is harvested stock (think of fish). We can assume a simple regenerate form

$$g(R_t) = \delta R_t - \gamma R_t^2 \text{ where } \delta > 1, \gamma < 1$$

Household's budget constraint when young is

$$c_t + k_{t+1} + p_t R_t = q_t X_t + w_t$$

LHS: expenses, including hoarding renewable resources. LHS: harvest then sell + wage. When old, his constraint is

$$d_{t+1} = (1+r)k_{t+1} + p_{t+1}R_{t+1}$$

Utility function is $\ln(c_t) + \beta\ln(d_{t+1})$. Household's choice variables are c_t, d_{t+1}, X_t, R_t. Find the optimal solutions by forming the Lagrangian and take the FOC wrt all the choice variables.

Ex. 2.11 (New Technology)**.** Without technology, a country solves

$$\begin{aligned} \max \log(c_0) &+ \beta\log(c_1), \\ s.t. c_0 + k_1 &= f(k_0), \\ c_1 &= f(k_1) \end{aligned}$$

If she invests in new technology, she solves

$$\begin{aligned} \max \log(c_0) &+ \beta\log(c_1), \\ s.t. c_0 + s_0 &= f(k_0), \\ s_0 &= k_1 + \lambda k_e \\ c_1 &= h(k_e)f(k_1). \end{aligned}$$

Equation (2) means capital is used to save and make New Tech.

Let us assume $f(k) = \gamma k$, $h(x) = ax + 1$.

1. Under which condition does the country invest in new technology?
2. Under what condition, investing in the New Technology is better?

2.3 Constraint Inequality Optimization

2.3.1 KKT First-order Conditions for MAX

In this branch of problems, the constraint has inequality signs.

$$\max f(x,y) \quad s.t. \quad g(x,y) \leq c.$$

We solve this problem by employing the cookbook method called KKT conditions (Karush-Kuhn-Tucker).

Theorem 2.3.1 (The KKT Conditions for MAX)**.** *Suppose we have 2 choice variables and 1 inequality constraint.*

$$\max f(x,y) \ s.t. \ g(x,y) \leq c$$

1. *Construct the Lagrangian*

$$\mathcal{L}(x,y) = f(x,y) - \lambda(g(x,y) - c)$$

2. *FOCs*

$$\begin{aligned}
&\mathcal{L}'_x = f'_x - \lambda g'_x = 0, \\
&\mathcal{L}'_y = f'_y - \lambda g'_y = 0, \\
&\lambda \cdot (g(x,y) - c) = 0, \\
&\lambda \geq 0, \\
&g(x,y) \leq c
\end{aligned}$$

3. *Complimentary slackness condition*

$$\begin{aligned}
&\lambda > 0, \ \textit{the constraint binds so that } g(x,y) = c \\
&\lambda = 0, \ \textit{the constraint does not bind so that } g(x,y) < c
\end{aligned}$$

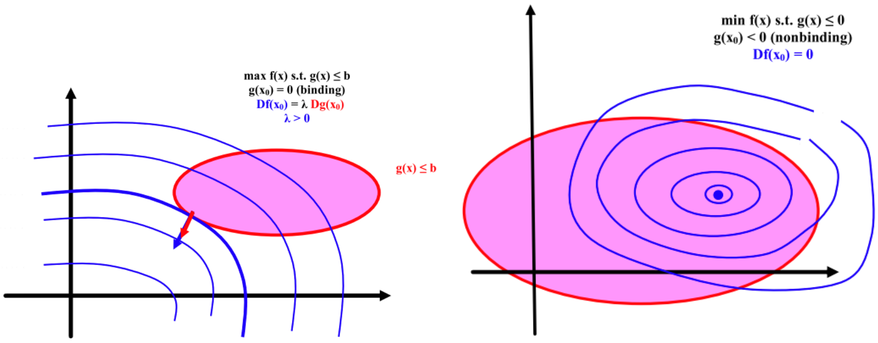
4. *For a minimum problem, the FOCs are the same, except that $\lambda \leq 0$.*

The two inequalities $\lambda \geq 0$ and $g(x,y) \leq c$ are complementary in the sense that at most one can be "slack" – that is, at most one can hold with inequality. Equivalently, at least one must be an equality. Failure to observe that it is possible to have both $\lambda = 0$ and $g(x,y) = c$ in the complementary slackness condition is the most common error when solving nonlinear programming problems.

2.3.2 KKT First-order Conditions for MIN

For a minimization problem, you have 3 options

1. **Flip the sign of the objective function**, then we will turn a Minimization problem into a Maximization problem, and its FOCs follow suit.
2. Keep the constraints as is (where all constraints are $\leq$), and the FOCs are the same as the MAXIMIZATION problem **except that** $\lambda \leq 0$.
3. Flip the **signs of the constraints so that they have the form** $\geq$, then the FOCs are the same as the MAXIMIZATION problem where $\lambda \geq 0$.



(a) Binding constraint **(b)** Non-binding constraint

Fig. 2.5. Constraint binding and non-binding cases

It is easier to follow the last option.

Example 2.6.

$$\begin{aligned} \min f(x,y) &= 2y - x^2 \\ s.t.\ x^2 + y^2 &\leq 1 \end{aligned}$$

Rewrite the problem as

$$\begin{aligned} \min f(x,y) &= 2y - x^2 \\ s.t.\ \ -x^2 - y^2 &\geq -1 \end{aligned}$$

The Lagrangian is

$$L(x,y,\lambda) = 2y - x^2 - \lambda(-x^2 - y^2 + 1)$$

FOCs:

$$\begin{aligned} &(i)\ \frac{\partial L}{\partial x} = 0 \iff -2x + 2\lambda x = 0 \\ &(ii)\ \frac{\partial L}{\partial y} = 0 \iff 2 + 2\lambda y = 0 \\ &(iii)\ \lambda \cdot (-x^2 - y^2 + 1) = 0, \\ &(iv)\ \lambda \geq 0 \text{ (if } \lambda > 0 \text{, constraint binds).} \end{aligned}$$

From $(i), (ii)$, we can derive $\lambda = 1, y = -1$. Since $\lambda > 0$, the constraint binds and we have $x^2 + y^2 = 1$. Since $y = -1$, we have $x = 0$, as the optimum.

2.3.3 Multiple Inequality Constraints

Consider an optimization problem of n choice variables and m inequality constraints

$$\begin{aligned} &\max f\underbrace{(x_1, \ldots, x_n)}_{\mathbf{x}} \qquad (2.1) \\ &s.t. g_1(\mathbf{x}) \leq c_1, \\ &\quad \ldots, \\ &\quad g_m(\mathbf{x}) \leq c_m \end{aligned}$$

Theorem 2.3.2 (KKT Formulation)**.** *Steps in solving the problem* (2.1)

1. *Write down the Lagrangian*

$$\mathcal{L}(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j (g_j(\mathbf{x}) - c_j)$$

2. *FOCs:*

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0,$$

   for each $i = 1, \ldots, n$.

3. *Complementary slackness*

$$\lambda_j \geq 0, g_j(\mathbf{x}) = c_j \;\; \textbf{\textit{or}} \;\; \lambda_j = 0, g_j(\mathbf{x}) < c_j$$

   for $j = 1, \ldots, m$. *Can also be summarized as*

$$\lambda_j \cdot g_j(\mathbf{x}) = 0.$$

4. *Find all* $\mathbf{x} = (x_1, \ldots, x_n)$ *associated with their* $\lambda_1, \ldots, \lambda_m$ *that satisfy FOCs and the complementary slackness. These are the solution candidates, and at least 1 of them solves the problem if it has a solution.*

Example 2.7. The problem is

$$\max\; x + 3y - 4e^{-x-y}$$
$$s.t. \begin{cases} 2 - x \geq 2y \\ x - 1 \leq -y \end{cases}$$

Write the problem as

$$\max\; x + 3y - 4e^{-x-y}$$
$$s.t. \begin{cases} x + 2y \leq 2 \\ x - 1 \leq -y \end{cases}$$

The Lagrangian is

$$\mathcal{L}(x, y) = x + 3y - 4e^{-x-y} - \lambda_1(x + 2y - 2) - \lambda_2(x + y - 1)$$

KKT conditions

$$\begin{aligned}
&(i)\; \mathcal{L}'_x = 1 + 4e^{-x-y} - \lambda_1 - \lambda_2 = 0, \\
&(ii)\; \mathcal{L}'_y = 3 + 4e^{-x-y} - 2\lambda_1 - \lambda_2 = 0, \\
&\lambda_1 \cdot (x + 2y - 2) = 0, \\
&\lambda_2 \cdot (x - 1 + y) = 0, \\
&(iii)\; \lambda_1 \geq 0\; (\lambda_1 = 0 \iff x + 2y < 2), \\
&(iv)\; \lambda_2 \geq 0\; (\lambda_2 = 0 \iff x + y < 1)
\end{aligned}$$

Since $\mathcal{L}$ is concave, the KKT conditions are both necessary and sufficient for optimality.

From (ii), (i), we get $\lambda_1 = 2 > 0$, thus making (iii) binds such that $x + 2y = 2$. Suppose $\lambda_2 = 0$, there is a contradiction. Suppose $\lambda_2 > 0$, from (iv) we deduce $x+y = 1$. Using (i) and (ii), we can find $\lambda_2 = e^{-1}(4-e) > 0$. Thus, the solution is

$$(x^*, y^*, \lambda_1, \lambda_2) = (0,\ 1, 2\ ,\ e^{-1}.(4-e))$$

2.3.4 Nonnegativity Constraints

Most oftentimes, in economics, we want to restrict the choice variables to take nonnegative values.

Theorem 2.3.3 (Reduced KKT conditions for Nonnegativity)**.** *Consider the problem*

$$\begin{aligned} \max\ & f(x,y) \\ s.t.\ & g(x,y) \le c, \\ & x \ge 0, \\ & y \ge 0 \end{aligned}$$

Rewrite the problem to

$$\begin{aligned} \max & f(x,y) \\ s.t. & g(x,y) \le c, \\ & -x \le 0, \\ & -y \le 0 \end{aligned}$$

The Lagrangian is [a]

$$\mathcal{L}(x,y) = f(x,y) - \lambda[g(x,y) - c] - \mu_1(-x) - \mu_1(-y)$$

The KKT conditions

$$\begin{aligned} &(i)\ \mathcal{L}'_x = f'_x - \lambda g'_x + \mu_1 = 0 \\ &(ii)\ \mathcal{L}'_y = f'_y - \lambda g'_y + \mu_2 = 0 \\ &(iii)\ \lambda \ge 0,\ \textit{with}\ \lambda = 0\ \ \textit{if}\ \ g(x,y) < c \\ &(iv)\ \mu_1 \ge 0,\ \textit{with}\ \mu_1 = 0\ \ \textit{if}\ \ x > 0 \\ &(v)\ \mu_2 \ge 0,\ \textit{with}\ \mu_2 = 0\ \ \textit{if}\ \ y > 0 \end{aligned}$$

Combining (i) and (iv) yields

$$f'_x - \lambda g'_x \le 0,\ \textbf{\textit{with equality if}}\ x > 0$$

Combining (ii) and (v) yields

$$f'_y - \lambda g'_y \le 0,\ \textbf{\textit{with equality if}}\ y > 0$$

So the KKT conditions are reduced to just

$$\begin{aligned}
&f'_x - \lambda g'_x \leq 0, \;\; \textbf{\textit{with equality if}}\; x > 0 \\
&f'_y - \lambda g'_y \leq 0, \;\; \textbf{\textit{with equality if}}\; y > 0 \\
&\lambda \geq 0, \;\; \textit{with}\; \lambda = 0 \;\; \textit{if} \;\; g(x,y) < c
\end{aligned}$$

[a]tips: You should denote the Lagrangian multipliers for the main constraint by λ and nonnegativity constraints by μ for easier handling.

2.3.5 Examples

Example 2.8. Consider the utility maximization problem where there are 2 goods x, y, price of good x is p and price of good y is normalized to 1, the budget is m. Find the optimal x, y.

$$\begin{aligned}
\max\; & x + \ln(1+y) \\
s.t.\; & px + y \leq m, \\
& x \geq 0 \\
& y \geq 0.
\end{aligned}$$

Solutions: The Lagrangian is

$$L = x + \ln(1+y) - \lambda(px + y - m)$$

Assume the solution (x^*, y^*) exists, it must satisfy the following KKT conditions

$$\begin{aligned}
&(i) && L'_x = 1 - p\lambda \leq 0, \;\textbf{with}\; 1 - p\lambda = 0 \iff x^* > 0, \\
&(ii) && L'_y = \frac{1}{1+y^*} - \lambda \leq 0 \;\textbf{with}\; \frac{1}{1+y^*} = 0 \iff y^* > 0 \\
&(iii) && \lambda \cdot (px^* + y^* - m) = 0, \\
&(iv) && \lambda \geq 0, px^* + y^* \leq m
\end{aligned}$$

The objective function is concave in (x, y), the constraint is linear, thus the Lagrangian is concave, so the FOC is also sufficient.

Observe from condition (i) that λ cannot be zero, then condition (iii) implies that $\lambda > 0$ and the constraint binds such that

$$(iv)\; px^* + y^* = m$$

Regarding which constraints $x \geq 0, y \geq 0$ bind, we need to consider 4 cases

1. $x^* = 0, y^* = 0$: Since $m > 0$, this is impossible.
2. $x^* > 0, y^* = 0$: From (ii), we get $\lambda \geq 1$, then (i) implies that

   $$p = \frac{1}{\lambda} \leq 1$$

   Then from (iv), we have

   $$\begin{aligned}
   x^* &= m/p, \\
   \lambda &= 1/p
   \end{aligned}$$

   if $0 < p < 1$.

3. $x^* = 0, y^* > 0$: By (iv), we have

$$y^* = m$$

Then (ii) yields

$$\lambda = \frac{1}{1+y^*} = \frac{1}{1+m}$$

Then from (i) we get the condition for this is that

$$p \geq m+1$$

4. $x^* > 0, y^* > 0$: With equality in both (i) and (ii), we have

$$\lambda = 1/p = 1/(1+y^*)$$

It follows that

$$\begin{aligned} &y^* = p-1, \\ &p > 1 \text{ (because } y^* > 0) \end{aligned}$$

Equation (iv) yields

$$\begin{aligned} &x^* = \frac{m+1-p}{p}, \\ &p < m+1 \text{ (because } x^* > 0) \end{aligned}$$

In summary

1. If $0 < p \leq 1$, then $(x^*, y^*) = (m/p, 0)$ with $\lambda = 1/p$
2. if $1 < p < m+1$, then $(x^*, y^*) = (\frac{m+1-p}{p}, p-1)$ with $\lambda = 1/p$
3. if $p \geq m+1$, then $(x^*, y^*) = (0, m)$ with $\lambda = 1/(1+m)$

In the 2 extreme cases (1) and (3), it is optimal to spend everything on only the cheaper good – x in case (1) and y in case (3).

2.3.6 Economic Applications

Ex. 2.12. Solve the problem

$$\begin{aligned} \max \ & f(x,y) = x^2 + y^2 + y - 1, \\ s.t. \ & g(x,y) = x^2 + y^2 \leq 1 \end{aligned}$$

Ex. 2.13 (Cost Function)**.** From Varian (1992), p.54–58.

1. Minimizing the cost function for the Cobb-Douglas technology

$$\begin{aligned} &\min_{x_1,x_2} c(\mathbf{w}, y) := w_1x_1 + w_2x_2, \\ &\text{s.t. } Ax_1^{\alpha}x_2^{\beta} = y. \end{aligned}$$

Derive the optimal demand for x_1, x_2. Let $A = 1, \alpha+\beta = 1$, find the cost function.