

SpringCamp 2024
Computational Macro

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Section 1: Two-period OLG (Diamond)

Motivation

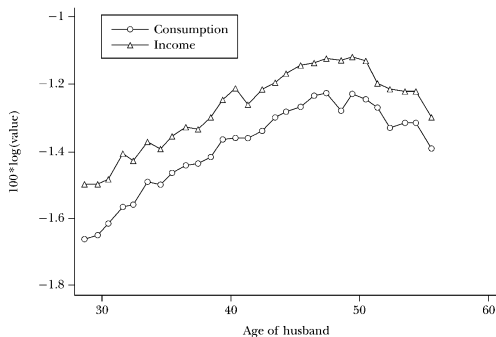
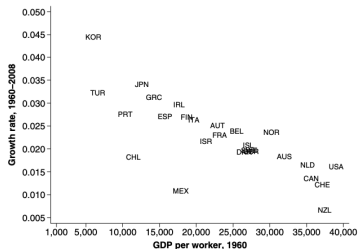


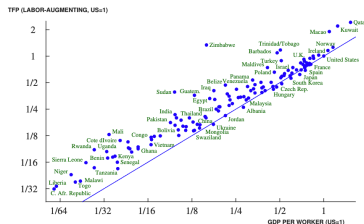
Figure: Life-cycle pattern of income and consumption (Browning and Crossley, 2001).

▷ People save when young and dissave when old.

Motivation



(a) Growth vs initial GDPpc



(b) TFP and GDPpw

Figure: Some facts on growth (Jones and Vollrath, 2013)

▷ Convergence theory (at least for OECD): if you start small, you grow fast.

Model

To do:

- ① Modeling Household
- ② Modeling Production sector.
- ③ State the equilibrium
- ④ See long-run dynamics of the model

In step 4, we calibrate and write a code to solve the model numerically.

Household

Preferences

$$U(c_t, d_{t+1}) = u(c_t) + \beta u(d_{t+1}), \quad (1)$$

subject to constraints

$$\begin{aligned} c_t + s_t &= w_t, \\ d_{t+1} &= R_{t+1}s_t. \end{aligned}$$

Possible functional forms

$$u(c) = \begin{cases} \frac{c^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}} & \text{if } \sigma > 0, \sigma \neq 1 \text{ (CIES) ,} \\ \frac{c^{1-\sigma} - 1}{1 - \sigma} & \text{if } \sigma \geq 0, \sigma \neq 1 \text{ (CRRA) ,} \\ \ln(c) & \text{if } \sigma = 1. \end{cases}$$

- 1 Solve for s_t using a Lagrangian.

Production

Technology

$$Y = AF(K_t, L_t)$$

Possible functional form

$$F(K, L) = \begin{cases} (\alpha K_t^{-\rho} + (1 - \alpha)L_t^{-\rho})^{-1/\rho} & \text{CES with } \rho \in (-1, +\infty), \rho \neq 0 \\ K_t^\alpha L_t^{1-\alpha} & \text{Cobb-Douglas } (\rho \rightarrow 0) \end{cases}$$

Profit:

$$\Pi_t = Y_t - w_t L_t - R_t K_t$$

- 1 Define the capital-labor ratio $k = K/L$, and solve for the optimal L, K .

Equilibrium

Labor market

$$L_t = N_t$$

Assuming full depreciation. The capital market clears

$$K_{t+1} = s_t N_t$$

Goods market clears

$$Y_t = N_{t-1} d_t + N_t (c_t + s_t)$$

- ① With Cobb-Douglas technology and log utility:

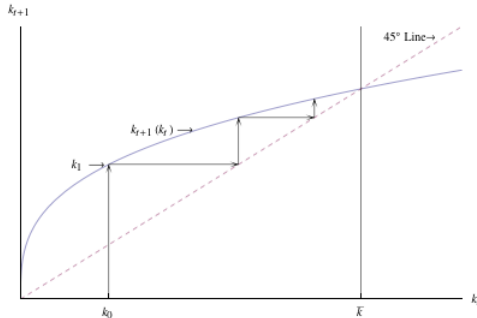
$$k_{t+1} = \phi(k_t) = \frac{\beta A(1 - \alpha)}{(1 + n)(1 + \beta)} k_t^\alpha. \quad (2)$$

- ② With CES technology ($\rho < 0$) and log utility:

$$k_{t+1} = \phi(k_t) = \frac{\beta A(1 - \alpha)(\alpha k_t^{-\rho} + 1 - \alpha)^{-(1+\rho)/\rho}}{(1 + n)(1 + \beta)}. \quad (3)$$

Steady State

Find the steady state of the capital-labor ratio k^* by solving $k_{t+1} = k_t = k^*$.



One can be found by hand, the other cannot.

Table: Calibration

β	α	ρ	A	n
0.99 ³⁰	0.3	-1.5	10	0.3

Section 2: Numerical techniques

The idea

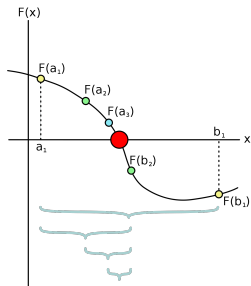
Rewrite the dynamics to rewrite it to

$$k - \frac{\beta A(1 - \alpha)}{(1 + n)(1 + \beta)} k^\alpha = 0. \quad (4)$$

Solving for k is the same as finding the root of this equation.

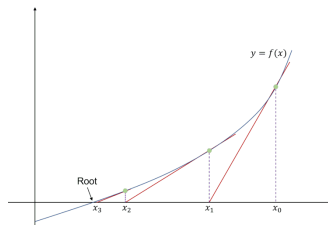
- ① Bisection
- ② Newton-Raphson
- ③ Secant Method
- ④ Gauss-Seidel
- ⑤ Let Python do its thing.

Bisection



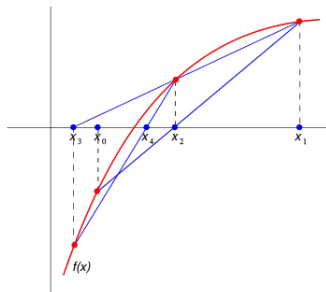
- ① Choose a and b such that $f(a) \times f(b) < 0$.
- ② Set $i = 1$.
- ③ Calculate $f(a)$.
- ④ Choose a point c where $c = a + \frac{b-a}{2}$
- ⑤ Calculate $f(c)$ and evaluate its sign.
 - ① if $|f(c)| < tol$, end the program and output c .
 - ② Else: if $f(c) \cdot f(a) > 0$, assign: $a = c$
 - ③ Else, assign $b = c$.
- ⑥ Repeat step 2 by setting $i = i + 1$.

Newton-Raphson



- 1 Provide an initial guess c_0 .
- 2 Set $i = 1$.
- 3 Set $c = c_0 - \frac{f(c_0)}{f'(c_0)}$.
- 4 Calculate $\varepsilon = |c - c_0|$:
 - 1 if $\varepsilon < tol$, end the program and output c .
 - 2 Else, set $i = i + 1$, update the guess: $c_0 = c$.

Secant



- ① Provide two initial guesses c_0, c_1 .
- ② Find $f(c_0)$ and $f(c_1)$.
- ③ Update the guess (look at Eq.(??)).

$$c = c_1 - f(c_1) \frac{c_1 - c_0}{f(c_1) - f(c_0)}.$$

- ④ Calculate $\varepsilon = |c - c_1|$:
 - ① if $\varepsilon < tol$, end the program and output c .
 - ② Else, set $i = i + 1$, $c_1 = c$.

Gauss-Seidel

This algorithm is actually very simple. Just repeat the dynamics until convergence.

- ① Guess the initial value of k_0
- ② Calculate other endogenous variables w, R based on (??) and (??)
- ③ Solve optimal saving decision s based on (??).
- ④ Calculate again the capital-labor ratio and get k_1 based on (??).
- ⑤ Calculate the error and verify if the algorithm has converged

$$\text{error} = \frac{k_1 - k_0}{k_0}.$$

If $\text{error} > 0$, update the capital-labor ratio with $0 < \lambda < 1$ as the update parameter

$$k_{0,\text{new}} = \lambda k_1 + (1 - \lambda)k_0.$$

and repeat step 2. Otherwise, if $\text{error} = 0$, stop.

Section 3: Infinitely lived agent (Ramsey)

Motivation

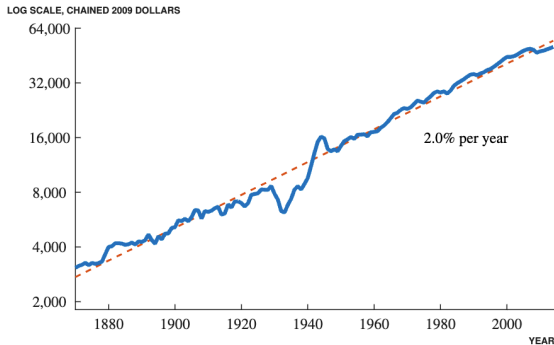


Figure: GDP per capita in the US. Source: Jones (2016).

- ▷ Small disturbances happen, but they don't change the long-run trend.
- ▷ The Ramsey model is the core of this model but without stochasticity.

Households

An infinite-lived household's problem

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t).$$

subject to

$$\begin{aligned} c_t + k_{t+1} &= f(k_t) + (1 - \delta)k_t, \\ k_0 &> 0. \end{aligned}$$

Assume the following:

$$\begin{aligned} u(c) &= \ln c, \\ f(k) &= k^\alpha \end{aligned}$$

Solve the model. You should get the Euler equation (intertemporal FOC)

$$\frac{c_{t+1}}{c_t} = \beta(f'(k_{t+1}) + (1 - \delta))$$

Find the policy function

First, find the steady states (\bar{c}, \bar{k}) :

$$c_t = c_{t+1} = \bar{c},$$

$$k_t = k_{t+1} = \bar{k}$$

The problem is, with the Euler equation and the steady state known, how do we find the solution - a sequence of $\{c_t, k_t\}_{t=0}^{\infty}$.

The goal is to find the policy function

$$k_{t+1} = h(k_t)$$

We solve by three methods:

- ① undetermined coefficients (guess and verify)
- ② perturbations (linear approximations)
- ③ value function iteration

We will learn each method in class. However, what you would get the policy something like this

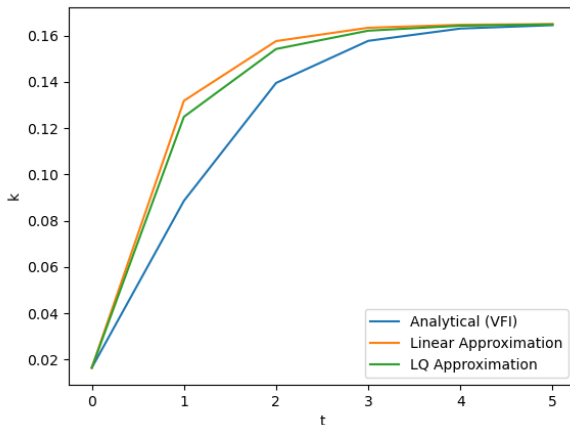


Figure: Comparisons of the accuracy of different perturbation methods.

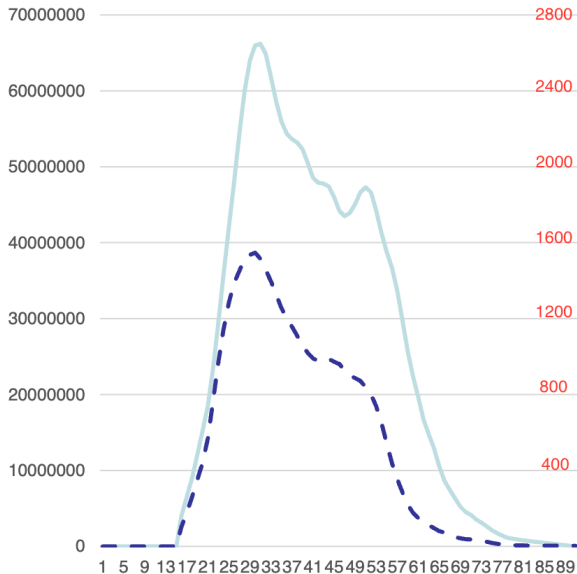
Section 4: Large-scale OLG (AK60)

Motivation



Figure: Life cycle Wealth in Japan

Compensation of employees per capita USD



Environment

The representative household's life span is 60, such that

$$T + TR = 40 + 20 = 60.$$

where T is the working length and TR is the retirement length. Labor supply n_t^s follows

$$l_t^s = 1 - n_t^s \text{ for } t \in \{1, 2, \dots, 40\}, \quad (5)$$

$$l_t^s = 1 \text{ for } t \in \{41, 42, \dots, 60\}. \quad (6)$$

where l_t^s is leisure. After T years, retirement is mandatory. The agent's maximization problem is

$$\sum_{s=1}^{T+TR} \beta^{s-1} u(c_{s+t-1}^s, l_{t+s-1}^s). \quad (7)$$

where β is the discount factor. The instantaneous utility function:

$$u(c, l) = \frac{((c + \psi)l^\gamma)^{1-\eta} - 1}{1-\eta}. \quad (8)$$

An agent is born without wealth and leaves no bequests upon death, thus $k_t^1 = k_t^{61} = 0$. The real budget constraint of the working agent is given by

$$k_{t+1}^{s+1} = (1 + r_t)k_t^s + (1 - \tau_t)w_t n_t^s - c_t^s \text{ for } s = 1, \dots, T. \quad (9)$$

where r_t, w_t are the interest and wage rates, while τ is the social security contribution tax. Once retired, the agents receive public pensions b and no labor earnings. The budget constraint for a retiree is

$$k_{t+1}^{s+1} = (1 + r_t)k_t^s + b - c_t^s \text{ for } s = T + 1, \dots, TR. \quad (10)$$

Production is Cobb-Douglas technology:

$$Y_t = N_t^{1-\alpha} K_t^\alpha.$$

Let $\delta \in [0, 1]$ be the depreciation rate. The factor prices are

$$\begin{aligned} w_t &= (1 - \alpha)K_t^\alpha N_t^{1-\alpha}, \\ r_t &= \alpha K_t^{\alpha-1} N_t^{-\alpha} - \delta. \end{aligned}$$

Its budget is balanced every period such that

$$\tau w_t N_t = \frac{TR}{T + TR} b.$$

Equilibrium

The FOC:

$$\frac{u_l(c_t^s, l_t^s)}{u_c(c_t^s, l_t^s)} = \gamma \frac{c_t^s + \psi}{l_t^s} = (1 - \tau_t) w_t. \quad (11)$$

The Euler:

$$\frac{1}{\beta} = \frac{u_c(c_{t+1}^{s+1}, l_{t+1}^{s+1})}{u_c(c_t^s, l_t^s)} (1 + r_{t+1}) = \frac{(c_{t+1}^{s+1} + \psi)^{-\eta} (l_{t+1}^{s+1})^{\gamma(1-\eta)}}{(c_t^s + \psi)^{-\eta} (l_t^s)^{\gamma(1-\eta)}} (1 + r_{t+1}). \quad (12)$$

Equilibrium

- Individual and aggregate behaviors are consistent

$$N_t = \sum_{s=1}^{T+TR} \frac{n_t^s}{T + TR}, \quad (13)$$

$$K_t = \sum_{s=1}^{T+TR} \frac{k_t^s}{T + TR}. \quad (14)$$

- Goods market clear

$$N_t^{1-\alpha} K_t^\alpha = \sum_{s=1}^{T+TR} \frac{c_t^s}{T + TR} + K_{t+1} - (1 - \delta) K_t.$$

Steady state

The concept of a steady state can be characterized by a constant distribution of capital stock over generations.

$$\{k_t^s\}_{s=1}^{60} = \{k_{t+1}^s\}_{s=1}^{60} = \{\bar{k}^s\}_{s=1}^{60}.$$

As a consequence, every other variable, such as r, w, b, τ , also becomes a constant.

Calibration

Parameters	β	η	α	δ	γ	ψ
Value	0.98	2	0.36	0.1	2	0.001

To calculate the tax rate, set the replacement ratio $\xi = 0.3$ such that

$$\xi = \frac{b}{(1 - \tau)w\bar{n}}.$$

with \bar{n} is the average labor supply, which equals to $N(T + TR)/T$. Since we want a realistic value, you can set the target values for the steady states of

$$\bar{n} = 0.35, \quad \tau = \frac{\xi}{2 + \xi}, \quad r = 0.045.$$

Finding the Steady state

The computation of the steady states is more complex than Section 1. However, since our functions are well-behaved, we can use the simple iteration method to reach the steady state by following the algorithm below.

- ① Make initial guesses of the steady state values of K and N .
- ② Compute w, r, τ, b that solve the firm's problem and the government's budget set.
- ③ Compute the optimal path for consumption, savings, and labor supply by backward iteration.
 - ① We know $k^1 = k^{61} = 0$. Make a guess of k^{60} .
 - ② With k^{61}, k^{60} known, solve for $k^{59}, k^{58}, \dots, k^2, k^1$.
 - ③ Compute k^1 :
 - ① if $k^1 = 0$. Stop the loop and output the series of k^1, \dots, k^{60} .
 - ② else, if $k^1 \neq 0$, update k^{60} using the secant method.
- ④ Recompute the new aggregate K and N .
- ⑤ If they are the same as the initial guess. Stop. Otherwise, update a new K and N and go back to step 2 until convergence.

If you do things correctly, you should find the steady state of K and N as 0.913 and 0.221, respectively.

