SpringCamp 2024
Computational Macro

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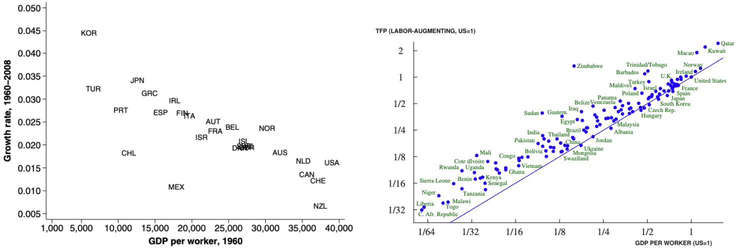
Section 1: Two-period OLG (Diamond)

Motivation

Figure: Life-cycle pattern of income and consumption (Browning and Crossley, 2001).

▷ People save when young and dissave when old.

Motivation



(a) Growth vs initial GDPpc

(b) TFP and GDPpw

Figure: Some facts on growth (Jones and Vollrath, 2013)

▷ Convergence theory (at least for OECD): if you start small, you grow fast.

Model

To do:

1. Modeling Household
2. Modeling Production sector.
3. State the equilibrium
4. See long-run dynamics of the model

In step 4, we calibrate and write a code to solve the model numerically.

Household

Preferences

$$U(c_t, d_{t+1}) = u(c_t) + \beta u(d_{t+1}), \tag{1}$$

subject to constraints

$$\begin{aligned} c_t + s_t &= w_t, \\ d_{t+1} &= R_{t+1} s_t. \end{aligned}$$

Possible functional forms

$$u(c) = \begin{cases} \dfrac{c^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}} & \text{if } \sigma > 0, \sigma \neq 1 \text{ (CIES)}, \\ \dfrac{c^{1-\sigma} - 1}{1 - \sigma} & \text{if } \sigma \geq 0, \sigma \neq 1 \text{ (CRRA)}, \\ \ln(c) & \text{if } \sigma = 1. \end{cases}$$

1. Solve fo s_t using a Lagrangian.

Production

Technology

$$Y = AF(K_t, L_t)$$

Possible functional form

$$F(K,L) = \begin{cases} \left(\alpha K_t^{-\rho} + (1-\alpha) L_t^{-\rho}\right)^{-1/\rho} & \text{CES with } \rho \in (-1, +\infty), \rho \neq 0 \\ K_t^{\alpha} L_t^{1-\alpha} & \text{Cobb-Douglas } (\rho \to 0) \end{cases}$$

Profit:

$$\Pi_t = Y_t - w_t L_t - R_t K_t$$

1. Define the capital-labor ratio $k = K/L$, and solve for the optimal L, K.

Equilibrium

Labor market

$$L_t = N_t$$

Assuming full depreciation. The capital market clears

$$K_{t+1} = s_t N_t$$

Goods market clears

$$Y_t = N_{t-1} d_t + N_t (c_t + s_t)$$

1. With Cobb-Douglas technology and log utility:

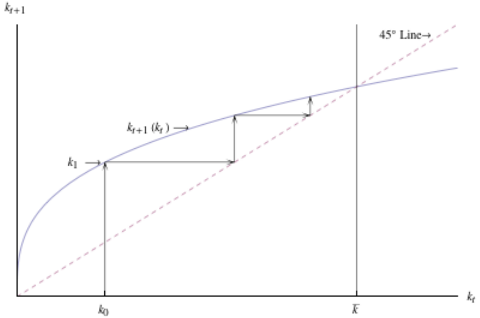
$$k_{t+1} = \phi(k_t) = \frac{\beta A(1-\alpha)}{(1+n)(1+\beta)} k_t^{\alpha}. \tag{2}$$

2. With CES technology ($\rho < 0$) and log utility:

$$k_{t+1} = \phi(k_t) = \frac{\beta A(1-\alpha)(\alpha k_t^{-\rho} + 1 - \alpha)^{-(1+\rho)/\rho}}{(1+n)(1+\beta)}. \tag{3}$$

Steady State

Find the steady state of the capital-labor ratio k^* by solving $k_{t+1} = k_t = k^*$.



One can be found by hand, the other cannot.

Table: Calibration

β	α	ρ	A	n
0.99^{30}	0.3	-1.5	10	0.3

Section 2: Numerical techniques

The idea

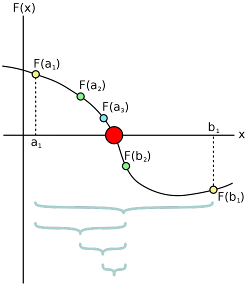
Rewrite the dynamics to rewrite it to

$$k - \frac{\beta A(1-\alpha)}{(1+n)(1+\beta)} k^{\alpha} = 0. \tag{4}$$

Solving for k is the same as finding the root of this equation.

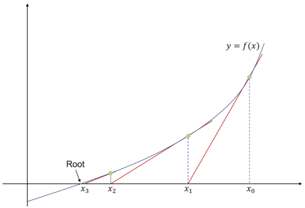
1. Bisection
2. Newton-Raphson
3. Secant Method
4. Gauss-Seidel
5. Let Python do its thing.

Bisection



1. Choose a and b such that $f(a) \times f(b) < 0$.
2. Set $i = 1$.
3. Calculate $f(a)$.
4. Choose a point c where $c = a + \frac{b-a}{2}$
5. Calculate $f(c)$ and evaluate its sign.
   1. if $|f(c)| < tol$, end the program and output c.
   2. Else: if $f(c) \cdot f(a) > 0$, assign: $\mathrm{a} = \mathrm{c}$
   3. Else, assign $b = c$.
6. Repeat step 2 by setting $i = i + 1$.

Newton-Raphson

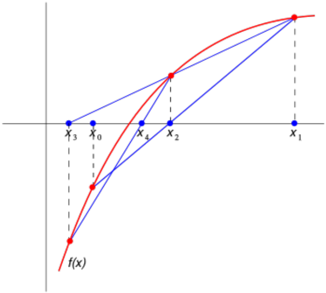


We approximate a function by its tangent line to get a successively better estimate of the roots. Let n be the n^{th} iteration, the formula for the next guess

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{5}$$

1. Provide an initial guess c_0.
2. Set $i = 1$.
3. Set $c = c_0 - \dfrac{f(c_0)}{f'(c_0)}$.
4. Calculate $\varepsilon = |c - c_0|$:
    1. if $\varepsilon < tol$, end the program and output c.
    2. Else, set $i = i + 1$, update the guess: $c_0 = c$.

Secant



1. Provide two initial guesses c_0, c_1.
2. Find $f(c_0)$ and $f(c_1)$.
3. Update the guess (look at Eq.(5)).

$$c = c_1 - f(c_1)\frac{c_1 - c_0}{f(c_1) - f(c_0)}.$$

4. Calculate $\varepsilon = |c - c_1|$:
   1. if $\varepsilon < tol$, end the program and output c.
   2. Else, set $i = i + 1$, $c_1 = c$.

Gauss-Seidel

This algorithm is actually very simple. Just repeat the dynamics until convergence.

1. Guess the initial value of k_0
2. Calculate other endogenous variables w, R based on (**??**) and (**??**)
3. Solve optimal saving decision s based on (**??**).
4. Calculate again the capital-labor ratio and get k_1 based on (**??**).
5. Calculate the error and verify if the algorithm has converged

$$\text{error} = \frac{k_1 - k_0}{k_0}.$$

If error > 0, update the capital-labor ratio with $0 < \lambda < 1$ as the update parameter

$$k_{0,new} = \lambda k_1 + (1 - \lambda) k_0.$$

and repeat step 2. Otherwise, if error $= 0$, stop.

Section 3: Infinitely lived agent (Ramsey)

Motivation

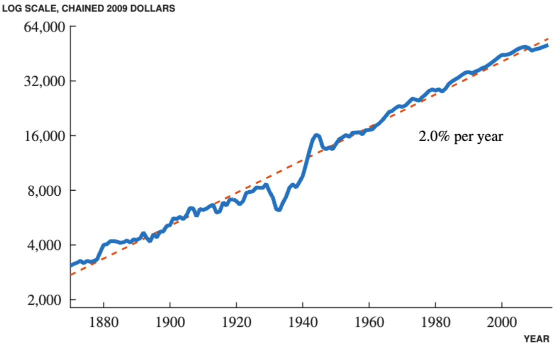


Figure: GDP per capita in the US. Source: Jones (2016).

- ▷ Small disturbances happen, but they don't change the long-run trend.
- ▷ The Ramsey model is the core of this model but without stochasticity.

Households

An infinite-lived household's problem

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t).$$

subject to

$$\begin{aligned} c_t + k_{t+1} &= f(k_t) + (1-\delta)k_t, \\ k_0 &> 0. \end{aligned}$$

Assume the following:

$$\begin{aligned} u(c) &= \ln c, \\ f(k) &= k^{\alpha} \end{aligned}$$

Solve the model. You should get the Euler equation (intertemporal FOC)

$$\frac{c_{t+1}}{c_t} = \beta(f'(k_{t+1}) + (1-\delta))$$

Find the policy function

First, find the steady states $(\bar{c}, \bar{k})$:

$$c_t = c_{t+1} = \bar{c},$$
$$k_t = k_{t+1} = \bar{k}$$

The problem is, with the Euler equation and the steady state known, how do we find the solution - a sequence of $\{c_t, k_t\}_{t=0}^{\infty}$.
The goal is to find the policy function

$$k_{t+1} = h(k_t)$$

We solve by three methods:

1. undetermined coefficients (guess and verify)
2. perturbations (linear approximations)
3. value function iteration

We will learn each method in class. However, what you would get the policy something like this

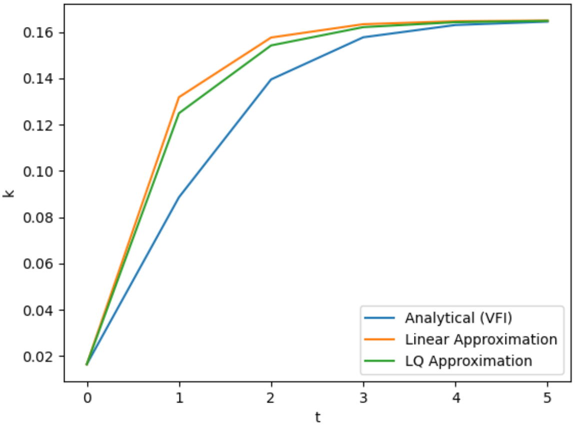


Figure: Comparisons of the accuracy of different perturbation methods.

Section 4: Large-scale OLG (AK60)

Motivation

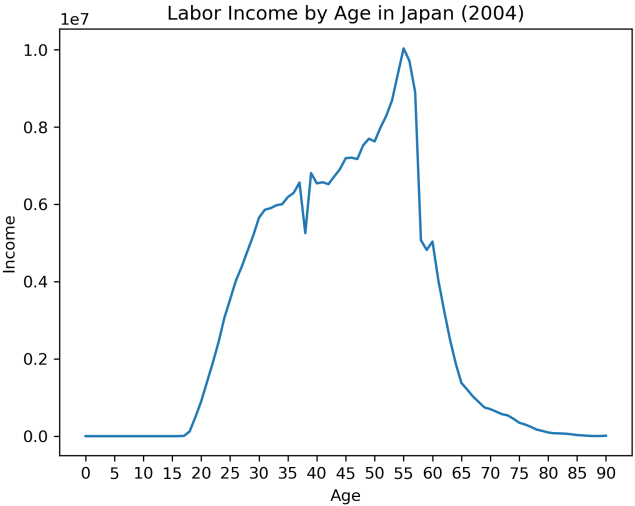
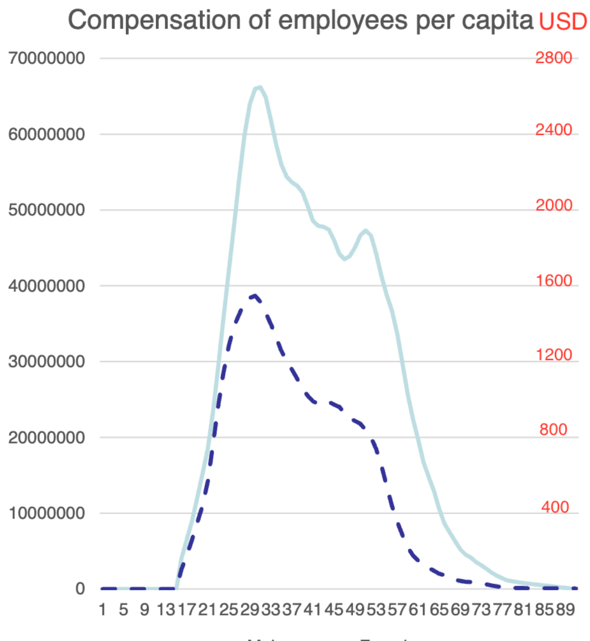


Figure: Life cycle Wealth in Japan


Compensation of employees per capita USD
70000000
60000000
50000000
40000000
30000000
20000000
10000000
0
2800
2400
2000
1600
1200
800
400
1 5 9 13 17 21 25 29 33 37 41 45 49 53 57 61 65 69 73 77 81 85 89

Environment

The representative household's life span is 60, such that

$$T + TR = 40 + 20 = 60.$$

where T is the working length and TR is the retirement length. Labor supply n_t^s follows

$$l_t^s = 1 - n_t^s \text{ for } t \in \{1, 2, \ldots, 40\}, \tag{6}$$
$$l_t^s = 1 \text{ for } t \in \{41, 42, \ldots, 60\}. \tag{7}$$

where l_t^s is leisure. After T years, retirement is mandatory. The agent's maximization problem is

$$\sum_{s=1}^{T+TR} \beta^{s-1} u(c_{s+t-1}^s, l_{t+s-1}^s). \tag{8}$$

where β is the discount factor. The instantaneous utility function:

$$u(c, l) = \frac{((c+\psi)l^\gamma)^{1-\eta} - 1}{1-\eta}. \tag{9}$$

An agent is born without wealth and leaves no bequests upon death, thus $k_t^1 = k_t^{61} = 0$. The real budget constraint of the working agent is given by

$$k_{t+1}^{s+1} = (1 + r_t)k_t^s + (1 - \tau_t)w_t n_t^s - c_t^s \text{ for } s = 1, \dots, T. \tag{10}$$

where r_t, w_t are the interest and wage rates, while τ is the social security contribution tax. Once retired, the agents receive public pensions b and no labor earnings. The budget constraint for a retiree is

$$k_{t+1}^{s+1} = (1 + r_t)k_t^s + b - c_t^s \text{ for } s = T + 1, \dots, TR. \tag{11}$$

Production is Cobb-Douglas technology:

$$Y_t = N_t^{1-\alpha} K_t^{\alpha}.$$

Let $\delta \in [0, 1]$ be the depreciation rate. The factor prices are

$$w_t = (1 - \alpha)K_t^{\alpha} N_t^{1-\alpha},$$
$$r_t = \alpha K_t^{\alpha-1} N_t^{-\alpha} - \delta.$$

Its budget is balanced every period such that

$$\tau w_t N_t = \frac{TR}{T + TR} b.$$

Equilibrium

The FOC:

$$\frac{u_l(c_t^s, l_t^s)}{u_c(c_t^s, l_t^s)} = \gamma \frac{c_t^s + \psi}{l_t^s} = (1 - \tau_t) w_t. \tag{12}$$

The Euler:

$$\frac{1}{\beta} = \frac{u_c(c_{t+1}^{s+1}, l_{t+1}^{s+1})}{u_c(c_t^s, l_t^s)}(1 + r_{t+1}) = \frac{(c_{t+1}^{s+1} + \psi)^{-\eta}(l_{t+1}^{s+1})^{\gamma(1-\eta)}}{(c_t^s + \psi)^{-\eta}(l_t^s)^{\gamma(1-\eta)}}(1 + r_{t+1}). \tag{13}$$

Equilibrium

1. Individual and aggregate behaviors are consistent

$$N_t = \sum_{s=1}^{T+TR} \frac{n_t^s}{T + TR}, \tag{14}$$

$$K_t = \sum_{s=1}^{T+TR} \frac{k_t^s}{T + TR}. \tag{15}$$

2. Goods market clear

$$N_t^{1-\alpha} K_t^{\alpha} = \sum_{s=1}^{T+TR} \frac{c_t^s}{T + TR} + K_{t+1} - (1 - \delta) K_t.$$

Steady state

The concept of a steady state can be characterized by a constant distribution of capital stock over generations.

$$\{k_t^s\}_{s=1}^{60} = \{k_{t+1}^s\}_{s=1}^{60} = \{\bar{k}^s\}_{s=1}^{60}.$$

As a consequence, every other variable, such as r, w, b, τ, also becomes a constant.

Calibration

Parameters	β	η	α	δ	γ	ψ
Value	0.98	2	0.36	0.1	2	0.001

To calculate the tax rate, set the replacement ratio $\xi = 0.3$ such that

$$\xi = \frac{b}{(1-\tau)w\bar{n}}.$$

with $\bar{n}$ is the average labor supply, which equals to $N(T+TR)/T$. Since we want a realistic value, you can set the target values for the steady states of

$$\bar{n} = 0.35, \qquad \tau = \frac{\xi}{2+\xi}, \qquad r = 0.045.$$

Finding the Steady state

The computation of the steady states is more complex than Section 1. However, since our functions are well-behaved, we can use the simple iteration method to reach the steady state by following the algorithm below.

1. Make initial guesses of the steady state values of K and N.
2. Compute w, r, τ, b that solve the firm's problem and the government's budget set.
3. Compute the optimal path for consumption, savings, and labor supply by backward iteration.
    1. We know $k^1 = k^{61} = 0$. Make a guess of k^{60}.
    2. With k^{61}, k^{60} known, solve for $k^{59}, k^{58}, \ldots, k^2, k^1$.
    3. Compute k^1:
        1. if $k^1 = 0$. Stop the loop and output the series of $k^1, \ldots, k^{60}$.
        2. else, if $k^1 \neq 0$, update k^{60} using the secant method.
4. Recompute the new aggregate K and N.
5. If they are the same as the initial guess. Stop. Otherwise, update a new K and N and go back to step 2 until convergence.

If you do things correctly, you should find the steady state of K and N as 0.913 and 0.221, respectively.