Project 1 for 2024 Spring Camp Microeconomics

Model

Consider a static economy consisting of n competitive firms denoted by $\{1, 2, ..., n\}$, each of which produces a distinct product. Each product can be either consumed by the households or used as an intermediate input for production of other goods. Firms employ nested CES production technologies with constant returns to transform labour, firm-specific capital, and intermediate goods into output. In particular, the output of firm i is given by

$$y_i = \left[\chi(1-\mu)^{\frac{1}{\sigma}} \left((z_i k_i)^{\alpha} l_i^{1-\alpha}\right)^{\frac{\sigma-1}{\sigma}} + \mu^{\frac{1}{\sigma}} M_i^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{1}$$

where l_i is the amount of labour hired by the firm, k_i denotes firm-specific capital, and $z_i \leq 1$ is a capital-augmenting shock to firm i with a steady-state value normalised to 1.

In the expression, α captures the share of capital in the bundle of primary factors (i.e., labour and capital), μ parametrises the share of material inputs in firms' production technology, σ represents the elasticity of substitution between primary factors and the intermediate input bundle.

The intermediate input bundle, M_i, is itself a CES aggregate of inputs purchased from other firms:

$$M_i = \left[\sum_{j=1}^{n} a_{ij}^{\frac{1}{\xi}} x_{ij}^{\frac{\xi-1}{\xi}}\right]^{\frac{\xi}{\xi-1}} \tag{2}$$

where x_{ij} is the amount of good j used in the production of good i and ξ is the elasticity of substitution between different intermediate goods. The coefficient $a_{ij} \geq 0$ designates the importance of good j as an intermediate input for the production of good i: a larger a_{ij} means that good j is a more important input in the production technology of firm i, whereas $a_{ij} = 0$ if firm i does not rely on good j as an intermediate input for production. Throughout, we normalise these coefficients by assuming that $\sum_{j=1}^{n} a_{ij} = 1$ for all i.

In addition to the firms described above, the economy is populated by a unit mass of identical households, who supply L units of labour and firm-specific capital $(k_1, ..., k_n)$ inelastically to the firms. Households have logarithmic preferences over the n goods given by

$$u(c_1, ..., c_n) = \sum_{i=1}^{n} \beta_i log\left(\frac{c_i}{\beta_i}\right) \tag{3}$$

where c_i denotes the amount of good i consumed. The constants $\beta_i \geq 0$ measure various goods' shares in the household's utility function, normalised such that $\sum_{i=1}^{n} \beta_i = 1$.

The competitive equilibrium of this economy is defined in the usual way: it consists of a collection of prices and quantities such that

i) the representative consumer maximises her utility;

ii) all firms maximise their profits while taking all prices as given;

iii) all markets clear.

Questions

1. Solve the households' utility maximisation problem to obtain the optimal consumption c_i (as a function of β_i, p_i and I).

2. To solve the firms' profit maximisation problem, start with deriving the FOCs by assuming first χ is arbitrary.

   (Hint: You should obtain an expression for each of the 3 endogeneous parameters l_i, k_i and x_{ij} and a simpler relationship below:)
$$\frac{k_i}{l_i} = \frac{w}{r}\frac{\alpha}{1-\alpha}$$

3. Remember from the exercise done in the course, the price index of the intermediate good bundle M_i is:
$$Q_i = \left(\sum_{j=1}^{n} a_{ij} p_j^{1-\xi}\right)^{\frac{1}{1-\xi}}$$
   Prove the following relationship:
$$M_i = \mu p i^{\sigma} Q_i^{-\sigma} y_i$$
   (Hint: Take natural logarithm of Eq.(2) and use the last FOC.)

4. Using the last FOC and the relation you just obtained in Q3, prove the following by defining $\omega_{sh} = \frac{p_h x_{sh}}{p_s y_s}$
$$\omega_{sh} = \mu a_{sh} p_s^{\sigma-1} p_h^{1-\xi} Q_s^{\xi-\sigma}$$

5. Now define $\chi = (\alpha^{\alpha}(1-\alpha)^{1-\alpha})^{\frac{1}{\sigma}-1}$. Use the FOCs and the relationships above to obtain the price:
$$p_i^{1-\sigma} = (1-\mu)(z_i^{-\alpha} r_i^{\alpha} w^{1-\alpha})^{1-\sigma} + \mu Q_i^{1-\sigma}$$
   Now can you see why we defined χ in this way?

6. Use the market clearing condition ($y_i = c_i + \sum_{j=1}^{n} x_{ji})$) and the relations above to obtain an expression for the domar's weight, a firm's sales as a share of GDP, defined as $\lambda_i = \frac{p_i y_i}{GDP}$.

   (Hint: You can assume the national income I you used in the budget constraint is equivalent to GDP.)