INSEIKAI Tohoku Spring Camp 2023
Mathematics

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0.1 Introduction

The course is aimed at equipping learners the essential mathematical tools to do research in economics. In that regards, basic algebra, calculus and optimization techniques are the core concepts that every student of economics must master, and we must abstract away from advanced topics such as matrix algebra, proofs, control theory. We also try to blend in as many applications of mathematics in economics as possible in the last sessions, to apply what you have learned in the Computation course prior to this one.

0.2 Agenda

Textbooks used: Simon and Blume (1994), Sydsæter and Hammond (2008), Sydsæter et al. (2008).

(1) Algebra	(2) Equations
(3) Functions of One Variable	(4) Properties of Functions
(5) Differentiation	(6) Differentiation in Use
(7) Single-Variable Optimization	(8) Integration
(9) Functions of Many Variables	(10) Comparative Statics
(11) Constrained Optimization	(12) Some Economic Problems
(13) Some Economic Models	(14) Project Planning
(15) Project Presentation	

Table 1: Agenda of the course

	Monday	Tuesday	Wednesday	Thursday	Friday
MARCH (2023)	6 [1]	7 [2]	8 [3]	9 [4]	10 [5]
	13 [7]	14 [8]	15 [9]	16 [10]	17 [11]
	20 [12]	21 [13]	22 [14]	23 [15]	
Class hours	Morning: 9:30 – 12:00 (problem solving)				
	Afternoon: 13:30 – 15:00 (lecture)				

Fig. 1. Schedule in Spring 2023

0.3 Project

At the end of the section, students form a group of 2–3to solve an Economic Problem, and present to the class on the following day their results. (10 minutes presentation)

α	`\alpha`	θ	`\theta`	o	`o`	τ	`\tau`
β	`\beta`	ϑ	`\vartheta`	π	`\pi`	υ	`\upsilon`
γ	`\gamma`	γ	`\gamma`	ϖ	`\varpi`	ϕ	`\phi`
δ	`\delta`	κ	`\kappa`	ρ	`\rho`	φ	`\varphi`
ϵ	`\epsilon`	λ	`\lambda`	ϱ	`\varrho`	χ	`\chi`
ε	`\varepsilon`	μ	`\mu`	σ	`\sigma`	ψ	`\psi`
ζ	`\zeta`	ν	`\nu`	ς	`\varsigma`	ω	`\omega`
η	`\eta`	ξ	`\xi`				
Γ	`\Gamma`	Λ	`\Lambda`	Σ	`\Sigma`	Ψ	`\Psi`
Δ	`\Delta`	Ξ	`\Xi`	Υ	`\Upsilon`	Ω	`\Omega`
Θ	`\Theta`	Π	`\Pi`	Φ	`\Phi`		

Fig. 2. Greek Letters

Chapter 1

Algebra

1.1 Summary

Arithmetic Operations

The denominator must always be $\neq 0$

$$\frac{p}{0} \textbf{ is undefined} \qquad ab + bc = (a + c)b,$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c}, \qquad \frac{a/b}{c} = \frac{a}{bc},$$

$$\frac{a}{b/c} = \frac{ac}{b}, \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd},$$

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{d},$$

Exponent Properties

$$a^{-n} = \frac{1}{a^n},$$

$$a^0 = 1 (a \neq 0), \qquad a^n a^m = a^{n+m},$$

$$\frac{a^n}{a^m} = a^{n-m} = \frac{1}{a^{m-n}}, \qquad (ab)^n = a^n b^n,$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \qquad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n},$$

$$a^{\frac{m}{n}} = (a^{1/m})^n = (a^n)^{1/m}$$

Properties of Radicals

$$\sqrt[n]{a} = a^{1/n}, \qquad \sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b},$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[nm]{a}, \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Some Properties

If $a < b$ then

$$a + c < b + c$$
$$a - c < b - c$$

If $a < b$ and $c > 0$ then

$$ac < bc$$
$$\frac{a}{c} < \frac{b}{c}$$

If $a < b$ and $c < 0$ then

$$ac > bc$$
$$\frac{a}{c} > \frac{b}{c}$$

If $\frac{a}{b} = \frac{c}{d}$ then

$$ad = bc$$

If $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$ are 2 points, the distance between them is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Properties of Absolute Value

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{cases}, \qquad |a| \geq 0,$$

$$|-a| = |a|, \qquad |ab| = |a||b|,$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \qquad |a + b| \leq |a| + |b| \textbf{ (triangle inequality)}$$

1.2 Summation Notation

1.2.1 Basics

Shorthand for

$$x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x_i$$

Ex. 1.1. Evaluate:

$$(a)\ \sum_{i=1}^{10} i^2 \qquad (b)\ \sum_{j=1}^{4} \frac{j+1}{j}$$

$$(c)\ \sum_{j=0}^{2} \frac{(-1)^j}{(j+1)(j+3)} \qquad (d)\ \sum_{k=3}^{6} (5k - 3)$$

Ex. 1.2. Expand

$$(a)\ \sum_{i=1}^{5} p_i q_i \qquad (b)\ \sum_{i=1}^{N} (x_{ij} - \bar{x}_j)^2$$

$$(c)\ \sum_{k=-2}^{2} 2\sqrt{k+2} \qquad (d)\ \sum_{k=1}^{n} a_{ki} b^{k+1}$$

Ex. 1.3. Express in summation

$$(a)\ 1^3 + 2^3 + \cdots + n^3 \qquad (b)\ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^n \frac{1}{2n+1}$$

1.2.2 Properties

Additivity property

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

Homogeneity property

$$\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$$

Ex. 1.4. Evaluate

$$\sum_{m=2}^{n} \frac{1}{(m-1)m} = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(n-1)m}$$

knowing that

$$\frac{1}{(m-1)m} = \frac{1}{m-1} - \frac{1}{m}$$

Ex. 1.5. The mean $\bar{x}$ of T numbers $x_1, x_2, \ldots, x_n$ is the sum of all numbers dividded by T:

$$\bar{x} = \frac{1}{T} \sum_{i=1}^{T} x_i$$

Prove that

$$(a)\ \sum_{i=1}^{T} (x_i - \bar{x}) = 0$$

$$(b)\ \sum_{i=1}^{T} (x_i - \bar{x})^2 = \sum_{i=1}^{T} x_i^2 - T\bar{x}^2$$

1.2.3 Double Sums

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}$$

Ex. 1.6. Compute

$$(a)\ \sum_{i=1}^{3}\sum_{j=1}^{4}(i+2j) \qquad (b)\ \sum_{i=1}^{3}\sum_{j=1}^{4} i\cdot 3^j$$

$$(c)\ \sum_{s=0}^{2}\sum_{r=2}^{4}\left(\frac{rs}{r+s}\right)^2 \qquad (d)\ \sum_{i=1}^{m}\sum_{j=1}^{n}(i+j^2)$$

$$(e)\ \sum_{i=1}^{m}\sum_{j=1}^{2} i^j$$

1.3 Exercises

Ex. 1.7 (Expand and simplify)**.**

$$(a)\ \frac{24x^3y^2z^3}{4x^2yz^2}, \qquad (b)\ [-(-ab^3)^{-3}(a^6b^6)^2]^3,$$

$$(c)\ \frac{a^5a^3a^{-2}}{a^{-3}a^6}, \qquad (d)\ \left[\left(\frac{x}{2}\right)^3\cdot\frac{8}{x^{-2}}\right]^{-3},$$

$$(e)\ \frac{a(a-1)}{b}+\frac{(b+1)(b-1)}{a}, \qquad (f)\ (x-3)(x+7),$$

$$(g)\ -\sqrt{3}(\sqrt{3}-\sqrt{6}), \qquad (h)\ \frac{p^\gamma(pq)^\sigma}{p^{2\gamma+\sigma}q^{\sigma-2}}$$

$$(i)\ \frac{2+a}{a^2b}+\frac{1-b}{ab^2}-\frac{2b}{a^2b^2} \qquad (k)\ \frac{x^0x^{-\beta\alpha}}{x^{2\alpha+1}x^{\beta-2}}$$

Ex. 1.8 (Expand)**.** Prove the Quadratic Identities

$$\begin{aligned}
&(a)\ (a+b)^2 = a^2+2ab+b^2,\\
&(b)\ (a-b)^2 = a^2-2ab+b^2\\
&(c)\ a^2-b^2 = (a+b)(a-b)\\
&(e)\ (a+b)^3 = a^3+3a^2b+3ab^2+b^3,\\
&(f)\ (a-b)^3 = a^3-3a^2b+3ab^2-b^3,\\
&(g)\ a^3+b^3 = (a+b)(a^2-ab+b^2),\\
&(h)\ a^3-b^3 = (a-b)(a^2+ab+b^2)
\end{aligned}$$

Ex. 1.9. The surface area of a sphere with a radius r is

$$S = 4\pi r^2.$$

(a) If the radius is tripled, how much will the area increase?
(b) If the radius increases by 16%, how many % will the surface area increase?
(c) How much must the radius increase for the surface area to increase by 1.5?

Ex. 1.10 (Compound Interest)**.** You have in total $A = \$1000$. The annual interest rate is $i = 5\%$ each year. At time t, you deposit everything at a bank. Answer the following:

(a) How much will you have in your account at time $t + 1$?

(b) How much will you have in your account at time $t + 2$?

(b) How much will you have in your account at time $t + s$? (where s is just a number)

Now, assume that the annual inflation rate is $\pi = 8\%$. You still choose to keep your money at the bank. Calculate:

(c) The real interest rate r. (Hint: it is $i - \pi$)

(d) How much will you have in your account at time $t+1$? at $t+2$? at $t+s$?

(e) Assume that inflation only happens at $t + 3$ and stays that way from that moment onward, how much will you have in your account at $t + 5$? (Hint: inflation rates at $t + 1$ and $t + 2$ are zero)

Ex. 1.11 (Expand. Mind the signs)**.**

$(a)\ p^2(2-q)+p(q-1)$, $(b)\ 2p(p+3)-(q+1)(q+2)$,

$(c)\ (2pq-3p^2)(p+2q)-(q^2-2pq)(2p-q)$, $(d)\ -3(n^2-2n+3)$,

$(e)\ (a^2b-ab^2)(a+b)$ $(f)\ (x-y)(x-2y)(x-3y)$,

$(g)\ 6a^2b(5ab-3ab^2)$, $(h)\ (4n-3)(n-2)$,

$(i)\ (2-t^2)(2+t^2)$, $(k)\ (u-v)^2(u+v)^2$,

$(l)\ (2t-1)(t^2-2t+1)$, $(m)\ (a+1)^2+(a-1)^2-2(a+1)(a-1)$,

$(n)\ (x+y+z)^2$, $(o)\ (x+y+z)^2-(x-y-z)^2$,

$(p)\ (x+2y)^2$, $(q)\ \left(\frac{1}{x}-x\right)^2$,

Ex. 1.12 (Factoring)**.**

$(a)\ a^3-a^2b$, $(b)\ 8x^2y^2-16xy$,

$(c)\ 16-b^2$, $(d)\ 4t^2s-8ts^2$,

$(e)\ K^3-K^2L$, $(f)\ K^2L-L^2K$,

$(g)\ K^{-4}-LK^{-5}$, $(h)\ -\frac{1}{5}x^2+2xy-5y^2$

Ex. 1.13 (Fractions)**.**

$$(a)\ \frac{1}{x-2}-\frac{1}{x+2}, \qquad (b)\ \frac{6x+25}{4x+2}-\frac{6x^2+x-2}{4x^2-1},$$
$$(c)\ \frac{18b^2}{a^2-9b^2}-\frac{a}{a+3b}+2, \qquad (d)\ \frac{1}{8ab}-\frac{1}{8b(a+2)},$$
$$(e)\ \frac{2t-t^2}{t+2}\left(\frac{5t}{t-2}-\frac{2t}{t-2}\right), \qquad (f)\ 2-\frac{a(1-\frac{1}{2a})}{1/4},$$
$$(g)\ \frac{2}{x}+\frac{1}{x+1}-3, \qquad (h)\ \frac{3x}{x+2}-\frac{4x}{2-x}-\frac{2x-1}{x^2-4},$$
$$(i)\ \frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{xy}}, \qquad (k)\ \frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x^2}+\frac{1}{y^2}},$$
$$(l)\ \left(\frac{1}{4}-\frac{1}{5}\right)^{-2}, \qquad (m)\ n-\frac{n}{1-\frac{1}{n}},$$
$$(n)\ \frac{1}{1+x^{a-b}}+\frac{1}{1+x^{b-a}}, \qquad (o)\ \frac{\frac{1}{x-1}+\frac{1}{x^2-1}}{x-\frac{2}{x+1}}$$

Ex. 1.14 (Radicals and Roots)**.**

$$(a)\ \frac{6}{\sqrt{7}},\ (b)\ \frac{\sqrt{32}}{\sqrt{2}},\ (c)\ \frac{\sqrt{54}-\sqrt{24}}{\sqrt{6}}, \quad (d)\ \frac{2}{\sqrt{3}\sqrt{8}},\ (e)\ 16^{3/2},\ (f)\ (1/27)^{-2/3}$$
$$(g)\ \frac{a^{3/8}}{a^{1/8}},\ (h)\ (x^{1/2}x^{3/2}x^{-2/3})^{3/4}, \qquad (i)\ \left(\frac{10p^{-1}q^{2/3}}{80p^2q^{-7/3}}\right)^{-2/3},$$
$$(k)\ \frac{x\sqrt{y}-y\sqrt{x}}{x\sqrt{y}+y\sqrt{x}}, \qquad (l)\ \frac{h}{\sqrt{x+h}-\sqrt{x}}$$
$$(m)\ (((a^{1/2})^{2/3})^{3/4})^{4/5}, \qquad (n)\ a^{1/2}a^{2/3}a^{3/4}a^{4/5},$$
$$(o)\ (((3a)^{-1})^{-2}.(2a^{-2})^{-1})/a^{-3}, \qquad (p)\ \frac{\sqrt[3]{a}a^{1/12}\sqrt[4]{a^3}}{a^{5/12}\sqrt{a}}$$

Ex. 1.15 (Economic problems)**.** Answer the following

(a) The population of a nation increased from 40 mil. to 60 mil. in 12 years. What is the yearly percentage rate of the population growth p?

(b) An item initially costs \$2. Its price is then increased by 5%, and afterward, it decreased by 5%. What is the final price?

Ex. 1.16 (Inequalities)**.** Determine x such that

(a) $3x-5>x-3$,

(b) $(x-1)(3-x)>0$,

(b) $\dfrac{2x-3}{x-1}>3-x$

(c) $\dfrac{2x-4}{3}\leq 7$

(d) $\dfrac{x}{24}-(x+1)+\dfrac{3x}{8}<\dfrac{5}{12}(x+1)$,

(e) $\dfrac{x+2}{x-1}<0$,

(f) $\dfrac{2x+1}{x-3}>1$,

(g) $5x^2\leq 125$,

(h) $x^2-2x\leq 0$,

(i) $\dfrac{1}{x-2}+\dfrac{3}{x^2-4x+4}\geq 0$,

(k) $\dfrac{-x-2}{x+2}>2$,

(l) $x^4<x^2$,

(m) $(x-1)(x+4)>0$,

(n) $(x-1)^2(x+4)>0$,

(o) $\dfrac{3x-1}{x}>x+3$,

(p) $-5<\dfrac{1}{x}<0$,

(r) $1\leq 2x-1<16$,

(s) $\dfrac{\frac{1}{x}-1}{\frac{1}{x}+1}\geq 1$

Ex. 1.17 (Absolute Value)**.** Determine x

(a) $|x-2|$ for $x=-3,-1,0,2,4$

(b) $|3-2x|=5$,

(c) $|x|\leq 2$,

(d) $|x-2|\leq 1$,

(e) $|3-8x|\leq 5$,

(f) $|x|>\sqrt{2}$,

(g) $|x^2-2|\leq 1$

Chapter 2

Equations

Solving an equation is finding all variable values to satisfy a mathematical expression.

2.1 Simple Equations

Example 2.1. Find all x such that

- (a) $3x + 10 = x + 4$
- (b) $\frac{x+2}{x-2} - \frac{8}{x^2 - 2x} = \frac{2}{x}$
- (c) A firm manufactures a good that costs \$20 per unit to produce. In addition, the firm has fixed costs of \$2000. Each unit is sold for \$75. How many units must be sold for the firm to meet its target profit of \$14,500?

A linear equation has the form

$$ax + b = 0, \tag{2.1}$$

where x is a variable, $a \neq 0$. To find the solutions, follow these steps

1. Identify the domain (For instance, in a fraction, the denominator must $\neq 0$; the expression inside square root $\sqrt{u}$ must ≥ 0)
2. Transform any equation to match the form (2.1)
3. Put all variables on the LHS, all parameters on the RHS so that $ax = -b$
4. Solutions: $x = -b/a$ and cross out values that do not belong to the domain

To make the expression on both sides equivalent, you can

- add or subtract the same number on both sides
- multiply both sides by the same number $\neq 0$x

Ex. 2.1. Solve:

$(a)\ \dfrac{x-3}{4}+2=3x$ $\qquad (b)\ 3x=\dfrac{1}{4}x-7$

$(c)\ \dfrac{1}{2x+1}=\dfrac{1}{x+2}$ $\qquad (d)\ \sqrt{2x+14}=16$

$(e)\ \dfrac{x-3}{x+3}=\dfrac{x-4}{x+4},$ $\qquad (f)\ \dfrac{3}{x-3}-\dfrac{2}{x+3}=\dfrac{9}{x^2-9},$

$(g) 4x+2(x-4)-3=2(3x-5)-1,$ $\qquad (h)\ \dfrac{6x}{5}-\dfrac{5}{x}=\dfrac{2x-3}{3}+\dfrac{8x}{15}$

$(i)\ \dfrac{2-\dfrac{x}{1-x}}{1+x}=\dfrac{6}{2x+1}$ $\qquad (j)\ \dfrac{1}{2}\left(\dfrac{x}{2}-\dfrac{3}{4}\right)-\dfrac{1}{4}\left(1-\dfrac{x}{3}\right)-\dfrac{1}{3}(1-x)=-\dfrac{1}{3}$

2.2 Equations with Parameters

The form:

$$y=ax+b$$

Example 2.2. Basic macroeconomic model

$$\begin{cases} Y=C+I, \\ C=a+bY \end{cases} \tag{2.2}$$

where Y is GDP, C is consumption, I is investment, a, b are positive parameters ($b<1$). By substituting, we can express (2.2) as

$$Y=a+bY+I$$

so that

$$Y=\frac{a}{1-b}+\frac{1}{1-b}I, \tag{2.3}$$

where Y is the **endogenous variable** and I is the **exogenous variable**. Given a, b, and I, we can find any value of Y. Both (2.2) and (2.3) are equivalent. But (2.2) is called the **structural form** of the model while (2.3) is called the **reduced form** of the model

Ex. 2.2. Find the reduced form of C from (2.2).

Ex. 2.3. Suppose the total demand for money of the economy

$$M=\alpha Y+\beta(r-Y)^{-\delta}$$

where M is quantity of money, Y is GDP, r is interest rate, the rest are positive parameters.

- Solve the equation for r

- Estimated parameters for the US during 1929–1952 is $\alpha=0.14, \beta=76.03, \gamma=2, \delta=0.84$, find r

Ex. 2.4. Solve for x and K

$(a)\ \dfrac{1}{ax} + \dfrac{1}{bx} = 2$ $\qquad (b)\ \dfrac{ax+b}{cx+d} = A$

$(c)\ \dfrac{1}{2}px^{-1/2} - w = 0$ $\qquad (d)\ \dfrac{ax}{\sqrt{1+x}} + \sqrt{1+x} = 0$

$(e)\ a^2x^2 - b^2 = 0$ $\qquad (f)\ (3+a^2)^x = 1,$

$(g)\ \alpha x - \alpha = \beta x - \beta$ $\qquad (h)\ \sqrt{qx} - 3q = 5,$

$(i)\ x = 94 + 0.3(x - (20 + 0.5x))$ $\qquad (j)\ K^{1/2}\left(\dfrac{1}{2}\dfrac{r}{w}K\right)^{1/4} = Q$

$(k)\ \dfrac{\frac{1}{2}L^{-1/2}K^{1/4}}{\frac{1}{4}K^{-3/4}L^{1/2}} = \dfrac{r}{w}$ $\qquad (l)\ \dfrac{1}{2}pK^{-1/4}\left(\dfrac{1}{2}\dfrac{r}{w}\right)^{1/4} = r$

$(m)\ \dfrac{x - 2y + xz}{x - z} = 4y$ $\qquad (n)\ Y = C\left(1 - \dfrac{K}{N}\right)$

2.3 Quadratic Equations

The form:

$$ax^2 + bx + c = 0 \tag{2.4}$$

Example 2.3. Solve

$$x^2 + 8x - 9 = 0$$

1. Method 1: Completing the square by adding 16 to both sides

$$\begin{aligned} & x^2 + 8x + 16 = 9 + 16 \\ & \Leftrightarrow (x+4)^2 = 25 \\ & \Leftrightarrow x + 4 = \pm 5 \\ & \Leftrightarrow x = 1 \text{ or } x = -9 \end{aligned}$$

2. Method 2: Factorization (the solutions often are divisors of c)

$$\begin{aligned} & x - x + 9x - 9 = 0 \\ & \Leftrightarrow (x-1)(x+9) = 0 \end{aligned}$$

3. Method 3: (more general, works in all cases) find roots with the quadratic formula

$$\triangle = b^2 - 4ac$$

If $\triangle \geq 0$ and $a \neq 0$, then there are 2 solutions

$$x = \frac{-b \pm \sqrt{\triangle}}{2a}$$

If $\triangle = 0$, there is only 1 solution

$$x = -\frac{b}{2a}$$

If $\triangle < 0$, there is no **real** solution.

Ex. 2.5. Verify Method 3 for the general form (2.4).

Ex. 2.6. Solve for x

$$\begin{array}{ll} (a)\ 15x - x^2 = 0 & (b)\ x^2 - 16 = 0 \\ (c)\ (x-3)(x+4) = 0 & (d)\ x^2 - 4x + 4 = 0 \\ (e)\ x^2 - 5x + 6 = 0 & (f)\ x^2 - x - 12 = 0 \\ (g)\ -\dfrac{1}{4}x^2 + \dfrac{1}{2}x + \dfrac{1}{2} = 0 & (h)\ x^2 + 11x - 26 = 0 \\ (i)\ x^3 - 4x = 0 & (k)\ x^4 - 5x^2 + 4 = 0 \\ (l)\ x^{-2} - 2x^{-1} - 15 = 0 & (m)\ 3x^2 = 5x - 1 \end{array}$$

2.4 System of 2 Linear Equations

The form

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

Example 2.4. Find x, y that satisfy both equations

$$\begin{cases} 2x + 3y = 18 \\ 3x - 4y = -7 \end{cases}$$

There are 2 methods: substitution and elimination.
Method 1 From the first equation, we have:

$$y = 6 - \frac{2}{3}x$$

Plugging into the second equation yields

$$3x - 4\left(6 - \frac{2}{3}x\right) = -7$$
$$\Leftrightarrow x = 3$$

which yields $y = 4$.

Method 2 Multiplying the second equation with 2/3 [1], the system becomes

$$\begin{cases} 2x + 3y = 18 \\ 2x - \frac{8}{3}y = -\frac{14}{3} \end{cases}$$

Subtract the second equation from the first one yields

$$\begin{cases} 2x + 3y = 18 \\ \frac{17}{3}y = \frac{68}{3} \end{cases}$$

which also yields $y = 4, x = 3$.

[1] A smarter way would be to multiply the first equation by 4 and the second equation by 3

Ex. 2.7. Solve the following system for x, y

$$(a)\ \begin{aligned} x - y &= 5 \\ x + y &= 11 \end{aligned} \qquad (b)\ \begin{aligned} 4x - 3y &= 1 \\ 2x + 9y &= 4 \end{aligned}$$

$$(c)\ \begin{aligned} 23x + 45y &= 181 \\ 10x + 15y &= 65 \end{aligned} \qquad (d)\ \begin{aligned} 0.01x + 0.21y &= 0.042 \\ -0.25x + 0.55y &= -0.47 \end{aligned}$$

Ex. 2.8. Answer the following questions by formulating the right equations

(a) 5 tables and 20 chairs cost \$1800, but 2 tables and 3 chair cost \$420. What are the unit prices for a chair and a table?

(b) A firm makes 2 goods, A and B. The estimated output of A is 50% higher than that of B. The profit per unit sold is \$300 for A and \$200 for B. If the profit target is \$13,000, how many units of A and B should the firm produce?

(c) At the beginning of the year, a person has a total of \$10,000 in 2 accounts. The first account yields an interest rate of 5% and the other yields 7.2% per year. At the end of the year, the total interest earned from both accounts is \$ 676. What was the initial balance in each of the 2 accounts?

Ex. 2.9. Harder problems

$$(a)\ \begin{aligned} \tfrac{2}{x} + \tfrac{3}{y} &= 4 \\ \tfrac{3}{x} - \tfrac{2}{y} &= 19 \end{aligned} \qquad (b)\ \begin{aligned} 3\sqrt{x} + 2\sqrt{y} &= 2 \\ 2\sqrt{x} - 3\sqrt{y} &= 1/4 \end{aligned}$$

$$(c)\ \begin{aligned} x^2 + y^2 &= 13 \\ 4x^2 - 3y^2 &= 24 \end{aligned}$$

2.5 Nonlinear Equations

In general

$$ab = ac \Leftrightarrow a = 0 \text{ or } b = c$$

Example 2.5.

$$(a)\ x^3\sqrt{x+2} = 0$$

$$(b)\ x(y+3)(z^2+1)\sqrt{w-3} = 0 \qquad (c)\ x^2 - 3x^3 = 0$$

$$(d)\ x(x+a) = x(2x+b) \qquad (e)\ xy^2(1-y) - 2\lambda(y-1) = 0$$

$$(f)\ \frac{1-K^2}{\sqrt{1+K^2}} = 0 \qquad (g)\ \frac{45+6r-3r^2}{(r^4+2)^{2/3}} = 0$$

$$(h)\ \frac{x^2-5x}{\sqrt{x^2-25}} = 0$$

Ex. 2.10. Solve

$$(a)\ \frac{5+x^2}{(x-1)(x+2)} = 0$$

$$(b)\ 1 + \frac{2x}{x^2+1} = 0 \qquad (c)\ \frac{(x+1)^{1/3} - \frac{1}{3}x(x+1)^{-2/3}}{(x+1)^{2/3}} = 0$$

$$(d)\ \frac{x}{x-1} + 2x = 0 \qquad (e)\ (x^2-4)\sqrt{5-x} = 0$$

$$(f)\ (x^4+1)(4+x) = 0 \qquad (g)\ k^\alpha = \left(\frac{w}{r}\right)^{\frac{\sigma}{1-\sigma}}$$

Chapter 3

Functions of One Variable

3.1 The Basics

Notation

$$y = f(x)$$

A function assigns a unique real number (y) to each number x in the domain **D**, based on some specific **rules** described in the function. The set of all possible values of y is called the range of f. We often use letters f, g, h, u, φ for functions.

Example 3.1.

$$\begin{aligned} &(a)\ f(x) = x^3, && (b)\ f(x) = 2x + 1 \\ &(c)\ C(x) = 100x\sqrt{x} + 500, && (d)\ C(x) = Ax^2 - B \\ &(e)\ f(x) = \frac{1}{x+3} && (f)\ g(x) = \sqrt{2x+4} \\ &(g)\ f(k) = Ak^\alpha && (h)\ u(c) = c^{1/2} \end{aligned}$$

Ex. 3.1. Find the domain of (e), (f), (g), (h) (The values for x so that its function is defined).

Ex. 3.2. Let

$$f(x) = x^2 + 1$$

(a) Compute $f(0), f(-1), f(1/2), f(\sqrt{2})$.
(b) For what values of x, it is true that

1. f(x) = f(-x)?
2. f(x+1) = f(x) + f(1)?
3. f(2x) = 2f(x)?

(c) Suppose $f(x) = a^2 - (x - a)^2$ where a is a constant, compute

1. $f(0), f(a), f(-a), f(2a)$
2. $3f(a) + f(-2a)$

Ex. 3.3. Find the domain and range of the functions

$(a)\ y = \sqrt{5-x}$ $\qquad$ $(b)\ y = \dfrac{2x-1}{x^2-x}$

$(c)\ y = \sqrt{\dfrac{x-1}{(x-2)(x+3)}}$ $\qquad$ $(d)\ y = 1-\sqrt{x+2}$

3.2 Graphs

One of the most important aspects of a function is to graph because graphs provide some visual cues and intuition before we want to prove something analytically. For example, the following are the graphs of $y = x^2 - 2x - 1$ and $y = x^2 + 2x + 1$. From here, we can visually see that there is no value of x such that $x^2 + 2x + 1 = 0$. Also, both functions have a global minimum. The function is decreasing before it reaches the minimum and then turns to an increasing function.

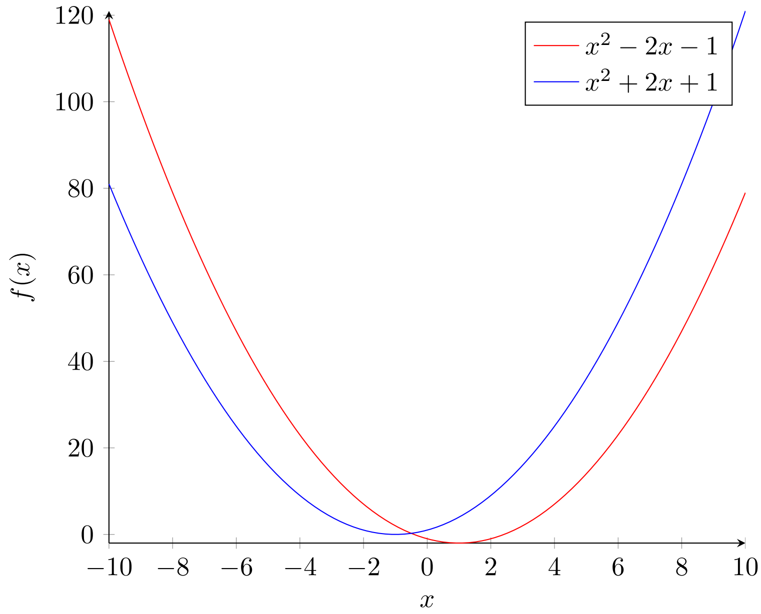


Fig. 3.1. Examples of graphs

3.3 Linear Functions

The form

$$y = f(x) = ax + b$$

The graph is a **straight line**.

How much would the function changes when we increase x by 1 unit?

$$\triangle f = f(x+1) - f(x) = a(x+1) + b - (ax+b) = a$$

i.e., f changes by a unit. For this reason, the number a is the **slope** of the line (or the function).

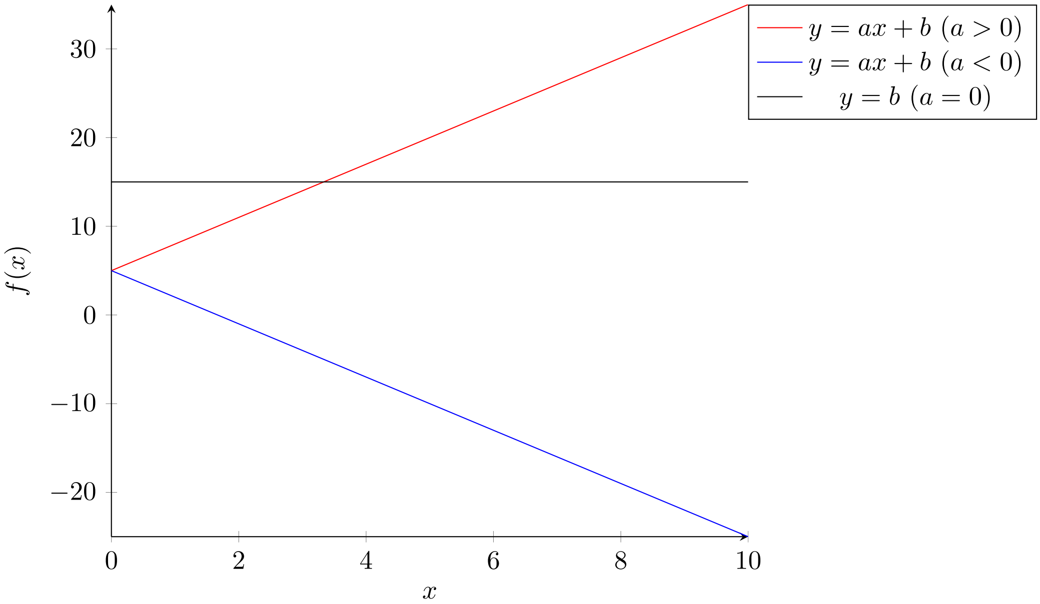


Fig. 3.2. Slope is positive (a¿0), the line slants upward right. The line slants downward if the slope is negative (a¡0). The line is horizontal when there is no steepness (a=0). The point where $x = 0$ is called **y-intercept**

The slope of a straight line L is

$$a = \frac{y_2 - y_1}{x_2 - x_1} \quad (x_1 \neq x_2)$$

where $(x_1, y_1), (x_2, y_2)$ are 2 distinct points on L.

To draw the graph of a linear function, given 2 points [(1)] on that line

1. First, calculate the slope

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

2. Plug into the following formula

$$y - y_1 = a(x - x_1)$$

then rearrange to get the form $y = ax + b$.

Ex. 3.4. Draw graphs of the following

(a) $3x + 4y = 12$ (b) $\dfrac{x}{10} - \dfrac{y}{5} = 1$

(c) $x = 3$

[(1)]Normally, the easiest 2 points are $(0, f(0))$ and $(x, f(x) = 0)$

3.4 Quadratic Functions

The form

$$y = f(x) = ax^2 + bx + c \qquad (a \neq 0)$$

In general, the graph of a quadratic function is a parabola.

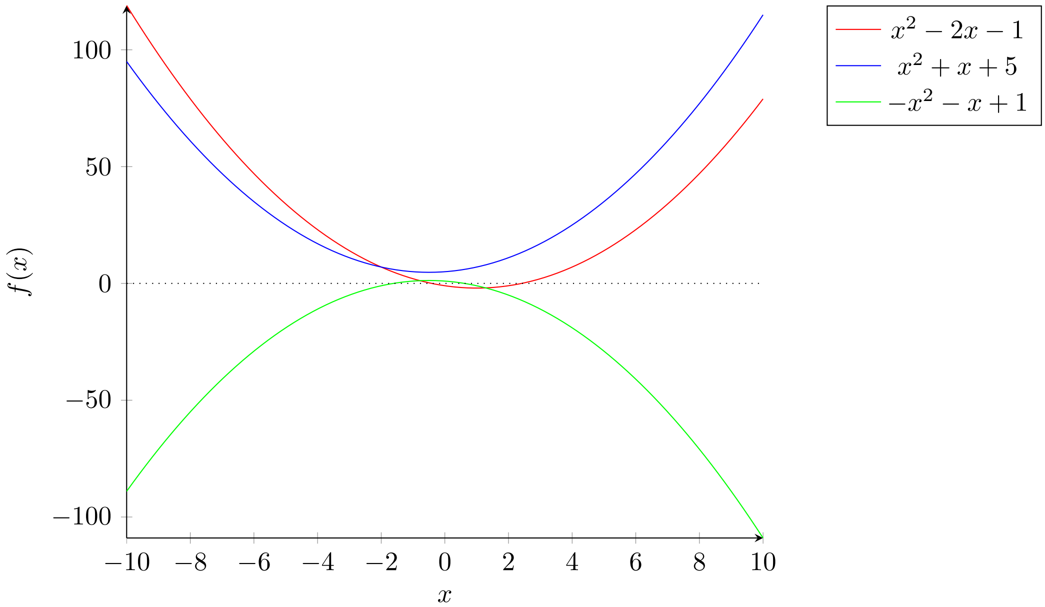


Fig. 3.3. Some graphs of different quadratic functions

It's easy to see that [blue] and [red] graphs have a min, while [green] has a max.

Ex. 3.5. Verify that: For a quadratic function with the form $f(x) = a^2 + bx + c$,

- If $a > 0$, the minimum is at $x = -b/2a$
- If $a < 0$, the maximum is at $x = -b/2a$

In economics, the cost function usually is a quadratic function.

Example 3.2. A firm set the price as follows

$$P = 102 - 2Q$$

where P is the unit price and Q is quantity. The cost of producing Q units is

$$C = \frac{1}{2}Q^2 + 2Q$$

Suppose the profit is $\pi = PQ - C$, find the optimal Q that maximizes profit.

Ex. 3.6. Sandmo model of efficient loan markets

$$U(x) = 72 - (4 + x^2)^2 - (4 - rx^2)$$

where r is a constant. Find x that maximizes $U(x)$.

3.5 Polynomials

A cubic function has the form

$$y = f(x) = ax^3 + bx^2 + cx + d \quad (a \neq 0)$$

From here, we can generalize a polynomial function of degree n as

$$y = f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Linear, quadratic, and cubic functions are polynomial functions of degrees 1, 2, and 3 respectively.

To find the integer roots of a polynomial $f(x)$ is to find the value of x such that $f(x) = 0$. Then, if such integer roots exist, it must be a factor of a_0.

Ex. 3.7. Find integer roots of

$$x^3 - 2x^2 + x - 2 = 0$$

All integer solutions of this equation must be factors of 2. The candidates are $\pm 1, \pm 2$. Simple checks show that $x = 2$ is the only integer solution. Knowing this root, we can factorize the equation

$$\begin{aligned} & x^3 - 2x^2 + x - 2 \\ &= (x-2)(x^2+1) \end{aligned}$$

Polynomial Division

Example 3.3. Divide

$$\begin{aligned} &(a)\ (-x^3 + 4x^2 - x - 6) \div (x - 2) \\ &(b)\ (x^4 + 3x^2 - 4) \div (x^2 + 2x) \end{aligned}$$

Results:

$$\begin{aligned} &(a)\ -x^2 + 2x + 3 && \text{remainder: } 0 \\ &(b)\ x^2 - 2x + 7 && \text{remainder: } -14x - 4 \end{aligned}$$

Ex. 3.8. Find integer roots then factorize the following equations

$$\begin{aligned} &(a)\ x^4 - x^3 - 7x^2 + x + 6 = 0 && (b)\ 2x^3 + 11x^2 - 7x - 6 = 0 \\ &(c)\ x^4 + x^3 + 2x^2 + x + 1 = 0 && (d)\ \frac{1}{4}x^3 - \frac{1}{4}x^2 - x + 1 = 0 \\ &(e)\ x^5 - 4x^3 - 3 = 0 \end{aligned}$$

Ex. 3.9. Divide

$$\begin{aligned} &(a)\ (2x^3 + 2x - 1) \div (x - 1) && (b)\ (x^6 + x^3 + x^2 + x) \div (x^2 + x) \\ &(c)\ (x^5 - 3x^4 + 1) \div (x^2 + x + 1) && (d)\ (3x^8 + x^2 + 1) \div (x^3 - 2x + 1) \end{aligned}$$

Ex. 3.10. In demand theory, we have the function

$$E = \alpha \frac{x^2 - \gamma x}{x + \beta}$$

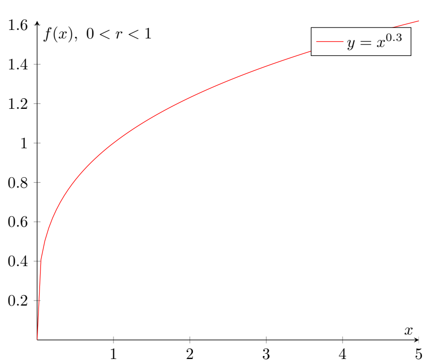
Perform the division $(x^2 - \gamma x) \div (x + \beta)$ and express E as a sum of a linear function and a fraction. (Hint: $x - (\beta + \gamma)$ remainder $\beta(\beta + \gamma)$.

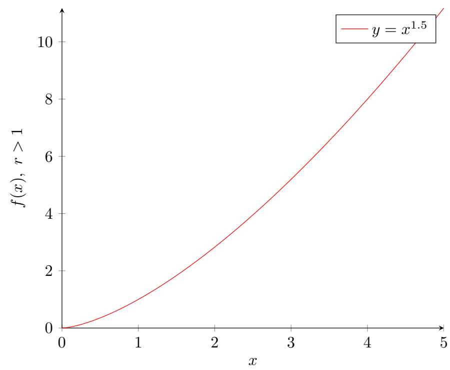
3.6 Power Functions

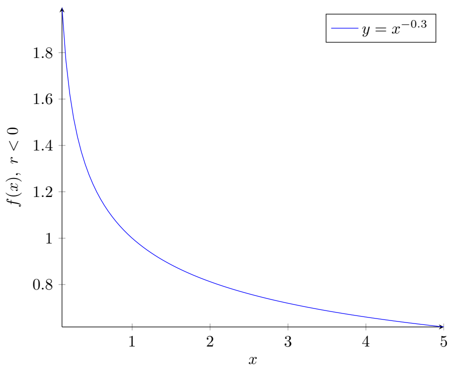
In general form (say: x **raised to the power of** r)

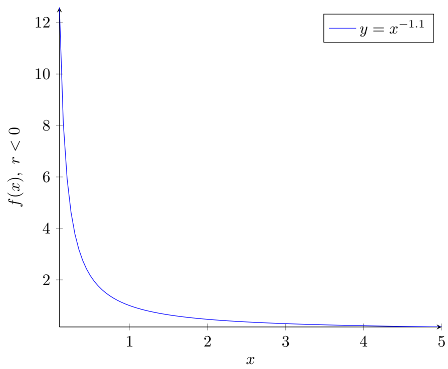
$$f(x) = Ax^r \quad (x > 0, \quad r \text{ and } A \text{ are constants})$$

The graph of power functions. Note that the signs of the **exponent** r influence the shape of the function.









Ex. 3.11. Sketch the graph

$$(a)\ y = x^{-3} \qquad\qquad (b)\ y = x^{-1/2}$$

Ex. 3.12. Solving for x

$$(a)\ 2^{2x} = 8 \qquad\qquad (b)\ 3^{3x+1} = 1/81$$

$$(c)\ 10^{x^2-2x+2} = 100 \qquad\qquad (d)\ 3^{5x}9^x = 27$$

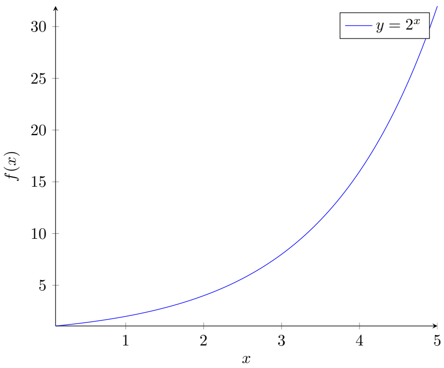
The power function is important since most of the utility and production-per-capita functions normally take this form. For instance, $U(c) = c^{1-\sigma}/(1-\sigma)$ or $y = k^{\alpha}$. The shapes of these functions (which depends on the sign of the exponent) are also worth paying attention to.

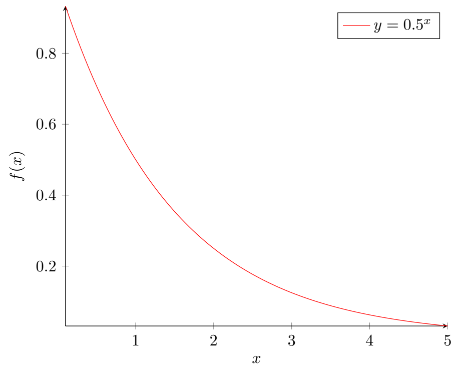
3.7 Exponential Functions

In general form:

$$y = f(x) = Aa^x \quad (A, a \text{ are positive integer })$$

Exponential functions appear in many important topics of economics such as economic and population growth, compound interest, exponential decay, etc.





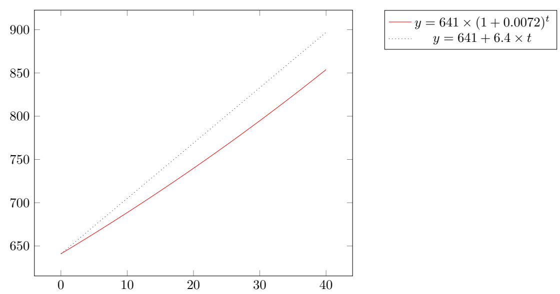
We can consider a the factor by which $f(x)$ changes when x increases by 1.

- If $a = 1 + p(\%)$, where $0 < p < 1$, $A > 0$, then $f(x)$ will increase by $p(\%)$ for each unit increase in x.
- If $a = 1 - p(\%)$, where $0 < p < 1$, $A > 0$, then $f(x)$ will decrease by $p(\%)$ for each unit increase in x.

Example 3.4. (Population Growth)
In the year 1960, the population of Europe is 641 mil. people. If the population grows at the annual rate of $g = 0.72\%$. How is the population in the year 2000? Plot the graph. Compared to a linear function $P = 641 + 6.4t$, which one do you think is a more reasonable estimation?
Let $t = 2000 - 1960 = 40$, the exponentially growing population is

$$P_{t+40} = P_t(1+g)^t$$
$$\Rightarrow P_{2000} = P_{1960}\ (1 + 0.0072)^{40} = 854$$

If we applied the linear function, the population would be 897.



Obviously, the exponential growth function provides more accuracy. Furthermore, a linear function implies that the population is negative for $t \leq 101$.

The Natural Exponential Function
The base e is called the natural exponential function.

$$f(x) = e^x = \exp(x)$$

where e is estimated to be

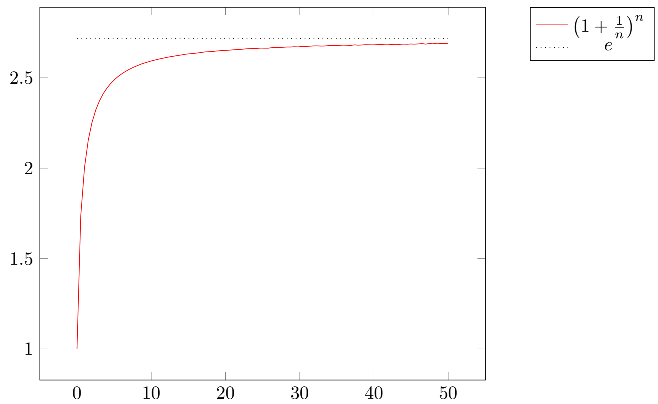
$$e = 2.718281828459045\ldots$$

e is also known as the Euler number. It's natural for many special reasons

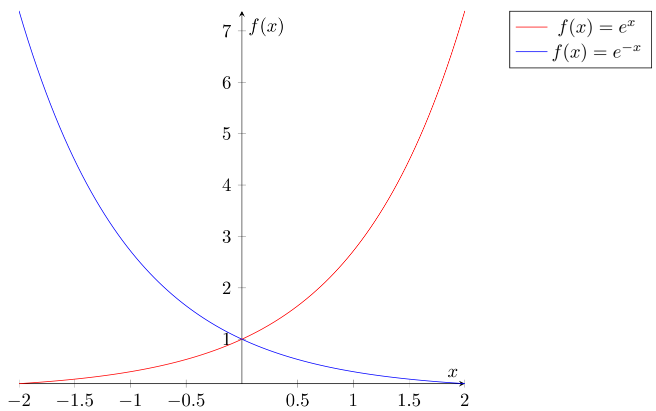
- Many things in nature grow with this base
- Its derivative is its own function $(e^x)' = e^x$.

The functional form of e is actually

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$



The graphs of $f(x) = e^x$ and $f(x) = e^{-x}$ are



Ex. 3.13. To know why e is important, calculate Example 3.4 using the following formula (other parameters remain the same, $P_t = 641, t = 40, g = 0.72/100$)

$$P_{t+40} = P_t \times e^{gt}$$

Compare the two results.

3.8 Logarithmic Functions

In general, if $a^x = b$ and $a, b > 0$

$$\log_a b = x$$

to **find the solution for a raised to the power of WHAT to get b.**

Thus, we can derive some important properties.

$$\log_a 1 = 0 \qquad \log_a a = 1$$
$$\log_a a^m = m \qquad \log_a a^{\log_a n} = n$$
$$\log_a(m \times n) = \log_a m + \log_a n \qquad \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$
$$\log_a b^m = m \log_a b \qquad \log_a b = \frac{1}{\log_b a}$$

Change of Base Formula

Ex. 3.14. Prove that for $a, b, c > 0$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Hint: First, take a raised to both sides, then takes log_c of both sides.

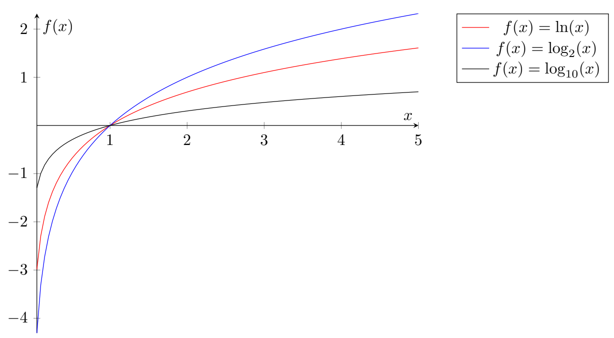
Some special properties of log base e (also written as ln)

$$\ln(1) = 0, \;\ln(e) = 1,$$
$$e^{\ln(x)} = x \quad (x > 0)$$
$$\ln(x) = \log_e x \Rightarrow \ln(e^x) = x$$

The log of any base can be converted to a natural base log with the Change of Base formula

$$\log_a b = \frac{\ln(b)}{\ln(a)}$$

The graphs of log with different bases.



We can see that the result of $x = \log_a b$ for any positive $b < 1$ is an $x < 0$.

Ex. 3.15. Solve for x:

$$\begin{array}{ll}
(a)\ 3^x = 8, & (b)\ \ln(x) = 3, \\
(c)\ \ln(x^2 - 4x + 5) = 0 & (d)\ \ln(x(x-2)) = 0 \\
(e)\ \dfrac{x\ln(x+3)}{x^2+1} = 0 & (f)\ \ln(\sqrt{x} - 5) = 0 \\
(g)\ 3^x 4^{x+2} = 8 & (h)\ 3\ln(x) + 2\ln(x^2) = 6 \\
(i)\ 4^x - 4^{x-1} = 3^{x+1} - 3^x & (k)\ \log_2(x) = 2 \\
(l)\ \log_x e^2 = 2 & (m)\ \log_3 x = -3
\end{array}$$

Ex. 3.16. Prove the Rule of 70 (doubling time)

Given the growth rate $g(\%)$, for an exponential growth function of the form

$$A(t) = A_0 \times (1+g)^t$$

prove that if $A(t) = 2A_0$, then

$$t = \frac{\ln(2)}{\ln(1+g)}$$

Note: For small growth g close to 0, why is

$$A_0 \times (1+g)^t \approx A_0 \times e^{gt}\ ? \tag{3.1}$$

Proof. We need to show that

$$(1+g)^t \approx e^{gt}$$

or equivalently

$$\begin{aligned} \ln((1+g)^t) &= \ln e^{gt} \\ \Leftrightarrow t\ln(1+g) &= gt \end{aligned}$$

which can be reduced to proving

$$\ln(1+g) \approx g$$

for small $g \approx 0$.

This can be seen from linear approximation using the first-order Taylor formula. In general, the approximation of $f(x)$ at a is

$$f(x) \approx f(a) + f'(a)(x-a)$$

Let $u = 1+g$, for very small g, then u is very close to 1. The approximation of $f(u) = \ln(u)$ at 1 is

$$f(1) \approx f(1) + f'(u)(u-1)$$

We know that

$$\begin{aligned} f(1) &= \ln(1) = 0, \\ f'(u) &= (\ln(1+g))' = \frac{1}{1+g} \approx 1 \end{aligned}$$

So that

$$f(1) \approx 0 + 1 \times (u-1) = 1 + g - 1 = g$$

Thus, for small g close to 0, $\ln(1+g) \approx g$ and we conclude (3.1). ∎

Chapter 4

Properties of Functions

4.1 Shifting graphs

In the last chapter we learned how to draw the graph of a function. Now we consider how changes in a function relate to shifts in its graph.

The general rules for shifting the graph of $y = f(x)$ is:

(i) If $y = f(x)$ is replaced by $y = f(x) + c$, the graph is moved upwards by c units if $c > 0$; it is moved downwards if $c < 0$.

(ii) If $y = f(x)$ is replaced by $y = f(x + c)$, the graph is moved c units to the left if $c > 0$; it is moved to the right if $c < 0$.

(iii) If $y = f(x)$ is replaced by $y = cf(x)$, the graph is stretched vertically if $c > 0$; it is stretched vertically and reflected about the x-axis if $c < 0$.

(iv) If $y = f(x)$ is replaced by $y = f(-x)$, the graph is reflected about the y-axis.

Example 4.1. Consider the simple example when $c = 2$ for the function $y = \sqrt{x}$, then the 4 transformations above would look like the followings:

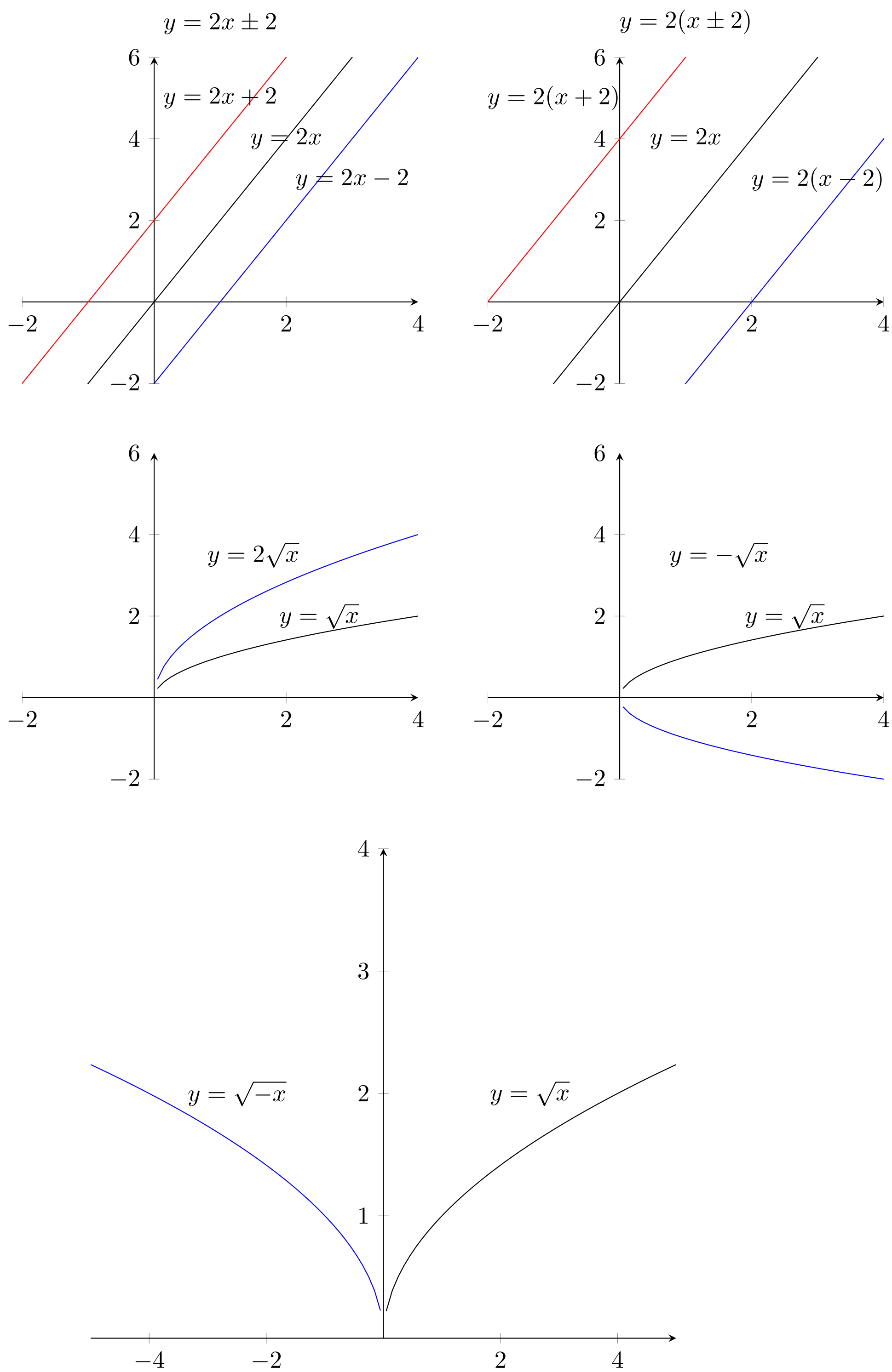


Fig. 4.1. Transformation of graphs

Ex. 4.1. Use the rules for shifting graphs and the following graph of $y = |x|$ to sketch the graph of $y = 2 - |x + 2|$.

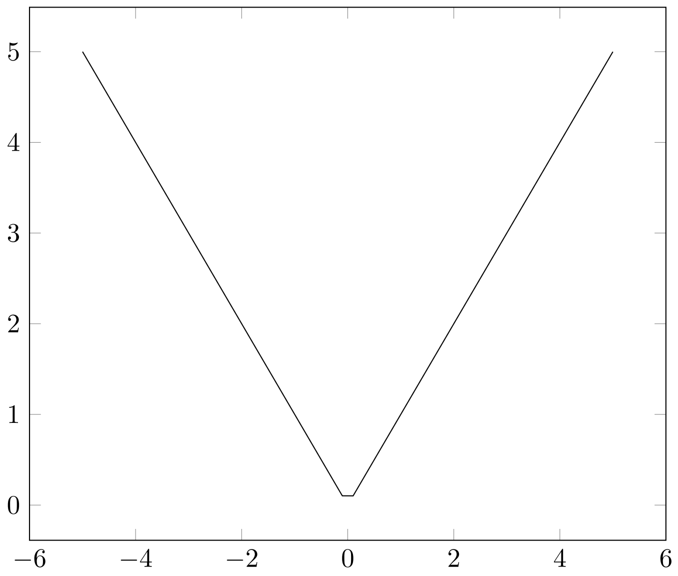


Fig. 4.2. Exercise

Ex. 4.2. Use the rules for shifting graphs to sketch the graphs of the following functions:

(a) $y = x^2 + 1$

(b) $y = (x + 3)^2$

(c) $y = 3 - (x + 1)^2$

Ex. 4.3. Starting from the graph of $f(x) = \frac{1}{x^2}$, sketch the graph of $g(x) = 2 - (x + 2)^{-2}$.

4.2 New functions from the old

4.2.1 Sum and difference

Suppose in general that f and g are functions which are both defined in a set A of real numbers. The function F defined by the formula $F(x) = f(x) + g(x)$ is called the **sum** of f and g, and we write $F = f + g$. The function G defined by $G(x) = f(x) - g(x)$ is called the **difference** between f and g, and we write $G = f - g$.

Example 4.2. Suppose the cost of producing $Q > 0$ units of a commodity is $C(Q)$ defined as:

$$C(Q) = a * Q^3 + b * Q^2 + c * Q + d$$

A commonly used measure in Economics is **average cost**, defined as $A(Q) = C(Q)/Q$:

$$A(Q) = a * Q^2 + b * Q + c + \frac{d}{Q}$$

Thus $A(Q)$ can be expressed as the sum of a quadratic function and a hyperbola. The sum graph is obtained by piling one graph on top of the other as shown in Figure 4.3.

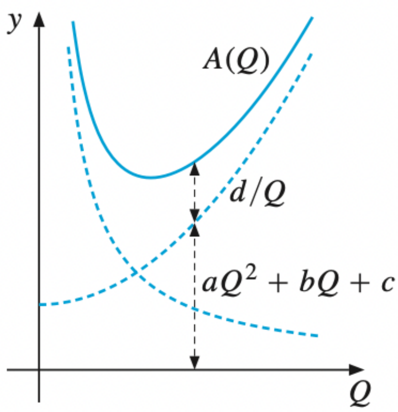


Fig. 4.3. Sum function

Example 4.3. Now let $R(Q)$ denote the revenue obtained by selling Q units. Then, the profit $\pi(Q)$ is given by $\pi(Q) = R(Q) - C(Q)$.
In the case of the firm getting a fixed price p per unit, so that the graph of $R(Q)$ is a straight line through the origin. The graph of $C(Q)$ must be subtracted from that of $R(Q)$. The production level that maximises profit is Q^*.
Similar to the sum function, subtracting the two graphs gives the difference graph as shown in Figure 4.4.

4.2.2 Products and quotients

If f and g are defined in a set A, the function F defined by $F(x) = f(x) \cdot g(x)$ is called the **product** of f and g, and we put $F = f \cdot g$ (or fg).

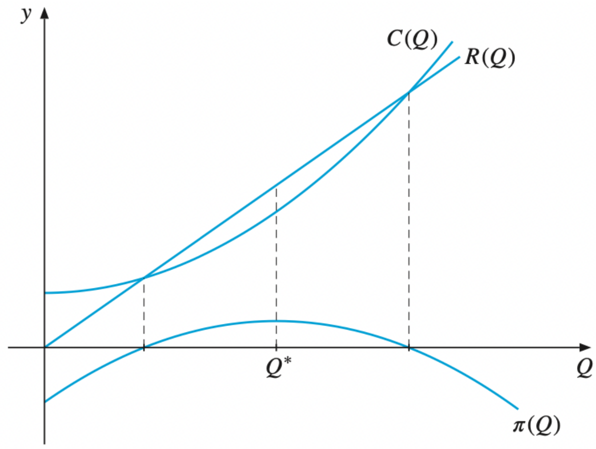


Fig. 4.4. Difference function

The function F defined where $g(x) \neq 0$ by $F(x) = f(x)/g(x)$ is called the **quotient** of f and g, and we write $F = f/g$.

4.2.3 Composite functions

Suppose the demand for a commodity is a function x of its price p. Suppose that price p is not constant, but depends on time t. Then it is natural to regard x as a function of t.

In general, if y is a function of u, and u is a function of x, then y can be regarded as a function of x. We call y a **composite function** of x. If we denote the two functions involved by f and g, with $y = f(u)$ and $u = g(x)$, then we can replace u by $g(x)$ and so write y in the form:

$$y = f(g(x))$$

When computing y, we first apply g to x to obtain $g(x)$, and then we apply f to $g(x)$. Here $g(x)$ is called the **kernel**, or **interior function**, while f is called the **exterior function**.

The function that maps x to $f(g(x))$ is often denoted by $f \circ g$. This is read as "f of g" or "f after g", and is called the **composition** of f with g. Correspondingly, $g \circ f$ denotes the function that maps x to $g(f(x))$. Thus, we have:

$$(f \circ g)(x) = f(g(x)) \quad \textit{and} \quad (g \circ f)(x) = g(f(x))$$

Note that $f \circ g$ and $g \circ f$ are usually quite different functions. For instance, if $g(x) = 2 - x^2$ and $f(u) = u^3$, then $(f \circ g)(x) = (2 - x^2)^3$, whereas $(g \circ f)(x) = 2 - (x^3)^2 = 2 - x^6$; the two resulting polynomials are not the same.

Example 4.4. Write the followings as composite functions:

(a) $y = (x^3 + x^2)^{50}$

(b) $y = e^{-(x-\mu)^2}$

Solution:

(a) Given a value of x, first compute x^3+x^2, which gives the interior function, $g(x) = x^3 + x^2$. Then take the 50th power of the result, so the exterior function is $f(u) = u^{50}$. Hence,

$$f(g(x)) = f(x^3 + x^2) = (x^3 + x^2)^{50}$$

(b) We can choose the interior function as $g(x) = -(x-\mu)^2$ and the exterior function as $f(u) = e^u$. Alternatively, we could choose $g(x) = (x-\mu)^2$ and $f(u) = e^{-u}$.

4.2.4 Symmetry

The function $f(x) = x^2$ satisfies $f(-x) = f(x)$, as indeed does any even power x^2n , with n an integer, positive or negative. So if $f(-x) = f(x)$ for all x in the domain of f , then f is called an **even function**. This condition implies that the graph of f is **symmetric** about the y-axis as shown in Fig. 4.5.

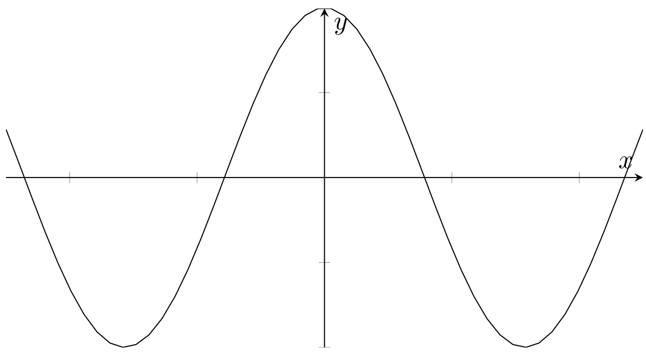


Fig. 4.5. Even function

On the other hand, any odd power x^{2n+1} such as $f(x) =^3$ satisfies $f(-x) = -f(x)$. So if $f(-x) = -f(x)$ for all x in the domain of f, implying that the graph of f is symmetric about the origin, as shown in Fig. 4.6, then f is called an **odd function**.

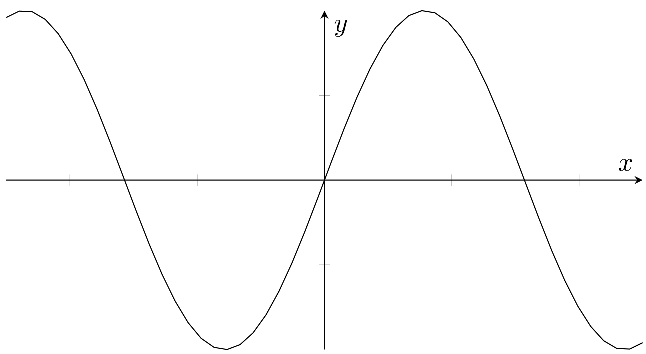


Fig. 4.6. Odd function

Finally, f is symmetric about a if $f(a+x) = f(a-x)$ for all x.The graph of f is then symmetric about the line $x = a$. For example, the quadratic function $f(x) = ax^2 + bx + c$ is symmetric about $x = -b/2a$. The function $y = e^{-(x-\mu)^2}$ is symmetric about $x = \mu$.

Ex. 4.4. Assuming $x > 0$, show graphically how to find the graph of $y = \frac{1}{4}x^2 + \frac{1}{x}$, by adding the graph of 1/x to the graph of $y = \frac{1}{4}x^2$.

Ex. 4.5. Sketch the graphs of the following functions:

(a) $y = \sqrt{x} - x$

(b) $y = e^x + e^{-x}$

(c) $y = e^{-x^2} + x$

Ex. 4.6. If $f(x) = 3x - x^3$ and $g(x) = x^3$, compute:$(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, $(f/g)(x)$, $f(g(1))$, and $g(f(1))$.

Ex. 4.7. Let $f(x) = 3x + 7$. Compute $f(f(x))$, and find the value x^* when $f(f(x^*)) = 100$.

4.3 Inverse functions

Suppose that the demand quantity D for a commodity depends on the price per unit P according to $D = 30/P^{1/3}$. This formula means that the demand D is corresponding to a given price P.
If we look at the matter from a producer's point of view, however, it may be more natural to treat the output as something it can choose and consider the resulting price. This functional relationship is obtained by solving $D = 30/P^{1/3}$ for P.
The two variables D and P in this example are related in a way that allows each to be regarded as a function of the other. In fact, the two functions:

$$f(P) = 30p^{-1/3} \quad and \quad g(D) = 27000D^{-3}$$

are **inverses** of each other.
In general, Let f be a function with domain A and range B. If and only if f is one-to-one, it has an inverse function g with domain B and range A. The function g is given by the following rule: For each y in B, the value $g(y)$ is the unique number x in A such that $f(x) = y$. Then

$$g(y) = x \Leftrightarrow y = f(x)(x \in A, y \in B)$$

Example 4.5. Solve the following equations for x and find the corresponding inverse functions:

(a) $y = 4x - 3$

(b) $y = \sqrt[5]{x+1}$

(c) $y = \frac{3x-1}{x+4}$

Solution:

(a) Solving the equation for x, we have the following equivalences:

$$y = 4x - 3 \Leftrightarrow 4x = y + 3 \Leftrightarrow x = \frac{1}{4}y + \frac{3}{4}$$

for all x and y. We conclude that $y = 4x - 3$ and $g(y) = \frac{1}{4}y + \frac{3}{4}$ are inverses of each other.

(b) We begin by raising each side to the fifth power and so obtain the equivalences

$$y = \sqrt[5]{x+1} \Leftrightarrow y^5 = x + 1 \Leftrightarrow x = y^5 - 1$$

These are valid for all x and all y. Hence, we have shown that $f(x) = \sqrt[5]{x+1}$ and $g(y) = y^5 - 1$ are inverses of each other.

(c) Here we begin by multiplying both sides of the equation by the denominator and rearranging. Then we will obtain

$$x = \frac{4y + 1}{3 - y}$$

We conclude that $f(x) = (3x - 1)/(x + 4)$ and $g(y) = (4y + 1)/(3 - y)$ are inverses of each other. Observe that f is only defined for $x \neq -4$, and g is only defined for $y \neq 3$.

Graphically, suppose two functions f and g are inverses of each other. Provided that the scales of the coordinate axes are the same, the graphs of $y = f(x)$ and $y = g(x)$ are symmetric about the line $y = x$.

Example 4.6. Consider the functions $f(x) = 4x - 3$ and $g(x) = \frac{1}{4}x + \frac{3}{4}$ that are inverses of each other.

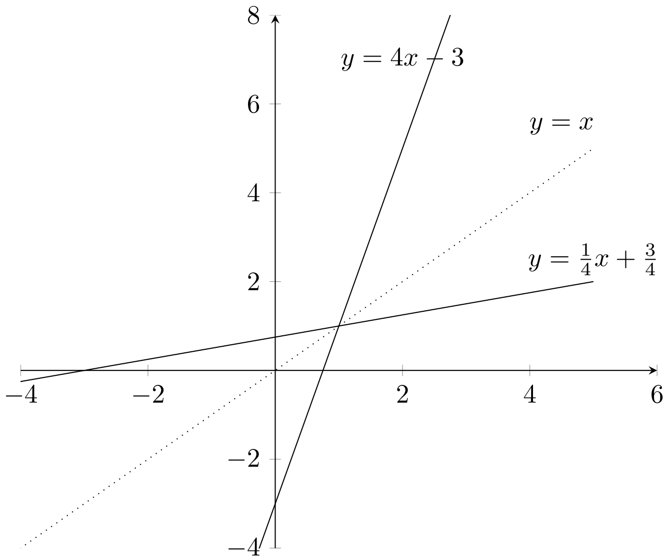


Fig. 4.7. Geometric characterisation of inverse functions

Ex. 4.8. Demand D as a function of price P is given by $D = \frac{32}{5} - \frac{3}{10}P$. Solve the equation for P and find the inverse function.

Ex. 4.9. Find the domains, ranges, and inverses of the functions given by the following formulas:

(a) $y = -3x$

(b) $y = 1/x$

(c) $y = x^3$

(d) $y = \sqrt{\sqrt{x} - 2}$

Ex. 4.10. Why does $f(x) = x^2$, for x in $(-\infty, \infty)$, have no inverse function? Show that f restricted to $[0, \infty)$ has an inverse, and find that inverse.

Ex. 4.11. Find inverses of the following functions, where x is the independent variable:

(a) $f(x) = (x^3 - 1)^{1/3}$

(b) $y = \frac{x+1}{x-2}$

(c) $f(x) = (1 - x^3)^{1/5} + 2$

Ex. 4.12. The functions defined by the following formulas are strictly increasing in their domains. Find the domain of each inverse function, and a formula for the corresponding inverse.

(a) $y = e^{x+4}$

(b) $y = ln\, x - 4$, $x > 0$

(c) $y = ln(2 + e^{x-3}$

Ex. 4.13. Find the inverse of $f(x) = \frac{1}{2}(e^x - e^{-x})$. (Hint: Solve a quadratic equation in $z = e^x$.)

4.4 Graphs of equations

The equations $x\sqrt{y} = 2$, $x^2 + y^2 = 16$, $and\, y^3 + 3x^2y = 13$ are three examples of equations in two variables x and y. A solution of such an equation is an ordered pair (a, b) such that the equation is satisfied when we replace x by a and y by b. The solution set of the equation is the set of all solutions. Representing all pairs in the solution set in a Cartesian coordinate system gives a set called the graph of the equation.

Example 4.7. Find some solutions of the first two of the equations, and try to sketch their graphs.
From $x\sqrt{y} = 2$ we obtain $y = 4/x^2$. Hence it is easy to find corresponding values for x and y as given in Table 4.1:

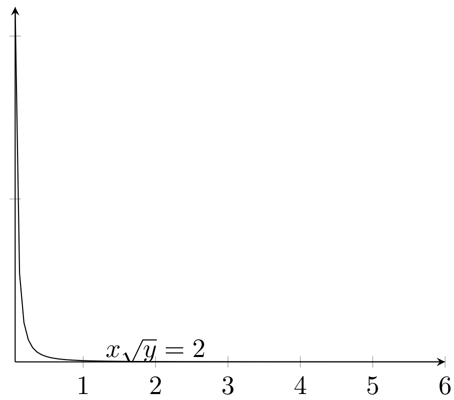
x	1	2	4	6
y	4	1	1/4	1/9

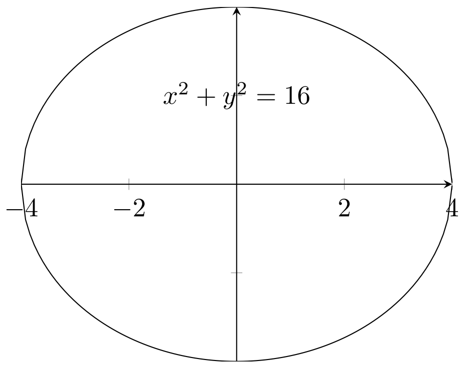
Table 4.1: Solutions of $x\sqrt{y} = 2$

For $x^2 + y^2 = 16$, if $y = 0$, $x^2 = 16$, so $x = \pm 4$. Thus (4,0) and (-4,0) are two solutions. Table 4.2 combines these with some other solutions.

x	-4	-3	-1	0	1	3	4
y	0	$\pm\sqrt{7}$	$\pm\sqrt{15}$	± 4	$\pm\sqrt{15}$	$\pm\sqrt{7}$	0

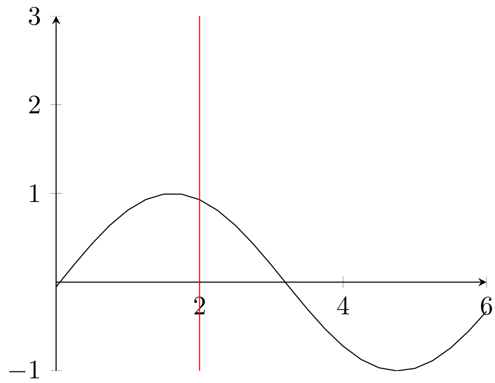
Table 4.2: Solutions of $x^2 + y^2 = 16$



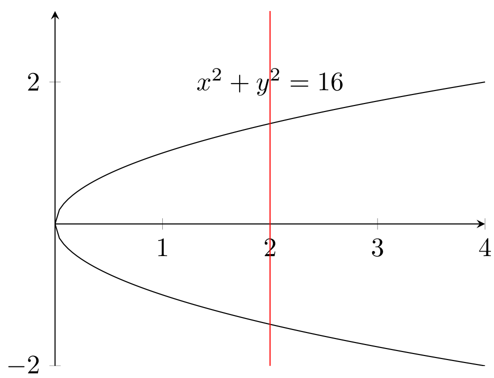


4.4.1 Vertical line test

By definition, a function assigns to each point x in the domain only one y-value. The graph of a function, therefore, has the property that a vertical line through any point on the x-axis has at most one point of intersection with the graph.



A function



Not a function

4.4.2 Compound functions

A function may be defined in several pieces, by giving a separate formula for each of a number of disjoint parts of the domain. Two examples of such compound functions are presented.

Example 4.8. Draw the graph of the function f defined by

$$f(x) = \begin{cases} -x \; for \; x \leq 0 \\ x^2 \; for \; 0 < x \leq 1 \\ 1.5 \; for \; x > 1 \end{cases}$$

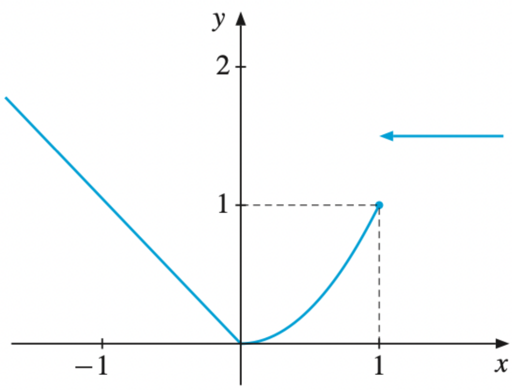


Fig. 4.8. Example of compound function

Ex. 4.14. Find some particular solutions of the following two equations, then sketch their graphs:

$(a)\; x^2 + 2y^2 = 6$ $\qquad$ $(b)\; y^2 - x^2 = 1$

Ex. 4.15. The function F is defined for all $r \geq 0$ by the following formulas:

$$F(r) = \begin{cases} 0 & for \; r \leq 7500 \\ 0.044(r - 7500) & for \; r > 7500 \end{cases}$$

Compute $F(100000)$, and sketch the graph of F.

4.5 Distance in the plane

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in the xy-plane. Then by Pythagoras's theorem, the distance d between P_1 and P_2 satisfies the equation $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.
In other words, the distance between the points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 4.9. Find the distance d between $P_1 = (-4, 3)$ and $P_2 = (5, -1)$. **Solution**:

$$d = \sqrt{(5 - (-4))^2 + (-1 - 3)^2} = \sqrt{9^2 + (-4)^2} = \sqrt{97}$$

4.5.1 Circles

The equation of a circle with centre at (a, b) and radius r is $(x-a)^2+(y-b)^2 = r^2$.

Example 4.10. Find the equation of the circle with centre at $(-4, 1)$ and radius 3. **Solution**: the equation of the circle is:

$$(x + 4)^2 + (y - 1)^2 = 9$$

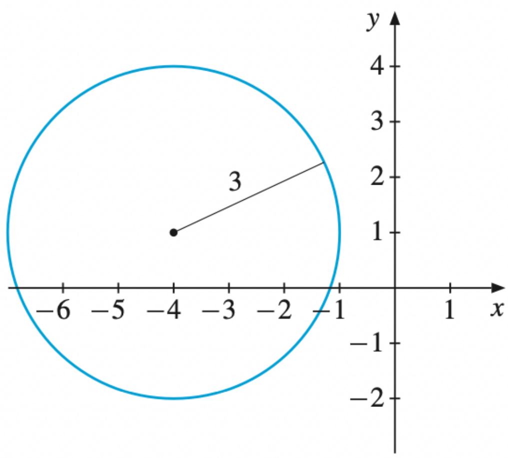


Fig. 4.9. Circle with centre at $(-4, 1)$ and radius 3

4.5.2 Ellipses and hyperbolas

The simplest type of **ellipse** has the equation:

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$

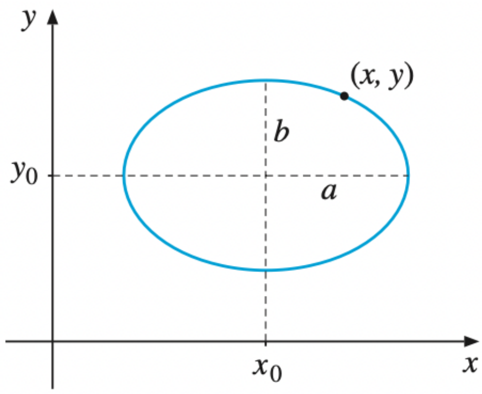


Fig. 4.10. Example of Ellipse

Figure 4.11 shows the graphs of two **hyperbolas**:

$$(a)\ \frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1 \qquad (b)\ \frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=-1$$

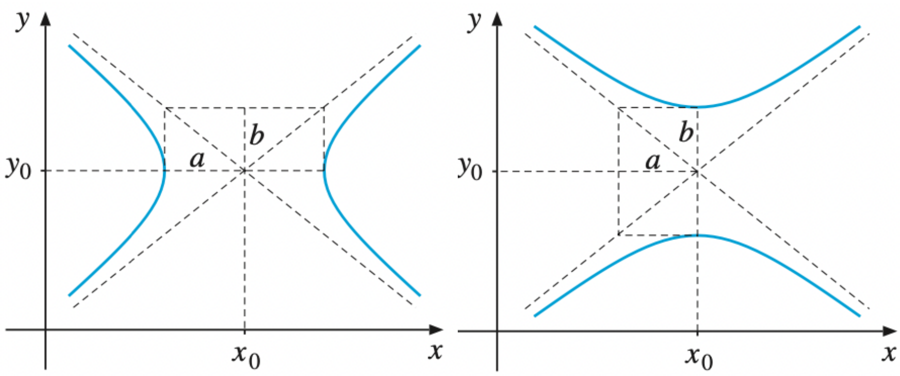


Fig. 4.11. Examples of Hyperbola

These have **asymptotes** which are like tangents at infinity, represented by the same two dashed lines in both figures. Their equations are $y - y_0 = \pm(b/a)(x - x_0)$.

Finally, consider the graph of the general quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, and C are not all 0. This will have one of the following shapes:

(i) If $4AC > B^2$, either an ellipse (possibly a circle), or a single point, or empty.

(ii) If $4AC = B^2$, either a parabola, or one line or two parallel lines, or empty.

(iii) If $4AC < B^2$, either a hyperbola, or two intersecting lines.

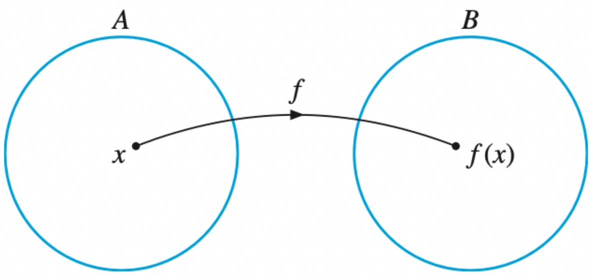


Fig. 4.12. A function from A to B

Ex. 4.16. Determine the distances between the following pairs of points:

$$(a)(1,3) \text{ and } (2,4) \qquad (b)(-1,2) \text{ and } (-3,3)$$
$$(c)(3/2,-2) \text{ and } (-5,1) \qquad (d)(x,y) \text{ and } (2x,y+3)$$
$$(e)(a,b) \text{ and } (-a,b) \qquad (f)(a,3) \text{ and } (2+a,5)$$

Ex. 4.17. The distance between (2, 4) and (5, y) is $\sqrt{13}$. Find y, and explain geometrically why there must be two values of y.

Ex. 4.18. Find the equations of:

(a) The circle with centre at (2, 3) and radius 4.

(b) The circle with centre at (2, 5) and one point at (-1, 3).

Ex. 4.19. Find the centre and the radius of the two circles with equations:

(a) $x^2 + y^2 + 10x - 6y + 30 = 0$

(b) $3x^2 + 3y^2 + 18x - 24y = -39$

4.6 General functions

So far we have studied functions of one variable. Yet a realistic description of many economic phenomena requires considering a large number of variables simultaneously.
An extensive discussion of functions of several variables begins in Chapter 9. This section introduces an even more general type of function.
The general definition of a **function** is a rule which to each element in a set A associates one, and only one, element in a set B.

The particular value $f(x)$ is often called the **image** of the element x by the function f. If each element of B is the image of at most one element in A, the function f is called **one-to-one**. Otherwise, if one or more elements of B are the images of more than one element in A, the function f is **many-to-one**.

Ex. 4.20. Which of the following rules define functions?

(a) The rule that assigns to each person in a classroom his or her height.

(b) The rule that assigns to each mother her youngest child alive today.

(c) The rule that assigns the perimeter of a rectangle to its area.

(d) The rule that assigns the surface area of a spherical ball to its volume.

(e) The rule that assigns the pair of numbers (x + 3, y) to the pair of numbers (x, y).

4.7 Review Exercises

Ex. 4.21. Use the graph of $y = |x|$ and the rules for shifting graphs to sketch the graphs of the following functions:

(a) $y = |x| + 1$

(b) $y = |x + 3|$

(c) $y = 3 - |x + 1|$

Ex. 4.22. If $f(x) = x^3 - 2$ and $g(x) = (1 - x)x^2$, compute the six functions: $(f + g)(x)$; $(f - g)(x)$; $(fg)(x)$; $(f/g)(x)$; $f(g(1))$; and $g(f(1))$.

Ex. 4.23. Consider the demand and supply curves $D = 150 - 12P$ and $S = 20 + 2P$.

(a) Find the equilibrium price P^*, and the corresponding quantity Q^*.

(b) Suppose a tax of \$2 per unit is imposed on the producer's output. How will this influence the equilibrium price?

(c) Compute the total revenue obtained by the producer before the tax is imposed(R^*) and after($\hat{R}$).

Ex. 4.24. The following functions are strictly increasing in their domains. Find the domains of their inverses and formulas for the inverses, using x as the free variable.

(a) $f(x) = 3 + ln(e^x - 2)$, for $x > ln2$

(b) $f(x) = \frac{a}{e^{-\lambda x}+a}$, where a and λ are positive, for $x \in (-\infty, \infty)$

Ex. 4.25. Determine the distances between the following pairs of points:

$$(a)(-4, 4) \; and \; (3, 8) \qquad\qquad (b)(2a, 3b) \; and \; (2 - a, 3b)$$

Ex. 4.26. Find the equations of the circles with:

(a) centre at (2,-3) and radius 5

(b) centre at (-2,2) and passing through(-10,1)

Ex. 4.27. A point P moves in the plane so that it is always equidistant between the two points A = (3, 2) and B = (5, -4). Find a simple equation that the coordinates (x, y) of P must satisfy. (Hint: Compute the square of the distance from P to the points A and B, respectively.)

Chapter 5

Differentiation

5.1 Summary of Differentiation

Note that x is a single variable while u can be a composite variable or function.

$$
\begin{aligned}
(x^\alpha)' &= \alpha x^{\alpha-1} & (u^\alpha)' &= \alpha u^{\alpha-1}u' \\
(\sqrt{x})' &= \frac{1}{2\sqrt{x}} & (\sqrt{u})' &= \frac{u'}{2\sqrt{u}} \\
\left(\frac{1}{x}\right)' &= -\frac{1}{x^2} & \left(\frac{1}{u}\right)' &= -\frac{u'}{u^2} \\
(e^x)' &= e^x & (e^u)' &= u'e^u \\
(a^x)' &= a^x\ln(a) & (a^u)' &= u'a^u\ln(u) \\
(\ln x)' &= \frac{1}{x} & (\ln u)' &= \frac{u'}{u} \\
(\log_a x)' &= \frac{1}{x\ln(a)} & (\log_a u)' &= \frac{u'}{u\ln(a)}
\end{aligned}
$$

5.2 Slopes of curves

When we study the graph of a function, we would like to have a precise measure of its steepness at a point. We know for linear functions in the form of $y = px + q$, its slope is denoted by p.
However, for an arbitrary function, we define the **slope** of a curve *at a particular point* as the slope of the tangent to the point.
As shown in the graph below, the same curve will have different slopes at different points. Suppose point P has coordinates $(a, f(a)$. The slope of the curve at point P is called the **derivative** of $f(x)$ at $x = a$, denoted by $f'(a)$.

5.3 Tangents and derivatives

Consider a point P on a curve in the xy-plane. Take another point Q on the curve. The entire straight line through P and Q is called a **secant**. If we keep P fixed, but let Q move along the curve toward P, then the secant will rotate

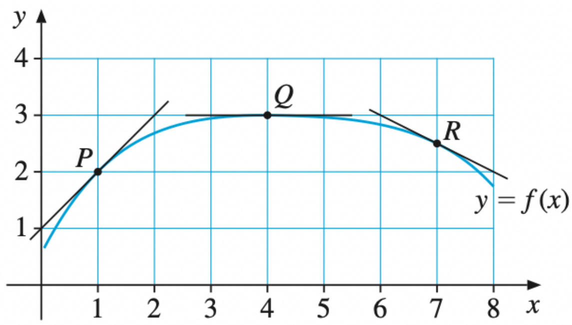


Fig. 5.1. Example of slopes at different points

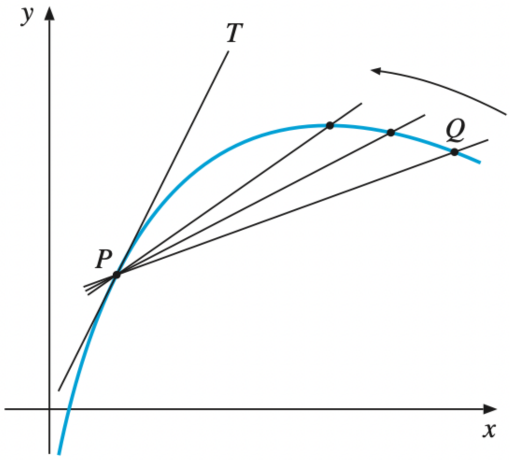


Fig. 5.2. Secants and the tangent

around P. The limiting straight line PT toward which the secant tends is called the **tangent** to the curve at P.

Now suppose the x-coordinate of point Q is $a+h$, where h is a small number different from zero. The slope of the secant PQ is therefore:

$$\frac{f(a+h)-f(a)}{h}$$

This fraction is called a **Newton quotient** of f. When h tends to zero, the secant PQ tends to the tangent. This suggests that we can define the slope of the tangent at P as the number that the slope of the secant approaches as h tends to 0. Formally:

The derivative of function f at point a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

The equation for the tangent to the graph of $y = f(x)$ at the point $(a, f(a))$ is

$$y - f(a) = f'(a)(x - a)$$

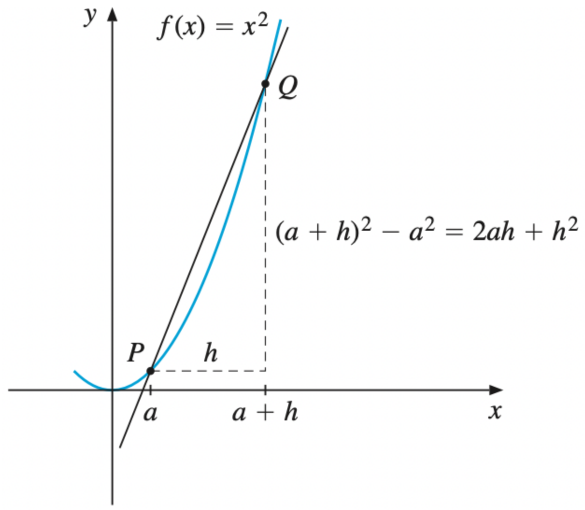


Fig. 5.3. Geometric interpretation

Example 5.1. Use the definition of slope to compute $f'(x)$ when $f(x) = x^2$. Find in particular $f'(1/2)$ and $f'(-1)$. Give geometric interpretations, and find the equation for the tangent at each of the points $(1/2, 1/4)$ and $(-1, 1)$.
Solution:
For $f(x) = x^2$, we have $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$, so

$$f(x+h) - f(x) = (x^2 + 2xh + h^2) - x^2 = 2xh + h^2$$

For all $h \neq 0$, the Newton quotient is:

$$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h} = 2x + h$$

Therefore we can obtain

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0}(2x + h) = 2x$$

When $x = 1/2$, we obtain $f'(1/2) = 2 \times 1/2 = 1$. Similarly, $f'(-1) = 2 \times (-1) = -2$.

At $(1/2, 1/4)$, the equation of the tangent is $y - 1/4 = 1 \times (x - 1/2)$ or $y = x - 1/4$. Similarly, the equation of the tangent at point $(-1, 1)$ is $y = -2x - 1$.

A commonly used form of notation for derivatives is the **differential notation** due to Leibniz. If $y = f(x)$, then in place of $f'(x)$, we write $\frac{dy}{dx}$, $\frac{df(x)}{dx}$ or $\frac{d}{dx}f(x)$.
We can think of the symbol "d/dx" as an instruction to differentiate what follows with respect to x. Differentiation occurs so often in mathematics that it has become standard to use w.r.t. as an abbreviation for with respect to.

When we use letters other than f , x, and y, the notation for the derivative changes accordingly. For example: $P(t) = t^2 \Rightarrow P'(t) = 2t$; $Y = K^3 \Rightarrow Y' = 3K^2$; and $A = r^2 \Rightarrow dA = 2r$.

Ex. 5.1. Let $f(x) = 3x^2 + 2x - 1$.

(a) Show that $f(x+h) - f(x) = 6x + 2 + 3h$ for $h \neq 0$, and use this result to find $f'(x)$.

(b) Find in particular $f'(0)$, $f'(-2)$, and $f'(3)$. Find also the equation of the tangent to the graph at the point $(0, -1)$.

Ex. 5.2. The demand function for a commodity with price P is given by the formula $D(P) = a - bP$ and the cost of producing x units of a commodity is given by the formula $C(x) = p + qx^2$. Use the rule of differentiation to find $dD(P)/dP$ and $C'(x)$.

Ex. 5.3. Show that $[f(x+h) - f(x)]/h = -1/x(x+h)$, and use this to show that

$$f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = \frac{-1}{x^2} = -x^{-2}$$

Ex. 5.4. In each case below, find the slope of the tangent to the graph of f at the specified point:

$$(a) f(x) = 3x + 2 \ at \ (0, 2) \qquad (b) f(x) = x^2 - 1 \ at \ (1, 0)$$
$$(c) f(x) = 2 + 3/x \ at \ (3, 3) \qquad (d) f(x) = x^3 - 2x \ at \ (0, 0)$$
$$(e) f(x) = x + 1/x \ at \ (-1, -2) \qquad (f) f(x) = x^4 \ at \ (1, 1)$$

Ex. 5.5. Figure 5.4 shows the graph of a function f. Determine the sign of the derivative $f'(x)$ at each of the four points a, b, c, and d.

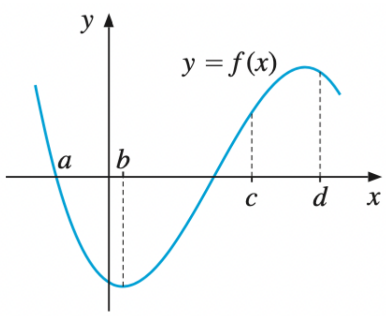


Fig. 5.4. Exercise 5.5

Ex. 5.6. Let $f(x) = \sqrt{x}$.

(a) Show that $(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x}) = h$

(b) Use the result in part (a) to show that the Newton quotient of $f(x)$ is $1/(\sqrt{x+h} + \sqrt{x})$.

(c) Use the result in part (b) to show that for $x > 0$, $f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$

Ex. 5.7. Apply the results of Ex 5.5 to prove first that

$$[(x+h)^{1/3} - x^{1/3}][(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}] = h$$

Then follow the argument used to solve Ex 5.5 to show that $f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3}$.

5.4 Increasing and decreasing functions

Assume that f is defined in an interval I and that x_1 and x_2 are numbers from that interval.

(i) If $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$, then f is *increasing* in I.

(ii) If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, then f is *strictly increasing* in I.

(iii) If $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$, then f is *decreasing* in I.

(iv) If $f(x_2) < f(x_1)$ whenever $x_2 > x_1$, then f is *strictly decreasing* in I.

We can also test whether a function is increasing or decreasing by checking the sign of its derivative:

$$\begin{aligned}
f'(x) \geq 0 \text{ for all } x \text{ in the interval } I &\Leftrightarrow f \text{ is increasing in } I \\
f'(x) \leq 0 \text{ for all } x \text{ in the interval } I &\Leftrightarrow f \text{ is decreasing in } I \\
f'(x) = 0 \text{ for all } x \text{ in the interval } I &\Leftrightarrow f \text{ is constant in } I
\end{aligned}$$

Example 5.2. Find the derivative of $f(x) = \frac{1}{2}x^2 - 2$.Then examine whether f is increasing or decreasing.
Solution:
We find that $f'(x) = x$, which is non-negative for $x \geq 0$, and non-positive if $x \leq 0$, and thus $f'(0) = 0$. We conclude that f is increasing in $[0, \infty)$ and decreasing in $(-\infty, 0]$.

Ex. 5.8. Examine whether $f(x) = x^2 - 4x + 3$ is increasing or decreasing.

Ex. 5.9. Examine whether $f(x) = -x^3 + 4x^2 - x - 6$ is increasing or decreasing.

Ex. 5.10. Show algebraically that $f(x) = x^3$ is strictly increasing by studying the sign of

$$x_2^3 - x_1^3 = (x_2 - x_1)(x_1^2 + x_1 x_2 + x_2) = (x_2 - x_1)\left[\left(x_1 + 2x_2)^2\right) + \frac{3}{4}x_2^2\right]$$

5.5 Rates of change

The derivative can be interpreted in many ways, one of which is the **rate of change**. The Newton quotient can be interpreted as the *average rate of change of f over the interval from a to a+h.*
Taking the limit as h tends to 0 gives the derivative of f at a, which we interpret as the **instantaneous rate of change**.
Sometimes we are interested in studying the proportion $f'(a)/f(a)$. This proportion can be interpreted as the **relative rate of change**.

Example 5.3. Consider a firm producing some commodity in a given period, and let $C(x)$ denote its cost of producing x units. The derivative $C'(x)$ at x is called the **marginal cost** at x. According to the definition, it is equal to

$$C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}$$

When h is small in absolute value, we obtain the approximation

$$C'(x) \approx \frac{C(x+h)-C(x)}{h}$$

The difference $C(x+h)-C(x)$ is called the **incremental cost** of producing h units of extra output. For h small, a linear approximation to this incremental cost is $hC'(x)$, the product of the marginal cost and the change in output. This is true even when $h < 0$, signifying a decrease in output and, provided that $C'(x) > 0$, a lower cost.
Note that putting $h = 1$ makes marginal cost approximately equal to

$$C'x) \approx C(x+1) - C(x)$$

Marginal cost is then approximately equal to the incremental cost $C(x+1) - C(x)$, that is, the additional cost of producing one more unit than x.

Example 5.4. Let $C(x)$ denote the cost in millions of dollars for removing x% of the pollution in a lake. Give an economic interpretation of the equality $C'(50) = 3$.
Solution:
Because of the linear approximation $C(50+h) - C(50) \approx hC'(50)$, the precise interpretation of $C'(50) = 3$ is that, starting at 50%, for each extra 1% of pollution that is removed, the extra cost is about 3 million dollars. Much less precisely, $C'(50) = 3$ means that it costs about 3 million dollars extra to remove 51% instead of 50% of the pollution.

Ex. 5.11. Let $C(x) = x^2 + 3x + 100$ be the cost function of a firm. Show that when x is changed from 100 to $100 + h$, where $h \neq 0$, the average rate of change per unit of output is

$$\frac{C(100+h) - C(100)}{h} = 203 + h$$

What is the marginal cost $C'(100)$? Use the definition of derivatives to find $C'(x)$ and in particular $C'(100)$.

Ex. 5.12. If the cost function of a firm is $C(x) = \bar{C} + cx$, give economic interpretations of the parameters c and $\bar{C}$.

Ex. 5.13. If the total saving of a country is a function $S(Y)$ of the national product Y, then $S'(Y)$ is called the **marginal propensity to save**, or MPS. Find the MPS for the following functions:

$$(a) S(Y) = \bar{S} + sY \qquad (b) S(Y) = 100 + 0.1Y + 0.0002Y^2$$

5.6 Limits

Suppose, in general, that a function f is defined for all x near a, but not necessarily at $x = a$. Then we say that the number A is the limit of $f(x)$ as x tends to a if $f(x)$ tends to A as x tends to (but is not equal to) a. We write:

$$\lim_{x \to a} f(x) = A, \quad or\ f(x) \to A\ as\ x \to a$$

Consider the function

$$F(x) = \frac{e^x - 1}{x}$$

The function is not defined for $x = 0$ but we can still find the values of $F(x)$ when x is close to zero.

x	-1	-0.1	-0.001	-0.0001	0	0.0001	0.001	0.1	1
$F(x)$	0.632	0.956	0.999	1.000	Not defined	1.000	1.001	1.052	1.718

Table 5.1: Values of $F(x)$ when x is close to 0

So we can write:

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1 \ or \ \frac{e^x - 1}{x} \to 1 \ as \ x \to 0$$

.

General rules for limits:
If $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, then:

$$\lim_{x\to a} [f(x) \pm g(x)] = A \pm B$$
$$\lim_{x\to a} [f(x) \cdot g(x)] = A \cdot B$$
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{A}{B}, \text{ if } B \neq 0$$
$$\lim_{x\to a} [f(x)]^r = A^r, \text{ if } A^r \text{ is defined and } r \text{ is a real number}$$

Example 5.5. Use the rules for limits to compute the following limits:

$$(a) \lim_{x\to -2} (x^2 + 5x) \quad (b) \lim_{x\to 4} \frac{2x^{3/2} - \sqrt{x}}{x^2 - 15} \qquad (c) \lim_{x\to a} Ax^n$$

Solution:

(a) The limit can be expressed as:

$$\lim_{x\to -2} x \cdot \lim_{x\to -2} x + \lim_{x\to -2} 5 \cdot \lim_{x\to -2} x$$

So

$$\lim_{x\to -2} (x^2 + 5x) = (-2)(-2) + 5(-2) = -6$$

(b) Similar to (a), the limit can be expressed as:

$$\lim_{x\to 4} \frac{2x^{3/2} - \sqrt{x}}{x^2 - 15} = \frac{2\lim_{x\to 4} x^{3/2} - \lim_{x\to 4} \sqrt{x}}{\lim_{x\to 4} x^2 - 15} = \frac{2 \cdot 4^{3/2} - \sqrt{4}}{4^2 - 15} = 14$$

(c) Finally,

$$\lim_{x\to a} Ax^n = \lim_{x\to a} A \cdot \lim_{x\to a} x^n = A \cdot (\lim_{x\to a} x)^n = A \cdot a^n$$

where n is a natural number.

Ex. 5.14. Determine the following by using the rules for limits:

$$(a)\lim_{x\to 0}(3+2x^2) \qquad (b)\lim_{x\to -1}\frac{3+2x}{x-1} \qquad (c)\lim_{x\to 2}(2x^2+5)^3$$

$$(d)\lim_{t\to 8}(5t+t^2-\frac{1}{8}t^3) \qquad (e)\lim_{y\to 0}\frac{(y+1)^5-y^5}{y+1} \qquad (f)\lim_{z\to -2}\frac{1/z+2}{z}$$

Ex. 5.15. If $f(x)=x^2+2x$, compute the following limits:

$$(a)\lim_{x\to 1}\frac{f(x)-f(1)}{x-1} \qquad (b)\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} \qquad (c)\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{h}$$

Ex. 5.16. Compute the following limits, where in part (c) n denotes any natural number:

$$(a)\lim_{x\to 2}\frac{x^2-2x}{x^3-8} \qquad (b)\lim_{h\to 0}\frac{\sqrt[3]{27+h}-3}{h} \qquad (c)\lim_{x\to 1}\frac{x^n-1}{x-1}$$

5.7 Simple rules for differentiation

If f is a constant function, then its derivative is 0:

$$f(x)=A \Rightarrow f'(x)=0$$

When taking derivatives, additive constants disappear while multiplicative constants are preserved:

$$y=A+f(x) \Rightarrow y'=f'(x)$$
$$y=Af(x) \Rightarrow y'=Af'(x)$$

The **power rule** is that given any constant a,

$$f(x)=x^a \Rightarrow f'(x)=ax^{a-1}$$

Example 5.6. Use the power rule to compute:

$$(a)\frac{d}{dx}(\frac{x^{100}}{100}) \qquad (b)\frac{d}{dx}(x^{-0.33}) \qquad (c)\frac{d}{dr}(-5r^{-3})$$

$$(d)\frac{d}{dp}(Ap^{\alpha}+B) \qquad (e)\frac{d}{dx}(\frac{A}{\sqrt{x}})$$

Solution:

(a) $\frac{d}{dx}(\frac{x^{100}}{100})=\frac{1}{100}100x^{100-1}=x^{99}$

(b) $\frac{d}{dx}(x^{-0.33})=-0.33x^{-0.33-1}=-0.33x^{-1.33}$

(c) $\frac{d}{dr}(-5r^{-3})=(-5)(-3)r^{-3-1}=15r^{-4}$

(d) $\frac{d}{dp}(Ap^{\alpha}+B)=A\alpha p^{\alpha-1}$

(e) $\frac{d}{dx}(\frac{A}{\sqrt{x}})=\frac{d}{dx}(Ax^{-1/2})=A(-\frac{1}{2})x^{-1/2-1}=-\frac{1}{2}Ax^{-3/2}$

Ex. 5.17. Compute the following:

$$(a)\frac{d}{dr}(4\pi r^2) \qquad (b)\frac{d}{dy}(Ay^{b+1}) \qquad (c)\frac{d}{dA}\left(\frac{1}{A^2\sqrt{A}}\right)$$

5.8 Sums, Products, and Quotients

5.8.1 Sums and Differences

Suppose f and g are both defined on a set A of real numbers. If both f and g are differentiable at x, then the sum $f+g$ and the difference $f-g$ are both differentiable at x, with

$$F(x) = f(x) \pm g(x) \Rightarrow F'(x) = f'(x) \pm g'(x)$$

Example 5.7. Compute $\frac{d}{dx}(3x^8 + x^{100}/100)$.
Solution:

$$\frac{d}{dx}(3x^8 + x^{100}/100) = \frac{d}{dx}(3x^8) + \frac{d}{dx}(x^{100}/100) = 24x^7 + x^{99}$$

5.8.2 Products

If both f and g are differentiable at the point x, then so is $F = f \cdot g$, and

$$F(x) = f(x) \cdot g(x) \Rightarrow F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Example 5.8. Find $h'(x)$ when $h(x) = (x^3 - x)(5x^4 + x^2)$. Confirm the answer by expanding $h(x)$ as a single polynomial, then differentiating the result.

Solution:

We see that $h(x) = f(x) \cdot g(x)$ with $f(x) = x^3 - x$ and $g(x) = 5x^4 + x^2$. Here $f'(x) = 3x^2 - 1$ and $g'(x) = 20x^3 + 2x$. Thus from the products rule:

$$h'(x) = (3x^2 - 1) \cdot (5x^4 + x^2) + (x^3 - x) \cdot (20x^3 + 2x) = 35x^6 - 20x^4 - 3x^2$$

Alternatively, expanding $h(x)$ as a polynomial gives $h(x) = 5x^7 - 4x^5 - x^3$, which gives the same derivative.

5.8.3 Quotients

If f and g are differentiable at x and $g(x) \neq 0$, then $F = f/g$ is differentiable at x, and

$$F(x) = \frac{f(x)}{g(x)} \Rightarrow F'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

Example 5.9. Compute $F'(x)$ and $F'(4)$ when

$$F(x) = \frac{3x-5}{x-2}$$

Solution:

We apply the quotient rule with $f(x) = 3x - 5$ and $g(x) = x - 2$. Then $f'(x) = 3$ and $g'(x) = 1$. So we obtain, for $x \neq 2$:

$$F'(x) = \frac{3 \cdot (x-1) - (3x-5) \cdot 1}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

To find $F'(4)$, we put $x = 4$ in the formula and obtain $F'(4) = -1/4$.

Ex. 5.18. Differentiate w.r.t x the following functions:

$$(a)\frac{3}{5}x^2 - 2x^7 + \frac{1}{8} - \sqrt{x} \qquad (b)(2x^2-1)(x^4-1) \qquad (c)\left(x^5 + \frac{1}{x}\right)(x^5+1)$$

Ex. 5.19. Differentiate w.r.t x the following functions:

$$(a)\frac{\sqrt{x}-2}{\sqrt{x}+1} \qquad (b)\frac{x^2-1}{x^2+1} \qquad (c)\frac{x^2+x+1}{x^2-x+1}$$

Ex. 5.20. For each of the following functions, determine the intervals where it is increasing.

$$(a)y = 3x^2 - 12x + 13 \qquad (b)y = \frac{1}{4}(x^4 - 6x^2) \qquad (c)y = \frac{x^2 - x^3}{2(x+1)}$$

Ex. 5.21. Find the equations for the tangents to the graphs of the following functions at the specified points:

$$(a)y = 3 - x - x^2 \ at\ x = 1 \qquad (b)y = \frac{x^2-1}{x^2+1} \ at\ x = 1$$

$$(c)y = \left(\frac{1}{x^2} + 1\right)(x^2 - 1) \ at\ x = 2 \qquad (d)y = \frac{x^4+1}{(x^2+1)(x+3)} \ at\ x = 0$$

5.9 Chain rule

The **chain rule** states that if y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of x, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The generalised power rule is that if $y = u^a$ and u is a differentiable function of x, then

$$y' = au^{a-1}u'$$

Example 5.10. Find dy/dx when:

(a) $y = u^5$ and $u = (1 - x^3)$

(b) $y = \frac{10}{(x^2+4x+5)^7}$

Solution:

(a) Here we apply the chain rule directly. Since $dy/du = 5u^4$ and $du/dx = -3x^2$, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4(-3x^2) = -15x^2u^4 = -15x^2(1-x^3)^4$$

(b) If we write $u = x^2 + 4x + 5$, then $y = 10u^{-7}$. By the generalised power rule, one has

$$\frac{dy}{dx} = 10(-7)u^{-8}u' = 5u^4(-3x^2) = -70u^{-8}(2x+4) = \frac{-140(x+2)}{(x^2+4x+5)^8}$$

5.9.1 Alternative formulation of the chain rule

Consider a composite function $y = f(g(x))$. If g is differentiable at x_0 and f is differentiable at $u_0 = g(x_0)$, then the composite function $F(x) = f(g(x))$ is differentiable at x_0, and

$$F'(x_0) = f'(u_0)g'(x_0) = f'(g(x_0))g'(x_0)$$

Example 5.11. Find the derivative of the compound function $F(x) = f(g(x))$ at $x_0 = -3$ in case $f(u) = u^3$ and $g(x) = 2 - x^2$.

Solution:

In this case we have $f'(u) = 3u^2$ and $g'(x) = -2x$. So according to the alternative formulation of the chain rule, one has $F'(-3) = f'(g(-3))g'(-3)$. Now $g(-3) = 2 - (-3)^2 = 2 - 9 = -7$; $g'(-3) = 6$; and $f'(g(-3)) = f'(-7) = 3(-7)^2 = 3 \cdot 49 = 147$. So $F'(-3) = f'(g(-3))g'(-3) = 147 \cdot 6 = 882$.

Ex. 5.22. Find the derivatives of the following functions:

$$(a) y = \frac{1}{(x^2 + x + 1)^5} \qquad (b) y = \sqrt{x + \sqrt{x + \sqrt{x}}}$$

Ex. 5.23. Suppose that $C = 20q - 40q(25 - \frac{1}{2}x)^{\frac{1}{2}}$, where q is a constant and $x < 50$. Find $\mathrm{d}C/\mathrm{d}x$.

5.10 Higher-order derivatives

The derivative f' of a function f is often called the first derivative of f. If f' is also differentiable, then we can differentiate f' in turn. The result $(f')'$ is called the second derivative, written more concisely as f''.

Similarly, a function $f(x)$ may have third, forth derivatives and so on. In general, the n-th derivative of f at x is expressed as:

$$y^{(n)} = f^{(n)}(x) \; or \; \frac{d^n y}{dx^n}$$

The number n is the **order** of the derivative.

A different form of notation is:

$$f''(x) = \frac{d^2 f(x)}{dx^2} \; or \; y'' = \frac{d^2 y}{dx^2}$$

Example 5.12. Find $f'(x)$ and $f''(x)$ when $f(x) = 2x^5 - 3x^3 + 2x$

Solution:

The rules for differentiating polynomials imply that $f'(x) = 10x^4 - 9x^2 + 2$. Then we differentiate each side of this equality to get $f''(x) = 40x^3 - 18x$.

Recall that the sign of the first derivative determines whether a function is increasing or decreasing on an interval I. Then the second derivative implies:

$$f''(x) \geq 0 \text{ on } I \Leftrightarrow f' \text{ is increasing in } I$$
$$f''(x) \leq 0 \text{ on } I \Leftrightarrow f' \text{ is decreasing in } I$$

Graphical visualisations are:

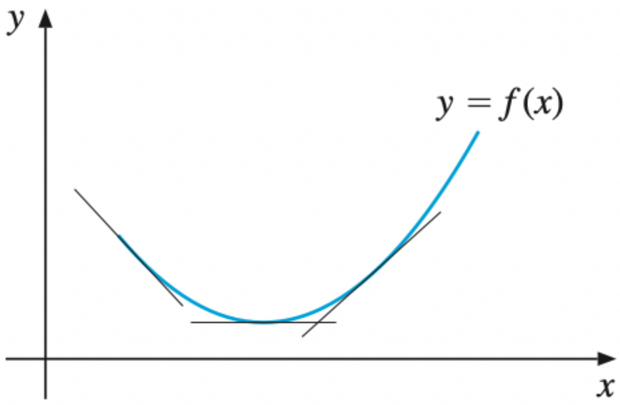


Fig. 5.5. The slope of the tangent line increases as x increases

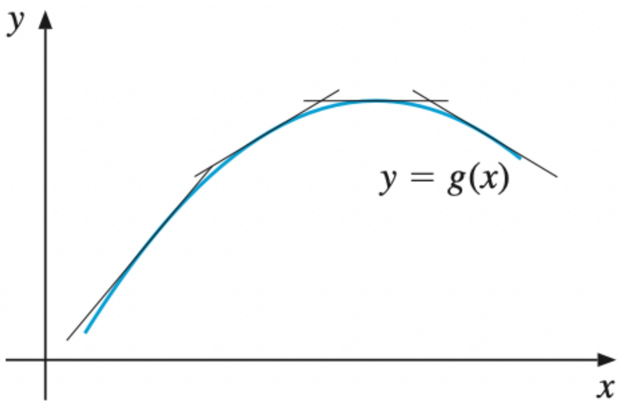


Fig. 5.6. The slope of the tangent line decreases as x increases

Suppose that f is continuous in the interval I and twice differentiable in the interior of I. Then we can introduce the following definitions:

$$f \text{ is } \textit{convex} \text{ on } I \Leftrightarrow f'' \geq 0 \text{ for all } x \text{ in } I$$
$$f \text{ is } \textit{concave} \text{ on } I \Leftrightarrow f'' \leq 0 \text{ for all } x \text{ in } I$$

Example 5.13. Check the convexity/concavity of the following functions:

$$(a) f(x) = x^2 - 2x + 2 \qquad (b) f(x) = ax^2 + bx + c$$

Solution:

(a) Here $f'(x) = 2x - 2$ so $f''(x) = 2$. Because $f''(x) > 0$ for all x, the function f is convex.

(b) Here $f'(x) = 2ax + b$, so $f''(x) = 2a$. If $a = 0$, then f is linear, so it is both concave and convex. If $a > 0$,then $f''(x) > 0$, so f is convex. If $a < 0$,then $f''(x) < 0$, so f is concave.

Ex. 5.24. Compute the second derivative of:

(a) $y = x^5 - 3x^4 + 2$

(b) $y = \sqrt{x}$

(c) $y = (1 + x^2)^{\frac{1}{2}}$

Ex. 5.25. Find $g''(2)$ when $g(t) = \frac{t^2}{t-1}$

5.11 Exponential functions

Derivative of the natural exponential function:

$$f(x) = e^x \;\Rightarrow\; f'(x) = e^x$$

Example 5.14. Find the first and second derivatives of:

$$(a) y = x^3 + e^x \qquad (b) y = x^5 e^x \qquad (c) y = e^x/x$$

Solution:

(a) We can easily find that $y' = 3x + e^x$ and $y'' = 6x + e^x$.

(b) By the product rule, $y' = 5x^4e^x + x^5e^x$. To find the second derivative, differentiate y' once more to obtain $y'' = 20x^3e^x + 5x^4e^x + 5x^4e^x + x^5e^x = 20x^3e^x + 10x^4e^x + x^5e^x$.

(c) The quotient rule yields:

$$y' = \frac{e^x x - e^x \cdot 1}{x^2} = \frac{e^x(x-1)}{x^2}$$

Differentiating again gives:

$$y'' = \frac{(e^x x + e^x - e^x)x^2 - (e^x x - e^x)2x}{(x^2)^2} = \frac{e^x(x^2 - 2x + 2)}{x^3}$$

.

Example 5.15. For each of the following functions, find the intervals where they are increasing:

$$(a) y = \frac{e^x}{x} \qquad (b) y = x^4 e^{-2x} \qquad (c) y = xe^{-\sqrt{x}}$$

Solution:

(a) As shown in the previous example, $y' = e^x(x - 1)/x^2$, so $y' \geq 0$ if and only if $x \geq 1$. Thus y is increasing in $[1, \infty)$.

(b) We can easily obtain $y' = x^3e^{-2x}(4 - 2x)$. A sign diagram reveals that y is increasing in [0,2].

(c) The function is only defined for $x \geq 0$. Using the chain rule and product rule, the derivative of y is:

$$y' = 1 \cdot e^{-\sqrt{x}} - \frac{xe^{-\sqrt{x}}}{2\sqrt{x}} = e^{-\sqrt{x}}\left(1 - \frac{1}{2}\sqrt{x}\right)$$

. It follows that y is increasing when $x > 0$ and $1 - \frac{1}{2}\sqrt{x} \geq 0$. Therefore y is increasing in [0,4].

Now consider in general the power function where the base is no longer the constant e. We have the following rule for differentiation:

$$y = a^x = e^{(ln\, a)x} \Rightarrow y' = ln\, a e^{(ln\, a)x} = a^x ln\, a$$

Example 5.16. Find the derivatives of: (a) $f(x) = 10^{-x}$; and (b) $g(x) = x2^{3x}$.
Solution:

(a) Using the rule above we obtain $f'(x) = -10^{-x}ln10$.

(b) Rewrite $y = 2^{3x} = 2^u$, where $u = 3x$. By the chain rule:

$$y' = (2^u ln2)u' = (2^{3x}ln2) \cdot 3 = 3 \cdot 2^{3x}ln2$$

Finally, using the product rule we obtain

$$g'(x) = 1 \cdot 2^{3x} + x \cdot 3 \cdot 2^{3x}ln2 = 2^{3x}(1 + 3xln2)$$

Ex. 5.26. Find the first and second derivatives of:

$$(a) y = e^{-3x} \qquad (b) y = 2e^{x^3}$$
$$(c) y = e^{1/x} \qquad (d) 5e^{2x^2-3x+1}$$

Ex. 5.27. Find the intervals where the following functions are increasing:

$$(a) y = x^3 + e^{2x} \qquad (b) y = 5x^2e^{-4x} \qquad (c) y = x^2e^{-x^2}$$

Ex. 5.28. Find:

$$(a) \frac{d}{dx}\left(e^{e^x}\right) \qquad (b) \frac{d}{dt}\left(e^{t/2} + e^{-t/2}\right)$$
$$(c) \frac{d}{dt}\left(\frac{1}{e^t + e^{-t}}\right) \qquad (d) \frac{d}{dz}\left((e^{z^3} - 1)^{1/3}\right)$$

5.12 Logarithmic Functions

Derivative of the natural logarithmic function:

$$g(x) = ln\, x \ \Rightarrow g'(x) = \frac{1}{x}$$

Example 5.17. Compute y' and y'' when:

$$(a)y = x^3 + lnx \qquad (b)y = x^2 lnx \qquad (c)y = lnx/x$$

Solution:

(a) We find easily that $y' = 3x^2 + 1/x$. Furthermore, $y'' = 6x - 1/x^2$.

(b) The product rule gives $y' = 2xlnx + x^2(1/x) = 2xlnx + x$. Then $y'' = 2lnx + 2x(1/x) + 1 = 2lnx + 3$.

(c) Here we use the quotient rule:

$$y' = \frac{(1/x)x - lnx \cdot 1}{x^2} = \frac{1 - lnx}{x^2}$$

Differentiating again gives:

$$y'' = \frac{-(1/x)x^2 - (1 - lnx)2x}{(x^2)^2} = \frac{2lnx - 3}{x^3}$$

In general, suppose that $y = lnh(x)$, where $h(x)$ is differentiable and positive. Then:

$$y = lnh(x) \ \Rightarrow y' = \frac{h'(x)}{h(x)}$$

Example 5.18. Find the domains of the following functions and compute their derivatives:

$$(a)y = ln(1 - x) \qquad (b)y = ln\left(\frac{x-1}{x+1}\right) - \frac{1}{4}x$$

Solution:

(a) $ln(1 - x)$ is defined if $1 - x > 0$, that is if $x < 1$. To find its derivative, we use the rule earlier, with $h(x) = 1 - x$. Then $h'(x) = -1$, and

$$y' = \frac{-1}{1 - x} = \frac{1}{x - 1}$$

(b) We can write $y = lnu - \frac{1}{4}x$,where $u = (x - 1)/(x + 1)$. For the function to be defined, we require that $u > 0$. A sign diagram shows that this is satisfied if $x < -1$ or $x > 1$. Then we obtain:

$$y' = \frac{u'}{u} - \frac{1}{4}$$

where

$$u' = \frac{1 \cdot (x + 1) - 1 \cdot (x - 1)}{(x + 1)^2} = \frac{2}{(x + 1)^2}$$

So

$$y' = \frac{2(x + 1)}{(x + 1)^2(x - 1)} - \frac{1}{4} = \frac{9 - x^2}{4(x^2 - 1)} = \frac{(3 - x)(3 + x)}{4(x - 1)(x + 1)}$$

Similar to power functions, when the base of the logarithmic function is not e, differentiating gives:

$$y = log_a x \ \Rightarrow y' = \frac{1}{lna}\frac{1}{x}$$

5.12.1 Approximating the number e

If $g(x) = lnx$, then $g'(x) = 1/x$, and, in particular, $g'(1) = 1$. We use in turn: (i) the definition of $g'(1)$; (ii) the fact that $ln1 = 0$; (iii) the rule $lnx^p = plnx$. The result is

$$1 = g'(1) = \lim_{h\to 0} \frac{ln(1+h) - ln1}{h} = \lim_{h\to 0} \frac{1}{h} ln(1+h) = \lim_{h\to 0} ln(1+h)^{1/h}$$

Taking exponential of both sides gives:

$$e = \lim_{h\to 0} (1+h)^{1/h}$$

Ex. 5.29. Find the derivatives of:

$$(a) y = x^3(lnx)^2 \qquad (b) y = \frac{x^2}{lnx}$$
$$(c) y = (lnx)^{10} \qquad (d) y = (lnx + 3x)^2$$

Ex. 5.30. Determine the domains of the functions defined by:

$$(a) y = ln(x^2 - 1) \qquad (b) y = ln(lnx) \qquad (c) y = \frac{1}{ln(lnx) - 1}$$

Ex. 5.31. Prove that if u and v are differentiable functions of x, and $u > 0$, then

$$y = u^v \Rightarrow y' = u^v \left(v' lnu + \frac{vu'}{u} \right)$$

5.13 Review Exercises

Ex. 5.32. Let $f(x) = x^2 - x + 2$. Show that $[f(x+h) - f(x)]/h = 2x - 1 + h$, and use this result to find $f'(x)$.

Ex. 5.33. Let $C(Q)$ denote the cost of producing Q units per month of a commodity. What is the interpretation of $C'(1000) = 25$? Suppose the price obtained per unit is fixed at 30 and that the current output per month is 1000. Is it profitable to increase production?

Ex. 5.34. For each of the following functions, find the equation for the tangent to the graph at the specified point:

$$(a) y = \sqrt{x} - x^2 \ at \ x = 4 \qquad (b) y = \frac{x^2 - x^3}{x+3} \ at \ x = 1$$

Ex. 5.35. If $R = S^\alpha$, $S = 1 + \beta K^\gamma$, and $K = At^p + B$, find an expression for dR/dt.

Ex. 5.36. Find the intervals where the following functions are increasing:

$$(a) y = ln(x)^2 - 4 \qquad (b) y = ln(e^x + e^{-x}) \qquad (c) y = x - \frac{3}{2} ln(x^2 + 2)$$

Ex. 5.37. (a) Suppose $\pi(Q) = QP(Q) - cQ$, where P is a differentiable function and c is a constant. Find an expression for dπ/dQ.

(b) Suppose $\pi(L) = PF(L) - wL$, where F is a differentiable function and P and w are constants. Find an expression for dπ/dL.

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