Homework 1.2

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1.

$$V_0 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \; ; \; V_1 = \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix} \; ; \; \langle v_0, v_1 \rangle = v_0^T v_1 = 0.25 + 0.25 - 0.25 - 0.25 = 0$$

2.

Same v_0, v_1; $v_2 = \begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{bmatrix}$; v_0, v_1, v_2 are mutually orthogonal and have unit norm

Let $\mathcal{B} = \{v_0, v_1, v_2, v_3\}$ be a orthonormal basis in R^4

$\Rightarrow$ $v_3 \perp \{v_0, v_1, v_2\}$ and have unit norm

+ Let $v_3 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, we have $\begin{cases} 0.5(a+b+c+d) = 0 & (1) \\ 0.5(a+b-c-d) = 0 & (2) \\ 0.5(a-b+c-d) = 0 & (3) \\ \sqrt{a^2+b^2+c^2+d^2} = 1 & (4) \end{cases}$

$$(1)(2) \Leftrightarrow \begin{cases} a+b = -(c+d) \\ a+b = c+d \end{cases}$$

$$\Leftrightarrow \begin{cases} a+b = 0 \\ c+d = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a = -b & (5) \\ c = -d & (6) \end{cases}$$

(3) becomes: $-b-b-d-d = 0$

$\Leftrightarrow -2b-2d = 0$

$\Leftrightarrow b = -d \; (7)$

$(5)(6)(7) \Leftrightarrow a = d = -b = -c$

(4) becomes: $\sqrt{4a^2} = 1$

$$\Leftrightarrow \left[\begin{array}{l} 2a = 1 \\ 2a = -1 \end{array} \right.$$

$a = 0.5$

$$\Leftrightarrow \begin{cases} \ldots \\ 2a = -1 \end{cases}$$

$$\Leftrightarrow \left[\begin{array}{l} a = 0.5 \\ a = -0.5 \end{array}\right.$$

We have 2 different vectors for v_3: $\begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{bmatrix}$; $\begin{bmatrix} -0.5 \\ 0.5 \\ 0.5 \\ -0.5 \end{bmatrix}$

3. $y = \begin{bmatrix} 1.5 \\ -0.5 \\ 2.5 \\ 0.5 \end{bmatrix}$; $\mathcal{B} = \{v_0, v_1, v_2, v_3\}$; $y_{\mathcal{B}} = \begin{bmatrix} y^T v_0 \\ y^T v_1 \\ y^T v_2 \\ y^T v_3 \end{bmatrix}$

+ $y^T v_0 = 0.75 - 0.25 + 1.25 + 0.25 = 2$
+ $y^T v_1 = 0.75 - 0.25 - 1.25 - 0.25 = -1$
+ $y^T v_2 = 0.75 + 0.25 + 1.25 - 0.25 = 2$
+ $y^T v_3 = 0.75 + 0.25 - 1.25 + 0.25 = 0$

4. Check the linearly independence of each set

+ $\{y, v_0, v_2, v_3\}$. Because v_0, v_2, v_3 are mutually orthogonal, if the linear system

$$\begin{cases} 0.5a + 0.5b + 0.5c = 1.5 & (1) \\ 0.5a - 0.5b - 0.5c = -0.5 & (2) \\ 0.5a + 0.5b - 0.5c = 2.5 & (3) \\ 0.5a - 0.5b + 0.5c = 0.5 & (4) \end{cases}$$

has solution $\Leftrightarrow$ the set is not a basis of R^4

$$\left.\begin{array}{l} (1)(2) \Leftrightarrow a = 1 \\ (3)(4) \Leftrightarrow a = 3 \end{array}\right\} \Rightarrow$$ the system doesn't have solution $\Leftrightarrow$ the set is a basis

+ $\{y, v_0, v_1, v_2\}$

$$\begin{cases} 0.5a + 0.5b + 0.5c = 1.5 & (5) \\ 0.5a + 0.5b - 0.5c = -0.5 & (6) \\ 0.5a - 0.5b + 0.5c = 2.5 & (7) \\ 0.5a - 0.5b - 0.5c = 0.5 & (8) \end{cases}$$

$$\left.\begin{array}{l} (6)(7) \Leftrightarrow a = 2 \\ (5)(6) \Leftrightarrow 2 + b = 1 \Leftrightarrow b = -1 \\ (5)(7) \Leftrightarrow 2 + c = 4 \Leftrightarrow c = 2 \end{array}\right\} \Rightarrow$$ the set is not a basis

+ $\{y, v_1, v_2, v_3 - 2v_1\}$

$$v_3 - 2v_1 = \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{bmatrix} - 2 \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -1.5 \\ 0.5 \\ 1.5 \end{bmatrix}$$

$$\begin{cases} 1.5a + 0.5b + 0.5c - 0.5d = 0 & (9) \\ -0.5a + 0.5b - 0.5c - 1.5d = 0 & (10) \\ 2.5a - 0.5b + 0.5c + 0.5d = 0 & (11) \\ 0.5a - 0.5b - 0.5c + 1.5d = 0 & (12) \end{cases}$$

$$\left.\begin{array}{l} (10)(12) \Leftrightarrow c = 0 \\ (9)(11) \Leftrightarrow a = 0 \\ (10)(11) \Leftrightarrow b = d = 0 \end{array}\right\} \Rightarrow \text{the system has only trivial solution} \Leftrightarrow \text{the set is a basis}$$

+ $\{y, v_1, v_2, v_3\}$

$$\begin{cases} 0.5a + 0.5b + 0.5c = 1.5 & (13) \\ 0.5a - 0.5b - 0.5c = -0.5 & (14) \\ -0.5a + 0.5b - 0.5c = 2.5 & (15) \\ -0.5a - 0.5b + 0.5c = 0.5 & (16) \end{cases}$$

$$\left.\begin{array}{l} (13)(14) \Leftrightarrow a = 1 \\ (14)(15) \Leftrightarrow c = -2 \text{ and } b = 4 \\ (14)(16) \Leftrightarrow b = 0 \end{array}\right\} \Rightarrow \text{the set doesn't have solution} \Leftrightarrow \text{the set is a basis}$$

5.
$$F = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

6.
$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} . \quad A^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$