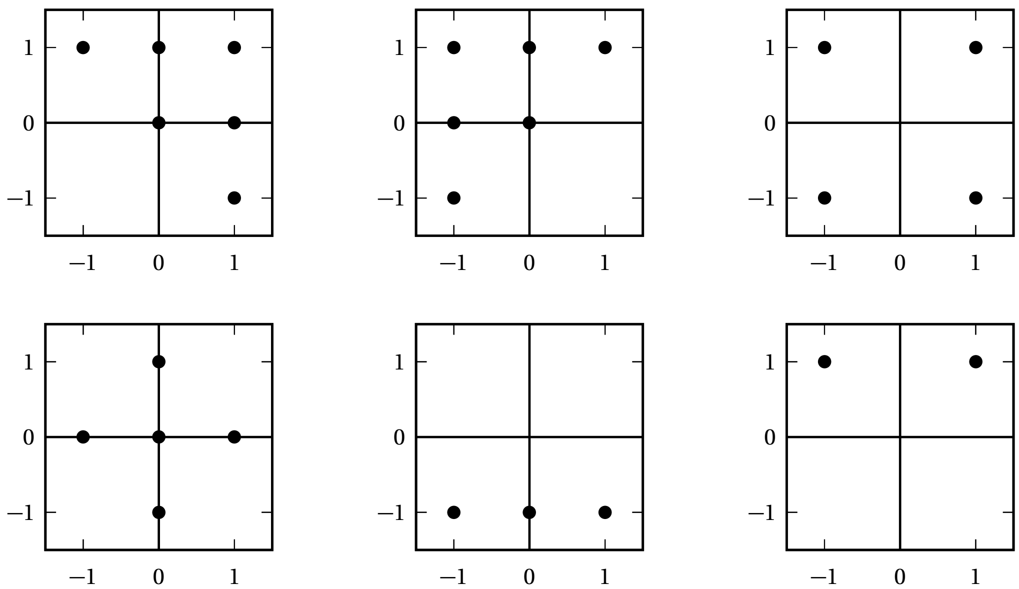
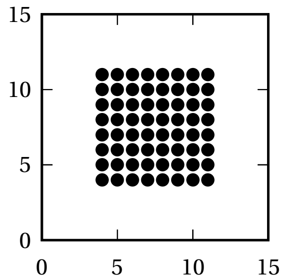
DSP4 – Practice Homework
Solutions

Exercise 1. 2D FIRs.
Consider the set of 2D FIR filters whose impulse responses have the support shown here:

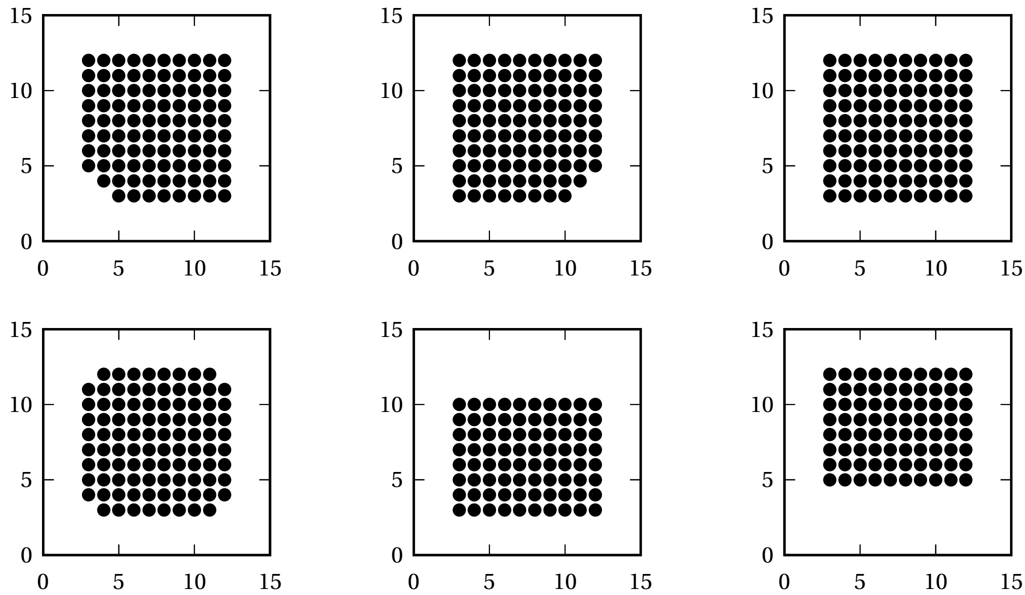


Now consider applying each of the filter to an image with an 8×8 square support (all dots represent nonzero values for the signals):



Draw the support of the output for each filtering operation.

Solution 1.



If in doubt, you can implement the filters in Python but remember that the axis convention in numerical packages for image processing may be flipped with respect to our notation (i.e., the origin [0, 0] may be the upper left corner of the image).

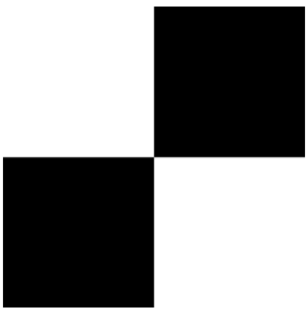
Exercise 2. Image interpolation.
Consider a $N \times N$ square image A in which

- $A[0, N-1]$ (upper left corner) and $A[N-1, 0]$ (lower right corner) are equal to 255 (white)
- $A[N-1, N-1]$ (upper right corner) and $A[0, 0]$ (lower left corner) are equal to 0 (black)
- all the other pixels are obtained using an interpolation from the four corners.

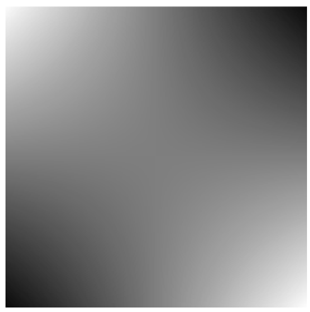
(a) Sketch the image A when the pixels are obtained via nearest-neighbor interpolation

(b) Sketch the image A when the pixels are obtained via bilinear interpolation.

Solution 2.



(a)



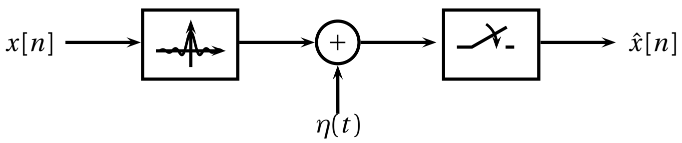
(b)

Exercise 3. Edge detection
Which of the following filters can be used for edge detection?

$$\text{(a)}\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{(b)}\ \begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix} \quad \text{(c)}\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{(d)}\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(e)}\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution 3.
Filter (a) is the standard Laplacian filter, which is derived exactly for edge detection. Filter (c) is the Laplacian filter rotated by 45 degrees, so that also works for edge detection. Filter (b) is the sum of (a) and (b), so that also works.
Filter (d) simply encodes a delta impulse response, so it has no useful properties. Filter (e) is a non-normalized moving average, that is, a lowpass filter which cannot be used for edge detection.

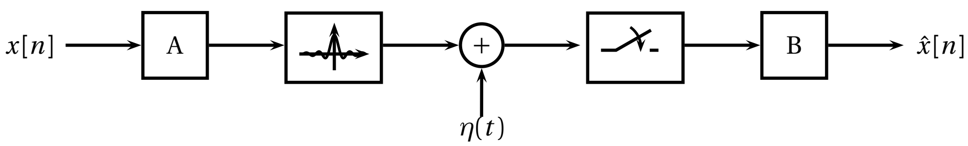
Exercise 4. Data Transmission.
Let $x[n]$ be the discrete-time version of an audio signal, originally bandlimited to 20KHz and sampled at 40KHz; assume that we can model $x[n]$ as an i.i.d. process with variance σ_x^2 . The signal is converted to continuous time, sent over a noisy analog channel and resampled at the receiving end using the following scheme, where both the ideal interpolator and sampler work at a frequency $F_s = 40KHz$.



The channel introduces zero-mean, additive white Gaussian noise. At the receiving end, after the sampler, assume that the effect of the noise introduced by the channel can be modeled as a zero-mean white Gaussian stochastic signal $\eta[n]$ with power spectral density $P_\eta(e^{j\omega}) = \sigma_0^2$.

(a) What is the signal to noise ratio (SNR) of $\hat{x}[n]$, i.e. the ratio of the power of the “good” signal and the power of the noise?

The SNR obtained with the transmission scheme above is too low for our purposes. Unfortunately the power constraint of the channel prevents us from simply amplifying the audio signal (in other words: the total power $\int_{-\pi}^{\pi} P_x(e^{j\omega})$ cannot be greater than $2\pi\sigma_x^2$). In order to improve the quality of the received signal, we modify the transmission scheme by adding pre-processing and post-processing digital blocks at the transmitting and receiving ends, while F_s is still equal to 40000Hz:



(b) Design the processing blocks A and B so that the signal to noise ratio of $\hat{x}[n]$ is at least twice that of the simple scheme above. You should use upsamplers, downsamplers and lowpass filters only.

(c) Using your new scheme, how long does it take to transmit a 3-minute song signal?

Solution 4.

(a) The power of the good signal is simply

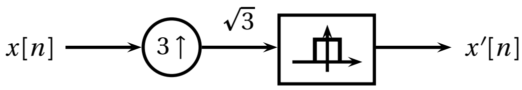
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(e^{j\omega})d\omega = \sigma_x^2$$

The power of the noise is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_\eta(e^{j\omega})d\omega = \sigma_0^2$$

so that the SNR is simply σ_x^2/σ_0^2. Just as in oversampling, the idea is to send the signal more slowly as to occupy less bandwidth. If the signal has a smaller bandwidth, we can increase its amplitude without exceeding the power constraint, which will allow us to have a better SNR over the band of interest.

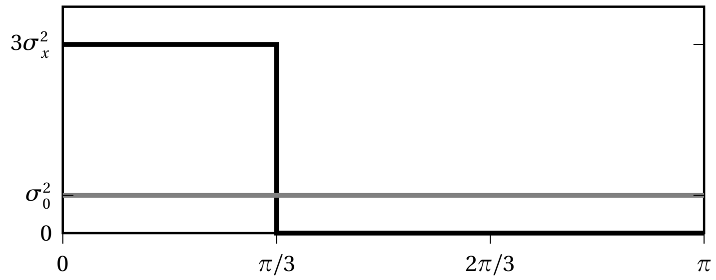
(b) Consider the following preprocessing chain, where the lowpass filter has a cutoff frequency $\frac{\pi}{3}$



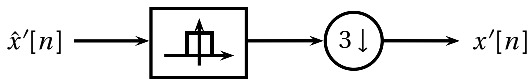
The power of $x'[n]$ is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_{x'}(e^{j\omega})d\omega = \frac{1}{2\pi}\int_{-\pi/3}^{\pi/3} 3\sigma_x^2 = \sigma_x^2$$

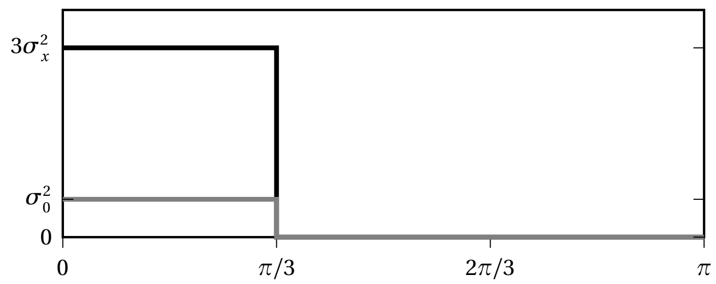
so that the power constraint is fulfilled. The signal and the noise at the receiver after the sampler have the following PSDs, where the PSD of the signal is in black and that of noise in gray:



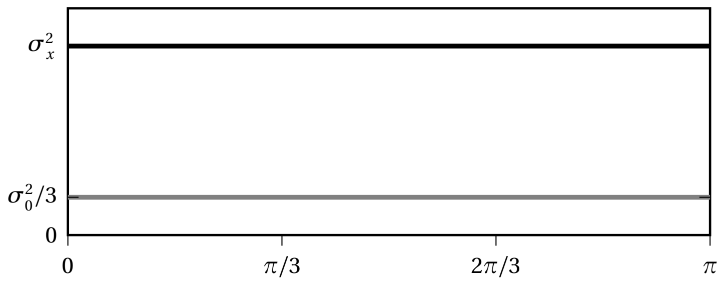
At the receiver we can filter out the out-of-band noise with the following scheme where, once again, the lowpass has cutoff frequency $\pi/3$:



After the filter the psd is like so:



and after the downsampler it becomes



so that the signal-to-noise ratio is

$$\mathrm{SNR}_2 = \sigma_x^2/(\sigma_0^2/3) = 3\mathrm{SNR}_1$$

(c) 9 minutes

Exercise 5. Phase Modulation.

In this exercise we will study a data transmission scheme known as *phase modulation* (PM). Consider a discrete-time signal $x[n]$, with the following properties:

- $|x[n]| < 1$ for all n
- $X(e^{j\omega}) = 0$ for $|\omega| < \alpha$, with α small.

A PM transmitter with carrier frequency ω_c works by producing the signal

$$y[n] = \mathscr{P}_{\omega_c}\{x[n]\} = \cos(\omega_c n + k\,x[n])$$

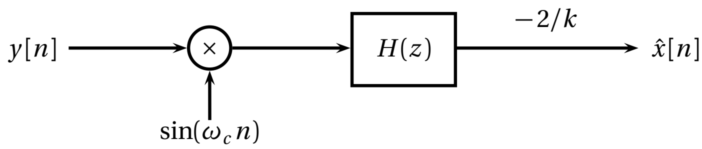
where k is a small positive constant; in other words, the data signal $x[n]$ is used to modify the instantaneous *phase* of a sinusoidal carrier. The advantage of this modulation technique is that it builds a signal with constant envelope (namely, a sinusoid with fixed amplitude) which results in a greater immunity to noise; this is the same principle behind the better quality of FM radio versus AM radio. However phase modulation is less "user friendly" than standard amplitude modulation because it is nonlinear.

(a) Show that phase modulation is *not* a linear operation.

Because of nonlinearity, the spectrum of the signal produced by a PM transmitter cannot be expressed in simple mathematical form. For the purpose of this exercise you can simply assume that the PM signal occupies the frequency band $[\omega_c - \gamma, \omega_c + \gamma]$ (and, obviously, the symmetric interval $[-\omega_c - \gamma, -\omega_c + \gamma]$) with

$$\gamma \approx 2(k+1)\alpha.$$

To demodulate a PM signal the following scheme is proposed, in which $H(z)$ is a lowpass filter with cutoff frequency equal to α:



(b) Show that $\hat{x}[n] \approx x[n]$. Assume that $\omega_c \gg \alpha$ and that k is small, say $k = 0.2$. (You may find it useful to express trigonometric functions in terms of complex exponentials if you don't recall the classic trigonometric identities. Also, remember that $\sin x \approx x$ for x sufficiently small).

Solution 5.

(a) Given a signal $x[n]$ fulfilling the magnitude and bandwidth requirements, if PM was a linear operation, for any scalar $\beta \in \mathbb{R}$ we should have

$$\mathscr{P}_{\omega_c}\{\beta x[n]\} = \beta \mathscr{P}_{\omega_c}\{x[n]\}.$$

However, irrespective of $x[n]$, $|\mathscr{P}_{\omega_c}\{\cdot\}| \leq 1$. Since we can always pick a value for β so that the right-hand side of the equality takes values larger than one, the equality cannot hold in general.

(b) Nonlinear operators make it impossible to proceed analytically in the frequency domain. In the time domain, however, the signal after the multiplier is

$$\begin{aligned} d[n] &= y[n]\sin(\omega_c n) \\ &= \cos(\omega_c n + k x[n])\sin(\omega_c n) \\ &= (1/2)\sin(\omega_c n + k x[n] + \omega_c n) - (1/2)\sin(\omega_c n + k x[n] - \omega_c n) \\ &= (1/2)\sin(2\omega_c n + k x[n]) - (1/2)\sin(k x[n]) \\ &\approx (1/2)\sin(2\omega_c n + k x[n]) - (k/2)x[n] \end{aligned}$$

where we have used the small-angle approximation for the sine since $|k x[n]| < 0.2$. The signal $d[n]$ now contain a baseband component and a PM component at twice the carrier frequency, which is eliminated by the lowpass filter:

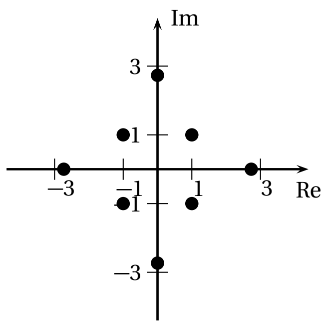
$$\hat{x}[n] = (-2/k)h[n] * d[n] \approx x[n].$$

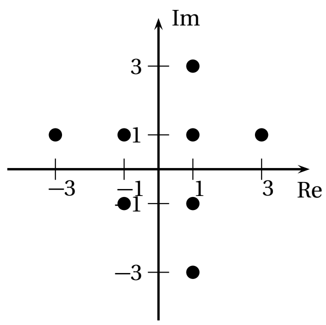
[Note: we used the trigonometric identity $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$. This can be easily derived by developing the product $(e^{j\alpha} + e^{-j\alpha})(e^{j\beta} - e^{-j\beta})/(2j)$.]

Exercise 6. The shape of a constellation.
In QAM the energy of the transmitted signal is directly proportional (via a gain factor) to the intrinsic power of the constellation σ^2, namely, the expected power of the transmitted symbol. For equiprobable symbols we have

$$\sigma^2 = \mathrm{E}\big[|a[n]|^2\big] = \sum_{\alpha \in \mathscr{A}} |\alpha|^2 \, p_a(\alpha)$$

where $\mathscr{A}$ is the set of all points in the constellation. The value of σ^2 depends on the arrangement of points on the complex plane and, by arranging the same number of alphabet symbols in a different manner, we can sometimes reduce σ^2 and therefore use a larger amplification gain while keeping the total output power constant, which in turn lowers the probability of error.
Consider the following two 8-point constellations, in which the outer points in the constellation on the left are at a distance of $1 + \sqrt{3}$ from the origin.





Compute the intrinsic power σ^2 for both constellations.

Solution 6.
In both constellations symbols are at only two possible distances from the origin. For the first constellation the distances are

- $|\alpha_1| = \sqrt{2}x$
- $|\alpha_2| = 1 + \sqrt{3}$

while for the second we have

- $|\alpha_1| = \sqrt{2}x$
- $|\alpha_2| = \sqrt{10}x$

Now, because of symmetry, it is simply

$$\sigma^2 = \sum_{\alpha \in \mathscr{A}} |\alpha|^2 p_a(\alpha) = \frac{4\alpha_1 + 4\alpha_2}{8}$$

so that, for the first constellation

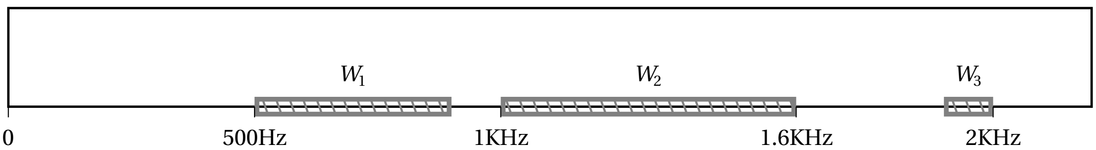
$$\sigma_1^2 = 4.73$$

and for the second

$$\sigma_2^2 = 6.$$

In other words, the irregular constellation on the left offers more than a 1 dB gain over the one on the right. This gain can be translated into a reliability gain by increasing the transmission gain while the transmitted signal remains within the power constraint.

Exercise 7. Transmitter design.
In this exercise you will need to design a data transmission system for the analog channel sketched in this picture (positive frequencies only):



The channel has three usable bands with the following characteristics:

- band W_1, from 500Hz to 900Hz
- band W_2, from 1000Hz to 1600Hz
- band W_3, from 1900Hz to 2000Hz

To transmit the data you can use one or more configurable passband data transmitters. For each transmitter you can set the following parameters:

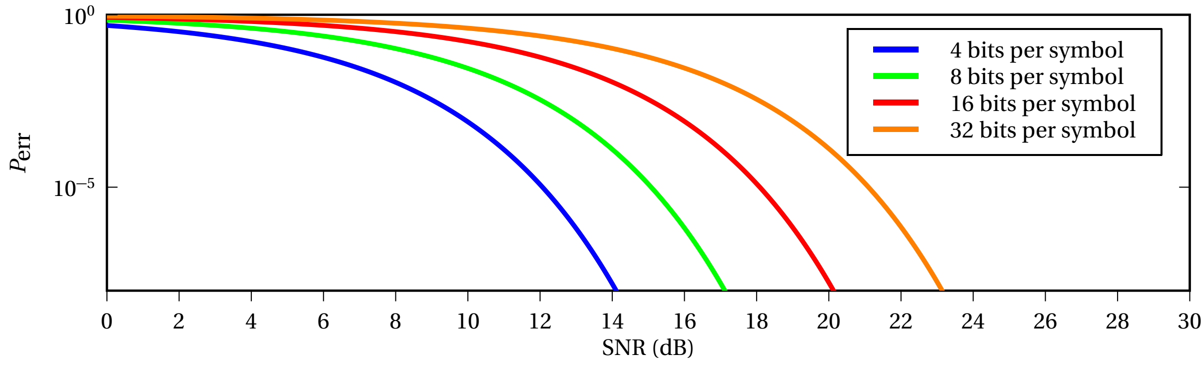
1- the center frequency for the trasmission band (in Hz)
2- the number of symbols per second (i.e. the symbol rate or Baud rate)
3- the number of bits per symbol (4, 8, 16, or 32)

Each transmitter can also adjust its gain so that its transmitted signal reaches the maximum power allowed by the channel's power constraint. Because of different noise levels across the spectrum the resulting SNR levels for each subband are

- band W_1: SNR of 16dBs
- band W_2: SNR of 10dBs

- band W_3: SNR of 20dBs

The operational characteristic for the data transmitters is shown in this chart, where each curve shows the attainable probability of error as a function of the SNR; each curve corresponds to a different bit-per-symbol operation mode:

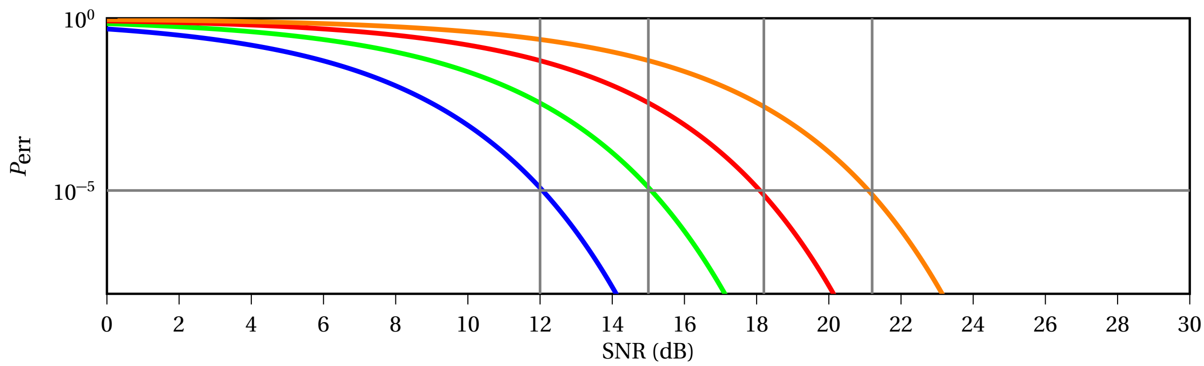


We want the transmission scheme to operate at an overall probability of error of 10^{-5} or less. Determine the maximum achievable transmission rate (in bits per second).

Solution 7.
Since we have three independent subbands, we will use a separate transmitter for each one. The number of symbols per second for each transmitter will be equal to the (positive) width of each subband while the transmission center frequency will be equal to each subband's center frequency

- band W_1: 400 symbols per second, center frequency $f_1 = 700$Hz
- band W_2: 600 symbols per second, center frequency $f_1 = 1300$Hz
- band W_3: 100 symbols per second, center frequency $f_1 = 1950$Hz

To compute the total throughput we need to determine how many bits per symbol can be sent over each subchannel at the given reliability figure. Since we need to operate at most at $P_e = 10^{-5}$ we can find the required minimum SNR for each operation curve by looking at the intersection with the line $P_e = 10^{-5}$:



From the plot we can see that:

- to transmit at 4 bits per symbol we need at least ≈ 12dB SNR
- to transmit at 8 bits per symbol we need at least ≈ 15dB SNR
- to transmit at 16 bits per symbol we need at least ≈ 18dB SNR

- to transmit at 32 bits per symbol we need at least ≈ 21dB SNR

therefore:

- band W_1: SNR of 16dBs $\Rightarrow$ we can transmit at 8 bits per symbol
- band W_2: SNR of 10dBs $\Rightarrow$ we cannot transmit since all possible rates will have $P_e > 10^{-5}$ at this SNR
- band W_3: SNR of 20dBs $\Rightarrow$ we can transmit at 16 bits per symbol

The final capacity of the transmission scheme is therefore $R = W_1 \cdot 8 + W_3 \cdot 16 = 4800$bps.