DSP2 – Practice Homework
Solutions

Exercise 1. LTI systems.
Consider the transformation $\mathscr{H}\{\mathbf{x}\}[n] = n\,x[n]$. Does $\mathscr{H}$ define an LTI system?

Solution 1.
The system is not time-invariant. To see this consider the following signals

$$\begin{aligned} x[n] &= \delta[n] \\ y[n] &= \delta[n-1] \end{aligned}$$

We have $\mathscr{H}\{\mathbf{x}\} = \mathbf{0}$. Clearly, it is $y[n] = x[n-1]$ but

$$\mathscr{H}\{\mathbf{y}\}[n] = \delta[n-1] \neq 0.$$

Exercise 2. Convolution.
Let $x[n]$ be a discrete-time sequence defined as

$$x[n] = \begin{cases} M-n & 0 \le n \le M, \\ M+n & -M \le n \le 0, \\ 0 & \text{otherwise.} \end{cases}$$

for some odd integer M.

(a) Show that $x[n]$ can be expressed as the convolution of two discrete-time sequences $x_1[n]$ and $x_2[n]$.

(b) Using the results found in (a), compute the DTFT of $x[n]$.

Solution 2.

(a) $x[n]$ can be written as the convolution of $x_1[n]$ and $x_2[n]$ defined as

$$\begin{aligned} x_1[n] = x_2[n] &= \begin{cases} 1 & -(M-1)/2 \le n \le (M-1)/2 \\ 0 & \text{otherwise.} \end{cases} \\ &= u[n+(M-1)/2] - u[n-(M+1)/2]. \end{aligned}$$

Then,

$$\begin{aligned} x_1[n] * x_2[n] &= \Sigma_k x_1[k] x_2[n-k] \\ &\stackrel{(1)}{=} \Sigma_k x_1[k] x_1[k-n] \\ &\stackrel{(2)}{=} x[n] \end{aligned}$$

(1) follows from the fact that $x_1[n] = x_2[n]$ and the symmetry of $x_1[n]$. (2) follows by noticing that the sum corresponds to the size of the overlapping area between $x_1[k]$ and its n-shifted version $x_1[k-n]$.

(b) The DTFT of $x_1[n]$ can be easily computed directly using the DTFT formula which, in this case, is a combination of simple geometric sums:

$$X_1(e^{j\omega}) = \sum_{n=-(M-1)/2}^{0} e^{j\omega n} - 1 + \sum_{n=0}^{(M-1)/2} e^{j\omega n}.$$

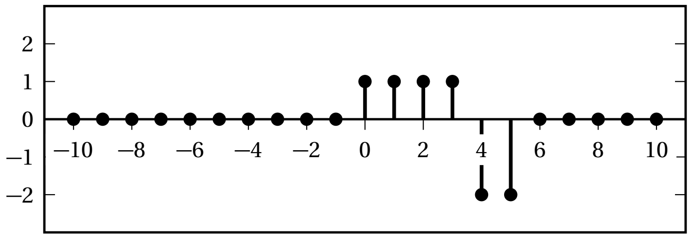
Alternatively, we can use the DTFT of the shifted step sequences $u[n]$:

$$\begin{aligned} X_1(e^{j\omega}) &\overset{(1)}{=} \left(\frac{1}{1-e^{-j\omega}} + \frac{1}{2}\tilde{\delta}(\omega)\right)\left(e^{j\omega(M-1)/2} - e^{-j\omega(M+1)/2}\right) \\ &\overset{(2)}{=} \frac{e^{j\omega(M-1)/2} - e^{-j\omega(M+1)/2}}{1-e^{-j\omega}} = \frac{e^{-j\omega/2}(e^{j\omega M/2} - e^{-j\omega M/2})}{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})} \\ &= \frac{\sin(\omega M/2)}{\sin(\omega/2)} \end{aligned}$$

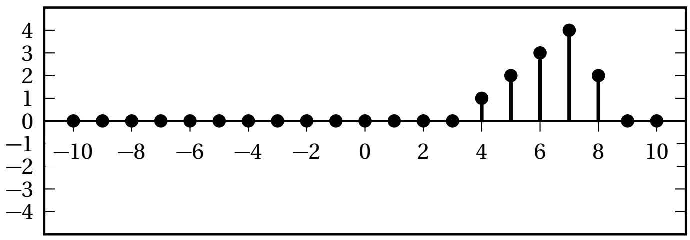
(1) follows from the DTFT of $u[n]$, (2) follows from $e^{j\omega(M-1)/2}\tilde{\delta}(\omega) = e^{-j\omega(M+1)/2}\tilde{\delta}(\omega)$. Finally, using the convolution theorem, we can write

$$\begin{aligned} X(e^{j\omega}) &= X_1(e^{j\omega})X_2(e^{j\omega}) \\ &= X_1(e^{j\omega})X_1(e^{j\omega}) \\ &= \left(\frac{sin(\omega M/2)}{sin(\omega/2)}\right)^2. \end{aligned}$$

Exercise 3. Impulse response.
Consider a system whose impulse response is shown below. Determine and carefully sketch the response of this system to the input $x[n] = u[n-4]$.



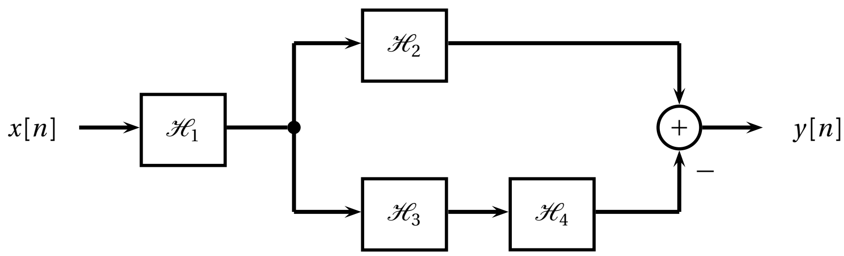
Solution 3.



Exercise 4. Series/Parallel.
Calculate the impulse response of the system shown below given that the impulse responses of the processing blocks are

- $h_1[n] = 3(-1)^n(\frac{1}{4})^n u[n-2]$
- $h_2[n] = h_3[n] = u[n+2]$
- $h_4[n] = \delta[n-1]$

Also determine the causality of the system and its BIBO stability.



Solution 4.
We have

$$\begin{aligned} h[n] &= h_1[n] * (h_2[n] - h_3[n] * h_4[n]) \\ &= h_1[n] * (u[n+2] - u[n+2] * \delta[n-1]) \\ &= h_1[n] * (u[n+2] - u[n+1]) \\ &= h_1[n] * \delta[n+2] \\ &= h_1[n+2] \\ &= 3(-1)^n(\frac{1}{4})^{n+2} u[n]. \end{aligned}$$

A discrete system is BIBO stable if the impulse response is absolutely summable. We have

$$\sum_{n=-\infty}^{\infty} |h[n]| = \frac{3}{4^2} \sum_{k\geq 0} 4^{-k} = \frac{3}{4^2} \frac{1}{1-1/4} = \frac{1}{4},$$

which means the system is BIBO stable. The system is causal because $h[n] = 0$ for $n < 0$.

Exercise 5. System properties.
For each of the input-output relationships listed below, determine if the corresponding transformation is linear, time-invariant, BIBO stable and causal. For LTI transformations, characterize the corresponding systems by their impulse response.

(a) $\mathcal{H}_1\{\mathbf{x}\}[n] = x[-n]$,

(b) $\mathcal{H}_2\{\mathbf{x}\}[n] = e^{-j\omega n} x[n]$,

(c) $\mathcal{H}_3\{\mathbf{x}\}[n] = \sum_{k=n-L}^{n+L} x[k]$, with $L > 0$

Solution 5.

(a) - $\mathcal{H}_1$ is linear:

$$\mathcal{H}_1\{a\mathbf{x}_1 + b\mathbf{x}_2\}[n] = a\,x_1[-n] + b\,x_2[-n] = a\mathcal{H}_1\{\mathbf{x}_1\}[n] + b\mathcal{H}_1\{\mathbf{x}_2\}[n].$$

- $\mathcal{H}_1$ is NOT time invariant: set $d[n] = x[n-N]$; then

$$\mathcal{H}_1\{\mathbf{d}\}[n] = d[-n] = x[-n-N] \neq \mathcal{H}_1\{\mathbf{x}\}[n-N] = x[-n+N].$$

- $\mathscr{H}_1$ is BIBO stable:

$$|x[n]| \leq M \Rightarrow |\mathscr{H}_1\{\mathbf{x}\}[n]| \leq M.$$

- $\mathscr{H}_1$ is not causal since producing the output for negative values of the index requires knowledge of the values of the input in the future.
- $\mathscr{H}_1$ is not LTI and therefore it cannot be characterized by an impulse response.

(b)

- $\mathscr{H}_2$ is linear:

$$\mathscr{H}_2\{a\mathbf{x}_1 + b\mathbf{x}_2\}[n] = e^{-j\omega n}(a x_1[n] + b x_2[n]) = a\mathscr{H}_2\{\mathbf{x}_1\}[n] + b\mathscr{H}_2\{\mathbf{x}_2\}[n].$$

- $\mathscr{H}$ is NOT time invariant (except in the trivial case $\omega = 0$): set $d[n] = x[n-N]$; then

$$\mathscr{H}_2\{\mathbf{d}\}[n] = e^{-j\omega n} d[n] = e^{-j\omega n} x[n-N] \neq \mathscr{H}_2\{\mathbf{x}\}[n-N] = e^{j\omega N} e^{-j\omega n} x[n-N].$$

- $\mathscr{H}$ is BIBO stable:

$$|x[n]| \leq M \Rightarrow |\mathscr{H}_2\{\mathbf{x}\}[n]| = |x[n]| \leq M.$$

- $\mathscr{H}_2$ is a memoryless transformation of the input and therefore it can be considered causal.
- $\mathscr{H}_2$ is not LTI and therefore it cannot be characterized by an impulse response.

(c)

- $\mathscr{H}_3$ is linear:

$$\mathscr{H}_3\{a\mathbf{x}_1 + b\mathbf{x}_2\}[n] = \sum_{k=n-L}^{n+L} (a x_1[k] + b x_2[k]) = a\mathscr{H}_3\{\mathbf{x}_1\}[n] + b\mathscr{H}_3\{\mathbf{x}_2\}[n].$$

- $\mathscr{H}$ is time invariant: set $d[n] = x[n-N]$; then

$$\mathscr{H}_3\{\mathbf{d}\}[n] = \sum_{k=n-L}^{n+L} d[k] = \sum_{k=n-L}^{n+L} x[k-N] = \sum_{k=(n-N)-L}^{(n-N)+L} x[k] = \mathscr{H}_3\{\mathbf{x}\}[n-N].$$

- $\mathscr{H}$ is BIBO stable:

$$|x[n]| \leq M \Rightarrow \mathscr{H}_3\{x\}[n] \leq |2L+1|M.$$

- $\mathscr{H}$ is not causal.
- The system impulse response is

$$h[n] = \begin{cases} 1 & \text{if } |n| \leq L, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 6. Zero phase filtering.

Let $\mathscr{R}$ be the time reversal operator for sequences so that

$$\mathscr{R}\{\mathbf{x}\}[n] = x[-n]$$

Let $\mathscr{H}$ be a linear time invariant system with a real-valued impulse response. What are the properties of a system implementing the following transformation, assuming that the input signal $\mathbf{x}$ is real-valued?

$$\mathscr{R}\{\mathscr{H}\{\mathscr{R}\{\mathscr{H}\{\mathbf{x}\}\}\}\}$$

Solution 6.

Let $X(e^{j\omega})$ be the DTFT of the sequence $\mathbf{x}$ and let $H(e^{j\omega})$ be the frequency response of the filter. Since both impulse response and signal are real, their DTFT are Hermitian-symmetric, i.e., $X(e^{j\omega}) = X^*(e^{-j\omega})$.
By the time reversal property of the DTFT, the DTFT of $x[-n]$ is $X(e^{-j\omega})$ which, because of the Hermitian symmetry, is also equal to $X^*(e^{j\omega})$. With this, we can write the following list of time-frequency correspondences

$$\begin{aligned}
\mathcal{H}\{\mathbf{x}\} &\overset{\text{DTFT}}{\longleftrightarrow} H(e^{j\omega})X(e^{j\omega})\\
\mathcal{R}\{\mathcal{H}\{\mathbf{x}\}\} &\overset{\text{DTFT}}{\longleftrightarrow} H^*(e^{j\omega})X^*(e^{j\omega})\\
\mathcal{H}\{\mathcal{R}\{\mathcal{H}\{\mathbf{x}\}\}\} &\overset{\text{DTFT}}{\longleftrightarrow} H(e^{j\omega})H^*(e^{j\omega})X^*(e^{j\omega})\\
\mathcal{R}\{\mathcal{H}\{\mathcal{R}\{\mathcal{H}\{\mathbf{x}\}\}\}\} &\overset{\text{DTFT}}{\longleftrightarrow} H^*(e^{j\omega})H(e^{j\omega})X(e^{j\omega}) = |H(e^{j\omega})|^2X(e^{j\omega})
\end{aligned}$$

Therefore the chain of operators implements an LTI system with zero phase delay and magnitude response equal to the squared magnitude response of the original filter.

Exercise 7. Pole-zero plot and stability

Consider a causal LTI system with the following transfer function

$$H(z) = \frac{3+4.5z^{-1}}{1+1.5z^{-1}} - \frac{2}{1-0.5z^{-1}}$$

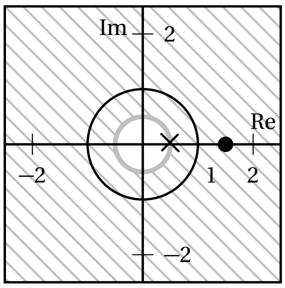
Sketch the pole-zero plot of the transfer function and specify its region of convergence. Is the system stable?

Solution 7.

To investigate the stability of the system, let's compute the overall transfer function and eliminate possible common factors; factorization in first-order terms yields

$$H(z) = \frac{3+4.5z^{-1}}{1+1.5z^{-1}} - \frac{2}{1-0.5z^{-1}} = \frac{(z-1.5)(z+1.5)}{(z-0.5)(z+1.5)}$$

We can see that the zeros of this system are in $z_{01} = 1.5$ and $z_{02} = -1.5$ and the poles in $z_{p1} = 0.5$ and $z_{p2} = -1.5$. From a theoretical point of view the zero in $z_{01} = -1.5$ cancels out the pole in $z_{p2} = -1.5$ and we obtain the following ROC, extending outwards from $z = 0.5$ and including the unit circle, so that the system is stable:



In other words, the second subsystem in the cascade has stabilized the overall processing function by canceling the zero in the first subsystem. Note however that, in practice, the stabilization of a system with this technique is a risky enterprise since the exact cancellation of the pole outside the unit circle will be extremely sensitive to numerical precision issues. If the coefficients of the filter (or the internal accumulators) are subject to truncation or rounding, the implicit position of the zero may drift ever so slightly from the implicit position of the pole and the system will no longer be stable.

Exercise 8. Stability.

Consider a causal discrete system represented by the following difference equation

$$y[n] - 3.25y[n-1] + 0.75y[n-2] = x[n-1] + 3x[n-2].$$

(a) Compute the transfer function and check the stability of this system.

(b) If the input signal is $x[n] = \delta[n] - 3\delta[n-1]$, compute the z-transform of the output signal and discuss the stability result you found before.

(c) Consider now the input signal $x[n] = \delta[n] - 0.25\delta[n-1]$; assume the filter is implemented in fixed-precision arithmetic and that the the processing unit can only handle numbers in the range [-1024, 1024]. Starting processing for $n = 0$, how many samples can we process before we hit overflow or underflow?

Solution 8.

(a) By taking the z-transform of the CCDE we have

$$Y(z)(1 - 3.25z^{-1} + 0.75z^{-2}) = X(z)(z^{-1} + 3z^{-2}),$$

from which we obtain the transfer function

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{z^{-1} + 3z^{-2}}{1 - 3.25z^{-1} + 0.75z^{-2}} \\ &= \frac{z^{-1}(1 + 3z^{-1})}{(1 - 0.25z^{-1})(1 - 3z^{-1})}. \end{aligned}$$

Since the system is causal, the region of convergence is $|z| > 3$. This does not include the unit circle and therefore the system is unstable.

(b) The z-transform of the output signal is

$$\begin{aligned} Y(z) = H(z)X(z) &= \frac{z^{-1}(1 + 3z^{-1})}{(1 - 0.25z^{-1})(1 - 3z^{-1})}(1 - 3z^{-1}) \\ &= \frac{z^{-1} + 3z^{-2}}{1 - 0.25z^{-1}}. \end{aligned}$$

The ROC for $Y(z)$ is $|z| > 0.25$, which includes the unit circle. This implies that $y[n]$ has a well-defined DTFT and therefore is finite-energy. What's happening is that the input signal has a spectral null exactly at the location of the pole of the system. Since in this case no energy "excites" the system pole, the output is a stable signal.

(c) The z-transform of the output signal is

$$Y(z) = H(z)X(z) = \frac{z^{-1} + 3z^{-2}}{1 - 3z^{-1}} = z^{-1}\frac{1}{1 - 3z^{-1}} + 3z^{-2}\frac{1}{1 - 3z^{-1}}$$

Since we know that

$$\mathscr{Z}\left\{3^n u[n]\right\} = \frac{1}{1 - 3z^{-1}}$$

we can easily invert $Y(z)$ to obtain

$$y[n] = 3^{n-1}u[n-1] + 3^{n-1}u[n-2] = \begin{cases} 0 & n < 1 \\ 1 & n = 1 \\ 2 \cdot 3^{n-1} & n \geq 2. \end{cases}$$

It appears that the input is monotonically increasing and, to see when we reach overflow, we need to find the first value of n for which

$$2 \cdot 3^{n-1} > 1000$$

which yields $n = 6$.

Exercise 9. Properties of the z-transform.
Let $x[n]$ be a discrete-time sequence and $X(z)$ its corresponding z-transform with appropriate ROC.

(a) Prove that the following relation holds:

$$n\,x[n] \stackrel{Z}{\longleftrightarrow} -z\frac{d}{dz}X(z).$$

(b) Show that

$$(n+1)\alpha^n u[n] \stackrel{Z}{\longleftrightarrow} \frac{1}{(1-\alpha z^{-1})^2}, \qquad |z| > |\alpha|.$$

(c) Suppose that the above expression corresponds to the impulse response of an LTI system. What can you say about the causality of such a system? About its stability?

(d) Let $\alpha = 0.8$, what is the spectral behavior of the corresponding filter? What if $\alpha = -0.8$?

Solution 9.

(a) Let $X(z) = \Sigma_n x[n]z^{-n}$. We have that

$$\begin{aligned}\frac{d}{dz}X(z) &= \frac{d}{dz}\left(\Sigma_n x[n]z^{-n}\right)\\ &= \Sigma_n(-n)x[n]z^{-n-1}\\ &= -z^{-1}\Sigma_n n\,x[n]z^{-n}\end{aligned}$$

and the relation follows directly.

(b) We have that

$$\alpha^n u[n] \stackrel{Z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}.$$

Using the previous result, we find

$$n\alpha^n u[n] \stackrel{Z}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-\alpha z^{-1}}\right) = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}.$$

Thus,

$$(n+1)\alpha^{n+1} u[n+1] \stackrel{Z}{\longleftrightarrow} z\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$$

and

$$(n+1)\alpha^n u[n+1] \stackrel{Z}{\longleftrightarrow} \frac{1}{(1-\alpha z^{-1})^2}.$$

The relation follows by noticing that

$$(n+1)\alpha^n u[n+1] = (n+1)\alpha^n u[n]$$

since when $n = -1$ both sides are equal to zero.

(c) The system is causal since the ROC corresponds to the outside of a circle of radius α (or equivalently since the impulse response is zero when $n < 0$). The system is stable when the unit circle lies inside the ROC, i.e. when $|\alpha| \leq 1$.

(d) When $\alpha = 0.8$, the angular frequency of the pole is $\omega = 0$. Thus the filter is lowpass. When $\alpha = -0.8$, $\omega = \pi$ and the filter is highpass.

Exercise 10. Interleaving sequences.
Consider two two-sided sequences $h[n]$ and $g[n]$ and consider a third sequence $x[n]$ which is built by interleaving the values of $h[n]$ and $g[n]$:

$$x[n] = \ldots, h[-3], g[-3], h[-2], g[-2], h[-1], g[-1], h[0], g[0], h[1], g[1], h[2], g[2], h[3], g[3], \ldots$$

with $x[0] = h[0]$.

(a) Express the z-transform of $x[n]$ in terms of the z-transforms of $h[n]$ and $g[n]$.

(b) Assume that the ROC of $H(z)$ is $0.64 < |z| < 4$ and that the ROC of $G(z)$ is $0.25 < |z| < 9$. What is the ROC of $X(z)$?

Solution 10.

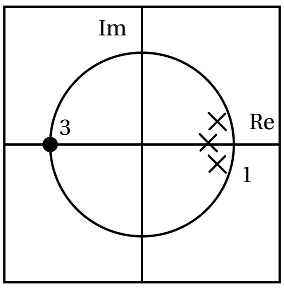
(a) We have that:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-2n} + g[n]z^{-(2n+1)} \\ &= H(z^2) + z^{-1}G(z^2) \end{aligned}$$

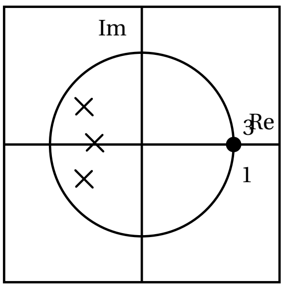
(b) The ROC is determined by the poles of the transform. Since the sequence is two sided, the ROC is a ring bounded by two poles z_L and z_R such that $|z_L| < |z_R|$ and no other pole has magnitude between $|z_L|$ and $|z_R|$. Consider $H(z)$; if z_0 is a pole of $H(z)$, $H(z^2)$ will have two poles at $\pm z_0^{1/2}$; however, the square root preserves the monotonicity of the magnitude and therefore no new poles will appear between the circles $|z| = \sqrt{|z_L|}$ and $|z| = \sqrt{|z_R|}$. Therefore the ROC for $H(z^2)$ is the ring $\sqrt{|z_L|} < |z| < \sqrt{|z_R|}$. The ROC of the sum $H(z^2) + z^{-1}G(z^2)$ is the intersection of the ROCs, and so

$$\text{ROC } = 0.8 < |z| < 2.$$

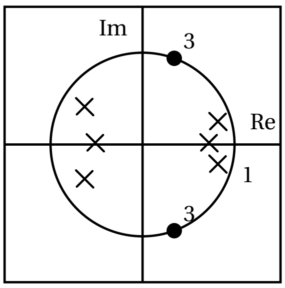
Exercise 11. Transfer function, poles, and zeros
The figures below show the zeros and poles of three different filters with the unit circle for reference. Each zero is represented with a 'o' and each pole with a 'x' on the plot. Multiple zeros and poles are indicated by the multiplicity number shown to the upper right of the zero or pole.Sketch the magnitude of each frequency response and determine the type of filter.



(a)

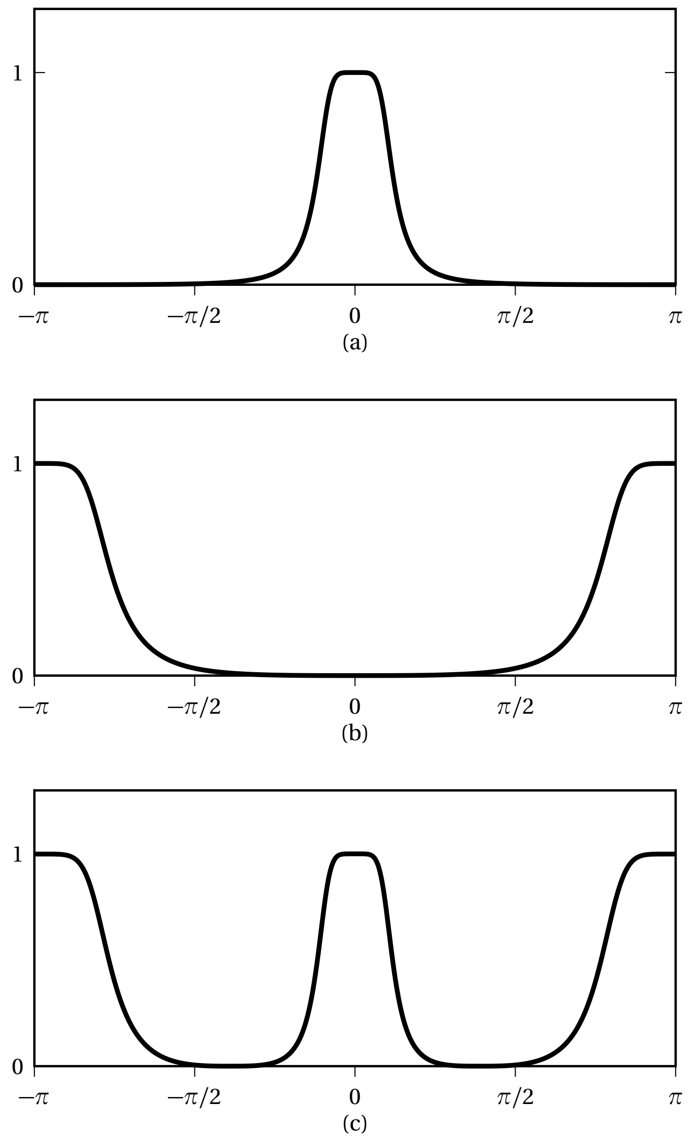


(b)



(c)

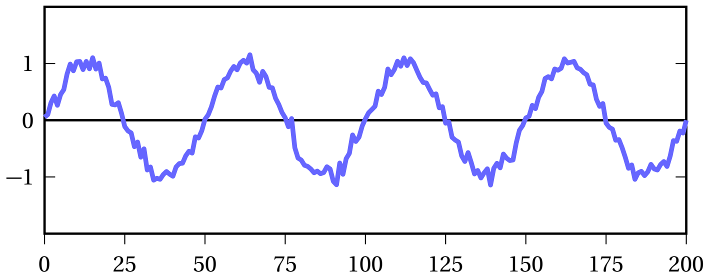
Solution 11.
To obtain the frequency response of a filter, we analyze the z-transform on the unit circle, that is, in $z = e^{j\omega}$. the following figures shows the exact magnitude of each frequency response.



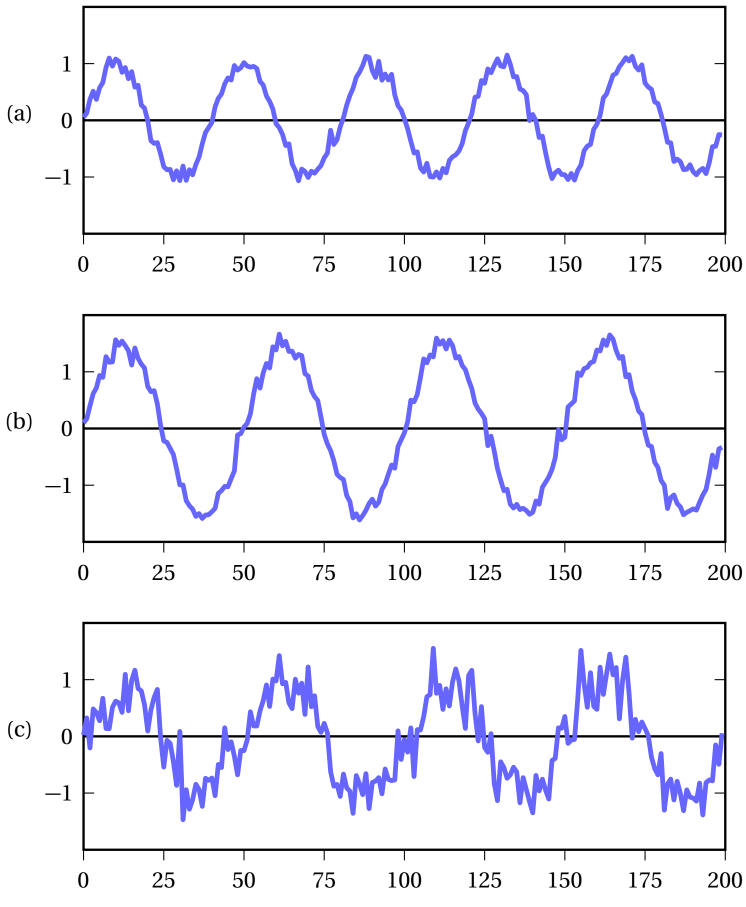
The first filter is a low-pass filter. Note that there are three poles located in low frequency (near $\omega = 0$), while there is a zero located in high frequency ($\omega = \pi$). The second filter is just the opposite. The zero is located in low frequency, while the influence of the three poles is maximum in high frequency ($\omega = \pi$). Therefore, it is a high-pass filter. In the third system, there are poles which affect low and high frequency and two zeros close to $w = \pi/2$. Therefore, this system is a stop-band filter. Incidentally, these are all Butterworth filters; this can be determined considering that

- they are IIR filters (they contain poles)
- they have monotonic frequency responses (i.e. no oscillations). Butterworth filters are the only filters that exhibit this property.

Exercise 12. Denoising.
Consider the signal $x[n] = \sin(\omega_0 n) + s[n]$ where $\omega_0 = 2\pi/50$ and $s[n]$ is a zero-mean, Gaussian white noise sequence with power spectral density $P_s(e^{j\omega}) = \sigma^2 = 10^{-2}$. A realization of the signal is plotted in the following figure:



The signal is filtered with a stable, real-valued, causal LTI system whose frequency response satisfies $|H(e^{j\omega})| \leq 1$ for all frequencies. For each of the following plots, explain if the signal in the plot could be the result of filtering $x[n]$ with $H(e^{j\omega})$; explain your answers in detail.



Solution 12.
The filter will act on the deterministic sinusoidal component and on the noise component independently;

therefore the output can be written as

$$\begin{aligned} y[n] &= h[n] * (\sin(\omega_0 n) + s[n]) \\ &= A\sin(\omega_c + \theta) + h[n] * s[n] \\ &= A\sin(\omega_c + \theta) + v[n] \end{aligned}$$

where $A = |H(e^{j\omega_c})|$ and $\theta = \angle H(e^{j\omega_c})$.

(a) NO: the frequency of the sinusoidal component in the output is clearly different than the frequency of the input; since a filter cannot change the frequency of a sinusoidal input this signal cannot be a valid output.

(b) NO: the amplitude of the sinusoidal component has increased, but this cannot happen since by design $|H(e^{j\omega})| \le 1$ for all ω.

(c) NO: the variance of the noise seems to have increased in the output. However the variance of the output noise is

$$\begin{aligned} \sigma_v^2 &= r_v[0] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 P_s(e^{j\omega}) d\omega \\ &= \sigma_s^2 \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega \\ &\le \sigma_s^2 \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \\ &\le \sigma_s^2, \end{aligned}$$

so the variance of the output noise cannot increase after filtering with the given $H(e^{j\omega})$.

Exercise 13. Autocorrelation.
Assume $w[n]$ is a white process with variance σ^2 and consider the WSS process

$$x[n] = w[n] + 0.5w[n-1]$$

(a) compute the autocorrelation $r_x[n]$ of the process $x[n]$

(b) write out the 3×3 autocorrelation matrix for $x[n]$

(c) compute the cross-correlation $r_{xw}[n]$ between $w[n]$ and $x[n]$

Solution 13.

(a) We have

$$\begin{aligned} r_x[k] &= \mathrm{E}[x[n]x[n-k]] \\ &= \mathrm{E}[(w[n] + 0.5w[n-1])(w[n-k] + 0.5w[n-1-k])] \\ &= \mathrm{E}[w[n]w[n-k]] + 0.5\mathrm{E}[w[n]w[n-1-k]] + 0.5\mathrm{E}[w[n-1]w[n-k]] + 0.25\mathrm{E}[w[n-1]w[n-1-k]] \\ &= \sigma^2[\delta[k] + 0.5\delta[k+1] + 0.5\delta[k-1] + 0.25\delta[k]] \\ &= \sigma^2[1.25\delta[k] + 0.5\delta[k+1] + 0.5\delta[k-1]] \end{aligned}$$

(b) The 3×3 autocorrelation matrix is

$$\sigma^2 \begin{bmatrix} 1.25 & 0.5 & 0 \\ 0.5 & 1.25 & 0.5 \\ 0 & 0.5 & 1.25 \end{bmatrix}$$

(c) $r_{xw}[k] = \mathrm{E}[x[n]w[n+k]] = \sigma^2\delta[k] + 0.5\sigma^2\delta[k+1]$

Exercise 14. Autocorrelation II.
Assume $w[n]$ is a white process with variance σ^2 and consider the WSS process

$$x[n] = w[n] + 0.5x[n-1]$$

(a) compute the autocorrelation $r_x[n]$ of the process $x[n]$

(b) write out the 3×3 autocorrelation matrix for $x[n]$

(c) compute the cross-correlation $r_{xw}[n]$ between $w[n]$ and $x[n]$

Solution 14.

(a) We can proceed as in the previous example, by computing expectations, or we can exploit the fundamental result on filtered random processes stating

$$r_x[n] = h[n] * h[-n] * r_w[n]$$

where $h[n]$ is the impulse response of the filter. In this case $H(z) = 1/(1-0.5z^{-1})$ so that $h[n] = 0.5^n u[n]$. With this, assuming $k > 0$,

$$\begin{aligned}(h[n] * h[-n])[k] &= \sum_{n=-\infty}^{\infty} h[n]h[n-k] \\ &= \sum_{n=0}^{\infty} h[n]h[n+k] \\ &= \sum_{n=0}^{\infty} 0.5^n 0.5^{n+k} \\ &= \frac{0.5^k}{1-0.5^2}.\end{aligned}$$

We can repeat the computation for $k < 0$: we obtain a symmetric sequence which, once convolved with $r_w[n] = \sigma_w^2\delta[n]$ yields again

$$r_x[k] = \frac{0.5^{|k|}}{0.75}\sigma_w^2.$$

(b) The 3×3 autocorrelation matrix is

$$\frac{\sigma_w^2}{0.75}\begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 1 \end{bmatrix}$$

(c) For non-negative values of the lag, the cross-correlation between input and output is

$$r_{xw}[k] = \mathrm{E}[x[n]w[n+k]] = \sigma_w^2\delta[k] \qquad (k \geq 0)$$

since the current output does not depend on future values of the input. For negative lags we have

$$\begin{aligned}r_{xw}[-k] &= \mathrm{E}[x[n]w[n-k]] \qquad (k > 0) \\ &= \mathrm{E}[w[n]w[n-k]] - 0.5\mathrm{E}[x[n-1]w[n-k]] \\ &= -0.5r_{xw}[-k+1]\end{aligned}$$

Using $r_{xw}[0] = \sigma_w^2$ and the recurrence relation above we finally have

$$r_{xw}[k] = 0.5^{-k}\sigma_w^2 u[-k].$$

Exercise 15. Linear Prediction.

The WSS stochastic process $\mathbf{x}$ has autocorrelation

$$r_x[k] = 0.5^{|k|+1}$$

(a) find the optimal third-order Linear Prediction filter that models $\mathbf{x}$

(b) how many nonzero coefficients does the filter have? Explain.

Solution 15.

A linear predictor of order 3 is a filter of the form

$$H(z) = \frac{1}{1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3}}$$

The normal equations to determine the coefficients are therefore

$$\begin{bmatrix} r_x[0] & r_x[1] & r_x[2] \\ r_x[1] & r_x[0] & r_x[1] \\ r_x[2] & r_x[1] & r_x[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} r_x[1] \\ r_x[2] \\ r_x[3] \end{bmatrix}$$

By replacing the values of the autocorrelation we have

$$\begin{bmatrix} 0.5 & 0.25 & 0.125 \\ 0.25 & 0.5 & 0.25 \\ 0.125 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.125 \\ 0.0625 \end{bmatrix}$$

By solving the system of equation we obtain

$$\begin{cases} a_1 = 0.5 \\ a_2 = a_3 = 0 \end{cases}$$

If we compare the autocorrelation sequence to the autocorrelation in the previous exercise, we can recognize that the input signal is an autoregressive signal obtained by filtering white noise with variance $\sigma_w^2 = 3/8$ via a filter with transfer function $H(z) = 1/(1 - 0.5z^{-1})$. The linear prediction algorithm therefore correctly identifies the order of the filter and its single nonzero coefficient.