DSP2 – Practice Homework

Exercise 1. LTI systems.
Consider the transformation $\mathscr{H}\{\mathbf{x}\}[n] = n\,x[n]$. Does $\mathscr{H}$ define an LTI system?

Exercise 2. Convolution.
Let $x[n]$ be a discrete-time sequence defined as

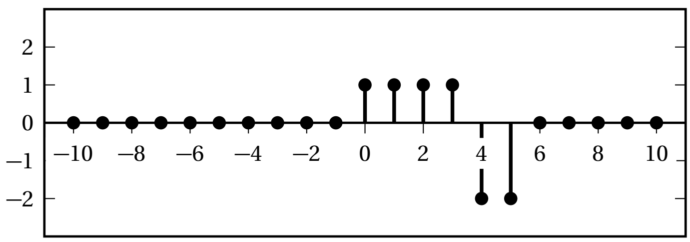
$$x[n] = \begin{cases} M-n & 0 \le n \le M, \\ M+n & -M \le n \le 0, \\ 0 & \text{otherwise.} \end{cases}$$

for some odd integer M.

(a) Show that $x[n]$ can be expressed as the convolution of two discrete-time sequences $x_1[n]$ and $x_2[n]$.

(b) Using the results found in (a), compute the DTFT of $x[n]$.

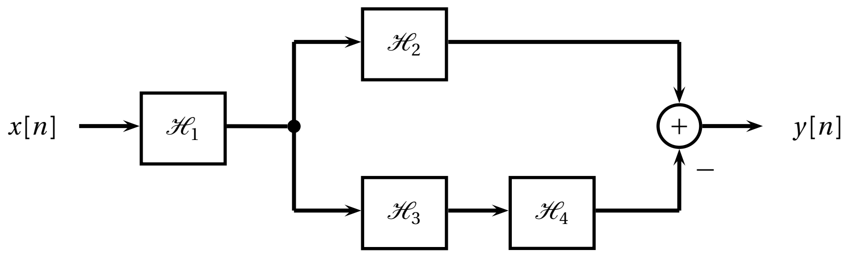
Exercise 3. Impulse response.
Consider a system whose impulse response is shown below. Determine and carefully sketch the response of this system to the input $x[n] = u[n-4]$.



Exercise 4. Series/Parallel.
Calculate the impulse response of the system shown below given that the impulse responses of the processing blocks are

- $h_1[n] = 3(-1)^n(\frac{1}{4})^n\,u[n-2]$
- $h_2[n] = h_3[n] = u[n+2]$
- $h_4[n] = \delta[n-1]$

Also determine the causality of the system and its BIBO stability.



Exercise 5. System properties.
For each of the input-output relationships listed below, determine if the corresponding transformation is linear, time-invariant, BIBO stable and causal. For LTI transformations, characterize the corresponding systems by their impulse response.

(a) $\mathcal{H}_1\{\mathbf{x}\}[n] = x[-n]$,

(b) $\mathcal{H}_2\{\mathbf{x}\}[n] = e^{-j\omega n}x[n]$,

(c) $\mathcal{H}_3\{\mathbf{x}\}[n] = \sum_{k=n-L}^{n+L} x[k]$,

Exercise 6. Zero phase filtering.
Let $\mathcal{R}$ be the time reversal operator for sequences so that

$$\mathcal{R}\{\mathbf{x}\}[n] = x[-n]$$

Let $\mathcal{H}$ be a linear time invariant system with a real-valued impulse response. What are the properties of a system implementing the following transformation, assuming that the input signal $\mathbf{x}$ is real-valued?

$$\mathcal{R}\{\mathcal{H}\{\mathcal{R}\{\mathcal{H}\{\mathbf{x}\}\}\}\}$$

Exercise 7. Pole-zero plot and stability
Consider a causal LTI system with the following transfer function

$$H(z) = \frac{3 + 4.5z^{-1}}{1 + 1.5z^{-1}} - \frac{2}{1 - 0.5z^{-1}}$$

Sketch the pole-zero plot of the transfer function and specify its region of convergence. Is the system stable?

Exercise 8. Stability.
Consider a causal discrete system represented by the following difference equation

$$y[n] - 3.25y[n-1] + 0.75y[n-2] = x[n-1] + 3x[n-2].$$

(a) Compute the transfer function and check the stability of this system.

(b) If the input signal is $x[n] = \delta[n] - 3\delta[n-1]$, compute the z-transform of the output signal and discuss the stability result you found before.

(c) Consider now the input signal $x[n] = \delta[n] - 0.25\delta[n-1]$; assume the filter is implemented in fixed-precision arithmetic and that the the processing unit can only handle numbers in the range [-1024, 1024]. Starting processing for $n = 0$, how many samples can we process before we hit overflow or underflow?

Exercise 9. Properties of the z-transform.

Let $x[n]$ be a discrete-time sequence and $X(z)$ its corresponding z-transform with appropriate ROC.

(a) Prove that the following relation holds:

$$n x[n] \stackrel{Z}{\longleftrightarrow} -z \frac{d}{dz} X(z).$$

(b) Show that

$$(n+1)\alpha^n u[n] \stackrel{Z}{\longleftrightarrow} \frac{1}{(1-\alpha z^{-1})^2}, \qquad |z| > |\alpha|.$$

(c) Suppose that the above expression corresponds to the impulse response of an LTI system. What can you say about the causality of such a system? About its stability?

(d) Let $\alpha = 0.8$, what is the spectral behavior of the corresponding filter? What if $\alpha = -0.8$?

Exercise 10. Interleaving sequences.

Consider two two-sided sequences $h[n]$ and $g[n]$ and consider a third sequence $x[n]$ which is built by interleaving the values of $h[n]$ and $g[n]$:

$$x[n] = \ldots, h[-3], g[-3], h[-2], g[-2], h[-1], g[-1], h[0], g[0], h[1], g[1], h[2], g[2], h[3], g[3], \ldots$$

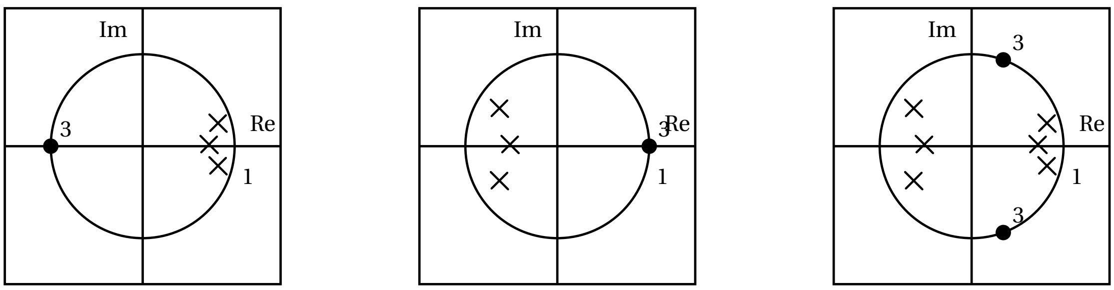
with $x[0] = h[0]$.

(a) Express the z-transform of $x[n]$ in terms of the z-transforms of $h[n]$ and $g[n]$.

(b) Assume that the ROC of $H(z)$ is $0.64 < |z| < 4$ and that the ROC of $G(z)$ is $0.25 < |z| < 9$. What is the ROC of $X(z)$?

Exercise 11. Transfer function, poles, and zeros

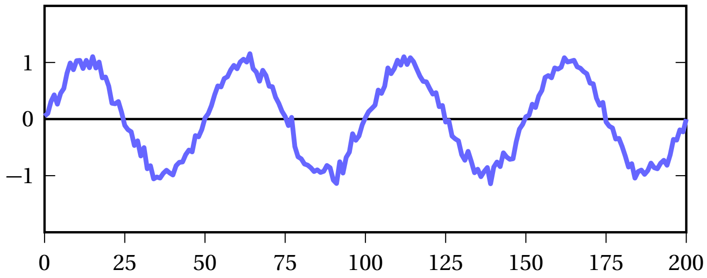
The figures below show the zeros and poles of three different filters with the unit circle for reference. Each zero is represented with a 'o' and each pole with a 'x' on the plot. Multiple zeros and poles are indicated by the multiplicity number shown to the upper right of the zero or pole.Sketch the magnitude of each frequency response and determine the type of filter.



(a) (b) (c)

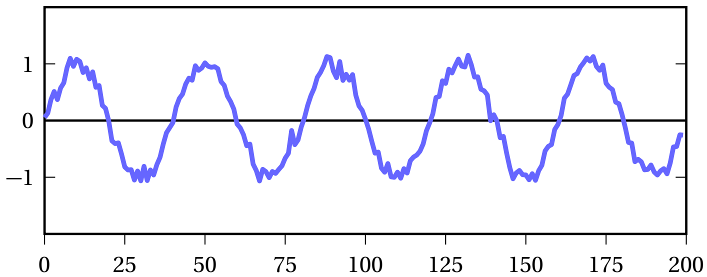
Exercise 12. Denoising.

Consider the signal $x[n] = \sin(\omega_0 n) + s[n]$ where $\omega_0 = 2\pi/50$ and $s[n]$ is a zero-mean, Gaussian white noise sequence with power spectral density $P_s(e^{j\omega}) = \sigma^2 = 10^{-2}$. A realization of the signal is plotted in the following figure:

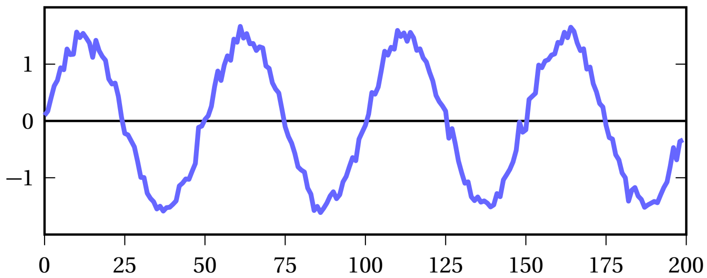


The signal is filtered with a stable, real-valued, causal LTI system whose frequency response satisfies $|H(e^{j\omega})| \leq 1$ for all frequencies. For each of the following plots, explain if the signal in the plot could be the result of filtering $x[n]$ with $H(e^{j\omega})$; explain your answers in detail.

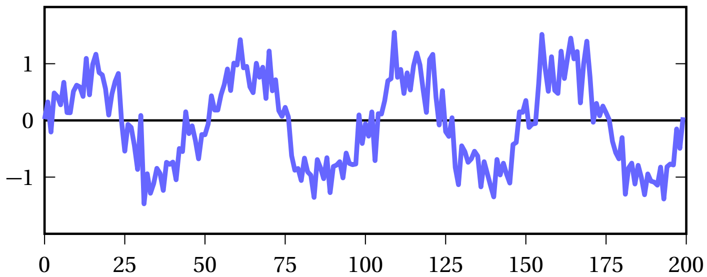
(a)



(b)



(c)



Exercise 13. Autocorrelation.

Assume $w[n]$ is a white process with variance σ^2 and consider the WSS process

$$x[n] = w[n] + 0.5w[n-1]$$

(a) compute the autocorrelation $r_x[n]$ of the process $x[n]$

(b) write out the 3×3 autocorrelation matrix for $x[n]$

(c) compute the cross-correlation $r_{xw}[n]$ between $w[n]$ and $x[n]$

Exercise 14. Autocorrelation II.

Assume $w[n]$ is a white process with variance σ^2 and consider the WSS process

$$x[n] = w[n] + 0.5x[n-1]$$

(a) compute the autocorrelation $r_x[n]$ of the process $x[n]$

(b) write out the 3×3 autocorrelation matrix for $x[n]$

(c) compute the cross-correlation $r_{xw}[n]$ between $w[n]$ and $x[n]$

Exercise 15. Linear Prediction.

The WSS stochastic process $\mathbf{x}$ has autocorrelation

$$r_x[k] = 0.5^{|k|+1}$$

(a) find the optimal third-order Linear Prediction filter that models $\mathbf{x}$

(b) how many nonzero coefficients does the filter have? Explain.