# I211E: Mathematical Logic

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**Schedule**

| | propositional logic | | predicate logic |
|---|---|---|---|
| 4/13 | syntax, semantics | 5/11 | syntax, semantics |
| 4/18 | normal forms | 5/16 | normal forms |
| 4/20 | examples | 5/18 | natural deduction I |
| 4/25 | natural deduction I | 5/23 | natural deduction II |
| 4/27 | natural deduction II | 5/25 | examples, properties |
| 5/2 | completeness | 5/30 | advanced topics |
| 5/9 | midterm exam | 6/1 | summary |
| | | 6/6 | exam |

**Evaluation**

midterm exam (40) + final exam (60)



## Contents

**Aim**

to learn how to write mathematical proofs

**Contents**

1. mathematical proofs
2. Killer Sudoku
3. Tic-Tac-Toe



## Mathematical Proofs

**Definition**

- $x \in A \cup B$ if $x \in A$ or $x \in B$
- $x \in A \cap B$ if $x \in A$ and $x \in B$
- $A \subseteq B$ if $x \in B$ for all $x \in A$

**Proposition**

$A \cup (A \cap B) \subseteq A$

Note: proof is same as derivation of $\vdash \forall x((x \in A \vee (x \in A \wedge x \in B)) \to x \in A)$

**Proof.**

Let $x$ be an arbitrary element. Suppose $x \in A \cup (A \cap B)$. By the definition of $\cup$ we have $x \in A$ or $x \in A \cap B$. We distinguish two cases. If $x \in A$ then $x \in A$ holds trivially. If $x \in A \cap B$ then by the definition of $\cap$ we have $x \in A$ and $x \in B$. So $x \in A$ holds. In any case the claim holds. □



**Proposition**

if $x, y \in \mathbb{R}$ and $x < y$ then $x < z < y$ for some $z \in \mathbb{R}$

Note: proof is same as derivation of $\vdash \forall x, y \in \mathbb{R}. (x < y \to \exists z \in \mathbb{R}. x < z < y)$

**Proof.**

Let $x$ and $y$ be arbitrary elements in $\mathbb{R}$. Suppose $x < y$. We show $x < z < y$ for some $z$. Take $z$ as follows:

$$z = \frac{x+y}{2}$$

Then $x < z < y$ is verified as follows:

$$z - x = \frac{x+y}{2} - x = \frac{y-x}{2} > 0 \qquad y - z = y - \frac{x+y}{2} = \frac{y-x}{2} > 0$$

Here the inequalities are derived from the assumption $x < y$. □



**Definition**

- $x \in \{a_1, \ldots, a_n\}$ if $x = a_1$ or ... or $x = a_n$
- $x \in \bigcup_{i \in I} A_i$ if $x \in A_i$ for some $i \in I$

**Proposition**

$\mathbb{N} \subseteq \bigcup_{i \in \mathbb{N}} \{i, i+1\}$ $\qquad \forall x \in \mathbb{N}. \exists i \in \mathbb{N}. x \in \{i, i+1\}$

**Proof.**

Let $x$ be an arbitrary element in $\mathbb{N}$. It is enough to show $x \in \{i, i+1\}$ for some $i \in \mathbb{N}$. Take $i = x$. Then $x \in \{i, i+1\}$ follows. □



**Definition**

- $x \in \{a_1, \ldots, a_n\}$ if $x = a_1$ or ... or $x = a_n$
- $x \in \bigcup_{i \in I} A_i$ if $x \in A_i$ for some $i \in I$

**Proposition**

$\bigcup_{i \in \mathbb{N}} \{i, i+1\} \subseteq \mathbb{N}$ $\qquad \forall x \in \bigcup_{i \in \mathbb{N}} \{i, i+1\}. x \in \mathbb{N}$

**Proof.**

Let $x$ be an arbitrary element in $\bigcup_{i \in \mathbb{N}} \{i, i+1\}$. By definition there exists $i \in \mathbb{N}$ such that $x \in \{i, i+1\}$. Thus, $x = i$ or $x = i+1$ for some $i \in \mathbb{N}$. We distinguish two cases. If $x = i$ then $x \in \mathbb{N}$ follows from $i \in \mathbb{N}$. If $x = i+1$ then $x \in \mathbb{N}$ follows from $i \in \mathbb{N}$. In either case, $x \in \mathbb{N}$ is concluded. □

**Theorem (mathematical induction)**

$(P(0) \land \forall n \in \mathbb{N}.\ (P(n) \to P(n+1))) \to \forall n \in \mathbb{N}.\ P(n)$

**Proposition**

$n! \geqslant 1$ for all $n \in \mathbb{N}$ $\qquad \forall n \in \mathbb{N}.\ n! \geqslant 1$

**Proof (faithful but verbose).**

By mathematical induction on $n$ we show $n! \geqslant 1$.

- Consider the base case $n = 0$. We have $n! = 1$. Thus, $n! \geqslant 1$.
- To show the inductive step, assume $n! \geqslant 1$. We show $(n+1)! \geqslant 1$. Using the assumption (the I.H.) we obtain:

$$(n+1)! = (n+1) \cdot n! \geqslant (n+1) \cdot 1 = n + 1 \geqslant 1$$

Hence, the claim holds. □



**Theorem (mathematical induction)**

$(P(0) \land \forall n \in \mathbb{N}.\ (P(n) \to P(n+1))) \to \forall n \in \mathbb{N}.\ P(n)$

**Proposition**

$n! \geqslant 1$ for all $n \in \mathbb{N}$ $\qquad \forall n \in \mathbb{N}.\ n! \geqslant 1$

**Proof (conventional style).**

We show $n! \geqslant 1$ by induction on $n$.

- If $n = 0$ then $n! = 1 \geqslant 1$.
- If $n > 0$ then

$$(n+1)! = (n+1) \cdot n! \geqslant (n+1) \cdot 1 = n + 1 \geqslant 1$$

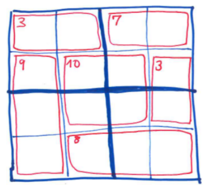
where the first inequality is due to the I.H. □



# Killer Sudoku (Encoding in Linear Arithmetic)



## Homework: $4 \times 4$ Killer Sudoku




1. Encode the Killer Sudoku problem into a linear integer arithmetic constraint:

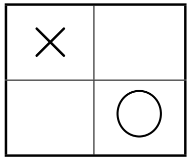
$$\begin{aligned}
&1 \leqslant x_{11} \land x_{11} \leqslant 4 \land \cdots \\
\land\ &\neg(x_{11} \doteq x_{12}) \land \neg(x_{12} \doteq x_{13}) \land \neg(x_{11} \doteq x_{14}) \land \cdots \\
\land\ &\cdots \\
\land\ &x_{11} + x_{12} \doteq 3 \land \cdots
\end{aligned}$$

2. Complete `killer.smt2` to solve the constraint by SMT solver (Z3).

# Tic-Tac-Toe (QBF Encoding)



## Tic-Tac-Toe



first player of $2 \times 2$ Tic-Tac-Toe has winning strategy!

$\boxed{1}$ $\exists$ 1st move (1st player wins or

$\boxed{2}$ $\forall$ 2nd move (2nd player does not win and

$\boxed{3}$ $\exists$ 3rd move first player wins))

**Q.** how to formalize and prove it?

**A.** QBF!



## Quantified Boolean Formulas (QBF)

**Syntax of QBF**

$\phi ::= p \mid \top \mid \bot \mid \neg\phi \mid \phi \vee \phi \mid \phi \wedge \phi \mid \phi \to \phi \mid \phi \leftrightarrow \phi \mid \forall x \phi \mid \exists x \phi$

**Semantics of QBF**

same as propositional case but

$$[\![\forall x \phi]\!]_v = \begin{cases} \mathsf{T} & \text{if } [\![\phi]\!]_{v[u/x]} = \mathsf{T} \text{ for all } u \in \{\mathsf{T},\mathsf{F}\} \\ \mathsf{F} & \text{otherwise} \end{cases}$$

$$[\![\exists x \phi]\!]_v = \begin{cases} \mathsf{T} & \text{if } [\![\phi]\!]_{v[u/x]} = \mathsf{T} \text{ for some } u \in \{\mathsf{T},\mathsf{F}\} \\ \mathsf{F} & \text{otherwise} \end{cases}$$

**Example:** $\forall x \exists y (x \leftrightarrow \neg y)$ is valid but $\forall x \forall y (x \leftrightarrow \neg y)$ is invalid



## QBF Encoding of $2 \times 2$ Tic-Tac-Toe (1/2)

let $X^i$ denote the $i$-th state:

$$X^i = \left( \begin{array}{cc|cc} x_1^i & x_2^i & x_5^i & x_6^i \\ x_3^i & x_4^i & x_7^i & x_8^i \end{array} \right) \qquad \begin{array}{|c|c|}\hline \times & \\ \hline & \bigcirc \\ \hline \end{array} = \left( \begin{array}{cc|cc} \mathsf{F} & \mathsf{F} & \mathsf{T} & \mathsf{F} \\ \mathsf{F} & \mathsf{T} & \mathsf{F} & \mathsf{F} \end{array} \right)$$

construct formulas:

$\mathsf{valid}(X) \iff X$ is valid state $\qquad \mathsf{valid}\left(\begin{array}{|c|c|}\hline \bigcirc\times & \\ \hline & \bigcirc \\ \hline \end{array}\right) \approx \bot$

$\mathsf{win}(X) \iff$ some player wins at $X$ $\qquad \mathsf{win}\left(\begin{array}{|c|c|}\hline \times & \times \\ \hline & \bigcirc \\ \hline \end{array}\right) \approx \top$

$\mathsf{next}(X, Y) \iff Y$ is next state of $X$ $\qquad \mathsf{next}\left(\begin{array}{|c|c|}\hline \bigcirc & \times \\ \hline & \\ \hline \end{array}, \begin{array}{|c|c|}\hline \bigcirc & \times \\ \hline & \bigcirc \\ \hline \end{array}\right) \approx \top$

## QBF Encoding of $2 \times 2$ Tic-Tac-Toe (2/2)

let $X^0 = \begin{array}{|c|c|}\hline \phantom{x} & \phantom{x} \\ \hline \phantom{x} & \phantom{x} \\ \hline\end{array}$

first player has winning strategy $\iff$

$\boxed{1}$ $\exists$ 1st move (1st player wins or

$$\exists X^1.\ \mathsf{valid}(X^1) \land \mathsf{next}(X^0, X^1) \land (\mathsf{win}(X^1)\ \lor \cdots$$

$\boxed{2}$ $\forall$ 2nd move (2nd player does not win and

$$\cdots \lor\ \forall X^2.\ \mathsf{valid}(X^2) \land \mathsf{next}(X^1, X^2) \land (\neg\mathsf{win}(X^2)\ \land \cdots$$

$\boxed{3}$ $\exists$ 3rd move first player wins))

$$\cdots \land\ \exists X^3.\ \mathsf{valid}(X^3) \land \mathsf{next}(X^2, X^3) \land \mathsf{win}(X^3)))$$

**Note:** modern QBF solvers can verify it, even for $4 \times 4$ Tic-Tac-Toe