

# Homework 12

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1.

(1)

$$\frac{\frac{\frac{[\forall x \forall y (x+s(y) = s(x+y)] \wedge \forall E}{\forall y (y+s(0) = s(y+0)) \wedge A}}{A \vdash 0+0 = 0} \text{ cong}}{0+0 = 0} \text{ long}}{s(0+0) = s(0)} \text{ trans}$$

$$0 + s(0) = s(0)$$

(2)

$$\frac{\frac{\frac{[\forall x \forall y (x+s(y) = s(x+y))] \wedge \forall E}{\forall y (s(0)+s(y) = s(s(0)+y)) \wedge \forall E}}{s(0)+s(0) = s(s(0)+0)} \text{ cong}}{s(0) + s(0) = s(s(0))} \text{ trans}$$

$$s(0) + s(0) = s(s(0))$$

(3)

$$\frac{\frac{\frac{s(0)+s(0) = s(s(0)) \text{ sym}}{s(s(0)) = s(0)+s(0)} \text{ trans}}{s(s(0)) = 0} \text{ I}}{\neg (s(0) + s(0) = 0) \text{ I} \rightarrow \neg (s(s(0)) = 0) \text{ I} \rightarrow \neg I_1}$$

(4)

$$\frac{\frac{\frac{\frac{[\forall x \forall y (x+s(y) = s(x+y))] \wedge \forall E}{\forall y (s(0)+s(y) = s(s(0)+y)) \wedge \forall E}}{s(0)+s(s(0)) = s(s(0)+s(0))} \text{ cong}}{s(0) + s(s(0)) = s(s(0)+s(0))} \text{ trans}}{s(0) + s(s(0)) = s(s(0)) + s(0)} \text{ sym}$$

$$s(0) + s(s(0)) = s(s(0)) + s(0) \text{ trans}$$

$$s(0) + s(s(0)) = s(s(0)) + s(0)$$

(5)

$$\frac{\frac{\frac{\frac{\frac{[\forall x (x+0 = x)] \wedge \forall E}{0 = 0} \text{ refl}}{s(0)+0 = s(0)} \text{ cong}}{0+(s(0)+0) = 0+s(0)} \text{ trans}}{0+(s(0)+0) = s(0)} \text{ sym}}{s(0)+0 = 0+(s(0)+0)} \text{ trans}$$

$$(0+s(0))+0 = 0+(s(0)+0)$$

$$(0+s(0))+0 = 0+(s(0)+0)$$

(6)

$$\frac{\frac{\frac{\frac{[\forall x (x.0 = 0)] \wedge \forall E}{0.0 = 0} \text{ refl}}{0 = 0} \text{ refl}}{0 = 0} \text{ refl}}{0.0 = 0} \text{ trans}$$

$$0.s(0) = 0$$

(7)

$\Gamma, \Gamma' \vdash n = m \vdash$

$$\frac{[\forall x \forall y (x+s(y) = s(x+y))] \wedge \forall E}{\dots}$$

(7)

$$\begin{array}{c}
 \frac{[\forall x(x+0=0)]_A \vee E}{s(s(0)).0 = 0} \quad \frac{[\forall y(x+sy=s(xy))]_A \vee E}{s(s(0)) \vdash s(s(0))} \text{REFL} \\
 \frac{s(s(0)) \vdash s(s(0))}{(s(s(0))).0 + s(s(0)) \vdash 0 + s(s(0))} \text{CONT} \quad \frac{\forall y(0+sy=s(0+y)) \vee E}{0+s(s(0)) \vdash s(0+y)} \text{REFL} \\
 \frac{0+s(s(0)) \vdash s(0+y)}{0+s(s(0)) \vdash s(s(0+y))} \text{TRANS} \quad \frac{0+s(s(0)) \vdash s(s(0))}{0+s(s(0)) \vdash s(s(0))} \text{TRANS} \\
 s(s(0)).s(0) \vdash (s(s(0)).0) + s(s(0)) \\
 (s(s(0)).0) + s(s(0)) \vdash s(s(0)) \text{TRANS} \\
 s(s(0)).s(0) \vdash s(s(0))
 \end{array}$$

(8)

$$\begin{array}{c}
 \frac{x \equiv y \quad z \equiv z}{x+z \equiv y+z} \text{REFL} \\
 \frac{x \equiv y \rightarrow x+z \equiv y+z}{\forall z(x \equiv y \rightarrow x+z \equiv y+z)} \rightarrow I_1 \\
 \frac{\forall z(x \equiv y \rightarrow x+z \equiv y+z)}{\forall x \forall y \forall z(x \equiv y \rightarrow x+z \equiv y+z)} \rightarrow A
 \end{array}$$

2.

(M)

a.

$$\frac{\forall x(x+0=x)}{0+0=0} \text{VE}$$

b.

$$\begin{array}{c}
 \frac{[\forall x \forall y(x+sy=s(xy))]_A \vee E}{\forall y(0+s(y)=s(0+y))} \\
 \frac{\forall y(0+s(y)=s(0+y))}{0+s(x)=s(0+x)} \quad \frac{[0+0=x]_A}{s(0+x)=s(x)} \text{TRANS} \\
 \frac{0+s(x)=s(x)}{0+x=x} \rightarrow I_1 \\
 0+x=x \rightarrow 0+s(x)=s(x)
 \end{array}$$

c.

$$\psi = (0+0=0) \wedge (\forall x(0+x=x \rightarrow 0+s(x)=s(x)))$$

$$\begin{array}{c}
 \frac{(a) \quad \frac{(b) \quad \frac{0+0=0}{\forall x(0+x=x \rightarrow 0+s(x)=s(x))}}{\psi} \wedge I}{\psi} \rightarrow E \\
 \frac{\psi}{\forall x(0+x=x)}
 \end{array}$$

$$(2) \forall x \forall y (x+sy = s(x)+y)$$

a.

$$\begin{array}{c}
 \frac{[\forall x(x+0=x)]_A \vee E}{x+0=x} \text{SYM} \\
 \frac{[\forall x(x+0=x)]_A \vee E \quad x+0=x \vdash x+0=x}{s(x)+0 \vdash s(x)} \text{CONG} \\
 \frac{s(x)+0 \vdash s(x)}{s(x)+0 \vdash s(x+0)} \text{TRANS} \\
 \frac{[\forall x \forall y (x+sy = s(x)+y)]_A \vee E \quad s(x)+0 \vdash s(x+0)}{s(x)+0 \vdash s(x+0)} \text{SYM} \\
 \frac{s(x)+0 \vdash s(x+0)}{x+s(0) \vdash s(x)+0} \text{TRANS} \\
 x+s(0) \vdash s(x)+0
 \end{array}$$

b.

$$\begin{array}{c}
 \frac{x+s(y) \vdash s(s(x)+y)}{s(x)+s(y) \vdash s(s(x)+y)} \text{SYM} \\
 \frac{s(x)+s(y) \vdash s(s(x)+y)}{\frac{s(x)+s(y) \vdash s(x+sy)}{x+s(s(y)) \vdash s(x+sy)}} \text{CONG} \\
 \frac{s(x)+s(y) \vdash s(x+sy)}{s(x+sy) \vdash s(x+sy)} \text{TRANS} \\
 \frac{s(x+sy) \vdash s(x+sy)}{x+s(s(y)) \vdash s(x+sy)} \text{SYM} \\
 \frac{x+s(s(y)) \vdash s(x+sy)}{x+s(s(y)) \vdash s(x)+s(y)} \text{TRANS} \\
 \frac{x+s(s(y)) \vdash s(x)+s(y)}{\psi} \rightarrow I_1
 \end{array}$$

$$\frac{\frac{x + s(s(y)) = s(x + s(y))}{x + s(s(y)) = s(x) + s(y)} \text{ SYM} \quad \text{TRANS}}{x + s(y) = s(x) + y} \rightarrow I_1$$

c.

$$\frac{\frac{\frac{(a) \quad x + s(0) = s(x) + 0}{\forall y (x + s(y) = s(x) + y) \rightarrow x + s(s(y)) = s(x) + s(y)} \wedge I}{\forall y (x + s(y) = s(x) + y) \rightarrow \forall y (\forall y (x + s(y) = s(x) + y))} \rightarrow E}{\forall y (x + s(y) = s(x) + y)} \rightarrow E$$

(d)

$$\frac{\frac{\frac{\frac{(\forall x(x+0=x)) \quad \forall x(x+0=x) \rightarrow \forall x(x+0=x)}{x+0=x} \text{ MS}}{x+0=x} \text{ TRANS}}{x+0=0+x} \text{ MS}}{x+0=0+x} \text{ TRANS}$$

$$\frac{\frac{\frac{\frac{\frac{(\forall y(x+y=s(x)+y)) \quad \forall y(x+y=s(x)+y) \rightarrow \forall y(y+s(x)=s(x)+y)}{y+s(x)=s(x)+y} \text{ MS}}{y+s(x)=s(x)+y} \text{ TRANS}}{y+s(x)=s(x)+y} \text{ MS}}{y+s(x)=s(x)+y} \text{ TRANS}}{y+s(x)=s(x)+y} \text{ TRANS}$$

$$\frac{s(y)+x = s(x+y)}{s(x+y) = s(y)+x} \text{ MS}$$

$$\frac{s(y)+x = s(y)+x}{s(y)+x = s(y)+x} \text{ TRANS}$$

$$\frac{x+y = y+x \rightarrow (x+y) = s(y)+x}{x+y = y+x \rightarrow (x+y) = s(y)+x} \rightarrow I_1$$

$$\frac{\frac{\frac{\frac{\frac{(\forall x(x+0=x)) \quad \forall x(x+0=x) \rightarrow \forall x(x+0=x)}{x+0=0+x} \text{ MS}}{x+0=0+x} \text{ TRANS}}{\forall y (x+y = y+x \rightarrow (x+y) = s(y)+x) \wedge I}{\psi \rightarrow \forall y (x+y = y+x)}}{\forall y (x+y = y+x) \rightarrow E}{\forall y (x+y = y+x) \rightarrow E}$$

e.

$$(1) \forall x (0 \cdot x = 0)$$

$$\frac{\frac{\frac{\frac{\frac{\forall x (x \cdot 0 = 0) \quad \forall x}{0 \cdot 0 = 0} \text{ MS}}{0 \cdot 0 = 0} \text{ CNT}}{0 \cdot x + 0 = 0 \cdot x} \text{ CNT}}{0 \cdot x + 0 = 0 \cdot x} \text{ TRANS}}{0 \cdot x = 0 \rightarrow 0 \cdot x = 0} \text{ TRANS}$$

$$\frac{\frac{\frac{\frac{\frac{\frac{(\forall x (x \cdot 0 = 0)) \quad \forall x (x \cdot 0 = 0) \rightarrow \forall x (x \cdot 0 = 0)}{0 \cdot 0 = 0} \text{ MS}}{0 \cdot x + 0 = 0 \cdot x} \text{ CNT}}{0 \cdot x + 0 = 0 \cdot x} \text{ TRANS}}{0 \cdot x + 0 = 0 \cdot x} \text{ TRANS}}{0 \cdot x = 0} \text{ TRANS}}{0 \cdot x = 0 \rightarrow 0 \cdot x = 0} \text{ TRANS}$$

$$\frac{\frac{\frac{\frac{\frac{\frac{(\forall x (0 \cdot x = 0)) \quad \forall x (0 \cdot x = 0) \rightarrow \forall x (0 \cdot x = 0)}{0 \cdot 0 = 0} \text{ MS}}{0 \cdot x + 0 = 0 \cdot x} \text{ CNT}}{0 \cdot x + 0 = 0 \cdot x} \text{ TRANS}}{0 \cdot x + 0 = 0 \cdot x} \text{ TRANS}}{0 \cdot x = 0} \text{ TRANS}}{\psi = (0 \cdot 0 = 0) \wedge \forall x (0 \cdot x = 0 \rightarrow 0 \cdot s(x) = 0)} \rightarrow E$$

$$(2) \forall x \forall y \forall z ((x+y)+z = x+(y+z))$$

a.

$$\frac{\forall x(x+0=x) \text{ } \forall E}{\frac{(x+y)+0=x+y \text{ } \forall E}{\frac{x=x \text{ } \text{REF}}{\frac{y+0=y \text{ } \text{SYM}}{\frac{y=y+0 \text{ } \text{COMM}}{\frac{x+0=(y+0)}{\frac{x+(y+0)=x+(y+0) \text{ } \text{TRANS}}{}}}}}}}$$

b.

$$\begin{aligned} & \frac{\forall x \forall y \forall z ((x+y)=s(x+y)) \text{ } Q}{\frac{x=x \text{ } \text{REF}}{\frac{y+s(z)=s(x+y+z) \text{ } \text{COMM}}{\frac{s(x+y+z)=s(x+s(y+z)) \text{ } \text{COMM}}{\frac{x+s(y+z)=s(x+s(y+z)) \text{ } \text{COMM}}{\frac{x+s(y+z)=s((x+y)+z) \text{ } \text{TRANS}}{\frac{x+(y+z)=x+s(y+z) \text{ } \text{TRANS}}{\frac{x+(y+z)=s((x+y)+z) \text{ } \text{TRANS}}{\frac{x+(y+z)=s(x+(y+z)) \text{ } \text{SYM}}{\frac{s(x+(y+z))=x+(y+s(z)) \text{ } \text{TRANS}}{\frac{(x+y)+z=x+(y+s(z)) \text{ } \text{TRANS}}{\frac{(x+y)+z=x+(y+z) \rightarrow (x+y)+s(z)=x+(y+s(z)) \text{ } \text{TRANS}}{\rightarrow I_1}}}}}}}}}} \end{aligned}$$

c. (a)

$$(x+y)+0=x+(y+0)$$

(b)

$$\begin{aligned} & \forall z \left[ (x+y)+z=x+(y+z) \rightarrow (x+y)+s(z)=x+(y+s(z)) \right] \text{ } \forall I \\ & \neg \exists \forall y \forall z \text{ } \emptyset \text{ } \forall A \end{aligned}$$

$$\left[ (\forall z (x+y)+z=x+(y+z)) \right] \rightarrow E$$