Homework 13

We learn how to write proofs in text.

1. Let A, B, C be sets. Prove each statement. See the slides for the definitions of $A \cup B$, $A \cap B$, $A \subseteq B$, and so on.

   (1) $A \subseteq A$.

   (2) $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$.

   (3) $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$.

   (4) $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$.

   (5) $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

   (6) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

2. For sets A, B we write $A = B$ if $A \subseteq B$ and $B \subseteq A$. The *union* $\bigcup_{i \in I} A_i$ denotes the set of all elements x that satisfy $x \in A_i$ for some $i \in I$. The *interval* $[a, b]$ between real numbers a, b is defined by $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$. Similarly $(a, b]$ is defined by $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$.

   (a) For each positive natural number i we define $A_i = [0, 1/i]$, $B_i = (0, 1/i]$, and $C_i = (-1/i, i]$. Compute $\bigcup_{i \in I} A_i$, $\bigcup_{i \in I} B_i$, and $\bigcup_{i \in I} C_i$.

   (b) Prove $(\bigcup_{i \in I} A_i) \cap B = \bigcup_{i \in I} (A_i \cap B)$.

3. We study the *big O notation*, which is used to describe complexity of algorithms. Let $f, g, h : \mathbb{N} \to \mathbb{N}$ be functions on $\mathbb{N}$. We write $f(n) \in O(g(n))$ if there are some natural numbers c and N such that $f(n) \leq cg(n)$ for all natural numbers $n \geq N$. In other words $f(n) \in O(g(n))$ is defined by:

   $$\exists c \exists N \forall n (n \geq N \to f(n) \leq cg(n))$$

   Note that $\in$ of the big O notation technically has nothing to do with that of set theory, and that the negation of $f(n) \in O(g(n))$ is:

   $$\forall c \forall N \exists n (n \geq N \wedge f(n) > cg(n))$$

   Show the following statements.

   (1) If $f(n) = n$ and $g(n) = n^2$ then $f(n) \in O(g(n))$.

   (2) If $f(n) = n + 10$ and $g(n) = n^2$ then $f(n) \in O(g(n))$.

   (3) If $f(n) = 100n$ and $g(n) = n^2$ then $f(n) \in O(g(n))$.

   (4) If $f(n) = 100n$ and $g(n) = n^2$ then $g(n) \in O(f(n))$ does not hold.

   (5) If $f(n) = n + 10$ and $g(n) = 5n$ then $f(n) \in O(g(n))$.

   (6) If $f(n) = n + 10$ and $g(n) = 5n$ then $g(n) \in O(f(n))$.

   (7) If $f(n) = n^3$ and $g(n) = 2^n$ then $f(n) \in O(g(n))$.

   (8) If $f(n) = n^3$ and $g(n) = 2^n$ then $g(n) \in O(f(n))$ does not hold.

   (9) If $f(n) \in O(h(n))$ and $g(n) \in O(h(n))$ then $f(n) + g(n) \in O(h(n))$.

   (10) If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ then $f(n) \in O(h(n))$.

4. Solve the homework about Killer Sudoku, see the slides.

5. Prove the soundness theorem of natural deduction for propositional logic.