Homework 13

30 May 2023 18:09

1.

(1) $A \subseteq A$.

Let x be an arbitrary element. Assume $x \in A$, then $x \in A$ holds trivially.

(2) $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$

Let x be an arbitrary element. Assume $x \in (A \cup B) \cap C$. By definition of $\cap$ we have $x \in A \cup B$ and $x \in C$. By definition of $\cup$ we have $x \in A$ or $x \in B$.

+ If $x \in A$ then by definition of $\cap$ we have $x \in A \cap C$
  As $x \in A \cap C$, by definition of $\cup$ we have $x \in (A \cap C) \cup (B \cap C)$
+

In any case the claim holds.

(3) $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$

Let x be an arbitrary element. Assume $x \in ((A \cap C) \cup (B \cap C))$. By definition of $\cup$ we have $x \in (A \cap C)$ or $x \in (B \cap C)$.

+ If $x \in (A \cap C)$ then by definition of $\cap$ we have $x \in A$ and $x \in C$.
  As $x \in A$, by definition of $\cup$ we have $x \in (A \cup B)$.
  As $x \in (A \cup B)$ and $x \in C$, by definition of $\cap$ we have $x \in (A \cup B) \cap C$
+

In any case the claim holds.

(4) $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$

Let x be an arbitrary element. Assume $x \in (A \cap B) \cup C$. By definition of $\cup$ we have $x \in A \cap B$ or $x \in C$.

+ If $x \in A \cap B$ then by definition of $\cap$ we have $x \in A$ and $x \in B$.
  As $x \in A$, by definition of $\cup$ we have $x \in A \cup C$.
  As $x \in B$, " " $x \in B \cup C$.
  As $x \in A \cup C$ and $x \in B \cup C$, by definition of $\cap$ we have $x \in (A \cup C) \cap (B \cup C)$
+ If $x \in C$ then by definition of $\cup$ we have $x \in (A \cup C)$ and $x \in (B \cup C)$.
  As $x \in A \cup C$ and $x \in B \cup C$, by definition of $\cap$ we have $x \in (A \cup C) \cap (B \cup C)$

In any case the claim holds.

(5) $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$

Let x be an arbitrary element. Assume $x \in (A \cup C) \cap (B \cup C)$. By definition of $\cap$ we have $x \in A \cup C$ and $x \in B \cup C$. As $x \in A \cup C$, by definition of $\cup$ we have $x \in A$ or $x \in C$.

+ If $x \in A$, as $x \in B \cup C$, by definition of $\cup$ we have $x \in B$ or $x \in C$.
  If $x \in B$, by definition of $\cap$ we have $x \in (A \cap B)$ and by definition of $\cup$ $x \in (A \cap B) \cup C$
  If $x \in C$ then by definition of $\cup$ we have $x \in (A \cap B) \cup C$.

If $x \in B$, by definition of $\cap$ we have $x \in (A \cap B)$ and by definition of $\cup$ $x \in (A \cap B) \cup C$

If $x \in C$ then by definition of $\cup$ we have $x \in (A \cap B) \cup C$.

If $x \in C$ then by definition of $\cup$ we have $x \in (A \cap B) \cup C$.

In any case the claim holds.

(6) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

$$\vdash \underbrace{(\forall x(x \in A \to x \in B) \wedge \forall x(x \in B \to x \in C))}_{(1)} \to \underbrace{\forall x(x \in A \to x \in C)}_{(2)}$$

Assume (1). We need to show (2). Let x be an arbitrary element. Assume $x \in A$.
It's enough to show $x \in C$. From (1) we have $x \in A \to x \in B$ for arbitrary x.
From assumption $x \in A$ we have $x \in B$. From (1) we have $x \in B \to x \in C$ for arbitrary x.
Since we have showed $x \in B$ we have $x \in C$.

2.

(a)

$$\bigcup_{i \in N} A_i = \bigcup_{i \in N} [0, 1/i] = \bigcup_{i \in N} \{x \in R \mid 0 \leq x \leq 1/i\} = [0, 1]$$

$$\bigcup_{i \in N} B_i = \bigcup_{i \in N} (0, 1/i] = \bigcup_{i \in N} \{x \in R \mid 0 < x \leq 1/i\} = (0, 1)$$

$$\bigcup_{i \in N} C_i = \bigcup_{i \in N} (-1/i, i] = \bigcup_{i \in N} \{x \in R \mid -1/i < x \leq i\} = N \cup (-1, 0)$$

(b) $(\bigcup_{i \in I} A_i) \cap B = \bigcup_{i \in I} (A_i \cap B)$

$\approx [(\bigcup_i A_i) \cap B \subseteq \bigcup_i (A_i \cap B)] \wedge [\bigcup_i (A_i \cap B) \subseteq (\bigcup_i A_i) \cap B]$

Let x be an arbitrary element. Assume $x \in (\bigcup_i A_i) \cap B$. We need to show
$x \in A_i \cap B$ for some $i \in I$. By definition of $\cap$ we have $x \in \bigcup_i A_i$ and $x \in B$.
As $x \in \bigcup_i A_i$ there exists $i \in I$ such that $x \in A_i$. As $x \in B$, by definition of $\cap$ we have
$x \in A_i \cap B$.

Let x be an arbitrary element. Assume $x \in \bigcup_i (A_i \cap B)$. We need to show
$x \in (\bigcup_i A_i) \cap B$. By definition there exists $i \in I$ s.t $x \in A_i \cap B$. By definition of $\cap$
we have $x \in A_i$ and $x \in B$. As $x \in A_i$ we have $x \in \bigcup_i (A_i)$ follows. By definition
of $\cap$ we have $x \in (\bigcup_i A_i) \cap B$.

3.

(1) Let n be an arbitrary element of N. Assume $n \geq N$ for some N.
We need to show $f(n) = n \leq c.g(n) = c.n^2$ for some c. Take $c = 1$ and $N = 2$, we have
$n \leq n^2$ for all $n \geq 2$, so $f(n) \in O(g(n))$ holds.

(2) Take $c = 10$ and $N = 2$

(3) $c = 100$, $N = 1$

(4) Let c and N be arbitrary elements of N. Take $n = \max(N, c.100 + 1)$
We have $n \geq N$ and

$$n \geq c.100 + 1$$
$$\Leftrightarrow n^2 \geq c.100n + n$$
$$\Leftrightarrow g(n) \geq c.f(n) + n$$
$$\Leftrightarrow g(n) > c\,f(n) \quad \text{as } n > 0$$

$\Leftrightarrow g(n) \geq c.f(n) + n$

$\Leftrightarrow g(n) > c\,f(n)$ as $n > 0$

so $g(n) \in O(f(n))$ doesn't hold.

(5) $c = 3$, $N = 1$

(6) $c = 5$, $N = 1$

(7) $c = 1$, $N = 10$

(8) $n = \max(N, \bar{n})$ with $\bar{n}$ satisfies $2^{\bar{n}} > c.\bar{n}^3$. We show there exists $\bar{n}$ which satisfied this condition:

$$2^n > c.n^3 \qquad (1)$$

$$\Leftrightarrow n > \log_2 c + 3\log_2 n$$

$$\Leftrightarrow n - \log_2 c - 3\log_2 n > 0$$

we distinguise 2 cases:

+ $\log_2 c \geq 3\log_2 n$: Take $n = \lceil 2\log_2 c \rceil + 1$
+ $\log_2 c < 3\log_2 n$: We have $n > 3\log_2 n$ with $n = 10$. Take $n = 20$

In either case there exists value of n which satisfies (1)

(9) $\vdash (\exists c \exists N \forall n (n \geq N \to f(n) \leq c\,h(n)) \wedge \exists c \exists N \forall n (n \geq N \to g(n) \leq c.h(n)) \to \exists c \exists N \forall n (n \geq N \to f(n) + g(n) \leq c.h(n))$

$$(1) \qquad \wedge \qquad (2) \qquad \to \qquad (3)$$

Assume $(1) \wedge (2)$. We need to show (3). From (3), take $c = \bar{c}$ and $N = \bar{N}$ and arbitrary n. Assume $n \geq \bar{N}$. It's enough to show $f(n) + g(n) \leq c.h(n)$. From assumption $(1) \wedge (2)$ we have (1) and (2)

+ From (1), take $c = c_1$ and $N = N_1$ s.t $N_1 \leq \bar{N}$ and arbitrary n. As $n \geq \bar{N}$ and $\bar{N} \geq N_1$ from assumptions we have $n \geq N_1$ and then $f(n) \leq c_1\,h(n)$ follows. (4)
+ $g(n) \leq c_2\,h(n)$ (4)
+ From (4) (5) we have $f(n) + g(n) \leq h(n)(c_1 + c_2)$ for arbitrary n. Take $\bar{c} = c_1 + c_2$ we have (3) holds.