I211E: Mathematical Logic

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Schedule

	propositional logic		predicate logic
4/13	syntax, semantics	5/11	syntax, semantics
4/18	normal forms	5/16	normal forms
4/20	examples	5/18	natural deduction I
4/25	natural deduction I	5/23	natural deduction II
4/27	natural deduction II	5/25	examples, properties
5/2	completeness	5/30	advanced topics
5/9	midterm exam	6/1	summary
		6/6	exam

Evaluation

midterm exam (40) + final exam (60)

Contents

Aim

to develop skill for manipulating quantified formulas

Contents

1. substitutions
2. logical equivalence
3. prenex normal form
4. Skolemization

Substitutions

Substitutions for Terms

Definition

- substitution θ (for terms) is mapping from variables to terms
- application $t\theta$ of θ to term t is defined as follows:

$$t\theta = \begin{cases} \theta(t) & \text{if } t \text{ is variable} \\ f(t_1\theta, \ldots, t_n\theta) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

- notation $[t_1/x_1, \ldots, t_n/x_n]$ is defined as in propositional logic

Example

for $\theta = [\mathsf{s}(y)/x,\ x/y]$: $\mathsf{f}(\mathsf{s}(x), y)\theta = \mathsf{f}(\mathsf{s}(\theta(x)), \theta(y)) = \mathsf{f}(\mathsf{s}(\mathsf{s}(y)), x)$

Definition

application $\phi\theta$ of θ to first-order formula ϕ is defined as follows:

$$\phi\theta = \begin{cases} P(t_1\theta, \ldots, t_n\theta) & \text{if } \phi = P(t_1, \ldots, t_n) \\ \phi & \text{if } \phi \in \{\top, \bot\} \\ \neg(\phi_1\theta) & \text{if } \phi = \neg\phi_1 \\ (\phi_1\theta) * (\phi_2\theta) & \text{if } \phi = \phi_1 * \phi_2 \text{ and } * \in \{\wedge, \vee, \rightarrow, \leftrightarrow\} \\ s\theta \doteq t\theta & \text{if } \phi = s \doteq t \\ Qx(\phi_1\theta') & \text{if } \phi = Qx\phi_1 \text{ with } Q \in \{\forall, \exists\} \end{cases}$$

where $\theta'(x) = x$ and $\theta'(y) = \theta(y)$ for all variables $y \neq x$

Exercise

compute $(\mathsf{P}(x) \wedge \mathsf{s}(y) \doteq z)\theta$ and $(\forall x \exists z\ (x + 0 > z))\theta$ for $\theta = [\mathsf{s}(y)/x,\ 0/y]$

Is Validity Closed under Substitutions?

Fact

$\vDash \phi \implies\ \vDash \phi\theta$ does not hold

Example

1. $(\exists x(x \doteq y))[\mathsf{s}(x)/y] = \exists x(x \doteq \mathsf{s}(x))$ variable x is captured
2. $\vDash \exists x(x \doteq y)$ but $\nvDash \exists x(x \doteq \mathsf{s}(x))$

Definition

t is free for x in ϕ if

- ϕ is of form $P(t_1, \ldots, t_n)$ or $s \doteq t$
- $\phi = \neg\phi_1$ and t is free for x in ϕ_1
- $\phi = \phi_1 * \phi_2$, $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, and t is free for x in ϕ_1 and in ϕ_2
- ϕ is of form $\forall y\phi_1$ or $\exists y\phi_1$, $y \notin \mathsf{FV}(t)$, t is free for x in ϕ_1, and $x \neq y$

Note

$\phi[t/x]$ causes no variable capture if t is free for x in ϕ

Example

- $\mathsf{s}(x)$ is not free for y in $\exists x(x \doteq y)$
- $\mathsf{s}(x)$ is free for y in $\exists z(z \doteq y)$ renaming resolves variable capture

Theorem

- $\models \phi \implies \models \phi[t/x]$ *if t is free for x in ϕ*
- $\models \phi \implies \models \phi\theta$ *if $x\theta$ is free for x in ϕ*

Example

1. $\models \exists x(x \doteq y)$ as we may take $x = y$
2. $\models \exists z(z \doteq y)$ by renaming x to z
3. $\exists z(z \doteq \mathsf{s}(x))$ by substituting $\mathsf{s}(x)$ into y

Logical Equivalence

Definition (logical equivalence)

$\phi \approx \psi$ if $\phi \leftrightarrow \psi$ is valid ($\models \phi \iff \models \psi$ in other words)

Note

$\approx$ behaves like equality as in propositional logic

Example

- $\forall x\phi \approx \forall y(\phi[y/x])$
- $\forall x\forall y\phi \approx \forall y\forall x\phi$
- $\forall x(\phi \wedge \psi) \approx (\forall x\phi) \wedge (\forall x\psi)$
- $\forall x\phi \approx \phi$ if $x \notin \mathsf{FV}(\phi)$
- $\neg\forall x\phi \approx \exists x\neg\phi$

Exercise

- give counterexample for $\forall x(\phi \vee \psi) \approx (\forall x\phi) \vee (\forall x\psi)$ and $\forall x\exists y\phi \approx \exists y\forall\phi$
- what happens to these facts if $\forall$ and $\exists$ are swapped?

Prenex Normal Form

Prenex Normal Form

Definition

prenex form is formula of form:

$$Q_1 x_1 \cdots Q_n x_n \underbrace{\phi}_{\text{matrix}}$$

where $Q_i \in \{\forall, \exists\}$ and ϕ is quantifier-free

Example

- $\forall x \exists y ((\neg \mathsf{P}(y) \vee \mathsf{Q}(\mathsf{a})) \wedge (\neg \mathsf{Q}(\mathsf{a}) \vee \mathsf{P}(x)))$ is prenex normal form
- $\forall x \forall y (x < y \to \exists z (x < z \wedge z < y))$ is not
- $\forall x \forall y \exists z (x < y \to (x < z \wedge z < y))$ is prenex normal form

Theorem

every formula has equivalent prenex normal form

Proof.

- use De Morgan laws and following rules

$$\begin{aligned} \neg \forall x \phi &\approx \exists x \neg \phi & \phi \leftrightarrow \psi &\approx (\phi \to \psi) \wedge (\psi \to \phi) \\ \neg \exists x \phi &\approx \forall x \neg \phi & \phi \to \psi &\approx \neg \phi \vee \psi \\ (\forall y \phi) * \psi &\approx \forall y (\phi * \psi) \\ (\exists y \phi) * \psi &\approx \exists y (\phi * \psi) \end{aligned}$$

  where $* \in \{\wedge, \vee\}$ and $y \notin \mathsf{FV}(\psi)$

- rename bound variables if necessary $\square$

Example

$$\begin{aligned} \forall x \mathsf{P}(x) \leftrightarrow \mathsf{Q}(\mathsf{a}) &\approx (\forall x \mathsf{P}(x) \to \mathsf{Q}(\mathsf{a})) \wedge (\mathsf{Q}(\mathsf{a}) \to \forall x \mathsf{P}(x)) \\ &\approx (\neg \forall x \mathsf{P}(x) \vee \mathsf{Q}(\mathsf{a})) \wedge (\neg \mathsf{Q}(\mathsf{a}) \vee \forall x \mathsf{P}(x)) \\ &\approx (\exists x \neg \mathsf{P}(x) \vee \mathsf{Q}(\mathsf{a})) \wedge (\neg \mathsf{Q}(\mathsf{a}) \vee \forall x \mathsf{P}(x)) \\ &\approx \exists x (\neg \mathsf{P}(x) \vee \mathsf{Q}(\mathsf{a})) \wedge (\neg \mathsf{Q}(\mathsf{a}) \vee \forall x \mathsf{P}(x)) \\ &\approx \exists x (\neg \mathsf{P}(x) \vee \mathsf{Q}(\mathsf{a})) \wedge \forall x (\neg \mathsf{Q}(\mathsf{a}) \vee \mathsf{P}(x)) \\ &\approx (\exists y (\neg \mathsf{P}(y) \vee \mathsf{Q}(\mathsf{a})) \wedge \forall x (\neg \mathsf{Q}(\mathsf{a}) \vee \mathsf{P}(x))) \\ &\approx \forall x (\exists y (\neg \mathsf{P}(y) \vee \mathsf{Q}(\mathsf{a})) \wedge (\neg \mathsf{Q}(\mathsf{a}) \vee \mathsf{P}(x))) \\ &\approx \forall x \exists y ((\neg \mathsf{P}(y) \vee \mathsf{Q}(\mathsf{a})) \wedge (\neg \mathsf{Q}(\mathsf{a}) \vee \mathsf{P}(x))) \end{aligned}$$

Exercise

$(\forall x \mathsf{P}(x)) \to \neg \exists y (\forall x \mathsf{Q}(x, y) \to \mathsf{P}(y))$

Observation

for every x there exists y such that $P(x, y)$
$\iff$ there exists function f such that for every x we have $P(x, f(x))$

Definition

Skolemaization of $\forall x_1 \cdots \forall x_n \exists y\ \phi$ is $\forall x_1 \cdots \forall x_n\ \phi[f(x_1, \ldots, x_n)/y]$,
where f is fresh n-ary function symbol

Theorem

ϕ is satisifiable $\iff$ ψ is satisfiable for every Skolemization ψ of ϕ

Exercise

Skolemize $\exists x \mathsf{P}(x)$, $\forall x \exists y \mathsf{Q}(x, y)$, $\forall x \forall y \exists z \mathsf{R}(x, y, z)$, and $\forall x \exists y \forall z \mathsf{R}(x, y, z)$

Skolem Normal Form

Definition

$\forall x_1 \dots \forall x_n \phi$ is Skolem normal form if ϕ is quantifier-free

Example

$\forall x \forall y \forall w\ \mathsf{P}(x, y, \mathsf{f}(x, y), w, \mathsf{g}(x, y, w))$

Theorem

every formula ϕ admits Skolem normal form whose satisfiability is equivalent to ϕ

Proof.

compute prenex normal form and then Skolemize it repeatedly □

Exercises for Skolemization

convert each formula to Skolem normal form

1. $\forall x \exists y \neg \exists z \forall w \mathsf{P}(x, y, z, w)$
2. $\forall x \forall y (x < y \to \exists z (x < z \land z < y))$
3. $\forall \epsilon (\epsilon > 0 \to \exists n (n \in \mathbb{N} \land |\mathsf{a}_n| < \epsilon))$ $\qquad \lim_{n \to \infty} \mathsf{a}_n = 0$

Supplementary Comments

- $[t/x,\ u/y]$ is written as $[t, u \,/\, x, y]$ in textbook
- Skolemization preserves (un)satisifiability but not validity
- ϕ is valid $\iff$ $\neg\phi$ is unsatisfiable; Skolemization can be used for validity!
- as in propositional logic, $\vDash \forall x(\mathsf{P}(x) \lor \neg \mathsf{P}(x))$ implies $\vDash \forall x(\phi \lor \neg\phi)$ for all ϕ
- but unfortunately first-order logic has no truth table method