

# Homework 10

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1.

$$(1) \frac{[\rho(x)]^1}{\rho(x) \rightarrow \rho(x)} \rightarrow I_1$$

$$\frac{\rho(x) \rightarrow \rho(x)}{\forall x (\rho(x) \rightarrow \rho(x))} \forall E$$

(2)

$$\frac{[\rho(x) \wedge \alpha(x)]^1}{\rho(x)} \wedge E$$

$$\frac{\rho(x)}{(\rho(x) \wedge \alpha(x)) \rightarrow \rho(x)} \rightarrow I_1$$

$$\frac{}{\forall x ((\rho(x) \wedge \alpha(x)) \rightarrow \rho(x))} \forall I$$

(3)

$$\frac{[\forall x (\rho(x) \rightarrow \alpha(x))]^1}{[\rho(x)]^1 \quad \rho(x) \rightarrow \alpha(x)} \quad \frac{[\forall x (\alpha(x) \rightarrow \beta(x))]^2}{\beta(x) \quad \alpha(x) \rightarrow \beta(x)}$$

$$\frac{\beta(x)}{\rho(x) \rightarrow \beta(x)} \rightarrow I_1$$

$$\frac{\rho(x) \rightarrow \beta(x)}{\forall x (\rho(x) \rightarrow \beta(x))} \forall I$$

$$\frac{\forall x (\rho(x) \rightarrow \beta(x))}{\forall x (\alpha(x) \rightarrow \beta(x)) \rightarrow \forall x (\rho(x) \rightarrow \beta(x))} \rightarrow I_2$$

$$\frac{\forall x (\rho(x) \rightarrow \alpha(x)) \rightarrow (\forall x (\alpha(x) \rightarrow \beta(x)) \rightarrow \forall x (\rho(x) \rightarrow \beta(x)))}{\forall x (\rho(x) \rightarrow \alpha(x)) \rightarrow (\forall x (\rho(x) \rightarrow \alpha(x)) \rightarrow \forall x (\rho(x) \rightarrow \beta(x)))} \rightarrow I_1$$

(4)

$$\frac{[\forall x (\rho(x) \rightarrow \alpha(x))]^1}{[\rho(x)]^1 \quad \rho(x) \rightarrow \alpha(x)} \rightarrow E$$

$$\frac{\rho(x) \rightarrow \alpha(x)}{\alpha(x) \wedge \beta(x)} \wedge E$$

$$\frac{\alpha(x) \wedge \beta(x)}{\rho(x) \rightarrow (\alpha(x) \wedge \beta(x))} \rightarrow I_1$$

$$\frac{\forall x (\rho(x) \rightarrow (\alpha(x) \wedge \beta(x)))}{\forall x (\rho(x) \rightarrow \beta(x)) \rightarrow \forall x (\rho(x) \rightarrow (\alpha(x) \wedge \beta(x)))} \rightarrow I_2$$

$$\frac{\forall x (\rho(x) \rightarrow \beta(x)) \rightarrow (\forall x (\rho(x) \rightarrow \beta(x)) \rightarrow \forall x (\rho(x) \rightarrow (\alpha(x) \wedge \beta(x))))}{\forall x (\rho(x) \rightarrow \alpha(x)) \rightarrow (\forall x (\rho(x) \rightarrow \beta(x)) \rightarrow \forall x (\rho(x) \rightarrow (\alpha(x) \wedge \beta(x))))} \rightarrow I_1$$

(5)

$$\frac{[\rho(a)]^2}{\rho(a) \vee \alpha(b)} \vee I$$

$$\frac{\rho(a) \vee \alpha(b)}{\exists x (\rho(x) \vee \alpha(x))} \exists I$$

$$\frac{\exists x (\rho(x) \vee \alpha(x))}{\exists x (\rho(x) \vee \alpha(x)) \vee \alpha(x)} \vee E_2$$

$$\frac{}{(\rho(a) \vee \alpha(b)) \rightarrow \exists x (\rho(x) \vee \alpha(x))} \rightarrow I_1$$

(6)

$$\frac{[\rho(a) \wedge \alpha(b)]^1}{\rho(a) \exists I}$$

$$\frac{[\rho(a) \wedge \alpha(b)]^1}{\alpha(b) \exists I}$$

$$\frac{\rho(a) \exists I \quad \alpha(b) \exists I}{\exists x \rho(x) \wedge \exists x \alpha(x)} \wedge I$$

$$\frac{}{(\rho(a) \wedge \alpha(b)) \rightarrow (\exists x \rho(x) \wedge \exists x \alpha(x))} \rightarrow I_1$$

(7)

$$\frac{[\forall x \rho(x) \wedge \forall x \alpha(x)]^1}{\forall x \rho(x) \vee E}$$

$$\frac{\forall x \rho(x) \vee E}{\rho(x)} \forall E$$

$$\frac{\rho(x)}{\alpha(x) \wedge \beta(x)} \wedge I$$

$$\frac{\alpha(x) \wedge \beta(x)}{\forall x (\rho(x) \wedge \alpha(x))} \rightarrow I_1$$

$$\frac{}{(\forall x \rho(x) \wedge \forall x \alpha(x)) \rightarrow \forall x (\rho(x) \wedge \alpha(x))} \rightarrow I_1$$

(8)

$$\begin{array}{c}
 \frac{\frac{\frac{\frac{[\forall x(P(x) \wedge Q(x))]}{P(x) \wedge Q(x)} \wedge E}{P(x)} \wedge E}{\forall x P(x)} \wedge I}{\forall x (P(x) \wedge Q(x))} \wedge I \\
 \frac{\frac{\frac{[\forall x(Q(x))]}{Q(x)} \wedge E}{Q(x)} \wedge I}{\forall x Q(x)} \wedge I \\
 \frac{\frac{[\forall x(P(x) \vee Q(x))]}{\forall x P(x) \vee \forall x Q(x)} \wedge I}{(\forall x P(x) \vee \forall x Q(x)) \rightarrow (\forall x (P(x) \vee Q(x)))} \rightarrow I_1
 \end{array}$$

(9)

$$\begin{array}{c}
 \frac{\frac{\frac{[\forall x P(x)]}{P(x)} \vee E}{P(x) \vee Q(x)} \vee E}{\frac{\frac{[\forall x Q(x)]}{Q(x)} \vee E}{P(x) \vee Q(x)} \vee E} \vee E_2 \\
 \frac{\frac{[\forall x (P(x) \vee Q(x))]}{P(x) \vee Q(x)} \vee I}{(\forall x P(x) \vee \forall x Q(x)) \rightarrow (\forall x (P(x) \vee Q(x)))} \rightarrow I_2
 \end{array}$$

(10)

$$\begin{array}{c}
 \frac{\frac{\frac{[\forall x (P(x) \rightarrow Q(x))]}{P(x) \rightarrow Q(x)} \wedge E}{\frac{[\forall x P(x)]}{P(x)} \wedge E_1}{\wedge E_2}}{\frac{\frac{Q(x)}{Q(x) \wedge I_A} \wedge I}{(\forall x P(x) \rightarrow Q(x)) \rightarrow I_2}} \rightarrow I_1 \\
 \frac{\frac{[\forall x (P(x) \rightarrow Q(x))]}{\forall x (P(x) \rightarrow Q(x))} \wedge I}{(\forall x (P(x) \rightarrow Q(x)) \rightarrow (\forall x P(x) \rightarrow \forall x Q(x)))} \rightarrow I_2
 \end{array}$$

(11)

$$\begin{array}{c}
 \frac{\frac{\frac{[\exists x (P(x) \wedge Q(x))]}{P(x) \wedge Q(x)} \exists E_1}{\frac{P(x)}{\exists x P(x)} \exists I}{\wedge E_2}}{\frac{\frac{Q(x)}{\exists x Q(x)} \exists I}{\exists x (P(x) \wedge Q(x)) \rightarrow (\exists x P(x) \wedge \exists x Q(x))}} \rightarrow I_1
 \end{array}$$

(12)

$$\begin{array}{c}
 \frac{\frac{\frac{[\exists x P(x)]}{P(x)} \vee I}{\frac{P(x) \vee Q(x)}{\exists x (P(x) \vee Q(x))} \exists I}{\wedge E_1}}{\frac{\frac{[\exists x Q(x)]}{Q(x)} \vee I}{\frac{P(x) \vee Q(x)}{\exists x (P(x) \vee Q(x))} \exists I} \vee I_2} \vee I_1 \\
 \frac{\frac{[\exists x (P(x) \vee Q(x))]}{\exists x (P(x) \vee Q(x))} \vee I}{(\exists x P(x) \vee \exists x Q(x)) \rightarrow (\exists x (P(x) \vee Q(x)))} \vee I_2
 \end{array}$$

(13)

$$\begin{array}{c}
 \frac{\frac{\frac{[\exists x (P(x) \vee Q(x))]}{P(x) \vee Q(x)} \exists E_1}{\frac{P(x)}{\exists x P(x)} \vee I}{\wedge E_2}}{\frac{\frac{Q(x)}{\exists x Q(x)} \vee I}{\exists x (P(x) \vee Q(x)) \rightarrow (\exists x P(x) \vee \exists x Q(x))}} \rightarrow I_1
 \end{array}$$

(14)

$$\begin{array}{c}
 \frac{\frac{\frac{[\exists x P(x)]}{P(x)} \exists I}{\frac{[\exists x P(x) \rightarrow \exists x Q(x)]}{\exists x P(x) \rightarrow \exists x Q(x)} \rightarrow E}{\frac{\frac{[\exists x Q(x)]}{Q(x)} \exists I}{\frac{P(x) \rightarrow Q(x)}{P(x) \rightarrow Q(x)} \exists I} \rightarrow I_1}{\frac{\frac{[\neg \exists x (P(x) \rightarrow Q(x))]}{\neg \exists x (P(x) \rightarrow Q(x))} \neg E}{\frac{\frac{\perp}{\frac{BFG}{P(x) \rightarrow Q(x)} \rightarrow I_2}{\frac{\frac{P(x) \rightarrow Q(x)}{\exists x (P(x) \rightarrow Q(x))} \exists I}{\frac{\frac{[\neg \exists x (P(x) \rightarrow Q(x))]}{\neg \exists x (P(x) \rightarrow Q(x))} \neg E}{\frac{\frac{\perp}{RAA_2}}{\frac{\frac{\perp}{\exists x (P(x) \rightarrow Q(x))} \rightarrow I_1}{(\exists x P(x) \rightarrow \exists x Q(x)) \rightarrow (\exists x (P(x) \rightarrow Q(x)))}} \rightarrow I_1
 \end{array}$$

(15)

S

$$(\exists x P(x) \rightarrow \exists x Q(x)) \rightarrow \exists x(P(x) \rightarrow Q(x))$$

(15)

$$\frac{\frac{\frac{\frac{[\neg \exists x \neg P(x)]^2}{\frac{[\neg P(x)]^1}{\frac{\perp}{P(x)} \text{ RAA}_1}} \frac{[\neg \exists x \neg P(x)]^1}{\frac{\perp}{\exists x \neg P(x)} \text{ RAA}_2}}{\frac{\perp}{\exists x \neg P(x)} \text{ RAA}_2}}{\frac{\perp}{\exists x \neg P(x)} \text{ RAA}_1}}{\rightarrow I_1}$$

(16)

$$\frac{\frac{\frac{[\exists x P(x)]^2}{\frac{[\forall x \rightarrow P(x)]^1}{\frac{[\neg P(y)]^1}{\frac{\perp}{\neg P(y)} \text{ R.E.}} \rightarrow E}} \frac{\perp}{\exists x \neg P(x)} \text{ R.E.}}{\frac{\perp}{\exists x \neg P(x)} \text{ R.E.}} \rightarrow I_2}}{\frac{\perp}{\exists x \neg P(x)} \text{ R.E.}} \rightarrow I_1$$

(17)

$$\frac{\frac{\frac{[\neg \exists x P(x)]^1}{\frac{[\exists x P(x)]^2}{\frac{\perp}{\neg P(x)} \text{ R.E.}} \rightarrow E}}{\frac{\perp}{\neg P(x)} \text{ R.E.}} \rightarrow I_2}}{\frac{\perp}{\neg P(x)} \text{ R.E.}} \rightarrow I_1$$

(18)

$$\frac{\frac{\frac{[\exists x \neg P(x)]^1}{\frac{[\neg P(a)]^2}{\frac{\perp}{\neg P(a)} \text{ R.E.}} \rightarrow E}}{\frac{\perp}{\neg P(a)} \text{ R.E.}} \rightarrow I_2}}{\frac{\perp}{\neg P(a)} \text{ R.E.}} \rightarrow I_1$$

(19)

$$\frac{\frac{\frac{[\forall x \forall y R(x,y)]^1}{\frac{[\forall y R(x,y)]^2}{\frac{\forall y R(x,y)}{\frac{R(x,y)}{\frac{\forall x R(x,y)}{\frac{R(x,y)}{\frac{\forall y \forall x R(x,y)}{\forall x \forall y R(x,y)}}} \text{ A.E.}} \text{ A.E.}} \text{ A.E.}} \text{ A.E.}} \text{ A.E.}} \text{ A.E.}} \rightarrow I_1$$

(20)

$$\frac{\frac{\frac{[\exists y \forall x R(x,y)]^1}{\frac{[\forall x R(x,y)]^2}{\frac{R(w,a)}{\frac{\exists y R(w,y)}{\frac{\forall x \exists y R(x,y)}{\frac{\forall x \forall y R(x,y)}{\forall y \forall x R(x,y)}}} \text{ A.E.}} \text{ A.E.}} \text{ A.E.}} \text{ A.E.}} \text{ A.E.}} \rightarrow I_1$$

2.

$$\frac{\frac{\frac{[\forall x (P(x) \rightarrow P(j(x)))]^1}{\frac{[\forall w (P(w) \rightarrow P(j(w)))]^2}{\frac{[\forall w P(w)]^2}{\frac{P(w) \rightarrow P(j(w))}{\frac{P(j(w))}{\frac{P(w) \rightarrow P(j(j(w)))}{\frac{\forall x (P(x) \rightarrow P(j(x)))}{\frac{\forall x (P(x) \rightarrow P(j(j(x))))}{\forall x (P(x) \rightarrow P(j(j(x))))}} \text{ A.E.}} \text{ A.E.}} \text{ A.E.}} \text{ A.E.}} \text{ A.E.}} \text{ A.E.}} \rightarrow I_1$$

Assume  $P(x) \rightarrow P(j(j(x)))$  (1) for all  $x$ . We show that  $P(x) \rightarrow P(j(j(j(x))))$  for all  $x$ . Assume  $P(w)$  for arbitrary  $w$ . It's enough to show  $P(j(j(j(w))))$ . By assumption (1)  $P(w) \rightarrow P(j(w))$  holds and with assumption  $P(w)$  we have  $P(j(w))$  holds. By assumption (1) we have  $P(j(w)) \rightarrow P(j(j(w)))$  holds and with  $P(j(w))$  holds we have  $P(j(j(w)))$  holds.

5.  $U = \mathbb{Z}$ ,  $\bar{R} = \left\{ m+n \in \mathbb{Z} \times \mathbb{Z} \mid m > n \right\}$

(a)  $y = x - 1$

(b)  $x = y$