I211E: Mathematical Logic

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Term 1-1, 2023

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Schedule

	propositional logic		predicate logic
4/13	syntax, semantics	5/11	syntax, semantics
4/18	normal forms	5/16	normal forms
4/20	examples	5/18	natural deduction I
4/25	natural deduction I	5/23	natural deduction II
4/27	natural deduction II	5/25	examples, properties
5/2	completeness	5/30	advanced topics
5/9	midterm exam	6/1	summary
		6/6	exam

Evaluation

midterm exam (40) $+$ final exam (60)

Contents

Aim

to learn how to prove formulas having equalities

Contents

1. natural deduction II ($\doteq$)
2. soundness and completeness
3. axioms

Natural Deduction II ($\doteq$)

Equality Axioms

Definition (inference rules for $\doteq$)

reflexivity, symmetry, transitivity, congruence, and substitution rules

$$\dfrac{}{t \doteq t}\ \text{REFL} \qquad \dfrac{s \doteq t}{t \doteq s}\ \text{SYM} \qquad \dfrac{s \doteq t \qquad t \doteq u}{s \doteq u}\ \text{TRANS}$$

$$\dfrac{s_1 \doteq t_1 \qquad \cdots \qquad s_n \doteq t_n}{f(s_1, \dots, s_n) \doteq f(t_1, \dots, t_n)}\ \text{CONG}$$

$$\dfrac{s_1 \doteq t_1 \qquad \cdots \qquad s_n \doteq t_n \qquad P(s_1, \dots, s_n)}{P(t_1, \dots, t_n)}\ \text{SUBST}$$

Proposition

$\vdash \forall x(x \cdot 0 = 0) \to 0 \doteq (x \cdot (y \cdot 0))$

Proof Sketch: $x \cdot \underline{(y \cdot 0)} = \underline{x \cdot 0} = 0$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x \doteq x}\ \text{REFL} \qquad \dfrac{[\forall x\,(x \cdot 0 \doteq 0)]^1}{y \cdot 0 \doteq 0}\ \forall\text{E}}{x \cdot (y \cdot 0) \doteq x \cdot 0}\ \text{CONG} \qquad \dfrac{[\forall x\,(x \cdot 0 \doteq 0)]^1}{x \cdot 0 \doteq 0}\ \forall\text{E}}{\dfrac{x \cdot (y \cdot 0) \doteq 0}{0 \doteq (x \cdot (y \cdot 0))}\ \text{SYM}}\ \text{TRANS}}{\forall x(x \cdot 0 = 0) \to 0 \doteq (x \cdot (y \cdot 0))}\ {\to}\text{I}$$

□

Soundness and Completeness (Gödel 1930)

Definition

- structure $\mathcal{A}$ is model of Γ if $\mathcal{A} \vDash \gamma$ holds for all $\gamma \in \Gamma$
- $\Gamma \vDash \phi$ if $\mathcal{A} \vDash \phi$ holds for all models $\mathcal{A}$ of Γ

Theorem ($\Rightarrow$ soundness; $\Leftarrow$ completeness)

$\Gamma \vdash \phi \iff \Gamma \vDash \phi$

Proof.

($\Rightarrow$) additional inference rules are valid ($\Leftarrow$) model existence lemma □

Corollary

$\Gamma \nvdash \phi$ if and only if there exists model $\mathcal{A}$ of Γ with $\mathcal{A} \nvDash \phi$ (countermodel)

Axioms

Terminology

axioms are formulas supposed to be true a priori

Definition

- ϕ is valid under set Γ of axioms if $\Gamma \vdash \phi$
- Γ is consistent if $\Gamma \nvdash \bot$
- Γ is inconsistent if $\Gamma \vdash \bot$

Example

- $\Gamma = \{\forall x\, \mathsf{P}(x), \forall x\, (\mathsf{P}(x) \to \mathsf{Q}(x))\}$ is consistent and $\Gamma \vdash \forall x\, \mathsf{Q}(x)$
- $\Gamma = \{\forall x(\neg(x \doteq x))\}$ is inconsistent and $\Gamma \vdash 0 \doteq \mathsf{s}(0)$

Fact

if Γ is inconsistent then $\Gamma \vdash \phi$ for all formulas ϕ

Fact ($\Rightarrow$: Model Existence Lemma)

Γ is consistent $\iff$ there exists model of Γ

Proposition

$\Gamma = \{\forall x\, \mathsf{P}(x), \forall x\, (\mathsf{P}(x) \to \mathsf{Q}(x))\}$ is consistent

Proof.

Γ admits the model $\mathcal{A} = (\{0\}, \bar{\mathsf{P}}, \bar{\mathsf{Q}})$ with $\bar{\mathsf{P}} = \bar{\mathsf{Q}} = \{0\}$. □

Definition (Robinson Arithmetic)

$$\mathbf{Q} \text{ is } \left\{ \begin{array}{ll} \forall x. & \neg(\mathsf{s}(x) \doteq 0) \\ \forall x. & x \doteq 0 \vee \exists y (x \doteq \mathsf{s}(y)) \\ \forall x, y. & \mathsf{s}(x) \doteq \mathsf{s}(y) \to x \doteq y \\ \forall x. & x + 0 \doteq x \\ \forall x, y. & x + \mathsf{s}(y) \doteq \mathsf{s}(x + y) \\ \forall x. & x \cdot 0 \doteq 0 \\ \forall x, y. & x \cdot \mathsf{s}(y) \doteq (x \cdot y) + x \end{array} \right\} \text{ over } \{+^{(2)}, \cdot^{(2)}, \mathsf{s}^{(1)}, 0^{(0)}\}$$

Exercise

- find natural model of $\mathbf{Q}$ whose universe is $\mathbb{N}$
- $\mathbf{Q} \vdash \forall x \exists y\, (x + y \doteq x)$
- $\mathbf{Q} \nvdash \forall x \forall y\, (x + y \doteq y + x)$

Proof of $\mathbf{Q} \vdash 0 + \mathsf{s}(0) \doteq \mathsf{s}(0)$

Proof Sketch: $\underline{0 + \mathsf{s}(0)} = \mathsf{s}(\underline{0 + 0}) = \mathsf{s}(0)$

$$\dfrac{\dfrac{\dfrac{[\forall x \forall y\, (x + \mathsf{s}(y) \doteq \mathsf{s}(x + y))]^{\mathbf{Q}}}{\forall y\, (0 + \mathsf{s}(y) \doteq \mathsf{s}(0 + y))}\,\forall\mathrm{E}}{0 + \mathsf{s}(0) \doteq \mathsf{s}(0 + 0)}\,\forall\mathrm{E} \qquad \dfrac{\dfrac{[\forall x\, (x + 0 \doteq x)]^{\mathbf{Q}}}{0 + 0 \doteq 0}\,\forall\mathrm{E}}{\mathsf{s}(0 + 0) \doteq \mathsf{s}(0)}\,\mathrm{CONG}}{0 + \mathsf{s}(0) \doteq \mathsf{s}(0)}\,\mathrm{TRANS}$$

Definition (Peano Arithmetic)

$\mathbf{PA} = \mathbf{Q} \cup \{\, \forall y_1, \ldots, y_n.\, ((\phi[0/x] \wedge \forall x\, (\phi \to \phi[\mathsf{s}(x)/x])) \to \forall x\, \phi) \mid (*) \,\}$

where $(*)$ means that ϕ is formula with $\mathsf{FV}(\phi) = \{x, y_1, \ldots, y_n\}$

Exercise

- $\mathbf{PA} \vdash \forall x \exists y\, (x + y \doteq x)$
- $\mathbf{PA} \vdash \forall x \forall y\, (x + y \doteq y + x)$

Set Theory

Definition (Zermelo–Fraenkel set theory with Axiom of Choice)

$\mathbf{ZFC}$ consists of following axioms over signature $\{\in^{(2)}\}$:

$$\exists X \forall z.\, \neg(z \in X)$$
$$\forall X, Y.\, (\forall x(z \in X \leftrightarrow z \in Y)) \to X \doteq Y$$
$$\forall X \exists Z \forall Y.\, (Y \in Z \leftrightarrow \exists W(W \in Z \wedge W \in X) \to X \doteq Y)$$
$$\vdots$$

Definition (von Neumann numbers)

$0 = \varnothing$, $1 = \{0\}$, $2 = \{0, 1\}$, ...

Proof of $\mathbf{PA} \vdash \forall x \exists y\, (x \cdot y \doteq x)$

Proof Idea: $0 + x = x$ and $x \cdot \mathsf{s}(0) = x$ are shown by mathemtical induciton on x

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots(a)}{0 \cdot \mathsf{s}(0) \doteq 0}\,\text{TRANS} \quad \dfrac{\vdots(b)}{\forall x\, (x \cdot \mathsf{s}(0) \doteq x \to \mathsf{s}(x) \cdot \mathsf{s}(0) \doteq \mathsf{s}(x))}\,\forall\text{I}}{\phi}\,\wedge\text{I} \qquad [\phi \to \forall x\, (x \cdot \mathsf{s}(0) \doteq x)]^{\mathbf{PA}}}{\forall x\, (x \cdot \mathsf{s}(0) \doteq x)}\,\to\text{E}}{x \cdot \mathsf{s}(0) \doteq x}\,\forall\text{E}}{\exists (x \cdot y \doteq x)}\,\exists\text{I}}{\forall x \exists (x \cdot y \doteq x)}\,\forall\text{I}}{}$$

where $\phi = 0 \cdot \mathsf{s}(0) \doteq 0 \wedge \forall x\, (x \cdot \mathsf{s}(0) \doteq x \to \mathsf{s}(x) \cdot \mathsf{s}(0) \doteq \mathsf{s}(x))$, and

(a) $\underline{0 \cdot \mathsf{s}(0)} = \underline{0 \cdot 0} + 0 = \underline{0 + 0} = 0$

(b) $\underline{\mathsf{s}(x) \cdot \mathsf{s}(0)} = \underline{\mathsf{s}(x) \cdot 0} + \mathsf{s}(x) = \underline{0 + \mathsf{s}(x)} = \mathsf{s}(\underline{0 + x}) = \mathsf{s}(x)$

Supplementary Comments

- alternative inference rules for $\doteq$ are:

$$\dfrac{}{t \doteq t}\,\doteq\text{I} \qquad \dfrac{s \doteq t \quad \phi[s/x]}{\phi[t/x]}\,\doteq\text{E} \quad (s \text{ and } t \text{ are free for } x)$$

- there are many (non-equivalent) variations of $\mathbf{Q}$, $\mathbf{PA}$, and $\mathbf{ZFC}$
- $\mathbf{PA}$ admits standard model on $\mathbb{N}$ and also non-standard models
- $\mathbf{ZFC}$ is powerful enough for formalizing most part of mathematics