# I211E: Mathematical Logic


Nao Hirokawa

JAIST


Term 1-1, 2023

https://www.jaist.ac.jp/~hirokawa/lectures/ml/



# Contents of This Course



### Schedule

| | propositional logic | | predicate logic |
|---|---|---|---|
| 4/13 | syntax, semantics | 5/11 | syntax, semantics |
| 4/18 | normal forms | 5/16 | normal forms |
| 4/20 | examples, properties | 5/18 | examples, properties |
| 4/25 | natural deduction I | 5/23 | natural deduction I |
| 4/27 | natural deduction II | 5/25 | natural deduction II |
| 5/2 | completeness | 5/30 | advanced topics |
| 5/9 | midterm exam | 6/2 | summary |
| | | 6/6 | exam |

### Evaluation

midterm exam (40) + final exam (60)



## Goal of This Course

### Goal

- able to **read and write** logical formulas
- able to **transform** formulas
- able to **prove/disprove** formulas

### Ultimate Goal

develop skills to **read textbooks** and to **write definitions/proofs in thesis**

# Reading/Writing Formulas and Proofs

- two plus three makes five:
$$2 + 3 = 5$$
- every number $x \in \mathbb{R}$ satisfies $x^2 \geqslant 0$:
$$\forall x \in \mathbb{R}. \quad x^2 \geqslant 0$$
- mathematical induction: $P(n)$ holds for all $n \in \mathbb{N}$ if next conditions hold: (i) $P(0)$, (ii) for every $k \in \mathbb{N}$ if $P(k)$ then $P(k+1)$:
$$(P(0) \wedge (\forall k \in \mathbb{N}.\ P(k) \to P(k+1))) \ \to \forall n \in \mathbb{N}.\ P(n)$$



**Proposition**

If $p$ implies $q$ and $q$ implies $r$ then $p$ implies $r$. $\quad ((p \to q) \wedge (q \to r)) \to (p \to r)$

**Proof.**

Assume that $p$ implies $q$ and $q$ implies $r$. Suppose that $p$ holds. By the first assumption we have $q$. By the second assumption we have $r$. Therefore, $p$ implies $r$. □

$$\dfrac{\dfrac{\dfrac{\dfrac{[p]^2 \quad \dfrac{[(p\to q)\wedge(q\to r)]^1}{p\to q}\,\wedge\text{-E}}{q}\,\to\text{-E} \qquad \dfrac{[(p\to q)\wedge(q\to r)]^1}{q\to r}\,\wedge\text{-E}}{r}\,\to\text{-E}}{p\to r}\,\to\text{-I}_2}{((p\to q)\wedge(q\to r))\to(p\to r)}\,\to\text{-I}_1$$



# Sudoku

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | 3 | | | 7 | | | | |
| 6 | | | 1 | 9 | 5 | | | |
| | 9 | 8 | | | | | 6 | |
| 8 | | | | 6 | | | | 3 |
| 4 | | | 8 | | 3 | | | 1 |
| 7 | | | | 2 | | | | 6 |
| | 6 | | | | | 2 | 8 | |
| | | | 4 | 1 | 9 | | | 5 |
| | | | | 8 | | | 7 | 9 |

How to solve it automatically?

The easiest way is formalizing the problem in propositional logic!



# Tic-Tac-Toe

| | | |
|---|---|---|
| | × | × |
| ○ | × | |
| | ○ | |

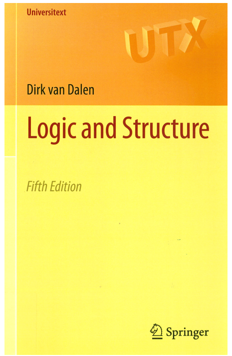
Is there a winning strategy for Tic-Tac-Toe?

Actually not. How to formalize and prove it?

# Propositional Logic: Syntax and Semantics



## Textbook for Propositional Logic




Dirk van Dalen

**Logic and Structure (5th edition)**

Springer Long Heidelberg, 2012

We will study Chapters 2, 3, and 4.



## Propositional Logic: Syntax

### Definition (propositional formulas)

| | atom | | verum | | falsum | | negation | | conjunction | | disjunction | | implication | | equivalence |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\phi ::=$ | $p$ | $\mid$ | $\top$ | $\mid$ | $\bot$ | $\mid$ | $\neg\phi$ | $\mid$ | $\phi \wedge \phi$ | $\mid$ | $\phi \vee \phi$ | $\mid$ | $\phi \rightarrow \phi$ | $\mid$ | $\phi \leftrightarrow \phi$ |
| | | | true | | contradiction | | not ... | | ... and ... | | ... or ... | | if ... then ... | | ... iff ... |

### Example

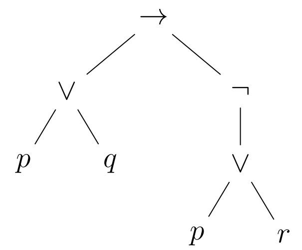
$p \rightarrow (q \vee r)$ means "if $p$ holds then $q$ or $r$ holds"

### Exercise

give formula for "if $p$ or $q$ then neither $p$ nor $r$ holds"



## Parse Tree of $(p \vee q) \rightarrow (\neg(p \vee q))$




### Exercise

1. draw parse tree of $\neg((\neg p) \vee (\neg q))$
2. how about $p \vee q \wedge r$ ? explain why this is not well-formed formula

## Propositional Logic: Semantics

**Definition**

- valuation $v$ is mapping from atoms to $\{\mathsf{T}, \mathsf{F}\}$
- valuation $[\![\cdot]\!]_v$ of propositional formula is defined as follows:

$$[\![p]\!]_v = v(p) \qquad [\![\phi \wedge \psi]\!]_v = \begin{cases} \mathsf{T} & \text{if } [\![\phi]\!]_v = \mathsf{T} \text{ and } [\![\psi]\!]_v = \mathsf{T} \\ \mathsf{F} & \text{otherwise} \end{cases}$$

$$[\![\top]\!]_v = \mathsf{T} \qquad [\![\phi \vee \psi]\!]_v = \begin{cases} \mathsf{T} & \text{if } [\![\phi]\!]_v = \mathsf{T} \text{ or } [\![\psi]\!]_v = \mathsf{T} \\ \mathsf{F} & \text{otherwise} \end{cases}$$

$$[\![\bot]\!]_v = \mathsf{F} \qquad [\![\phi \to \psi]\!]_v = \begin{cases} \mathsf{T} & \text{if } [\![\phi]\!]_v = \mathsf{F} \text{ or } [\![\psi]\!]_v = \mathsf{T} \\ \mathsf{F} & \text{otherwise} \end{cases}$$

$$[\![\neg\phi]\!]_v = \begin{cases} \mathsf{F} & \text{if } [\![\phi]\!]_v = \mathsf{T} \\ \mathsf{T} & \text{if } [\![\phi]\!]_v = \mathsf{F} \end{cases} \qquad [\![\phi \leftrightarrow \psi]\!]_v = \begin{cases} \mathsf{T} & \text{if } [\![\phi]\!]_v = [\![\psi]\!]_v \\ \mathsf{F} & \text{otherwise} \end{cases}$$



## Examples for Valuations

**Example**

for valuation $v$ with $v(p) = \mathsf{T}$ and $v(q) = \mathsf{F}$

$$[\![ \underbrace{\underbrace{\neg \underbrace{p}_{\mathsf{T}}}_{\mathsf{F}} \vee \underbrace{q}_{\mathsf{F}}}_{\mathsf{F}} ]\!]_v = \mathsf{F} \qquad [\![ \underbrace{\underbrace{(\neg\underbrace{(\neg \underbrace{p}_{\mathsf{T}})}_{\mathsf{F}})}_{\mathsf{T}} \to \underbrace{p}_{\mathsf{T}}}_{\mathsf{T}} ]\!]_v = \mathsf{T}$$

**Exercise**

what if $v(p) = \mathsf{F}$ and $v(q) = \mathsf{T}$?



# Validity and Satisfiability



## Validity and Satisfiability

**Definition**

- $\phi$ is valid if $[\![\phi]\!]_v = \mathsf{T}$ for all valuations $v$
- $\phi$ is satisfiable if $[\![\phi]\!]_v = \mathsf{T}$ for some valuations $v$

**Example**

$\phi = (p \vee q) \to (\neg(p \vee r))$ is satisfiable but not valid because:

- $[\![\phi]\!]_v = \mathsf{T}$ for $v = \{p \mapsto \mathsf{F},\ q \mapsto \mathsf{T},\ r \mapsto \mathsf{F}\}$
- $[\![\phi]\!]_v = \mathsf{F}$ for $v = \{p \mapsto \mathsf{T},\ q \mapsto \mathsf{T},\ r \mapsto \mathsf{T}\}$

**Exercise**

is $\neg(\neg p)) \to p$ valid? is it satisfiable?

## Truth Tables for Base Cases

| $\top$ | $\bot$ |
|---|---|
| T | F |

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

| $p$ | $q$ | $p \wedge q$ | $p \vee q$ | $p \to q$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | T | F | F |
| F | T | F | T | T | F |
| F | F | F | F | T | T |



## Truth Tables for Complex Formulas

| $p$ | $q$ | $r$ | $p \vee q$ | $p \vee r$ | $\neg(p \vee r)$ | $(p \vee q) \to (\neg(p \vee r))$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | F | F |
| T | T | F | T | T | F | F |
| T | F | T | T | T | F | F |
| T | F | F | T | T | F | F |
| F | T | T | T | T | F | F |
| F | T | F | F | F | T | T |
| F | F | T | F | T | F | T |
| F | F | F | F | F | T | T |

- is $(p \vee q) \to (\neg(p \vee r))$ satisfiable? why?
- is it valid? why?



## Truth Tables for Complex Formulas (Compact Form)

| $p$ | $q$ | $r$ | $(p \vee q)$ | $\to$ | $(\neg$ | $(p \vee r))$ |
|---|---|---|---|---|---|---|
| T | T | T | T | F | T | F |
| T | T | F | T | F | T | F |
| T | F | T | T | F | T | F |
| T | F | F | T | F | T | F |
| F | T | T | T | F | T | F |
| F | T | F | F | T | F | T |
| F | F | T | F | T | T | F |
| F | F | F | F | T | F | T |

**Exercise**

compute truth table for $\phi = \neg(p \vee q) \to ((\neg p) \wedge (\neg q))$; satisfiable? valid?



## Duality and Decidability Results

**Theorem**

*$\phi$ is valid $\iff$ $\neg\phi$ is unsatisfiable (not satisfiable)*

**Theorem**

*validity and satisfiability problems are decidable:*

| | | | |
|---|---|---|---|
| *instance:* | *propositional formula $\phi$* | *instance:* | *propositional formula $\phi$* |
| *question:* | *is $\phi$ valid?* | *question:* | *is $\phi$ satisfiable?* |

**Proof.**

compute truth table for $\phi$ $\square$

# Logical Equivalence



**Definition (logical equivalence)**

$\phi \approx \psi$ if $\phi \leftrightarrow \psi$ is valid

**Example**

$$p \approx \neg\neg p \qquad p \wedge q \not\approx p \vee q$$
$$p \to q \approx (\neg p) \vee q \qquad p \to q \not\approx q \to p$$
$$\neg(p \wedge q) \approx (\neg p) \vee (\neg q) \qquad (p \to q) \to r \not\approx p \to (q \to r)$$



## How To Prove Logical Equivalence

**Theorem (de Morgan's law)**

$\neg(p \wedge q) \approx (\neg p) \vee (\neg q)$

**Proof.**

The claim is shown by the truth table method:

| $p$ | $q$ | $\neg$ | $(p \wedge q)$ | $\leftrightarrow$ | $(\neg p)$ | $\vee$ | $(\neg r)$ |
|---|---|---|---|---|---|---|---|
| T | T | F | T | T | F | F | F |
| T | F | T | F | T | F | T | T |
| F | T | T | F | T | T | T | F |
| F | F | T | F | T | T | T | T |

□



## How To Disprove Logical Equivalence

**Proposition**

$p \to q \not\approx q \to p$

**Hint**

before writing proof, compute truth table for $(p \to q) \leftrightarrow (q \to p)$

**Proof.**

Consider the valuation $v = \{p \mapsto \mathsf{F}, q \mapsto \mathsf{T}\}$. We have:

$$[\![(p \to q) \leftrightarrow (q \to p)]\!]_v = \mathsf{F}$$

So $(p \to q) \leftrightarrow (q \to p)$ is invalid. Hence the claim holds. □

## Supplementary Comments

- T and F are written as $1$ and $0$ in textbook
- set of all propositional formulas is referred to as PROP in textbook
- valid formula $\phi$ is called **tautology** and denoted by $\models \phi$
- $\phi \approx \psi \iff \models \phi \leftrightarrow \psi$