I211E: Mathematical Logic

Nao Hirokawa

JAIST

Term 1-1, 2023

`https://www.jaist.ac.jp/~hirokawa/lectures/ml/`

Schedule

	propositional logic		predicate logic
4/13	syntax, semantics	5/11	syntax, semantics
4/18	normal forms	5/16	normal forms
4/20	examples	5/18	natural deduction I
4/25	natural deduction I	5/23	natural deduction II
4/27	natural deduction II	5/25	examples, properties
5/2	completeness	5/30	advanced topics
5/9	midterm exam	6/1	summary
		6/6	exam

Evaluation

midterm exam (40) + final exam (60)

Contents

Aim

to develop skill for manipulating formulas

Contents

1. tautologies
2. negation normal forms
3. conjunctive normal forms

Tautologies

Tautologies

Definition

propositional formula ϕ is tautology ($\models \phi$) if ϕ is valid

Example

- $\models (\neg(\neg p)) \to p$
- $\models (\neg(\neg(\neg q))) \to (\neg q)$
- $\models (\neg(\neg(p \vee q))) \to (p \vee q)$
- ...
- $\models (\neg(\neg \phi)) \to \phi$ for all propositional formulas ϕ

Q. doesn't $\models (\neg(\neg p)) \to p$ imply $\models (\neg(\neg \phi)) \to \phi$ for all propositions ϕ?

A. yes!

Definition

- substitution θ is mapping from atoms to propositional formulas
- application of θ to propositional formula ϕ:

$$\phi\theta = \begin{cases} \theta(\phi) & \text{if } \phi \text{ is atom} \\ \phi & \text{if } \phi \in \{\top, \bot\} \\ \neg(\phi_1\theta) & \text{if } \phi = \neg\phi_1 \\ (\phi_1\theta) * (\phi_2\theta) & \text{if } \phi = \phi_1 * \phi_2 \text{ and } * \in \{\vee, \wedge, \to, \leftrightarrow\} \end{cases}$$

Notation

$[\phi_1/p_1, \dots, \phi_n/p_n]$ is substitution θ defined as $\theta(p) = \begin{cases} \phi_i & \text{if } p = p_i \\ p & \text{otherwise} \end{cases}$

Example

for $\phi = (p \vee q) \vee r$ and $\theta = [q/p, \neg p/q]$

$$((p \vee q) \vee r)\theta = (\theta(p) \vee \theta(q)) \vee \theta(r) = (q \vee (\neg p)) \vee r$$

Theorem (tautologies are closed under substitutions)

if $\models \phi$ then $\models \phi\theta$ for all substitutions θ (to be proved in next lecture)

Fact

$\models (\neg(\neg \phi)) \to \phi$ for all propositional formulas ϕ

Proof.

As $\models (\neg(\neg p)) \to p$, the claim follows by applying the substitution $[\phi/p]$. □

How to Prove General Proposition: Rigorous Version

Theorem (de Morgan's Law, generalized version)

$\models (\neg(\phi \wedge \psi)) \to ((\neg \phi) \vee (\neg \psi))$ *for all propositional formulas* ϕ *and* ψ

Proof.

The next truth table proves $\models (\neg(p \wedge q)) \to ((\neg p) \vee (\neg q))$:

p	q	$\neg$	$(p \wedge q)$	$\to$	$((\neg p)$	$\vee$	$(\neg q))$
T	T	F	T	T	F	F	F
T	F	T	F	T	F	T	T
F	T	T	F	T	T	T	F
F	F	T	F	T	T	T	T

The claim follows by applying the substitution $[\phi/p, \psi/q]$ □

How to Prove General Proposition: Simplified Version

Theorem (de Morgan's Law, generalized version)

$\models (\neg(\phi \wedge \psi)) \to ((\neg\phi) \vee (\neg\psi))$ *for all propositional formulas ϕ and ψ*

Proof.

The claim is verified by the truth table:

ϕ	ψ	$\neg$	$(\phi \wedge \psi)$	$\to$	$((\neg\phi)$	$\vee$	$(\neg\psi))$
T	T	F	T	T	F	F	F
T	F	T	F	T	F	T	T
F	T	T	F	T	T	T	F
F	F	T	F	T	T	T	T

□

Negation Normal Forms

Logical Equivalence as Equality

Fact

logical equivalence $\approx$ behaves like equality

Example

recall that following equivalences hold:

$\boxed{1}$ $\neg(\neg\phi) \approx \phi$

$\boxed{2}$ $\neg(\phi \wedge \psi) \approx (\neg\phi) \vee (\neg\psi)$

we have:

$$\begin{aligned} \neg(\neg p \wedge \neg q) &\approx (\neg(\neg p)) \vee (\neg(\neg q)) && \text{by } \boxed{2} \\ &\approx p \vee (\neg(\neg q)) && \text{by } \boxed{1} \\ &\approx p \vee q && \text{by } \boxed{1} \end{aligned}$$

Properties of Implication

Fact

- $\phi \leftrightarrow \psi \approx (\phi \to \psi) \wedge (\psi \to \phi)$
- $\phi \to \psi \approx (\neg\phi) \vee \psi$ material implication

Observation

$\to$ and $\leftrightarrow$ can be replaced by $\neg$, $\vee$, and $\wedge$

Exercise

simplify formulas

$\boxed{1}$ $\top \to \phi$ $\boxed{2}$ $\bot \to \phi$ $\boxed{3}$ $\phi \to \top$ $\boxed{4}$ $\phi \to \bot$ $\boxed{5}$ $\phi \to \phi$

Properties of Negation

Exercise

simplify formulas so that negation $\neg$ do not occur at root:

1. $\neg\top$
2. $\neg\bot$
3. $\neg(\neg\phi)$
4. $\neg(\phi \wedge \psi)$
5. $\neg(\phi \vee \psi)$

Observation

$\neg$ can be pushed into front of atoms

Negation Normal Forms

Definition

formula is negation normal form if

- it consists of $\top$, $\bot$, $\neg$, $\vee$, $\wedge$, and atoms, and
- $\neg$ only occurs at front of atom

Example

$(\neg p) \vee (\neg q)$ is negation normal form, but $(\neg(\neg p)) \vee (\neg q)$ is not

Theorem

for every propositional formula there exists equivalent negation normal form

Exercise: compute negation normal form of $\neg(p \leftrightarrow q)$

Conjunctive Normal Forms

Properties of Conjunction and Disjunction

Fact

- $(\phi \wedge \psi) \wedge \chi \;\approx\; \phi \wedge (\psi \wedge \chi)$ associativity
- $\phi \wedge \psi \;\approx\; \psi \wedge \phi$ commutativity
- $\phi \vee (\psi \wedge \chi) \;\approx\; (\phi \vee \psi) \wedge (\phi \vee \chi)$ distributivity

Notation

- $\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \cdots \wedge \phi_n$ stands for $((\phi_1 \wedge \phi_2) \wedge \phi_3) \wedge \cdots \wedge \phi_n$
- $\bigwedge\{\phi_1, \ldots, \phi_n\}$ and $\bigwedge_{i=1}^{n} \phi_i$ stand for $\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_n$

Exercise: what about case of disjunction $\vee$?

Conjunctive Normal Forms

Definition

conjunctive normal form (CNF) is defined as follows:

$$\ell ::= p \mid \neg p \qquad (p\text{: atom}) \qquad \text{literal}$$
$$C ::= \bot \mid \ell \vee \cdots \vee \ell \qquad \text{clause}$$
$$\text{CNF} ::= \top \mid C \wedge \cdots \wedge C \qquad \text{CNF}$$

Notation

we assume that $\neg$ is more tightly bounded than $\wedge, \vee, \rightarrow, \leftrightarrow$

for example, $\neg p \wedge q = (\neg p) \wedge q$

Example

$(\neg p \vee \neg q) \wedge (p \vee q)$ is CNF, but $(\neg p \wedge \neg q) \vee (p \wedge q)$ is not

Transformation into CNF

Theorem

for every formula there exists equivalent CNF

Proof.

compute negation normal form; then distribute $\vee$ over $\wedge$ □

Exercise

compute CNF equivalent to $\neg(\neg p \vee q) \vee (p \wedge \neg q \wedge r)$

Efficient Algorithm to Decide Validity of CNF

decide validity of ϕ and ψ, and justify algorithm:

$$\begin{aligned}\phi = {} & (\neg x \vee \boxed{\neg y} \vee z \vee \boxed{y}) \\ & \wedge (\boxed{x} \vee w \vee \boxed{\neg x} \vee \neg w) \\ & \wedge (v \vee \boxed{\neg w} \vee \boxed{w} \vee y) \\ & \wedge (\boxed{\neg z} \vee \neg x \vee \neg w \vee \boxed{z})\end{aligned}$$

valid

$$\begin{aligned}\psi = {} & (\neg x \vee \boxed{\neg y} \vee z \vee \boxed{y}) \\ & \wedge (\boxed{x} \vee w \vee \boxed{\neg x} \vee \neg w) \\ & \wedge (x \vee \neg y \vee \neg z \vee w) \\ & \wedge (\neg z \vee \neg x \vee \neg w \vee y)\end{aligned}$$

invalid

Theorem

next problem is decidable in polynomial time:

instance: CNF ϕ

question: is ϕ valid?

Supplementary Comments

- $\bigwedge_{i=1}^{n} \phi_i$ is written as $\bigwedge\!\!\!\bigwedge_{i=1}^{n} \phi_i$ in textbook
- usually we assume $\bigwedge \varnothing = \top$ and $\bigvee \varnothing = \bot$