I211E: Mathematical Logic

Nao Hirokawa

JAIST

Term 1-1, 2023

`https://www.jaist.ac.jp/~hirokawa/lectures/ml/`

Schedule

	propositional logic		predicate logic
4/13	syntax, semantics	5/11	syntax, semantics
4/18	normal forms	5/16	normal forms
4/20	examples	5/18	natural deduction I
4/25	natural deduction I	5/23	natural deduction II
4/27	natural deduction II	5/25	examples, properties
5/2	completeness	5/30	advanced topics
5/9	midterm exam	6/1	summary
		6/6	exam

Evaluation

midterm exam (40) + final exam (60)

Sudoku

5	3			7				2
6			1	9	5			8
	9	8					6	7
8				6				3
4			8		3			1
7				2				6
	6					2	8	4
			4	1	9	6	3	5
				8		1	7	9

let's solve Sudoku as satisfiability problem

Contents

Aim

to develop formalization skill

Content

1. satisfiability problems (SAT)
2. Sudoku
3. other applications

Satisfiability Problem

Satisfiability Problem I

Question

is there valuation v that satisfies following formula?

$$\bigwedge \left\{ \begin{array}{rr} x \vee & y \\ \neg x \vee & \neg y \end{array} \right\}$$

Answer

- solution: $\left\{ \begin{array}{l} x \mapsto \mathsf{T} \\ y \mapsto \mathsf{F} \end{array} \right\}$
- another solution: $\left\{ \begin{array}{l} x \mapsto \mathsf{F} \\ y \mapsto \mathsf{T} \end{array} \right\}$

Satisfiability Problem II

Question

is there valuation that satisfies following formula?

$$\bigwedge \left\{ \begin{array}{r} \neg x \vee \neg y \vee z \\ x \vee z \\ \neg z \\ y \vee z \end{array} \right\}$$

Answer

no. unsatisfiability is shown by truth table

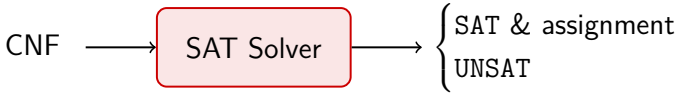
Satisfiability Problem III

Question

is there valuation that satisfies following formula?

$$\bigwedge \left\{ \begin{array}{rrr} & \neg x_3 \vee & \neg x_4 \\ \neg x_1 \vee & \neg x_3 \vee & x_4 \\ & x_3 \vee & x_4 \\ & x_3 \vee & \neg x_4 \\ x_1 \vee x_2 \vee & \neg x_3 & \end{array} \right\}$$

SAT Solvers



Example

```
p cnf 4 5
-3 -4  0
-1 -3  4  0
 3  4  0
 3 -4  0
 1  2 -3  0
```

CNF with 4 atoms and 5 clauses

$(\neg x_3 \vee \neg x_4)$

$\wedge\ (\neg x_1 \vee \neg x_3 \vee\ \ x_4)$

$\wedge\ (\ \ x_3 \vee\ \ x_4)$

$\wedge\ (\ \ x_3 \vee \neg x_4)$

$\wedge\ (\ \ x_1 \vee\ \ x_2 \vee \neg x_3)$

```
SAT
-1 2 3 -4 0
```

satisfiable assignment

$\{x_1 \mapsto \mathsf{F},\ x_2 \mapsto \mathsf{T},\ x_3 \mapsto \mathsf{T},\ x_4 \mapsto \mathsf{F}\}$

Sudoku

SAT Encoding

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

idea: x_{ijk} is true if cell at (i, j) is digit k

1. every cell (i, j) contains exactly one digit:

$$\text{one}(\{x_{ij1}, \ldots, x_{ij9}\})$$

2. every digit k occurs in every group of nine cells $(i_1, j_1), \ldots, (i_9, j_9)$:

$$\text{one}(\{x_{i_1 j_1 k}, \ldots, x_{i_9 j_9 k}\})$$

Can Enumeration Find Solution?

- 12750 clauses over 729 variables
- $2824013958708217496949108842204627863351353911851577524683401930862693830361198499905873920995229996970897865498283996578123296865878390947626553088486946106430796091482716120572632072492703527723757359478834530365734912$ variable assignments

fortunately efficient algorithm (DPLL) exists!

Cardinality Constraints

Idea

x_{ijd} is true $\iff$ cell at (i, j) contains digit d

Exercise

Let $A = \{x, y, z\}$. Describe following statements in propositional logic:

- at least one atom in A is true
- at most one atom in A is true

General Forms

$$\text{at-least-one}(A) = \bigvee A$$
$$\text{at-most-one}(A) = \bigwedge\{\neg x \vee \neg y \mid x, y \in A \text{ and } x \neq y\}$$
$$\text{one}(A) = \text{at-least-one}(A) \wedge \text{at-most-one}(A)$$

Propositional Encoding of Sudoku

$$\begin{aligned}
\phi_{\text{Sudoku}} = & \bigwedge_{i=1}^{9}\bigwedge_{j=1}^{9} \text{one}(\{x_{ij1}, \ldots, x_{ij9}\}) \\
& \wedge \bigwedge_{i=1}^{9}\bigwedge_{k=1}^{9} \text{one}(\{x_{i1k}, \ldots, x_{i9k}\}) \\
& \wedge \bigwedge_{j=1}^{9}\bigwedge_{k=1}^{9} \text{one}(\{x_{1jk}, \ldots, x_{9jk}\}) \\
& \wedge \bigwedge_{i=0}^{2}\bigwedge_{j=0}^{2}\bigwedge_{k=1}^{9} \text{one}\left(\left\{\begin{array}{lll} x_{3i+1,3j+1,k}, & x_{3i+1,3j+2,k}, & x_{3i+1,3j+3,k}, \\ x_{3i+2,3j+1,k}, & x_{3i+2,3j+2,k}, & x_{3i+2,3j+3,k}, \\ x_{3i+3,3j+1,k}, & x_{3i+3,3j+2,k}, & x_{3i+3,3j+3,k} \end{array}\right\}\right)
\end{aligned}$$

How To Solve Sudoku?

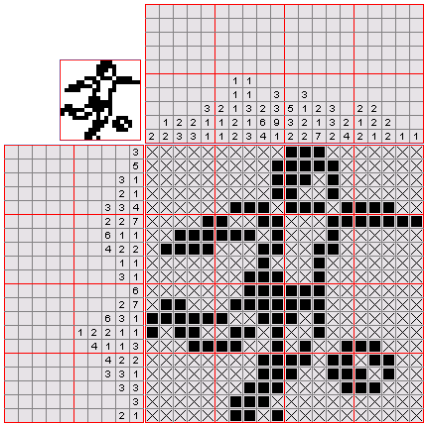
5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

SAT solver can find valuation v such that

$$[\![\phi_{\text{Sudoku}} \wedge x_{115} \wedge x_{123} \wedge \cdots \wedge x_{999})]\!]_v = \mathsf{T}$$

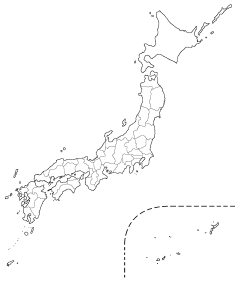
Other Applications

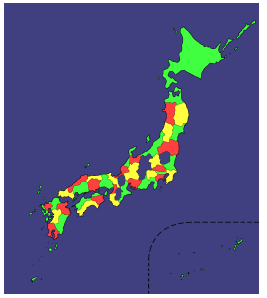
Nonogram



(C) Juraj Simlovic, CC BY-SA 3.0

Map Coloring (Noto Coloring is Homework!)





taken from `https://commons.wikimedia.org/wiki/File:Japan_template_large.png`

Software Verification

failure of Ariane 5 (C) ESA