Homework 8

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1)

(1) $\forall x \forall y\ (f(x,y) \doteq f(y,x))$

Let $\alpha : V \to A$ and $a, b \in A$. We have :

$[[f(x,y)]]_{A,\alpha[a/x, b/y]} = a+b \qquad [[f(y,x)]]_{A,\alpha[a/x,b/y]} = b+a$

so $A, \alpha[a/x, b/y] \models (f(x,y) \doteq f(y,x)) \ \forall a, b \in A$

$\Leftrightarrow A, \alpha \models \forall x \forall y (f(x,y) \doteq f(y,x))$

$\Leftrightarrow A \models \forall x \forall y (f(x,y) \doteq f(y,x))$

(3) $\forall x \forall y \forall z (f(x, g(y,z)) \doteq f(g(x,y), g(x,z))$

Let $\alpha : V \to A$ and arbitrary $a, b, c \in A$:

$[[f(x, g(y,z))]]_{A,\alpha[a/x, b/y, c/z]} = a + bc$;

$[[f(g(x,y), g(x,z))]]_{A,\alpha[a/x, b/y, c/z]} = ab + ac$

so $A, \alpha[a/x, b/y, c/z] \not\models (f(x, g(y,z)) \doteq f(g(x,y), g(x,z)) \ \forall a,b,c \in A \Leftrightarrow A, \alpha \not\models \phi \Leftrightarrow A \not\models \phi$

(5) $\exists x \forall y (f(x,y) \doteq y)$

Let $\alpha : V \to A$ and $a, b \in A$

$[[f(x,y)]]_{A,\alpha[b/x, a/y]} = a+b$; $[[y]]_{A,\alpha[b/x, a/y]} = a$

so $A, \alpha[0/x, a/y] \models (f(x,y) \doteq y) \ \forall a \in A$ and $b = 0$

$\Leftrightarrow A, \alpha \models \phi \Leftrightarrow A \models \phi$

(7) $\forall x (\forall y (f(x,y) \doteq y) \to x \doteq 0)$

Let $\alpha : V \to A$ and $a, b \in A$

$[[f(x,y)]]_{A,\alpha[b/x, a/y]} = a+b$; $[[y]]_{A,\alpha[b/x,a/y]} = a$; $[[x]]_{A,\alpha[b/x,a/y]} = b$.

+ If $b = 0$, we have $[[x]]_{A,\alpha[0/x, a/y]} = 0$

$\Leftrightarrow A, \alpha[b/x, a/y] \models ((f(x,y) \doteq y) \to x \doteq 0) \ \forall a \in A$

+ If $b \neq 0$, we have $A, \alpha[b/x, a/y] \models \neg (f(x,y) \doteq y)$

$\Leftrightarrow A, \alpha[b/x, a/y] \models ((f(x,y) \doteq y) \to x \doteq 0) \ \forall a \in A$

At either case, we have $A, \alpha[b/x, a/y] \models ((f(x,y) \doteq y) \to x \doteq 0) \ \forall a \in A$

$\Leftrightarrow A, \alpha[b/x, a/y] \models ((f(x,y) \doteq y) \to x \doteq 0) \ \forall a, b \in A$

$\Leftrightarrow A, \alpha \models \phi \Leftrightarrow A \models \phi$

(9) $\forall x \ \exists y \ (g(x,y) \doteq 1)$

Let $\alpha : V \to A$ and $a, b \in A$

$[[g(x,y)]]_{A,\alpha[a/x, b/y]} = a \cdot b = 1 \Leftrightarrow b = \frac{1}{a}$ only valid if $a, b \in \mathbb{Q}$ and $a \neq 0$. $\mathbb{N} \not\models \phi$, $\mathbb{Z} \not\models \phi$

$\Leftrightarrow \mathbb{Q}, \alpha[a/x, b/y] \models (g(x,y) \doteq 1)$ if $a \neq 0$ and $b = \frac{1}{a}$

$\Leftrightarrow \mathbb{Q}, \alpha \not\models \phi \Leftrightarrow \mathbb{Q} \not\models \phi$

(10) $\forall x (\neg (x \doteq 0) \to \exists y (g(x,y) \doteq 1))$

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Let $\alpha : V \to A$ and $a, b \in A$

(10) $\forall x\, (\neg (x \doteq 0) \to \exists y\, (g(x,y) \doteq 1))$ 7

Let $\alpha : V \to A$ and $a, b \in A$

$[[x]]_{A,\alpha[a|x,b|y]} = a$; $[[g(x,y)]]_{A,\alpha[a|x,b|y]} = a \cdot b = 1 \Leftrightarrow b = \frac{1}{a}$ valid if $\underset{a \neq 0}{a,b \in \mathbb{Q}}$. $\mathbb{N} \not\models \emptyset$, $\mathbb{Z} \not\models \emptyset$

+ If $a \neq 0$: $\mathbb{Q}, \alpha[a|x, b|y] \models (g(x,y) \doteq 1)$ with $b = \frac{1}{a}$

+ If $a = 0$: $\mathbb{Q}, \alpha[a|x, b|y] \models \neg(x \doteq 0)$ $\forall b$

so we have: $\mathbb{Q}, \alpha[a|x, b|y] \models (\neg(x \doteq 0) \to (g(x,y) \doteq 1))$ $\forall a \in \mathbb{Q}$ and $b = \frac{1}{a}$

$\Leftrightarrow \mathbb{Q}, \alpha \models \forall x (\neg(x \doteq 0) \to \exists y\, (g(x,y) \doteq 1))$

$\Leftrightarrow \mathbb{Q} \models \emptyset$

(11) $\forall x\, \neg P(x,x)$

Consider $x = a$ for $a \in A$. Because $(a,a) \notin P$ $\forall a$

$\Leftrightarrow A \not\models \forall x\, P(x,x) \Leftrightarrow A \models \forall x\, \neg P(x,x)$

2)

(1) $K(a,b) \wedge \neg K(b,a)$. $A = \{\mathbb{N}, \bar{K}, \bar{a}, \bar{b}\}$: $\bar{K} = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m > n\}$, $\bar{a} = 1$, $\bar{b} = 0$

(2) $\forall x\, K(a,x)$

(3) $\forall x\, \neg K(x,b)$

(4) $\forall x\, K(x,a) \to K(x,b)$

(5) $\exists x\, K(x,a) \wedge \neg K(x,b)$

(6) $\forall x\, K(x,x)$. A: $\bar{K} = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m = n\}$, B: $\bar{K} = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m > n\}$

(7) $\forall x\, \neg K(x,x)$

(8) $\forall x \forall y\, K(x,y) \to K(y,x)$

(9) $\forall x \exists y\, K(x,y) \wedge \neg K(y,x)$

(10) $\neg(x \doteq a) \to K(x,b)$

(11) $\forall x \forall y\, \neg(x \doteq y) \to (\neg K(x,b) \vee \neg K(y,b))$

(12) $\forall x \forall y \forall z\, (K(x,z) \wedge K(z,y)) \to K(x,y))$

3)

(1) $\neg \forall x\, P(x) \leftrightarrow \exists x\, \neg P(x)$ valid

(2) $\neg \exists P(x) \leftrightarrow \forall x\, \neg P(x)$ valid

(3) $(\forall x\, P(x) \wedge \forall x\, Q(x)) \leftrightarrow \forall x\, (P(x) \wedge Q(x))$ valid

(4) $(\forall x\, P(x) \vee \forall x\, Q(x)) \leftrightarrow \forall x\, (P(x) \vee Q(x))$ not valid. $P = \{0\}$, $Q = \{1\}$.

(5) $(\forall x\, P(x) \to \forall x\, Q(x)) \leftrightarrow \forall x\, (P(x) \to Q(x))$ not valid. $P = \{0\}$, $Q = \emptyset$

(6) $(\exists x\, P(x) \wedge \exists x\, Q(x)) \leftrightarrow \exists x\, (P(x) \wedge Q(x))$ not valid

$P = \{x \in \mathbb{Z} \mid x \geq 0\}$, $Q = \{x \in \mathbb{Z} \mid x < 0\}$

(7) $(\exists x\, P(x) \vee \exists x\, Q(x)) \leftrightarrow \exists x\, (P(x) \vee Q(x))$ valid

(8) $(\exists x\, P(x) \to \exists x\, Q(x)) \leftrightarrow \exists x\, (P(x) \to Q(x))$ not valid. $P = \{0\}$, $Q = \emptyset$

$(\forall x\, \neg P(x) \vee \exists x\, Q(x)) \leftrightarrow \exists x\, (\neg P(x) \vee Q(x))$