# I211E: Mathematical Logic

Nao Hirokawa

JAIST

Term 1-1, 2023

https://www.jaist.ac.jp/~hirokawa/lectures/ml/



### Schedule

| | propositional logic | | predicate logic |
|---|---|---|---|
| 4/13 | syntax, semantics | 5/11 | syntax, semantics |
| 4/18 | normal forms | 5/16 | normal forms |
| 4/20 | examples | 5/18 | natural deduction I |
| 4/25 | natural deduction I | 5/23 | natural deduction II |
| 4/27 | natural deduction II | 5/25 | examples, properties |
| 5/2 | completeness | 5/30 | advanced topics |
| 5/9 | midterm exam | 6/1 | summary |
| | | 6/6 | exam |

### Evaluation

midterm exam (40) + final exam (60)



## Contents

### Aim

to understand induction and notion of proof

### Contents

1. structural induction
2. logical equivalence as equality, proved
3. natural deduction I ($\rightarrow$, $\wedge$, $\vee$)



## Structural Induction

## Mathematical Induction

**Theorem (mathematical induction)**

*$P(n)$ holds for all natural numbers $n \in \mathbb{N}$ if*

- *$P(0)$, and*
- *for every $n \in \mathbb{N}$ if $P(n)$ then $P(n+1)$*

**Exercise**

show that equation

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

holds for all natural numbers $n \in \mathbb{N}$



## Strong Induction

**Theorem (strong induction)**

*$P(n)$ for all $n \in \mathbb{N}$ if next condition holds:*

*for every $n \in \mathbb{N}$ if $P(n')$ for all $n' < n$ then $P(n)$*

equivalently

**Theorem**

*$P(n)$ for all $n \in \mathbb{N}$ if next condition holds:*

*for every $n \in \mathbb{N}$ if $P(0)$, $P(1)$, $\ldots$, $P(n-1)$ then $P(n)$*



**Proposition**

$F_n + 1 - n \geqslant 0$ for all $n \in \mathbb{N}$, where: $F_n = \begin{cases} n & \text{if } n \leqslant 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geqslant 2 \end{cases}$

**Proof.**

We show the claim by strong induction on $n$.

- If $n \leqslant 1$ then by the definition of $F_n$ we obtain:

$$F_n + 1 - n = n + 1 - n = 1 \geqslant 0$$

- If $n \geqslant 2$ then we obtain:

$$\begin{aligned} F_n + 1 - n &= F_{n-1} + F_{n-2} + 1 - n && \text{by definition of } F_n \\ &\geqslant (n-1) + (n-2) + 1 - n && \text{by the I.H.} \\ &= n - 2 \geqslant 0 && \text{by } n \geqslant 2 \end{aligned}$$

$\square$



## Subformulas

**Definition**

set Sub($\phi$) of all subformulas of $\phi$ is inductively defined as follows:

$$\mathsf{Sub}(\phi) = \begin{cases} \{\phi\} & \text{if } \phi \text{ is atom or } \phi \in \{\top, \bot\} \\ \{\phi\} \cup \mathsf{Sub}(\phi_1) & \text{if } \phi = \neg\phi_1 \\ \{\phi\} \cup \mathsf{Sub}(\phi_1) \cup \mathsf{Sub}(\phi_2) & \text{if } \phi = \phi_1 * \phi_2 \text{ and } * \in \{\wedge, \vee, \rightarrow, \leftrightarrow\} \end{cases}$$

$\psi \in \mathsf{Sub}(\phi)$ is proper subformula of $\phi$ if $\phi \neq \psi$

**Example**

- $\mathsf{Sub}(p \vee \neg q) = \{p \vee \neg q,\ p,\ \neg q,\ q\}$
- $p$, $\neg q$, and $q$ are proper subformulas of $p \vee \neg q$, but $p \vee \neg q$ is not

## Structural Induction (Strong Version)

**Theorem**

*$P(\phi)$ for all formulas $\phi$ if next condition holds:*

*for every formula $\phi$ if $P(\psi)$ for all proper subformulas $\psi$ then $P(\phi)$*



**Proposition**

Let $\phi$ be formula containing no atom. We have $\phi \approx \top$ or $\phi \approx \bot$.

**Proof.**

We show the claim by structural induction on $\phi$.

- If $\phi$ is an atom then this contradicts the assumption.
- If $\phi = \top$ or $\phi = \bot$ then $\phi \approx \top$ or $\phi \approx \bot$, respectively.
- If $\phi = \neg\phi_1$ then $\phi_1$ contains no atom. By the I.H. $\phi_1 \approx \top$ or $\phi_1 \approx \bot$. Hence, $\phi \approx \neg\top = \bot$ or $\phi \approx \neg\bot = \top$ follows, respectively.
- If $\phi = \phi_1 \wedge \phi_2$ then $\phi_1$ and $\phi_2$ contain no atom. For each $i \in \{1, 2\}$ the I.H. yields $\phi_i \approx \top$ or $\phi_i \approx \bot$. If $\phi_1 \approx \top$ and $\phi_2 \approx \top$ then $\phi \approx \top$. Otherwise, $\phi \approx \bot$. In either case, the claim holds.
- The remaining cases are shown in a way similar to the last case. □



# Proofs of Logical Equvalence as Equality



## Tautologies are Closed under Substitutions

**Lemma**

*Let $\theta$ be substitution and $u$ valuations. We have $[\![\phi\theta]\!]_u = [\![\phi]\!]_v$ for valuation $v$ defined as $v(p) = [\![p\theta]\!]_u$ for all atoms $p$.*

**Proof.**

We show the claim by induction on $\phi$.

1. If $\phi$ is an atom, $[\![\phi\theta]\!]_u = v(\phi) = [\![\phi]\!]_v$.
2. If $\phi \in \{\top, \bot\}$ then $\phi\theta = \phi$. Thus, the claim holds.
3. If $\phi = \neg\phi_1$ then by the I.H. $[\![\phi_1\theta]\!]_u = [\![\phi_1]\!]_v$. Hence, $[\![\phi\theta]\!]_u = [\![\phi]\!]_v$.
4. If $\phi = \phi_1 * \phi_2$ and $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ then by the I.H. $[\![\phi_1\theta]\!]_u = [\![\phi_1]\!]_v$ and $[\![\phi_2\theta]\!]_u = [\![\phi_2]\!]_v$. Hence, $[\![\phi\theta]\!]_u = [\![\phi]\!]_v$. □

**Theorem**

*if $\models \phi$ then $\models \phi\theta$ for all substitutions $\theta$.*

**Proof.**

Suppose $\models \phi$. Let $u$ be an arbitrary valuation. We define the valuation $v$ as $v(p) = [\![p\theta]\!]_u$ for all atoms $p$. By the last lemma and $\models \phi$ we obtain:

$$[\![\phi\theta]\!]_u = [\![\phi]\!]_v = \mathsf{T}$$

Therefore, $\models \phi\theta$ follows. □



## Logical Equivalence $\approx$ is Equality

**Theorem**

1. $\phi \approx \phi$ *reflexivity*
2. $\phi \approx \psi \implies \psi \approx \phi$ *symmetry*
3. $\phi \approx \psi \approx \chi \implies \phi \approx \chi$ *transitivity*
4. $\phi \approx \psi \implies \phi\theta \approx \psi\theta$ *closure under substitutions*
5. $\theta \approx \theta' \implies \phi\theta \approx \phi\theta'$ *closure under contexts*

*where $\theta \approx \theta'$ denotes $p\theta \approx p\theta'$ for all atoms $p$*

**Proof Sketch.**

2. Straightforward from $\models (p \leftrightarrow q) \to (q \leftrightarrow p)$.
4. Suppose $\models \phi \leftrightarrow \psi$. We have $\models (\phi \leftrightarrow \psi)\theta$, which leads to $\models \phi\theta \leftrightarrow \psi\theta$.
5. One can show that $\theta \approx \theta'$ implies $\models \phi\theta \to \psi\theta$. □



# Natural Deduction I



**Informal Definition**

- inference rules are given by following forms:

$$\frac{\phi}{\psi} \qquad \frac{\begin{matrix}[\chi] \\ \vdots \\ \phi\end{matrix}}{\psi} \qquad \frac{\phi_1 \quad \phi_2}{\psi} \qquad \frac{\phi_1 \quad \begin{matrix}[\chi_2] \\ \vdots \\ \phi_2\end{matrix} \quad \begin{matrix}[\chi_3] \\ \vdots \\ \phi_3\end{matrix}}{\psi} \qquad \begin{matrix}\text{hypotheses} \\ \vdots \\ \text{premises} \\ \text{conclusion}\end{matrix}$$

- proof of $\phi$ is derivation tree whose root is $\phi$ and leaves are hypotheses, like:

$$\frac{[\phi_1] \qquad \dfrac{[\phi_4] \quad [\phi_5]}{\phi_2} \qquad \dfrac{[\phi_6]}{\phi_3}}{\phi}$$

- natural deduction consists of about $10$ inference rules (to be explained)
- $\phi$ is provable ($\vdash \phi$) if there exists proof of $\phi$ wrt natural deduction

**Definition (introduction and elimination rules for $\to$)**

$$\dfrac{\begin{matrix}[\phi]^i\\ \vdots\\ \psi\end{matrix}}{\phi\to\psi}\,{\to}\mathrm{I}_i \qquad\qquad \dfrac{\phi \quad \phi\to\psi}{\psi}\,{\to}\mathrm{E}$$

**Proposition**

$\vdash p \to ((p\to q)\to q)$

**Proof.**

$$\dfrac{\dfrac{\dfrac{[p]^1 \quad [p\to q]^2}{q}\,{\to}\mathrm{E}}{(p\to q)\to q}\,{\to}\mathrm{I}_2}{p\to((p\to q)\to q)}\,{\to}\mathrm{I}_1$$

□

**Proposition**

When $p$ holds, if $p$ implies $q$ then $q$ holds.

**Proof.**

We show $p\to((p\to q)\to q)$.
Assume $p$. It is enough to show $(p\to q)\to q$.
Assume $p\to q$. It is enough to show $q$.
Since $p$ and $p\to q$, we obtain $q$. □



**Definition (introduction and elimination rules for $\wedge$)**

$$\dfrac{\phi \quad \psi}{\phi\wedge\psi}\,\wedge\mathrm{I} \qquad \dfrac{\phi\wedge\psi}{\phi}\,\wedge\mathrm{E} \qquad \dfrac{\phi\wedge\psi}{\psi}\,\wedge\mathrm{E}$$

**Proposition**

$\vdash (p\wedge(p\to q))\to q$

**Proof.**

$$\dfrac{\dfrac{\dfrac{[p\wedge(p\to q)]^1}{p}\,\wedge\mathrm{E} \quad \dfrac{[p\wedge(p\to q)]^1}{p\to q}\,\wedge\mathrm{E}}{q}\,{\to}\mathrm{E}}{(p\wedge(p\to q))\to q}\,{\to}\mathrm{I}_1$$

□

**Proposition**

If $p$ and $p$ implies $q$ then $q$.

**Proof.**

We show $(p\wedge(p\to q))\to q$.
Assume $p\wedge(p\to q)$. It is enough to show $q$.
Since $p\wedge(p\to q)$, we have $p$.
Since $p\wedge(p\to q)$, we have $p\to q$.
As $p$ and $p\to q$, we obtain $q$. □



**Definition (introduction and elimination rules for $\vee$)**

$$\dfrac{\phi}{\phi\vee\psi}\,\vee\mathrm{I} \qquad \dfrac{\psi}{\phi\vee\psi}\,\vee\mathrm{I} \qquad \dfrac{\phi\vee\psi \quad \begin{matrix}[\phi]^i\\ \vdots\\ \chi\end{matrix} \quad \begin{matrix}[\psi]^i\\ \vdots\\ \chi\end{matrix}}{\chi}\,\vee\mathrm{E}_i$$

**Proposition**

$(p\vee(p\wedge q))\to p$

**Proof.**

$$\dfrac{\dfrac{[p\vee(p\wedge q)]^1 \quad [p]^2 \quad \dfrac{[p\wedge q]^2}{p}\,\wedge\mathrm{E}}{p}\,\vee\mathrm{E}_2}{(p\vee(p\wedge q))\to p}\,{\to}\mathrm{I}_1$$

□

**Proposition**

If $p$ or $p\wedge q$ then $p$.

**Proof.**

We show $(p\vee(p\wedge q))\to p$.
Assume $p\vee(p\wedge q)$. It is enough to show $p$.
To show $p$, we distinguish two cases.
If $p$ then $p$ holds trivially.
If $p\wedge q$ then $p$ follows.
In any case, $p$ holds. □



## Supplementary Comments

- Exercise: Prove $\vdash p\to(p\vee(p\wedge q))$.
- inference rule is also called derivation rule and proof rule
- derivation tree is also called proof tree and proof figure
- introduction/elimination rules for $\neg$ and $\perp$ will be explained in next lecture
- $\top$ and $\phi\leftrightarrow\psi$ are treated as $\neg\perp$ and $(\phi\to\psi)\wedge(\psi\to\phi)$, respectively
- hypotheses are formalized in form of $\Gamma\vdash\phi$

$$\dfrac{\phi\vee\psi \quad \begin{matrix}[\phi]^i\\ \vdots\\ \chi\end{matrix} \quad \begin{matrix}[\phi]^i\\ \vdots\\ \chi\end{matrix}}{\chi}\,{\to}\mathrm{E}_i \qquad \dfrac{\Gamma\vdash\phi\vee\psi \quad \Gamma\cup\{\phi\}\vdash\chi \quad \Gamma\cup\{\psi\}\vdash\chi}{\Gamma\vdash\chi}\,{\to}\mathrm{E}$$

- $\varnothing\vdash\phi$ is abbreviated to $\vdash\phi$; such $\phi$ is called theorem