I211E: Mathematical Logic

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Schedule

	propositional logic		predicate logic
4/13	syntax, semantics	5/11	syntax, semantics
4/18	normal forms	5/16	normal forms
4/20	examples	5/18	natural deduction I
4/25	natural deduction I	5/23	natural deduction II
4/27	natural deduction II	5/25	examples, properties
5/2	completeness	5/30	advanced topics
5/9	midterm exam	6/1	summary
		6/6	exam

Evaluation

midterm exam (40) + final exam (60)

Soundness and Completeness

Definition

- $v \vDash \phi$ if $[\![\phi]\!]_v = \mathsf{T}$
- $v \vDash \Gamma$ if $[\![\phi]\!]_v = \mathsf{T}$ for all $\phi \in \Gamma$
- $\Gamma \vDash \phi$ if for every valuation v we have: $v \vDash \Gamma \implies v \vDash \phi$

Example

$\{p, p \to q\} \vDash p \wedge q$ and $\{p, \neg p, p \to q\} \vDash \bot$

Definition (validity of inference rules)

$\dfrac{\Gamma_1 \vdash \phi_1 \quad \cdots \quad \Gamma_n \vdash \phi_n}{\Delta \vdash \psi}$ is valid if we have: $\Gamma_1 \vDash \phi_1, \ldots, \Gamma_n \vDash \phi_n \implies \Delta \vDash \psi$

Lemma

$$\frac{\Gamma \cup \{\phi\} \vdash \psi}{\Gamma \vdash \phi \to \psi} \to\mathrm{I}$$ *is valid*

Proof.

Assume $\Gamma \cup \{\phi\} \vDash \psi$. We show $\Gamma \vDash \phi \to \psi$. Let v be a valuation. Assume $v \vDash \Gamma$. It is enough to show $v \vDash \phi \to \psi$. As $v \vDash \phi$ or $v \nvDash \phi$, we distinguish two cases.

- If $v \vDash \phi$ then $v \vDash \Gamma \cup \{\phi\}$. By assumption $v \vDash \psi$. So $v \vDash \phi \to \psi$.
- If $v \nvDash \phi$ then $v \vDash \phi \to \psi$ is immediate.

In either case, the claim holds. □

Exercise

prove that $\vee\mathrm{E}$ is valid.

Lemma

all inference rules for natural deduction are valid

Soundness Theorem

$\Gamma \vdash \phi \implies \Gamma \vDash \phi$

Proof.

We show the claim by induction on proof tree of $\Gamma \vdash \phi$.

- If $\dfrac{\phi \in \Gamma}{\Gamma \vdash \phi}$ then $\Gamma \vDash \phi$ because $v \vDash \phi$ whenever $v \vDash \Gamma$.
- If $\dfrac{\Gamma \cup \{\phi\} \vdash \psi}{\Gamma \vdash \phi \to \psi} \to\mathrm{I}$ then $\Gamma \cup \{\phi\} \vDash \psi$ by the I.H. Lemma yields $\Gamma \vDash \phi \to \psi$.
- The other cases are also shown in the same way. □

Completeness of Natural Deduction

Completeness Theorem

$\Gamma \vDash \phi \implies \Gamma \vdash \phi$

Proof.

Let $\{\phi_0, \phi_1, \ldots\}$ be the set of all propositional formulas. Define Γ^* as follows:

$$\Gamma_0 = \Gamma \qquad \Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\phi_i\} & \text{if } \Gamma_i \cup \{\phi_i\} \nvdash \bot \\ \Gamma_i & \text{if } \Gamma_i \cup \{\phi_i\} \vdash \bot \end{cases} \qquad \Gamma^* = \bigcup_{i=0}^{\infty} \Gamma_i$$

Define v by $v \vDash p \iff p \in \Gamma^*$. Structural induction on ψ shows $v \vDash \psi \iff \psi \in \Gamma^*$. The contraposition of the claim is shown as follows:

$$\begin{aligned} \Gamma \nvdash \phi &\implies \Gamma \cup \{\neg\phi\} \nvdash \bot \implies \Gamma^* \cup \{\neg\phi\} \nvdash \bot \\ &\implies v \vDash \Gamma^* \cup \{\neg\phi\} \implies \Gamma^* \nvDash \phi \end{aligned}$$ □

Predicate Logic

Terms

- let $\mathcal{V}$ be set of variables $x, y, z, \ldots$
- let $\mathcal{F}$ be set of function symbols $f^{(n)}$ called signature

Definition

- terms over signature $\mathcal{F}$ are given by BNF: $t ::= x \mid f(t, \ldots, t)$
- $f()$ is abbreviated to f

Example

let $\mathcal{V} = \{x, y, \ldots\}$ and $\mathcal{F} = \{\mathsf{f}^{(2)}, \mathsf{s}^{(1)}, \mathsf{0}^{(0)}\}$

- $\mathsf{0}$, $\mathsf{s}(x)$, $\mathsf{f}(\mathsf{s}(x), y)$, and $\mathsf{s}(\mathsf{f}(x, y))$ are terms
- $\mathsf{0}(x)$, $x(\mathsf{0})$, and $\mathsf{f}(x, x, x)$ are not

First-Order Predicate Logic: Syntax

- let $\mathcal{V}$ be set of variables
- let $\mathcal{F}$ be set of function symbols $f^{(n)}$, where n is arity
- let $\mathcal{P}$ be set of predicate symbols $P^{(n)}$, where n is arity

Definition (first-order formulas)

first-order formulas over $\mathcal{P}$ and $\mathcal{F}$ are given by BNF:

$$\begin{array}{lll}
\phi ::= & \top \mid \bot \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \phi \leftrightarrow \phi & \text{logical connectives} \\
& \mid t \doteq t & \text{equality} \\
& \mid P(t_1, \ldots, t_n) & \text{predicate} \\
& \mid \forall x \phi & \text{universal quantifier} \\
& \mid \exists x \phi & \text{existential quantifier}
\end{array}$$

where $P^{(n)} \in \mathcal{P}$, $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, and $x \in \mathcal{V}$

Example

let $\mathcal{V} = \{x, y, \ldots\}$, $\mathcal{F} = \{\mathsf{s}^{(1)}, \mathsf{0}^{(0)}\}$ and $\mathcal{P} = \{\mathsf{P}^{(1)}, \mathsf{Q}^{(2)}, >^{(2)}\}$

- $\mathsf{P}(\mathsf{s}(x))$ and $\forall x(\mathsf{P}(x) \to \exists y \mathsf{Q}(x, y))$ are formulas
- $\mathsf{s}(x)$, $\mathsf{s}(\mathsf{P}(x))$, $\forall x$, $\forall \mathsf{s}(x)(\mathsf{P}(x))$, and $\exists \mathsf{Q}(x, y)$ are not

Example

- $\forall x(x > \mathsf{0} \to x^2 > \mathsf{0})$ — $x^2 > 0$ holds for all $x > 0$
- $\forall x(x > \mathsf{0} \vee x \doteq \mathsf{0})$ — $x \geqslant 0$ holds for all x
- $\forall x \exists y(x > y)$ — for every x there exists y such that $x > y$
- $\exists x \forall y(x > y)$ — there exists x such that $x > y$ for all y
- $\forall x \forall y(x > y \to \exists z(x > z \wedge z > y))$ — dense set like $\mathbb{Q}$ and $\mathbb{R}$

Exercise: Write down formula for mathematical/strong induction.

Bound and free variables

quantifiers bind variables in their scope:

$$\forall x \overbrace{(\forall y \overbrace{\mathsf{P}(y)}^{\text{scope for } y} \rightarrow \mathsf{Q}(x, y))}^{\text{scope for } x}$$

- such quantified variables are called bound variables
- variables not bound by quantifiers are free variables

Exercise

mark scopes. which variable occurrences are free/bound?

1. $(\forall x(\mathsf{P}(x, y) \rightarrow \exists y \mathsf{Q}(y, x, z))) \vee \mathsf{R}(y, z)$
2. $(\mathsf{P}(x) \rightarrow \exists y(\mathsf{Q}(y) \vee ((\forall x \mathsf{P}(x)) \rightarrow \exists x \mathsf{P}(x))))$
3. $\forall x \exists y \forall z(\mathsf{P}(x) \rightarrow (\mathsf{Q}(x, y) \vee \mathsf{Q}(y, z)))$

Sentences

Exercise

define set $\mathsf{FV}(\phi)$ of all free variables

$$\mathsf{FV}(\phi) = \begin{cases} \varnothing & \text{if } \phi \in \{\top, \bot\} \\ \boxed{?} & \text{if } \phi = \neg\phi_1 \\ \boxed{?} & \text{if } \phi = \phi_1 * \phi_2 \text{ with } * \in \{\wedge, \vee, \rightarrow, \leftrightarrow\} \\ \boxed{?} & \text{if } \phi \text{ is } s \doteq t \\ \boxed{?} & \text{if } \phi = \forall x \phi_1 \text{ or } \phi = \exists x \phi_1 \end{cases} \qquad \mathsf{FV}(t) = \begin{cases} \boxed{?} & \text{if } t \dots \\ \boxed{?} & \text{if } t \dots \end{cases}$$

Definition

formula ϕ is sentence if $\mathsf{FV}(\phi) = \varnothing$

First-Order Logic: Semantics

Definition

structure $\mathcal{A}$ is tuple $(U, \{\bar{P}\}_{P \in \mathcal{P}}, \{\bar{f}\}_{f \in \mathcal{F}})$, where

- U is non-empty set — universe
- $\bar{P}^{(n)} \subseteq U^n$ — interpretation of predicate symbol
- $\bar{f}^{(n)} : U^n \rightarrow U$ — interpretation of function symbol

$(U, \{\bar{P}_1, \dots, \bar{P}_m\}, \{\bar{f}_1, \dots, \bar{f}_n\})$ is abbreviated to $(U, \bar{P}_1, \dots, \bar{P}_m, \bar{f}_1, \dots, \bar{f}_n)$

Definition (valuation)

given assignment $\alpha : \mathcal{V} \rightarrow U$, valuation $[\![t]\!]_{\mathcal{A},\alpha}$ is defined as follows:

$$[\![t]\!]_{\mathcal{A},\alpha} = \begin{cases} \alpha(x) & \text{if } t \text{ is variable } x \\ \bar{f}([\![t_1]\!]_{\mathcal{A},\alpha}, \dots, [\![t_n]\!]_{\mathcal{A},\alpha}) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

Example for Structures

consider structure $\mathcal{A} = (\mathbb{N}, \bar{\mathsf{P}}, \bar{\mathsf{f}}, \bar{\mathsf{s}}, \bar{0})$ with

$$\bar{0} = 0 \quad \bar{\mathsf{s}}(n) = n + 1 \quad \bar{\mathsf{f}}(m, n) = m + n \quad \bar{\mathsf{P}} = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m > n\}$$

assignment α with $\alpha(x) = 3$

$$[\![\mathsf{s}(\mathsf{f}(0, x))]\!]_{\mathcal{A},\alpha} = \bar{\mathsf{s}}([\![\mathsf{f}(0, x)]\!]_{\mathcal{A},\alpha}) = \bar{\mathsf{s}}(\bar{\mathsf{f}}(\bar{0}, \alpha(x))) = (0 + 3) + 1 = 4$$
$$[\![\mathsf{f}(x, 0)]\!]_{\mathcal{A},\alpha} = \bar{\mathsf{f}}(\alpha(x), \bar{0}) = 3$$

therefore, $(4, 3) \in \bar{\mathsf{P}}$

Definition ($\mathcal{A}$ is model of ϕ)

$\mathcal{A} \vDash \phi$ if $\mathcal{A}, \alpha \vDash \phi$ for all α, where:

$$
\begin{array}{lcl}
\mathcal{A}, \alpha \vDash \top & & \\
\mathcal{A}, \alpha \nvDash \bot & & \\
\mathcal{A}, \alpha \vDash \neg\phi & \iff & \mathcal{A}, \alpha \nvDash \phi \\
\mathcal{A}, \alpha \vDash \phi \wedge \psi & \iff & \mathcal{A}, \alpha \vDash \phi \text{ and } \mathcal{A}, \alpha \vDash \psi \\
 & \vdots & \\
\mathcal{A}, \alpha \vDash s \doteq t & \iff & [\![s]\!]_{\mathcal{A},\alpha} = [\![t]\!]_{\mathcal{A},\alpha} \\
\mathcal{A}, \alpha \vDash P(t_1, \ldots, t_n) & \iff & ([\![t_1]\!], \ldots, [\![t_n]\!]) \in \bar{P} \\
\mathcal{A}, \alpha \vDash \forall x \phi & \iff & \mathcal{A}, \alpha[a/x] \vDash \phi \text{ for all } a \in U \\
\mathcal{A}, \alpha \vDash \exists x \phi & \iff & \mathcal{A}, \alpha[a/x] \vDash \phi \text{ for some } a \in U
\end{array}
$$

with $(\alpha[a/x])(x) = a$ and $(\alpha[a/x])(y) = \alpha(y)$ for all $y \neq x$

Proposition

For the structure $\mathcal{A} = (\mathbb{N}, \bar{\mathsf{P}}, \bar{\mathsf{f}}, \bar{\mathsf{s}}, \bar{0})$ with

$$\bar{0} = 0 \quad \bar{\mathsf{s}}(n) = n + 1 \quad \bar{\mathsf{f}}(m, n) = m + n \quad \bar{\mathsf{P}} = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m > n\}$$

we have $\mathcal{A} \vDash \forall x\, \mathsf{P}(\mathsf{s}(\mathsf{f}(0, x)), \mathsf{f}(x, 0))$.

Proof.

Let $\alpha : \mathcal{V} \to \mathbb{N}$ be an assignment. Let a be an arbitrary element in $\mathbb{N}$. We have:

$$[\![\mathsf{s}(\mathsf{f}(0, x))]\!]_{\mathcal{A},\alpha[a/x]} = a + 1 \qquad [\![\mathsf{f}(x, 0)]\!]_{\mathcal{A},\alpha[a/x]} = a$$

So $(a + 1, a) \in \bar{\mathsf{P}}$. Thus, $\mathcal{A}, \alpha[a/x] \vDash \mathsf{P}(\mathsf{s}(\mathsf{f}(0, x)), \mathsf{f}(x, 0))$ for all $a \in \mathbb{N}$.
Therefore, $\mathcal{A}, \alpha \vDash \forall x\, \mathsf{P}(\mathsf{s}(\mathsf{f}(0, x)), \mathsf{f}(x, 0))$.
Hence, $\mathcal{A} \vDash \forall x\, \mathsf{P}(\mathsf{s}(\mathsf{f}(0, x)), \mathsf{f}(x, 0))$ follows. □

Proposition

For the structure $\mathcal{A} = (\mathbb{N}, \bar{\mathsf{P}}, \bar{\mathsf{f}}, \bar{\mathsf{s}}, \bar{0})$ with

$$\bar{0} = 0 \quad \bar{\mathsf{s}}(n) = n + 1 \quad \bar{\mathsf{f}}(m, n) = m + n \quad \bar{\mathsf{P}} = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m > n\}$$

we have $\mathcal{A} \vDash \forall x \exists y\, \mathsf{P}(y, x)$.

Proof.

Let $\alpha : \mathcal{V} \to \mathbb{N}$ be an assignment. Let a be an arbitrary element in $\mathbb{N}$.
Take $b = a + 1$. As $b > a$, we obtain $(b, a) \in \bar{\mathsf{P}}$. Thus, $\mathcal{A}, \alpha[a/x][b/y] \vDash \mathsf{P}(y, x)$.
So we have $\mathcal{A}, \alpha[a/x] \vDash \exists y\, \mathsf{P}(y, x)$. Therefore, $\mathcal{A}, \alpha \vDash \forall x \exists y\, \mathsf{P}(y, x)$.
Hence, $\mathcal{A} \vDash \forall x \exists y\, \mathsf{P}(y, x)$ follows. □

Proof (shorter version).

Given $x \in \mathbb{N}$, take $y = x + 1$. Then $\bar{\mathsf{P}}(y, x)$ holds. Hence, the claim follows. □

Proposition

For the structure $\mathcal{A} = (U, \bar{\mathsf{P}}, \bar{\mathsf{f}}, \bar{\mathsf{s}}, \bar{0})$ with $U = \mathbb{N}$ and

$$\bar{0} = 0 \quad \bar{\mathsf{s}}(n) = n + 1 \quad \bar{\mathsf{f}}(m, n) = m + n \quad \bar{\mathsf{P}} = \{(m, n) \in U \times U \mid m > n\}$$

we have $\mathcal{A} \nvDash \forall x \exists y\, \mathsf{P}(x, y)$.

Proof.

Consider $x = 0$. For any $y \in \mathbb{N}$ the predicate $\bar{\mathsf{P}}(x, y)$ does not hold because 0 is the smallest number in $\mathbb{N}$. □

Exercise

1. what about $\forall x \forall y\, \mathsf{P}(x, y)$, $\exists x \exists y\, \mathsf{P}(x, y)$, and $\exists x \forall y\, \mathsf{P}(x, y)$?
2. what if $U = \mathbb{Z}$?

Example for Equality $\doteq$

let $\mathcal{A} = (\mathbb{N}, \{\bar{\mathsf{E}}\}, \{\bar{+}, \bar{\mathsf{s}}, \bar{0}\})$ be structure with

$$\bar{0} = 0 \qquad \bar{\mathsf{s}}(n) = n + 1 \qquad m \mathbin{\bar{+}} n = m + n \qquad \bar{\mathsf{E}} = \{2n \mid n \in \mathbb{N}\}$$

and α assignment such that $\alpha(x) = 3$

1. $\mathcal{A} \vDash \forall x \forall y\ (x \doteq y \to x + 0 \doteq 0 + y)$
2. $\mathcal{A} \vDash \mathsf{E}(0)$
3. $\mathcal{A} \nvDash \mathsf{E}(\mathsf{s}(0))$
4. $\mathcal{A} \vDash \forall x\ (\mathsf{E}(x) \to \mathsf{E}(\mathsf{s}(\mathsf{s}(x))))$
5. $\mathcal{A} \vDash \forall x \forall y\ ((\mathsf{E}(x) \wedge \mathsf{E}(y)) \to \mathsf{E}(x + y))$

Exercise

1. which of formulas hold if $\bar{0} = 1$?
2. find structure with universe $\{0, 1\}$ that plays same role

Validity and Satisfiability

Definition

- ϕ is valid ($\models \phi$) if $\mathcal{A} \models \phi$ for all $\mathcal{A}$
- ϕ is satisfiable if $\mathcal{A} \models \phi$ for some $\mathcal{A}$

Lemma

ϕ is valid $\iff$ $\neg\phi$ is unsatisfiable

Exercise

which are valid, and which are satisfiable?

1:	$\mathsf{P}(\mathsf{a})$	3:	$\mathsf{P}(\mathsf{a}) \to \exists x \mathsf{P}(x)$
2:	$(\forall x \mathsf{P}(x)) \to \mathsf{P}(\mathsf{a})$	4:	$(\forall x \mathsf{P}(x)) \ \wedge\ (\exists x \neg \mathsf{P}(x))$

Supplementary Comments

- first-order logic with set theory is considered as foundation of mathematics
- predicates are also called relations
- in textbook $[\![t]\!]_{\mathcal{A},\alpha}$ is written as $[\![t]\!]_{\mathcal{A}}$ and $t^{\mathcal{A}}$ where α is omitted
- $\forall x \mathsf{P}(x)\ \wedge\ \exists y \mathsf{Q}(y)$ is usually parsed as $(\forall x \mathsf{P}(x))\ \wedge\ (\exists y \mathsf{Q}(y))$
- $\forall x.\ \mathsf{P}(x) \wedge \mathsf{Q}(x)$ is often parsed as $\forall x (\mathsf{P}(x) \wedge \mathsf{Q}(x))$
- $\forall x, y, z.\ \phi$ stands for $\forall x \forall y \forall z.\ \phi$
- formulas like $\forall x \mathsf{P}(x)$ and $\forall y \mathsf{P}(y)$ are called α-equivalent