Final

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1)

$$\frac{\Gamma \vdash \phi \vee \psi \quad \Gamma \cup \{\psi\} \vdash \chi \quad \Gamma \cup \{\phi\} \vdash \chi}{\Gamma \vdash \chi}$$

Assume $\Gamma \models \phi \vee \psi$ (1), $\Gamma \cup \{\psi\} \models \chi$ (2) and $\Gamma \cup \{\phi\} \models \chi$ (3).
We show $\Gamma \models \chi$. Let v be a valuation. Assume $v \models \Gamma$. It's enough to show $v \models \chi$. By (1) we have $v \models \phi \vee \psi$. From (1) we have either $v \models \psi$ or $v \models \phi$. If $v \models \psi$ then $v \models \Gamma \cup \{\psi\}$ and by (2) we obtain $v \models \chi$. If $v \models \phi$ then we have $v \models \Gamma \cup \{\phi\}$ and by (3) we obtain $v \models \chi$

2)

a) $(P(x,y) \wedge \forall x \exists y P(x,y))[y/x]$

$= P(y,y) \wedge \forall x \exists y P(x,y)$

b) $[[\phi]]_{A,v} = \bar{P}(1,2) \wedge \forall x \exists y \bar{P}(x,y)$

because $1 < 2 \Rightarrow (1,2) \notin \bar{P} \Rightarrow A,v \not\models P(x,y)$ and $A,v \not\models \phi$

c) Consider $x = 0$. For any $y \in N$, $\bar{P}$ does not hold because 0 is the smallest number in N. So $A \not\models \forall x \exists y P(x,y)$

3) $\forall x \exists y R(x,y) \to \exists x \forall y R(x,y) = \phi$

Consider $A = (Z, \bar{R})$ with $\bar{R} = \{(a,b) \mid a,b \in N \times N \text{ and } a > b\}$

- Let x be an arbitrary number in Z. Take $y = x-1$. We have $A \models \forall x \exists y R(x,y)$.
- Let x be some number in Z. Take $y = x+1$. We have $A \not\models \exists x \forall y R(x,y)$.

As $A \not\models \phi$, ϕ is invalid

4) $P(x) \leftrightarrow \forall x \exists y Q(x,y)$

$\approx [P(x) \to \forall x \exists y Q(x,y)] \wedge [\forall x \exists y Q(x,y) \to P(x)]$

$\approx [\neg P(x) \vee \forall x \exists y Q(x,y)] \wedge [\neg \forall x \exists y Q(x,y) \vee P(x)]$

$\approx \forall u \exists v (\neg P(x) \vee Q(u,v)) \wedge (\exists x \forall y \neg Q(x,y) \vee P(x))$

$\approx \forall u \exists v (\neg P(x) \vee Q(u,v)) \wedge \exists z \forall t (\neg Q(z,t) \vee P(x))$

$\approx \forall u \exists v \exists z \forall t\ (\neg P(x) \vee Q(u,v)) \wedge (\neg Q(z,t) \vee P(x))$

Skolemization:

$\forall u \forall t \quad (\neg P(x) \vee Q(u, f(u))) \wedge (\neg Q(g(u),t) \vee P(x))$

5) $\phi\theta = \phi$

We show the claim by structure induction on ϕ

- Base cases:
  - $\phi \in \{\bot, \top\}$: $\phi\theta = \phi$
  - $\phi = Q_1 x_1 \dots Q_n x_n P^{(n)}$ for $Q_i \in \{\exists, \forall\}, n \in N$ and any predicate $P^{(n)}$ s.t. $FV(P^{(n)}) \subseteq \{x_1 \dots x_n\}$

+ $\phi \in \{\bot, \top\}$: $\phi\theta = \phi$
+ $\phi = Q_1x_1 \dots Q_nx_n P^{(n)}$ for $Q_i \in \{\exists, \forall\}, n \in \mathbb{N}$ and any predicate $P^{(n)}$ s.t $FV(P^{(n)}) \subseteq \{x_1 \dots x_n\}$
  $\phi\theta = (Q_1x_1 \dots Q_nx_n P^{(n)})\theta = Q_1x_1 \dots Q_nx_n(P^{(n)}\theta') = Q_1x_1 \dots Q_nx_n P^{(n)} = \phi$ with $\theta'(x_i) = x_i$ for $i = 1 \dots n$
+ $\phi = Q_1x_1 \dots Q_nx_n \; s \doteq t$ for $Q_i \in \{\exists, \forall\}$, $n \in \mathbb{N}$ and any terms s, t s.t
  $FV(s) \subseteq \{x_1 \dots x_n\}$ and $FV(t) \subseteq \{x_1 \dots x_n\}$
  $\phi\theta = (Q_1x_1 \dots Q_nx_n \; s \doteq t)\theta = Q_1x_1 \dots Q_nx_n((s \doteq t)\theta') = Q_1x_1 \dots Q_nx_n \; (s\theta' \doteq t\theta') = Q_1x_1 \dots Q_nx_n \; s \doteq t$
  with $\theta'(x_i) = x_i$ for $i = 1 \dots n$
+ $\phi = Q_1x_1 \dots Q_nx_n(\phi_1 * \phi_2)$ for $Q_i \in \{\exists, \forall\}, n \in \mathbb{N}$, $* \in \{\wedge, \vee, \to, \leftrightarrow\}$ and any formulars ϕ_1, ϕ_2 s.t $FV(\phi_1) \in \{x_1 \dots x_n\}$ and $FV(\phi_2) \in \{x_1 \dots x_n\}$
  $\phi\theta = (Q_1x_1 \dots Q_nx_n(\phi_1 * \phi_2))\theta = Q_1x_1 \dots Q_nx_n(\phi_1\theta' * \phi_2\theta') = Q_1x_1 \dots Q_nx_n(\phi_1 * \phi_2)$
  with $\theta'(x_i) = x_i$ for $i = 1 \dots n$
+ $\phi = Q_1x_1 \dots Q_nx_n \neg \phi_1$

+ Inductive cases:
  + $\phi = \phi_1 * \phi_2$ with ϕ_1, ϕ_2 are sentences
  + $\phi = \neg \phi_1$ with ϕ_1 is a sentence
    $\phi\theta = (\neg\phi_1)\theta = \neg(\phi_1\theta) = \neg\phi_1 = \phi$

6)

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{[P(f(x)) \wedge \forall x\, P(x)]^1}{\forall x\, P(x)}\;\to \forall E}{P(f(x))}}{\forall x\, P(f(x))}\;\to \forall I}{(P(f(x)) \wedge \forall x\, P(x)) \to \forall x\, P(f(x))}\;\to I_1$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg P(x,y)]}{\exists y \neg P(x,y)} \quad [\neg \exists y \neg P(x,y)]}{\bot}}{P(x,y)}}{\forall y\, P(x,y)}}{\exists x \forall y\, P(x,y)} \qquad \neg \exists x \forall y\, P(x,y)}{\bot}}{\exists y \neg P(x,y)}}{\forall x \exists y \neg P(x,y)}}{\neg \exists x \forall y\, P(x,y) \to \forall x \exists y \neg P(x,y)}$$

3.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\forall x \forall y\,(x + s(y) \doteq s(x+y))]^{Q}}{s(0)+s(0) \doteq s(s(0)+0)}\,\forall E}{s(s(0)+0) \doteq s(0)+s(0)}\,\text{SYM} \quad [s(0)+s(0) \doteq 0]}{s(s(0)+0) \doteq 0}\,\text{TRANS} \quad \dfrac{\forall x(\neg\, s(x) \doteq 0)}{\neg(s(s(0)+0) \doteq 0)}\,\forall E}{\dfrac{\bot}{\neg(s(0)+s(0) \doteq 0)}\,\neg I}\,\neg E$$

13.3

(7) $\exists c\, \exists N\, \forall n\,(n \geq N \rightarrow n^3 \leq c \cdot 2^n)$

Take $c = 6$ and $N = 3$. Let $n \in N$ s.t $n \geq N$.

$$2^n \geq \frac{1}{6} n^3 \quad \text{for } n \geq N$$

$$\Leftrightarrow \quad 6 \cdot 2^n \geq n^3$$

(8) $\forall c\, \forall N\, \exists n\,(n \geq N \wedge 2^n > c \cdot n^3)$

Let c and N be arbitrary numbers in N. We have

$$2^n \geq \frac{1}{24} n^4 - \frac{1}{12} n^3 \quad \text{for } n \geq 4$$

$$\Leftrightarrow 2^n \geq n^3\left(\frac{1}{24} n - \frac{1}{12}\right) >$$

Choose n s.t $\frac{1}{24} n - \frac{1}{12} > c$

$$\Leftrightarrow \quad n > 24c + 2$$

Take $n = 24c + N + 3$, we have $n \geq N$ and

$$2^n > c \cdot n^3$$

so $g(n) \in O(f(n))$ doesn't hold