Homework 4

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0.

a) $\phi = p$; $|\phi| = 1$

b) $\phi = \top$; $|\phi| = 1$

c) $\phi = \neg(p \wedge q)$; $|\phi| = 1 + |p \wedge q| = 1 + 1 + |p| + |q| = 4$

d) $\phi = p \vee p$; $|\phi| = 1 + |p| + |p| = 3$

e) $\phi = (p \to (q \wedge r))$; $|\phi| = 1 + |p| + |q \wedge r| = 1 + 1 + 1 + |q| + |r| = 5$

f) $\phi = (\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q))$; $|\phi| = 1 + |\neg(p \vee q)| + |\neg p \wedge \neg q|$

$= 1 + 1 + |p \vee q| + 1 + |\neg p| + |\neg q|$

$= 3 + 1 + |p| + |q| + 1 + |p| + 1 + |q|$

$= 10$

1.

a) i. $\phi = p$; $Atoms(\phi) = \{p\}$

ii. $\phi = \top$; $Atoms(\phi) = \emptyset$

iii. $\phi = \neg(p \wedge q)$; $Atoms(\phi) = \{p, q\}$

iv. $\phi = p \vee p$; $Atoms(\phi) = \{p\}$

v. $\phi = p \to (q \wedge r)$; $Atoms(\phi) = \{p, q, r\}$

vi. $\phi = (\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q))$; $Atoms(\phi) = \{p, q\}$

b)
+ $Atoms(p) = \{p\}$
+ $Atoms(\top) = \emptyset$
+ $Atoms(\phi \wedge \psi) = Atoms(\phi) \cup Atoms(\psi)$
+ $Atoms(\phi \to \psi) =$ "
+ $Atoms(\neg\phi) = Atoms(\phi)$
+ $Atoms(\bot) = \emptyset$
+ $Atoms(\phi \vee \psi) = Atoms(\phi) \cup Atoms(\psi)$
+ $Atoms(\phi \leftrightarrow \psi) =$ "

c) $|Atoms(\phi)| \leq |\phi|$

We show the claim by structural induction on ϕ:

* Base cases:
  + $\phi \in \{\top, \bot\}$

    (LHS) : $|Atoms(\phi)| = |\emptyset| = 0$

    (RHS) : $|\phi| = 1$

    As (LHS) < (RHS), the claims holds
  + ϕ is an atom : $\phi = p$

    (LHS): $|Atoms(p)| = |\{p\}| = 1$

    (RHS): $|\phi| = 1$

    As (LHS) = RHS, the claim holds
* Inductive cases: Let ϕ be a formula with at least 1 atom and 1 operator
  + $\phi = \neg\phi_1$

    (LHS): $|Atoms(\phi)| = |Atoms(\phi_1)|$

    (RHS): $|\phi| = |\neg\phi_1| = 1 + |\phi_1|$

    By I.H : $|Atoms(\phi_1)| \leq |\phi_1| \Rightarrow |Atoms(\phi_1)| < 1 + |\phi_1|$

    As (LHS) < (RHS), the claim holds
  + $\phi = \phi_1 * \phi_2$ with $* \in \{\wedge, \vee, \to, \leftrightarrow\}$

    (LHS) : $|Atoms(\phi)| = |Atoms(\phi_1 * \phi_2)| = |Atoms(\phi_1) \cup Atoms(\phi_2)|$

    $\leq |Atoms(\phi_1)| + |Atoms(\phi_2)|$

(LHS) : $|Atoms(\phi)| = |Atoms(\phi_1 * \phi_2)| = |Atoms(\phi_1) \cup Atoms(\phi_2)|$
$\leq |Atoms(\phi_1)| + |Atoms(\phi_2)|$

(RHS) : $|\phi| = |\phi_1 * \phi_2| = 1 + |\phi_1| + |\phi_2|$

By I.H: $|Atoms(\phi_1)| \leq |\phi_1| \Rightarrow |Atom(\phi_1) \cup Atoms(\phi_2)| \leq |\phi_1| + |\phi_2|$
$|Atoms(\phi_2)| \leq |\phi_2|$ $\qquad < 1 + |\phi_1| + |\phi_2|$

As (LHS) < (RHS), the claim holds

2.

a) i. $\phi = (\neg p)$; $\phi^* = \neg(p^*) = \neg(\neg p) = p$

ii. $\phi = (p \wedge q)$; $\phi^* = p^* \vee q^* = \neg p \vee \neg q$

iii. $\phi = (p \vee q)$; $\phi^* = p^* \wedge q^* = \neg p \wedge \neg q$

iv. $\phi = ((p \vee \neg q) \wedge (q \vee r))$; $\phi^* = (p \vee \neg q)^* \vee (q \vee r)^*$
$= (p^* \wedge (\neg q)^*) \vee (q^* \wedge r^*)$
$= (\neg p \wedge \neg(q^*)) \vee (\neg q \wedge \neg r)$
$= (\neg p \wedge q) \vee (\neg q \wedge \neg r)$

v. $\phi = (\neg p \vee \neg q)$; $\phi^* = (\neg p)^* \wedge (\neg q)^*$
$= \neg(p)^* \wedge \neg(q)^*$
$= p \wedge q$

vi. $\phi = (\neg p \wedge \neg q)$; $\phi^* = (\neg p)^* \vee (\neg q)^*$
$= \neg(p)^* \vee \neg(q)^*$
$= p \vee q$

b) $\phi^* \approx \neg \phi$

We show this claim by structural induction on ϕ:

* Base case : $\phi = p$
  (LHS): $\phi^* = p^* = \neg p$
  (RHS): $\neg \phi = \neg p$
  As (LHS) $\approx$ (RHS), the claim holds
* Inductive step:
  + $\phi = \neg \phi_1$
    (LHS): $\phi^* = (\neg \phi_1)^* = \neg(\phi_1^*) = \neg(\neg \phi_1) = \phi_1$ by I.H
    (RHS) : $\neg \phi = \neg(\neg \phi_1) = \phi_1$
    As (LHS) $\approx$ (RHS), the claim holds
  + $\phi = \phi_1 \wedge \phi_2$
    (LHS) : $\phi^* = (\phi_1 \wedge \phi_2)^* = \phi_1^* \vee \phi_2^*$
    $= \neg \phi_1 \vee \neg \phi_2$ by I.H
    (RHS) : $\neg \phi = \neg(\phi_1 \wedge \phi_2) = \neg \phi_1 \vee \neg \phi_2$
    As (LHS) $\approx$ (RHS), the claim holds
  + $\phi = \phi_1 \vee \phi_2$ can be proved using the same procedure

c) $\neg(\phi \vee \psi) \approx (\phi \vee \psi)^*$
$= \phi^* \wedge \psi^*$
$= \neg \phi \wedge \neg \psi$

3.

(1) $p \to (p \wedge p)$

$$\dfrac{\dfrac{[p]^1 \quad [p]^1}{p \wedge p}\ \wedge I}{p \to (p \wedge p)}\ \to I_1$$

(2) $(p \land q) \to (q \land p)$

$$\dfrac{\dfrac{\dfrac{[p \land q]^1}{p}\,\land E \qquad \dfrac{[p \land q]^1}{q}\,\land E}{q \land p}\,\land I}{(p \land q) \to (q \land p)}\,\to I_1$$

(3) $(p \land (q \land r)) \to ((p \land q) \land r)$

$$\dfrac{\dfrac{\dfrac{\dfrac{[p \land (q \land r)]^1}{p}\,\land E \qquad \dfrac{\dfrac{[p \land (q \land r)]^1}{q \land r}\,\land E}{q}\,\land E}{p \land q}\,\land I \qquad \dfrac{\dfrac{[p \land (q \land r)]^1}{q \land r}\,\land E}{r}\,\land E}{(p \land q) \land r}\,\land I}{(p \land (q \land r)) \to ((p \land q) \land r)}\,\to I_1$$

(4) $(p \lor p) \to p$

$$\dfrac{\dfrac{[p \lor p]^1 \quad [p]^2 \quad [p]^2}{p}\,\lor E_2}{(p \lor p) \to p}\,\to I_1$$

(5) $(p \lor q) \to (q \lor p)$

$$\dfrac{\dfrac{[p \lor q]^1 \qquad \dfrac{[p]^2}{q \lor p}\,\lor I \qquad \dfrac{[q]^2}{q \lor p}\,\lor I}{q \lor p}\,\lor E_2}{(p \lor q) \to (q \lor p)}\,\to I_1$$

(6) $(p \lor (q \lor r)) \to ((p \lor q) \lor r)$

$$\dfrac{\dfrac{[p \lor (q \lor r)]^1 \qquad \dfrac{\dfrac{[p]^2}{p \lor q}\,\lor I}{(p \lor q) \lor r}\,\lor I \qquad \dfrac{[q \lor r]^2 \qquad \dfrac{\dfrac{[q]^3}{p \lor q}\,\lor I}{(p \lor q) \lor r}\,\lor I \qquad \dfrac{[r]^3}{(p \lor q) \lor r}\,\lor I}{(p \lor q) \lor r}\,\lor E_3}{(p \lor q) \lor r}\,\lor E_2}{(p \lor (q \lor r)) \to ((p \lor q) \lor r)}\,\to I_1$$

(7) $(p \land (q \lor r)) \to ((p \land q) \lor (p \land r))$

$$\dfrac{\dfrac{\dfrac{[p \land (q \lor r)]^1}{q \lor r}\,\land E \qquad \dfrac{\dfrac{[q]^2 \quad \dfrac{[p \land (q \lor r)]^1}{p}\,\land E}{p \land q}\,\land I}{(p \land q) \lor (p \land r)}\,\lor I \qquad \dfrac{\dfrac{[r]^2 \quad \dfrac{[p \land (q \lor r)]^1}{p}\,\land E}{p \land r}\,\land I}{(p \land q) \lor (p \land r)}\,\lor I}{(p \land q) \lor (p \land r)}\,\lor E_2}{(p \land (q \lor r)) \to ((p \land q) \lor (p \land r))}\,\to I_1$$

(8) $((p \land q) \lor (p \land r)) \to (p \land (q \lor r))$

$[p \land q]^2 \quad [p \land r]^2$

$\dfrac{[p \land q]^3}{\ }\,\land E \qquad \dfrac{[p \land r]^3}{\ }\,\land E$

(8) $((p \wedge q) \vee (p \wedge r)) \to (p \wedge (q \vee r))$

$$
\begin{prooftree}
\AxiomC{$[(p\wedge q)\vee(p\wedge r)]^1$}
\AxiomC{$[p\wedge q]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{p}
\AxiomC{$[p\wedge r]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{p}
\RightLabel{$\vee E_2$}
\TrinaryInfC{p}
\AxiomC{$[(p\wedge q)\vee(p\wedge r)]^1$}
\AxiomC{$[p\wedge q]^3$}
\RightLabel{$\wedge E$}
\UnaryInfC{q}
\RightLabel{$\vee I$}
\UnaryInfC{$q\vee r$}
\AxiomC{$[p\wedge r]^3$}
\RightLabel{$\wedge E$}
\UnaryInfC{r}
\RightLabel{$\vee I$}
\UnaryInfC{$q\vee r$}
\RightLabel{$\vee E_3$}
\TrinaryInfC{$q\vee r$}
\RightLabel{$\wedge I$}
\BinaryInfC{$p\wedge(q\vee r)$}
\RightLabel{$\to I_1$}
\UnaryInfC{$((p\wedge q)\vee(p\wedge r))\to(p\wedge(q\vee r))$}
\end{prooftree}
$$

(9) $(p \vee (q \wedge r)) \to ((p \vee q) \wedge (p \vee r))$

$$
\begin{prooftree}
\AxiomC{$[p\vee(q\wedge r)]^1$}
\AxiomC{$[p]^2$}
\RightLabel{$\vee I$}
\UnaryInfC{$p\vee q$}
\AxiomC{$[q\wedge r]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{q}
\RightLabel{$\vee I$}
\UnaryInfC{$p\vee q$}
\RightLabel{$\vee E_2$}
\TrinaryInfC{$p\vee q$}
\AxiomC{$[p\vee(q\wedge r)]^1$}
\AxiomC{$[p]^3$}
\RightLabel{$\vee I$}
\UnaryInfC{$p\vee r$}
\AxiomC{$[q\wedge r]^3$}
\RightLabel{$\wedge E$}
\UnaryInfC{r}
\RightLabel{$\vee I$}
\UnaryInfC{$p\vee r$}
\RightLabel{$\vee E_3$}
\TrinaryInfC{$p\vee r$}
\RightLabel{$\wedge I$}
\BinaryInfC{$(p\vee q)\wedge(p\vee r)$}
\RightLabel{$\to I_1$}
\UnaryInfC{$(p\vee(q\wedge r))\to((p\vee q)\wedge(p\vee r))$}
\end{prooftree}
$$

(10) $((p \vee q) \wedge (p \vee r)) \to (p \vee (q \wedge r))$

$$
\begin{prooftree}
\AxiomC{$[(p\vee q)\wedge(p\vee r)]^1$}
\RightLabel{$\wedge E$}
\UnaryInfC{$p\vee r$}
\AxiomC{$[p]^2$}
\RightLabel{$\vee I$}
\UnaryInfC{$p\vee(q\wedge r)$}
\AxiomC{$[(p\vee q)\wedge(p\vee r)]^1$}
\RightLabel{$\wedge E$}
\UnaryInfC{$p\vee q$}
\AxiomC{$[p]^3$}
\RightLabel{$\vee I$}
\UnaryInfC{$p\vee(q\wedge r)$}
\AxiomC{$[q]^3$}
\AxiomC{$[r]^2$}
\RightLabel{$\wedge I$}
\BinaryInfC{$q\wedge r$}
\RightLabel{$\vee I$}
\UnaryInfC{$p\vee(q\wedge r)$}
\RightLabel{$\vee E_3$}
\TrinaryInfC{$p\vee(q\wedge r)$}
\RightLabel{$\vee E_2$}
\TrinaryInfC{$p\vee(q\wedge r)$}
\RightLabel{$\to I_1$}
\UnaryInfC{$((p\vee q)\wedge(p\vee r))\to(p\vee(q\wedge r))$}
\end{prooftree}
$$

(11) $p \to (q \to p)$

$$
\begin{prooftree}
\AxiomC{$[p]^1$}
\noLine
\UnaryInfC{$q\to p$}
\RightLabel{$\to I_1$}
\UnaryInfC{$p\to(q\to p)$}
\end{prooftree}
$$

(12) $p \to (q \to (p \wedge q))$

$$
\begin{prooftree}
\AxiomC{$[p]^1$}
\AxiomC{$[q]^2$}
\RightLabel{$\wedge I$}
\BinaryInfC{$p\wedge q$}
\RightLabel{$\to I_2$}
\UnaryInfC{$q\to(p\wedge q)$}
\RightLabel{$\to I_1$}
\UnaryInfC{$p\to(q\to(p\wedge q))$}
\end{prooftree}
$$

(13) $(p \to q) \to ((q \to r) \to (p \to r))$

$$
\dfrac{[p\to q]^1 \quad [p]^3}{q}\ \to E \qquad [q\to r]^2 \quad \to E
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p\to q]^1 \quad [p]^3}{q}\to E \quad [q\to r]^2}{r}\to E}{p\to r}\to I_3}{(q\to r)\to(p\to r)}\to I_2}{(p\to q)\to((q\to r)\to(p\to r))}\to I_1$$

(14) $(p\to(q\to r))\to((p\wedge q)\to r)$

$$\dfrac{\dfrac{\dfrac{\dfrac{[p\wedge q]^2}{q}\wedge E \qquad \dfrac{\dfrac{[p\wedge q]^2}{p}\wedge E \quad [p\to(q\to r)]^1}{q\to r}\to E}{r}\to E}{((p\wedge q)\to r}\to I_2}{(p\to(q\to r))\to((p\wedge q)\to r)}\to I_1$$

(15) $((p\wedge q)\to r)\to(p\to(q\to r))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]^2 \quad [q]^3}{p\wedge q}\wedge I \qquad [((p\wedge q)\to r)]^1}{r}\to E}{q\to r}\to I_3}{p\to(q\to r)}\to I_2}{((p\wedge q)\to r)\to(p\to(q\to r))}\to I_1$$

(16) $(p\to q)\to((p\to r)\to(p\to(q\wedge r)))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]^3 \quad [p\to q]^1}{q}\to E \qquad \dfrac{[p]^3 \quad [p\to r]^2}{r}\to E}{q\wedge r}\wedge I}{p\to(q\wedge r)}\to I_3}{(p\to r)\to(p\to(q\wedge r))}\to I_2}{(p\to q)\to((p\to r)\to(p\to(q\wedge r)))}\to I_1$$

(17) $(p\to r)\to((q\to r)\to((p\vee q)\to r))$

$$\dfrac{\dfrac{\dfrac{\dfrac{[p\vee q]^3 \qquad \dfrac{[p]^4 \quad [p\to r]^1}{r}\to E \qquad \dfrac{[q]^4 \quad [q\to r]^2}{r}\to E}{r}\vee E_4}{(p\vee q)\to r}\to I_3}{(q\to r)\to((p\vee q)\to r)}\to I_2}{(p\to r)\to((q\to r)\to((p\vee q)\to r))}\to I_1$$