I211E: Mathematical Logic

Nao Hirokawa

JAIST

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`https://www.jaist.ac.jp/~hirokawa/lectures/ml/`

Schedule

	propositional logic		predicate logic
4/13	syntax, semantics	5/11	syntax, semantics
4/18	normal forms	5/16	normal forms
4/20	examples	5/18	natural deduction I
4/25	natural deduction I	5/23	natural deduction II
4/27	natural deduction II	5/25	examples, properties
5/2	completeness	5/30	advanced topics
5/9	midterm exam	6/1	summary
		6/6	exam

Evaluation

midterm exam (40) + final exam (60)

Contents

Aim

to develop basic proof skill

Contents

Definition (introduction and elimination rules for $\neg$)

$$\dfrac{\begin{array}{c}[\phi]^i\\ \vdots\\ \bot\end{array}}{\neg\phi}\,\neg\mathrm{I}_i \qquad\qquad \dfrac{\phi \quad \neg\phi}{\bot}\,\neg\mathrm{E}_i$$

Proposition

$\vdash (p \to q) \to (\neg q \to \neg p)$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]^3 \quad [p \to q]^1}{q}\,{\to}\mathrm{E}_1 \quad [\neg q]^2}{\bot}\,\neg\mathrm{E}}{\neg p}\,\neg\mathrm{I}_3}{\neg q \to \neg p}\,{\to}\mathrm{I}_2}{(p \to q) \to (\neg q \to \neg p)}\,{\to}\mathrm{I}_1$$

□

Proposition

If p implies q then $\neg q$ implies $\neg p$.

Proof.

We show $(p \to q) \to (\neg q \to \neg p)$.
Assume $p \to q$. It is enough to show $\neg q \to \neg p$.
Assume $\neg q$. It is enough to show $\neg p$.
Assume to the contrary that p holds.
Since p and $p \to q$, we obtain q.
Since q and $\neg q$, we have contradiction.
Thus, $\neg p$ holds. □

Definition (elimination rule for $\bot$)

$$\dfrac{\bot}{\phi}\ \mathrm{EFQ} \qquad\qquad \text{EFQ} = \text{ex falso quodlibet}$$

Proposition

$\vdash (\neg p \vee q) \to (p \to q)$

Proof.

$$\dfrac{\dfrac{[\neg p \vee q]^1 \qquad \dfrac{\dfrac{[p]^2 \qquad [\neg p]^3}{\bot}\,\neg\mathrm{E}}{q}\,\mathrm{EFQ} \qquad [q]^3}{\dfrac{q}{p \to q}\,{\to}\mathrm{I}_2}\,\vee\mathrm{E}}{(\neg p \vee q) \to (p \to q)}\,{\to}\mathrm{I}_1$$

□

Proposition

If $\neg p \vee q$ then p implies q.

Proof.

We show $(\neg p \vee q) \to (p \to q)$.
Assume $\neg p \vee q$. It is enough to show $p \to q$.
Assume p. It is enough to show q.
Using $\neg p \vee q$, we distinguish two cases.
If $\neg p$ then we have $\neg p$ and p. Contradiction.
Hence, q holds.
If q then q holds trivially.
In any case, q holds. □

Definition (another elimination rule for $\bot$)

$$\dfrac{\begin{matrix}[\neg\phi]^i \\ \vdots \\ \bot\end{matrix}}{\phi}\ \mathrm{RAA}_i \qquad\qquad \text{RAA} = \text{reductio ad absurdum} = \text{proof by contradiction}$$

Proposition

$\vdash \neg\neg p \to p$

Proof.

$$\dfrac{}{\neg\neg p \to p}\,{\to}\mathrm{I}_1$$

□

Proposition

If $\neg\neg p$ then p.

Proof.

We show $\neg\neg p \to p$.
Assume $\neg\neg p$. It is enough to show p.
Assume to the contrary that $\neg p$ holds.
Since p and $\neg p$ hold, we have contradiction.
Hence, p holds. □

Supplementary Comments

$$\dfrac{\bot}{\phi}\ \mathrm{EFQ} \qquad\qquad \dfrac{\begin{matrix}[\neg\phi]^i \\ \vdots \\ \bot\end{matrix}}{\phi}\ \mathrm{RAA}_i \qquad\qquad \dfrac{\begin{matrix}[\phi]^i \\ \vdots \\ \bot\end{matrix}}{\neg\phi}\ \neg\mathrm{I}_i$$

- classical logic uses RAA
- intuitionistic logic forbids use of RAA
- mathematical proofs usually adopt classical logic
- in intuitionistic logic, neither $\neg\neg p \to p$ nor $p \vee \neg p$ hold

Exercise: Understanding Proof Structure

Proposition (law of excluded middle (or third in textbook))

p or $\neg p$ holds.

Proof.

We show $p \vee \neg p$. Assume to the contrary that $\neg(p \vee \neg p)$ holds.

1. As a preliminary, we show that $\neg p$ holds. Assume to the contrary that p holds. Since p holds, $p \vee \neg p$ holds. Because $p \vee \neg p$ and $\neg(p \vee \neg p)$ hold, we obtain contradiction. Therefore, $\neg p$ must hold.
2. Now we derive contradiction as follows. Since $\neg p$ holds, $p \vee \neg p$ follows. Because $p \vee \neg p$ and $\neg(p \vee \neg p)$ hold, we obtain contradiction. □

Exercise: Prove $\vdash p \vee \neg p$.

Exercise: Proof Writing

Proposition

$\vdash (p \land (\neg p \lor q)) \to q$

Proof.

$$\dfrac{\dfrac{\dfrac{[p \land (\neg p \lor q)]^1}{\neg p \lor q}\,\land\mathrm{E} \qquad \dfrac{\dfrac{\dfrac{[p \land (\neg p \lor q)]^1}{p}\,\land\mathrm{E} \qquad [\neg p]^2}{\bot}\,\neg\mathrm{E}}{q}\,\mathrm{EFQ} \qquad [q]^2}{q}\,\lor\mathrm{E}_2}{(p \land (\neg p \lor q)) \to q}\,{\to}\mathrm{I}_1 \qquad \square$$

Exercise: Write a proof for $(p \land (\neg p \lor q)) \to q$ in text.