Homework 8

Study Sections 3.1 and 3.2.

1. Let $\mathcal{V} = \{x, y, z, \ldots\}$, $\mathcal{F} = \{\mathsf{0}^{(0)}, \mathsf{1}^{(0)}, \mathsf{f}^{(2)}, \mathsf{g}^{(2)}\}$, and $\mathcal{P} = \{\mathsf{P}^{(2)}\}$. The universes of the structures $\mathcal{N}$, $\mathcal{Z}$ and $\mathcal{Q}$ are the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots, \}$, the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and the rationals $\mathbb{Q} = \{0, \pm 1, \pm \frac{1}{2}, \ldots\}$, respectively, and their interpretations on the universes U are given by $\bar{\mathsf{0}} = 0$, $\bar{\mathsf{1}} = 1$, $\bar{\mathsf{f}}(m, n) = m + n$, $\bar{\mathsf{g}}(m, n) = m \cdot n$ and:

$$\bar{\mathsf{P}} = \{(m, n) \in U \times U \mid m > n\}$$

   For each sentence ϕ and structure $\mathcal{A} \in \{\mathcal{N}, \mathcal{Z}, \mathcal{Q}\}$, determine if $\mathcal{A} \models \phi$.

   (1) $\forall x \forall y (\mathsf{f}(x, y) \doteq \mathsf{f}(y, x))$
   (2) $\forall x \forall y (\mathsf{g}(x, y) \doteq \mathsf{g}(y, x))$
   (3) $\forall x \forall y \forall z (\mathsf{f}(x, \mathsf{g}(y, z)) \doteq \mathsf{f}(\mathsf{g}(x, y), \mathsf{g}(x, z)))$
   (4) $\forall x \forall y \forall z (\mathsf{g}(x, \mathsf{f}(y, z)) \doteq \mathsf{g}(\mathsf{f}(x, y), \mathsf{f}(x, z)))$
   (5) $\exists x \forall y (\mathsf{f}(x, y) \doteq y)$
   (6) $\exists x \forall y (\mathsf{g}(x, y) \doteq y)$
   (7) $\forall x (\forall y (\mathsf{f}(x, y) \doteq y) \to x \doteq \mathsf{0})$
   (8) $\forall x \exists y (\mathsf{f}(x, y) \doteq \mathsf{0})$
   (9) $\forall x \exists y (\mathsf{g}(x, y) \doteq \mathsf{1})$
   (10) $\forall x (\neg (x \doteq \mathsf{0}) \to \exists y (\mathsf{g}(x, y) \doteq \mathsf{1}))$
   (11) $\forall x\ \neg \mathsf{P}(x, x)$
   (12) $\forall x \forall y \forall z ((\mathsf{P}(x, y) \wedge \mathsf{P}(y, z)) \to \mathsf{P}(x, z))$
   (13) $\forall x \exists y\ \mathsf{P}(y, x)$
   (14) $\forall x \exists y\ \mathsf{P}(x, y)$
   (15) $\exists x \forall y (x \doteq y \vee \mathsf{P}(y, x))$
   (16) $\forall x \forall y\ \mathsf{P}(\mathsf{f}(x, y), x)$
   (17) $\forall x \forall y\ (\neg (y \doteq \mathsf{0}) \to \mathsf{P}(\mathsf{f}(x, y), x))$
   (18) $\forall x \forall y\ \mathsf{P}(\mathsf{g}(x, y), x)$
   (19) $\forall x \forall y\ (\mathsf{P}(y, 1) \to \mathsf{P}(\mathsf{g}(x, y), x))$
   (20) $\forall x \forall y\ (\mathsf{P}(x, y) \to \exists z (\mathsf{P}(x, z) \wedge \mathsf{P}(z, y)))$

2. Let $\mathcal{V} = \{x, y, z, \ldots\}$, $\mathcal{F} = \{\mathsf{a}^{(0)}, \mathsf{b}^{(0)}\}$, and $\mathcal{P} = \{\mathsf{K}^{(2)}\}$. Suppose that a denotes Alice, b Bob, and $\mathsf{K}(x, y)$ reads as x knows y. Translate each sentence in English into a sentence ϕ of first-order logic, and then give structures $\mathcal{A}$ and $\mathcal{B}$ over the universe $\mathbb{N}$ such that $\mathcal{A} \models \phi$ and $\mathcal{B} \not\models \phi$.

   (1) Alice knows Bob, but Bob does not know Alice.
   (2) Alice knows everyone.
   (3) No one knows Bob.
   (4) Everyone who knows Alice knows Bob.
   (5) There is a person who knows Alice but doesn't know Bob.

(6) Everyone knows himself.

(7) No one knows himself.

(8) Whenever one knows another, they know each other.

(9) Even if one knows another, they not always know each other.

(10) Only Alice knows Bob.

(11) There are at most one person who knows Bob.

(12) If someone knows another person, and that person knows a third person, then the first person knows the third person.

3. Let $\mathcal{V} = \{x, y, \ldots\}$, $\mathcal{F}$ the empty set, and $\mathcal{P} = \{\mathsf{P}^{(1)}, \mathsf{Q}^{(1)}\}$. For each sentence ϕ, determine if ϕ is valid or not, and give a structure $\mathcal{A}$ such that $\mathcal{A} \not\models \phi$ if ϕ is not valid.

   (1) $\neg\forall x \mathsf{P}(x) \leftrightarrow \exists x \neg\mathsf{P}(x)$

   (2) $\neg\exists x \mathsf{P}(x) \leftrightarrow \forall x \neg\mathsf{P}(x)$

   (3) $(\forall x \mathsf{P}(x) \land \forall x \mathsf{Q}(x)) \leftrightarrow \forall x(\mathsf{P}(x) \land \mathsf{Q}(x))$

   (4) $(\forall x \mathsf{P}(x) \lor \forall x \mathsf{Q}(x)) \leftrightarrow \forall x(\mathsf{P}(x) \lor \mathsf{Q}(x))$

   (5) $(\forall x \mathsf{P}(x) \to \forall x \mathsf{Q}(x)) \leftrightarrow \forall x(\mathsf{P}(x) \to \mathsf{Q}(x))$

   (6) $(\exists x \mathsf{P}(x) \land \exists x \mathsf{Q}(x)) \leftrightarrow \exists x(\mathsf{P}(x) \land \mathsf{Q}(x))$

   (7) $(\exists x \mathsf{P}(x) \lor \exists x \mathsf{Q}(x)) \leftrightarrow \exists x(\mathsf{P}(x) \lor \mathsf{Q}(x))$

   (8) $(\exists x \mathsf{P}(x) \to \exists x \mathsf{Q}(x)) \leftrightarrow \exists x(\mathsf{P}(x) \to \mathsf{Q}(x))$