I211E: Mathematical Logic

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Schedule

	propositional logic		predicate logic
4/13	syntax, semantics	5/11	syntax, semantics
4/18	normal forms	5/16	normal forms
4/20	examples	5/18	natural deduction I
4/25	natural deduction I	5/23	natural deduction II
4/27	natural deduction II	5/25	examples, properties
5/2	completeness	5/30	advanced topics
5/9	midterm exam	6/1	summary
		6/6	exam

Evaluation

midterm exam (40) + final exam (60)

Contents

Aim

to learn how to prove formulas having quantifiers and equalities

Contents

1. natural deduction I (∀, ∃)
2. eigenvariable conditions

Natural Deduction

Informal Definition

1. underlying notions (e.g. proof tree) and terminologies are same as before
2. natural deduction for first-order logic consists of
   - all inference rules for propositional logic
   - introduction/elimination rules for $\forall$ and $\exists$, and
   - inference rules for $\doteq$

Definition (introduction and elimination rules for $\forall$)

$$\dfrac{\phi[y/x]}{\forall x \phi}\ \forall\mathrm{I}\ \ (y \text{ is eigenvariable}) \qquad \dfrac{\forall x \phi}{\phi[t/x]}\ \forall\mathrm{E}\ \ (t \text{ is free for } x \text{ in } \phi)$$

eigenvariable means that y is arbitrary element (to be defined later)

Proposition

$\vdash (\forall x\, \mathsf{P}(x)) \to (\forall x\, \mathsf{P}(\mathsf{f}(x)))$

Proof.

$$\dfrac{\dfrac{\dfrac{[\forall x\, \mathsf{P}(x)]^1}{\mathsf{P}(\mathsf{f}(x))}\ \forall\mathrm{E}}{\forall x\, \mathsf{P}(\mathsf{f}(x))}\ \forall\mathrm{I}}{(\forall x\, \mathsf{P}(x)) \to (\forall x\, \mathsf{P}(\mathsf{f}(x)))}\ {\to}\mathrm{I}_1$$

□

Proposition

if $\mathsf{P}(x)$ for all x then $\mathsf{P}(\mathsf{f}(x))$ for all x

Proof.

Assume $\mathsf{P}(x)$ for all x. We show $\mathsf{P}(\mathsf{f}(x))$ for all x. Let x be an arbitrary element. It is enough to show $\mathsf{P}(\mathsf{f}(x))$. By assumption $\mathsf{P}(\mathsf{f}(x))$ holds. □

Exercise:
$\vdash ((\forall x\, \mathsf{P}(x)) \land (\forall y\, \mathsf{Q}(y))) \to (\forall z\, (\mathsf{P}(z) \land \mathsf{Q}(z)))$

Definition (introduction and elimination rules for $\exists$)

$$\dfrac{\phi[t/x]}{\exists x \phi}\ \exists\mathrm{I}\ \ (t \text{ is free for } x \text{ in } \phi) \qquad \dfrac{\exists x \phi \qquad \begin{matrix}[\phi[y/x]]^i \\ \vdots \\ \psi\end{matrix}}{\psi}\ \exists\mathrm{E}_i\ \ (y \text{ is eigenvariable})$$

Proposition

$\vdash (\exists x\, \mathsf{P}(\mathsf{f}(x))) \to (\exists x\, \mathsf{P}(x))$

Proof.

$$\dfrac{\dfrac{[\exists x\, \mathsf{P}(\mathsf{f}(x))]^1 \qquad \dfrac{[\mathsf{P}(\mathsf{f}(y))]^2}{\exists x\, \mathsf{P}(x)}\ \exists\mathrm{I}}{\exists x\, \mathsf{P}(x)}\ \exists\mathrm{E}}{(\exists x\, \mathsf{P}(\mathsf{f}(x))) \to (\exists x\, \mathsf{P}(x))}\ {\to}\mathrm{I}_1$$

□

Proposition

if $\mathsf{P}(\mathsf{f}(x))$ for some x then $\mathsf{P}(x)$ for some x

Proof.

Assume $\mathsf{P}(\mathsf{f}(x))$ for some x. We show that there exists element x such that $\mathsf{P}(x)$. By assumption $\mathsf{P}(\mathsf{f}(y))$ for some y. Take $x = \mathsf{f}(y)$. We obtain $\mathsf{P}(x)$. □

Exercise:
$\vdash ((\exists x\, \mathsf{P}(x)) \lor (\exists y\, \mathsf{Q}(y))) \to (\exists z\, (\mathsf{P}(z) \lor \mathsf{Q}(z)))$

Proof of $\vdash ((\exists x\, \mathsf{P}(x)) \lor (\exists y\, \mathsf{Q}(y))) \to (\exists z\, (\mathsf{P}(z) \lor \mathsf{Q}(z)))$

$$\dfrac{\dfrac{[(\exists x\, \mathsf{P}(x)) \lor (\exists y\, \mathsf{Q}(y))]^1 \qquad \dfrac{[\exists x\, \mathsf{P}(x)]^2 \qquad \dfrac{\dfrac{[\mathsf{P}(x)]^3}{\mathsf{P}(x) \lor \mathsf{Q}(x)}\ \lor\mathrm{I}}{\exists z\, (\mathsf{P}(z) \lor \mathsf{Q}(z))}\ \exists\mathrm{I}}{\exists z\, (\mathsf{P}(z) \lor \mathsf{Q}(z))}\ \exists\mathrm{E}_3 \qquad \dfrac{[\exists y\, \mathsf{Q}(y)]^2 \qquad \dfrac{\dfrac{[\mathsf{Q}(y)]^4}{\mathsf{P}(y) \lor \mathsf{Q}(y)}\ \lor\mathrm{I}}{\exists z\, (\mathsf{P}(z) \lor \mathsf{Q}(z))}\ \exists\mathrm{I}}{\exists z\, (\mathsf{P}(z) \lor \mathsf{Q}(z))}\ \exists\mathrm{E}_4}{\exists z\, (\mathsf{P}(z) \lor \mathsf{Q}(z))}\ \lor\mathrm{E}_2}{((\exists x\, \mathsf{P}(x)) \lor (\exists y\, \mathsf{Q}(y))) \to (\exists z\, (\mathsf{P}(z) \lor \mathsf{Q}(z)))}\ {\to}\mathrm{I}_1$$

Eigenvariable Conditions Are Essential

Definition (precise version)

$$\dfrac{\Gamma \vdash \phi[y/x]}{\Gamma \vdash \forall x\,\phi}\ \forall\mathrm{I} \quad (y \text{ is eigenvariable})$$

y is eigenvariable in $\forall\mathrm{I}$ if $y \notin \mathsf{FV}(\gamma)$ for all $\gamma \in \Gamma \cup \{\forall x\phi\}$ (i.e. y is fresh)

Example

without eigenvariable condition, wrong proposition could be proved:

$$\dfrac{\dfrac{[\mathsf{P}(x)]^1}{\forall x\,\mathsf{P}(x)}\ \forall\mathrm{I}}{\mathsf{P}(x) \to \forall x\,\mathsf{P}(x)}\ {\to}\mathrm{I}_1$$

We show $\mathsf{P}(x) \to \forall x\,\mathsf{P}(x)$. Assume $\mathsf{P}(x)$.
Let x be an arbitrary element. (?!)
We show $\mathsf{P}(x)$.
By assumption, $\mathsf{P}(x)$ holds.

Definition (precise version)

$$\dfrac{\Gamma \vdash \exists x\,\phi \qquad \Gamma \cup \{\phi[y/x]\} \vdash \psi}{\psi}\ \exists\mathrm{E} \quad (y \text{ is eigenvariable})$$

y is eigenvariable in $\exists\mathrm{E}$ if $y \notin \mathsf{FV}(\gamma)$ for all $\gamma \in \Gamma \cup \{\exists x\phi, \psi\}$ (i.e. y is fresh)

Example

without eigenvariable condition, wrong proposition could be proved:

$$\dfrac{\dfrac{[\exists x\,\mathsf{P}(x)]^1 \qquad [\mathsf{P}(x)]^2}{\mathsf{P}(x)}\ \exists\mathrm{E}_2}{\exists x \mathsf{P}(x) \to \mathsf{P}(x)}\ {\to}\mathrm{I}_1$$

We show $\exists x\,\mathsf{P}(x) \to \mathsf{P}(x)$. Assume $\exists x\,\mathsf{P}(x)$.
It is enough to show $\mathsf{P}(x)$.
By assumption $\mathsf{P}(x)$ holds for some x.
Thus, $\mathsf{P}(x)$ is proved. ?!

How to Prove General Formulas

Proposition

$\vdash \forall x(\phi \land \psi) \to (\forall x\phi \land \forall x\psi)$ holds for all formulas ϕ and ψ

Proof.

Let ϕ and ψ be arbitrary formulas. Validity of $\forall x(\phi \land \psi) \to (\forall x\phi \land \forall x\psi)$ is shown as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x\,(\phi \land \psi)]^1}{\phi[y/x] \land \psi[y/x]}\ \forall\mathrm{E}}{\phi[y/x]}\ \land\mathrm{I}}{\forall x\phi}\ \forall\mathrm{I} \qquad \dfrac{\dfrac{\dfrac{[\forall x\,(\phi \land \psi)]^1}{\phi[z/x] \land \psi[z/x]}\ \forall\mathrm{E}}{\psi[z/x]}\ \land\mathrm{I}}{\forall x\phi}\ \forall\mathrm{I}}{\forall x\phi \land \forall x\psi}\ \land\mathrm{I}}{\forall x(\phi \land \psi) \to (\forall x\phi \land \forall x\psi)}\ {\to}\mathrm{I}_1$$

where y and z are eigenvariables. □

Exercises

1. Prove $\vdash ((\forall x\, \mathsf{P}(x, y)) \wedge \mathsf{Q}(y, z)) \;\rightarrow\; \forall x\, (\mathsf{P}(x, y) \wedge \mathsf{Q}(y, z))$.

2. Prove $\vdash ((\forall x\, \phi) \wedge \psi) \;\rightarrow\; \forall x\, (\phi \wedge \psi)$ for all formulas ϕ with $x \notin \mathsf{FV}(\psi)$.