Homework 4

Study Section 2.4.

0. The *size* $|\phi|$ of a formula ϕ is the number of *symbol occurrence* in ϕ, defined by *recursion*:

$$\begin{aligned} |p| &= 1 & |\neg\phi| &= 1 + |\phi| \\ |\top| &= 1 & |\bot| &= 1 \\ |\phi \wedge \psi| &= |\phi| + |\psi| + 1 & |\phi \vee \psi| &= |\phi| + |\psi| + 1 \\ |\phi \to \psi| &= |\phi| + |\psi| + 1 & |\phi \leftrightarrow \psi| &= |\phi| + |\psi| + 1 \end{aligned}$$

Note that parentheses do not count as symbols. For instance, $|(p \to q)| = 3$ since there are three symbols $p, q, \to$. Compute $|\phi|$ for each formula ϕ.

(a) $\phi = p$
(b) $\phi = \top$
(c) $\phi = \neg(p \wedge q)$
(d) $\phi = p \vee p$
(e) $\phi = (p \to (q \wedge r))$
(f) $\phi = (\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q))$

1. We write $\mathsf{Atoms}(\phi)$ for the set of all atoms in a formula ϕ.

(a) Compute $\mathsf{Atoms}(\phi)$ for each formula ϕ. Note that $\top, \bot$ are not atoms.
  i. $\phi = p$
  ii. $\phi = \top$
  iii. $\phi = \neg(p \wedge q)$
  iv. $\phi = p \vee p$
  v. $\phi = (p \to (q \wedge r))$
  vi. $\phi = (\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q))$

(b) Define the function Atoms by *recursion*.

$$\begin{aligned} \mathsf{Atoms}(p) &= \cdots & \mathsf{Atoms}(\neg\phi) &= \cdots \\ \mathsf{Atoms}(\top) &= \cdots & \mathsf{Atoms}(\bot) &= \cdots \\ \mathsf{Atoms}(\phi \wedge \psi) &= \cdots & \mathsf{Atoms}(\phi \vee \psi) &= \cdots \\ \mathsf{Atoms}(\phi \to \psi) &= \cdots & \mathsf{Atoms}(\phi \leftrightarrow \psi) &= \cdots \end{aligned}$$

(c) Show that $|\mathsf{Atoms}(\phi)| \leq |\phi|$ for all formulas ϕ by structural induction on ϕ. Here $|\mathsf{Atoms}(\phi)|$ denotes the *cardinality* of the set $\mathsf{Atoms}(\phi)$, and $|\phi|$ denotes the *size* of the formula ϕ. For instance, $|\mathsf{Atoms}((p \to q) \to r)| = 3$ and $|\mathsf{Atoms}(p \to p)| = 1$.

2. In this section we only consider formulas made of $\neg, \vee, \wedge$ and atoms:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi$$

For each formula ϕ the formula ϕ^* is defined as follows:

$$\begin{aligned} p^* &= \neg p & (\neg\phi)^* &= \neg(\phi^*) \\ (\phi \wedge \psi)^* &= \phi^* \vee \psi^* & (\phi \vee \psi)^* &= \phi^* \wedge \psi^* \end{aligned}$$

Here p denotes an atom.

(a) Compute ϕ^* for each formula ϕ.

i. $\phi = (\neg p)$
ii. $\phi = (p \land q)$
iii. $\phi = (p \lor q)$
iv. $\phi = ((p \lor \neg q) \land (q \lor r))$
v. $\phi = (\neg p \lor \neg q)$
vi. $\phi = (\neg p \land \neg q)$

(b) Show that $\phi^* \approx \neg\phi$ for all formulas ϕ by structural induction on ϕ. Note that the (syntactical) equality $=$ and the logical equivalence $\approx$ must be distinguished in the proof.

(c) Show that $\neg(\phi \lor \psi) \approx (\neg\phi \land \neg\psi)$ for all formulas ϕ, ψ using the previous result.

3. Write a derivation of each formula in the natural deduction.

(1) $p \to (p \land p)$
(2) $(p \land q) \to (q \land p)$
(3) $(p \land (q \land r)) \to ((p \land q) \land r)$
(4) $(p \lor p) \to p$
(5) $(p \lor q) \to (q \lor p)$
(6) $(p \lor (q \lor r)) \to ((p \lor q) \lor r)$
(7) $(p \land (q \lor r)) \to ((p \land q) \lor (p \land r))$
(8) $((p \land q) \lor (p \land r)) \to (p \land (q \lor r))$
(9) $(p \lor (q \land r)) \to ((p \lor q) \land (p \lor r))$
(10) $((p \lor q) \land (p \lor r)) \to (p \lor (q \land r))$
(11) $p \to (q \to p)$
(12) $p \to (q \to (p \land q))$
(13) $(p \to q) \to ((q \to r) \to (p \to r))$
(14) $(p \to (q \to r)) \to ((p \land q) \to r)$
(15) $((p \land q) \to r) \to (p \to (q \to r))$
(16) $(p \to q) \to ((p \to r) \to (p \to (q \land r)))$
(17) $(p \to r) \to ((q \to r) \to ((p \lor q) \to r))$