Note 2

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* Unconstrained Optimization

+ 1st-order necessary condition:

If x^* is a local optimum $\Rightarrow \nabla f(x^*) = 0$. ($\nabla^2 f(x^*)$ is "≥ 0" for 2nd-order)

+ 2nd-order sufficient condition:

If x^* satisfied the 1st-order necessary condition and $\nabla^2 f(x^*)$ is "≥ 0"

$\Rightarrow x^*$ is a local optimum

* Constrained Optimization

+ Lagrangian:

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{l} \lambda_i h_i(x) + \sum_{j=1}^{m} \mu_j g_j(x)$$

+ 1st-order necessary condition (KKT condition)

If x^* is a local optimum then there exists λ^* and μ^*:

$$\begin{cases} \nabla f(x^*) + \lambda^T h(x) + \mu^T g(x) = 0 \\ h(x^*) = 0 \\ g(x^*) \leq 0 \\ \mu \geq 0 \\ \mu^T g(x^*) = 0 \end{cases}$$

+ 2nd-order necessary condition:

Let $I(x^*)$ denotes the set of active constraints:

$$I(x^*) = \{ j \mid g_j(x^*) = 0 \}$$

Let M be the set of all vectors orthogonal to the gradient vectors of $I(x^*)$:

$$M = \left\{ y \in \mathbb{R}^n \,\middle|\, \begin{array}{ll} \nabla h_i(x^*)^T y = 0 & \text{for all } i \\ \nabla g_j(x^*)^T y = 0 & \text{for } j \in I(x^*) \end{array} \right\}$$

If (λ^*, μ^*) satisfied the KKT condition w.r.t a local minimum x^*, then

$$y^T \nabla_x^2 L(x^*, \lambda^*, \mu^*) y \geq 0 \quad \forall y \in M$$

≤ 0 for local maximum

+ 2nd-order sufficient condition:

If (x^*, λ^*, μ^*) satisfied the KKT conditions and

$$y^T \nabla_x^2 L(x^*, \lambda^*, \mu^*) y > 0 \quad \forall y \in M \setminus \{0\}$$

If (x^*, λ^*, μ^*) satisfied the KKT conditions and

$$y^T \nabla_x^2 L(x^*, \lambda^*, \mu^*)\, y > 0 \quad \forall y \in M \setminus \{0\}$$

then x^* is a local minimum

+ Method :
  1. Generate candidate : Try various combination of inactive and active constraints
     For each candidate, solve this to get the solution

$$\begin{cases} \nabla f(x) + \sum_{i=1}^{l} \lambda_i h_i(x) + \sum_{j=1}^{m} \mu_j g_j(x) = 0 \\ h_i(x) = 0 & \forall i \\ g_j(x) = 0 & \forall j \in I \\ \mu_j = 0 & \forall j \notin I \end{cases}$$

  2. Testing candidate : Check whether the solution satisfied the KKT condition and the 2nd-order condition

* Combinatorial Optimization
+ Greedy algorithm : Locally optimal decision at each step
+ Dynamic programming : Divide the main problem into sub-problems, the optimal solution contains optimal solutions to subproblems
  1. Characterize the structure of the optimal solution (Tabular form)
  2. Recursively define the value of the optimal solution (Define $p(i,j)$)
  3. Compute the optimal value (Calculate $p(i,j)$ iteratively)
  4. Construct the optimal solution

Common Derivatives

$\frac{d}{dx}(x) = 1$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\frac{d}{dx}(a^x) = a^x \ln(a)$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\frac{d}{dx}(\mathrm{e}^x) = \mathrm{e}^x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\ln(x)) = \frac{1}{x},\ x > 0$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x},\ x \neq 0$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a},\ x > 0$