I214 System Optimization
Chapter 4: Network Optimization (Part B)

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Outline

4.3 Maximum Flow Problem

4.3.1 Definition

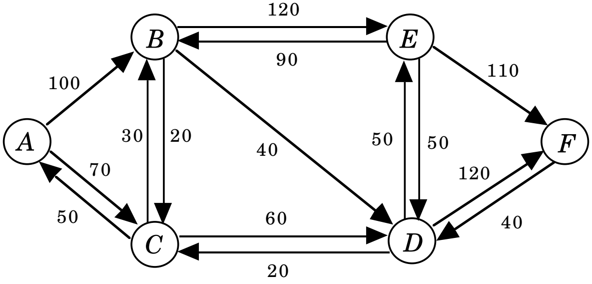
4.3.2 Max-Flow Min-Cut Theorem

4.3.3 Ford-Fulkerson Method

4.3 Maximum Flow Problem

Recall the example of a maximum flow problem we gave in Lecture 1.

Problem Consider the Stone River Shipping Company. The figure below shows a network of transportation routes. The number on each edge indicates the maximum amount of goods this company can transport per day. Find the maximum amount of goods this company can transport from city A to city F per day, where we are allowed to use multiple routes in parallel, and goods can be transshipped at each city.



4.3.1 Definition

A *flow network*, is a graphical network where each edge has a capacity and along each edge some amount of flow can be sent, but not exceeding the capacity.

For a flow network, let $(\mathcal{V}, \mathcal{E})$ be a graph:

- The source node is s.
- The sink node is t.
- $c_{i\to j} \geq 0$ is the *capacity* from node i to node j.
- $x_{i\to j}$ is the *flow* from node i to node j.
- Total flow from s to t is v, called the *flow value*.

Flow Rules

The flow $x_{i \to j}$ should observe the following rules:

- Flow cannot exceed capacity: $x_{i \to j} \leq c_{i \to j}$
- Flow balance: the flow into a node q matches the flow out of node q:

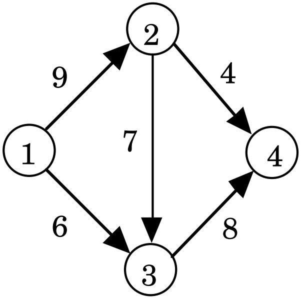
$$\sum_{i \in \mathcal{S}} c_{i \to q} = \sum_{j \in \mathcal{T}} c_{q \to j}$$

- Flow from source s matches flow to sink t, and is equal to v:

$$\sum_{i \in \mathcal{V}} x_{s \to i} = \sum_{j \in \mathcal{V}} x_{j \to t} = v$$

Example

The network shown below has four nodes, with capacities as indicated.



What is $c_{i\to j}$? What is a possible flow? $\bigstar$1

Two Questions for Flow Networks

For flow networks, two questions arise:

- What is the maximum possible flow?
  - The max-flow, min-cut theorem gives the maximum possible flow.
- How to find the maximum possible flow?
  - Linear programming ★2
  - Ford-Fulkerson method

4.3.2 Max-Flow Min-Cut Theorem

- The capacity of a network is the maximum flow on that network.
- The max-flow min-cut theorem gives the maximum flow.
- A cut partitions a network into two sets of nodes, and the capacity of a cut is the sum of the capacities between the two sets.
- The capacity of the network is the minimum capacity of all possible cuts.

Definition of A Cut and Its Capacity

Definition

For a network $G = (\mathcal{V}, \mathcal{E})$ let $\mathcal{S}$ and $\mathcal{T}$ be a partition of $\mathcal{V}$. A *cut* denoted $(\mathcal{S}, \mathcal{T})$ is

$$(\mathcal{S}, \mathcal{T}) = \{(i, j) \in \mathcal{E} \mid i \in \mathcal{S}, j \in \mathcal{T}\} \tag{1}$$

That is, a cut $(\mathcal{S}, \mathcal{T})$ is the set of edges (i, j) such that $i \in \mathcal{S}$ and $j \in \mathcal{T}$.

For a flow network with source s and sink t, assume that $s \in \mathcal{S}$ and $t \in \mathcal{T}$.

Definition

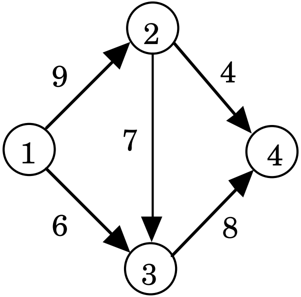
For a flow network $G = (\mathcal{V}, \mathcal{E})$ with capacities c_{ij}, let $\mathcal{S}$ and $\mathcal{T}$ be a partition of $\mathcal{V}$. The *capacity of cut* $(\mathcal{S}, \mathcal{T})$ is:

$$c(\mathcal{S}, \mathcal{T}) = \sum_{(i,j) \in (\mathcal{S}, \mathcal{T})} c_{ij}. \tag{2}$$

The capacity of a cut is the sum of the capacity of the edges in a cut.

Example — Cuts and Their Capacity

For the network below, find the capacity of all possible cuts.



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Max-flow min-cut theorem

Proposition

Max-flow min-cut theorem The maximum value of an s-t flow is equal to the minimum capacity over all s-t cuts.

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4.3.3 Ford-Fulkerson Method

- A path with available capacity is called an *augmenting path*.
- This available capacity is removed to form a *residual network*.

The Ford-Fulkerson method finds maximum flow for a given network.

- iteratively increases the flow
- replaces the original network with a residual network
- as long as there is a path from the source node to the to the sink node with available capacity, then additional flow is sent along this path.
- These steps repeat until there are no augmenting paths available.

Residual Network

Given a network G and a flow x, a *residual network* G_f is a network that shows how much the flow can be changed with respect to x:

- A label c_f on the residual network indicates how much additional flow is available.
- G_f has the same set of vertices as G, but the edges may be added and the capacities are different.
- The value c_f indicates the margin to increase the flow, and is called the *residual capacity*.

In the Ford-Fulkerson method, it may be necessary to *decrease* flow on certain edges.

- In order to represent the possible decrease of flow on an edge $i \to j$, the residual network G_f adds a new edge $j \to i$ which is not in the original graph G.

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Residual Capacity

More formally, if the flow is $x_{i \to j} > 0$, then the residual capacity is updated as:

$$c_f(e) = \begin{cases} c(e) - x_{i \to j} & \text{if } e = (i \to j) \\ x_{i \to j} & \text{if } e = (j \to i) \end{cases} \tag{3}$$

The important idea is that if there is a flow of value $x_{i \to j}$, then we can cancel it by adding the same amount of flow in the reverse direction.

Augmenting Path

An *augmenting path* is any path $\mathcal{P}$ on the residual network G_f from source s to the sink t with positive flow.

The flow on $\mathcal{P}$ is the minimum of the residual capacities:

$$\Delta = \min\{c(e), e \in \mathcal{P}\}$$

We can add at most Δ along the following amount of flow along the path.

Ford-Fulkerson Method

Ford-Fulkerson Method As long as an augmenting path exists, select one and add as much flow along it as possible.

- Terminates when there is no feasible flow from s to t.
- After termination, for any edge $i \to j$ in the original graph, the flow $x_{i \to j}$ is given by the residual capacity $c_f(j \to i)$ for the reverse link.

The above does not state *how* to select the augmenting path, so we call it a "method" rather than an algorithm.

Example

Apply the Ford-Fulkerson method to the flow network below.

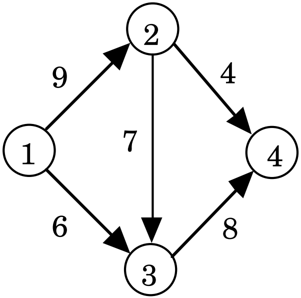

Figure 1: Example of a flow network where each edge has a capacity. The source node s is 1, the sink node t is 4.

Class Info

- Tutorial Hours: Today at 13:30.
- Homework 4. Deadline: Friday, January 13 at 18:00.
- Next lecture: Monday, January 16 at 10:50
- Homework 5 (Ford-Fulkerson; Exercise 4.5). Do not submit. Solutions will be published before midterm exam.
- Midterm exam will be Friday, January 20 13:30–15:10. Paper-based in the classroom.

Midterm Exam

The midterm is Friday, January 20 13:30–15:10. The exam is closed book. You may use:

- One page of notes, A4-sized paper, double-sided OK.
- Blank scratch paper

You may not use anything else: No printed materials, including books, lecture notes, and slides. No notes (except as above). No internet-connected devices. No calculators — largest matrix inverse will be 2×2.

Exam Content

- Problems similar to Homework 1–5
- Solutions to Homework 1–5 are provided.

Tutorial Hours — Problem

Consider the maximum flow problem for the network below, where the source is vertex 1 and the sink is vertex 4.

Find the minimum and maximum number of iterations in the execution of Ford-Fulkerson method. Note that the number of iterations is the number of times augmenting paths are added.

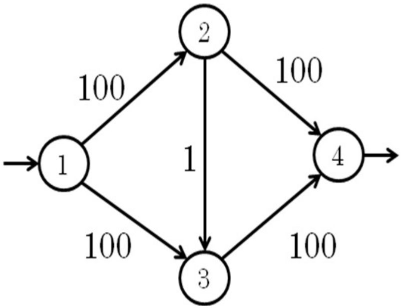

Figure 2: A network.