# I214 System Optimization
# Chapter 5: Nonlinear Programming (Part C)

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# Outline

## 5.6 Constrained Optimization Problems

Previously we considered equality-constrained optimization. Now we introduce inequalities.

The general nonlinear optimization problem is:

$$\begin{aligned} \textbf{Minimize} \quad & f(\mathbf{x}) \\ \textbf{subject to} \quad & h_i(\mathbf{x}) = 0, \;\; i = 1, 2, \ldots, \ell \\ & g_j(\mathbf{x}) \leq 0, \;\; j = 1, 2, \ldots, m \end{aligned} \tag{1}$$

## 5.6.1 Lagrangian Function

For the constrained optimization problem in (1), the *Lagrangian function* or simply *Lagrangian* is given as:

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) + \sum_{i=1}^{\ell} \lambda_i h_i(\mathbf{x}) + \sum_{j=1}^{m} \mu_j g_j(\mathbf{x}), \tag{2}$$

where $\mu_j \geq 0,\ j = 1, \cdots, m$ and there is no sign constraint on $\lambda_i$. The $\lambda_i$ and $\mu_j$ are called *Lagrangian multipliers*. The Lagrangian function $L$ is a function of variables $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_\ell) \in \mathbb{R}^\ell$ and $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_m)$.

## 5.6.2 Conditions for Optimality

This subsection introduces the Karush-Kuhn-Tucker condition, called KKT conditions for short. They are necessary conditions for optimality. In addition second-order necessary conditions and second-order sufficient conditions for optimality are given.

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### Definition

When a constraint holds with equality, then the constraint is called *active constraint*.

For example, suppose there exists a constraint $g_j(\mathbf{x}) \leq 0$ and $g_j(\mathbf{a}) = 0$ holds at a point $\mathbf{x} = \mathbf{a} \in \mathcal{S}$. Then the constraint $g_j(\mathbf{x}) \leq 0$ is called active at $\mathbf{x} = \mathbf{a}$.

# Recall the Visualizaiton of Lagrange Multipliers

This figure illustrates minimizing $f(x, y)$ subject to one equality constraint $g(x, y) = c$.

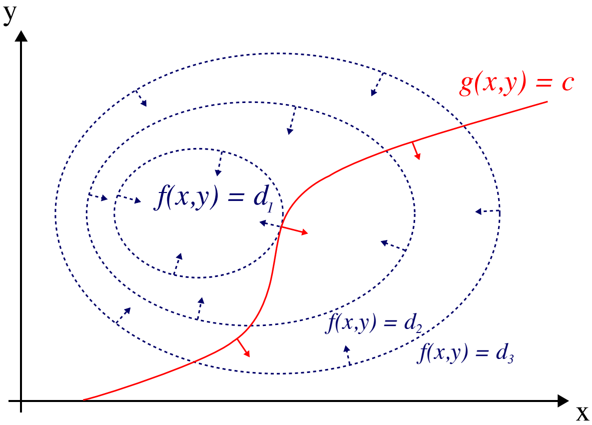
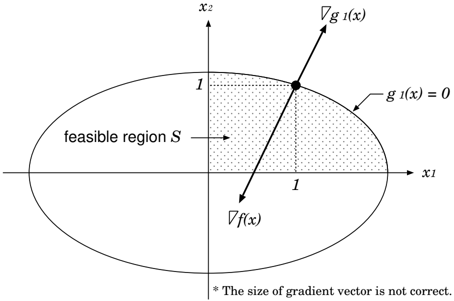



Figure 1: Credit: Wikipedia/Nexcis

# Example

$$
\begin{aligned}
\textbf{Minimize} \quad & f(\mathbf{x}) = -2x_1x_2^2 \\
\textbf{subject to} \quad & g_1(\mathbf{x}) = \tfrac{1}{2}x_1^2 + x_2^2 - \tfrac{3}{2} \le 0 \\
& g_2(\mathbf{x}) = -x_1 \le 0 \\
& g_3(\mathbf{x}) = -x_2 \le 0
\end{aligned}
\tag{3}
$$




Assume that a candidate solution $\mathbf{x}^* = (1, 1)^t$ has already been found. Check if it is optimal or not.

# 5.7 Methods to Solve Nonlinear Problem

The following shows a typical procedure to solve nonlinear programming problem using the optimality conditions.

**Generate Candidates** Using the Karush-Kuhn-Tucker condition, enumerate candidates for optimal solutions.

**Testing Candidates:** Using the second-order conditions, test whether each candidate is:

- an optimal solution — the sufficient condition is satisfied
- not an optimal solution — the necessary condition is not satisfied, or
- un-decidable — other ad hoc approach is utilized to check candidates.

# Find Active Constraints

Try various combinations of active and inactive, as shown in the following procedure:

1. For each inequality constraint, we assume it to be active or inactive. Let $\mathcal{I} \subseteq \{1, \cdots, m\}$ be a set of indices of active inequality constraints.
2. Solve the following simultaneous equations with respect to $\mathbf{x}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$.

$$\begin{cases} \nabla f(\mathbf{x}) + \displaystyle\sum_{i=1}^{\ell} \lambda_i \nabla h_i(\mathbf{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\mathbf{x}) = \mathbf{0} \\ h_i(\mathbf{x}) = 0, \quad i \in \{1, \cdots, \ell\} \\ g_j(\mathbf{x}) = 0, \quad j \in \mathcal{I} \\ \mu_k = 0, \qquad k \in \{1, \cdots, m\} \backslash \mathcal{I} \end{cases}$$

3. Check whether the solution satisfies constraints and assumptions. That is:

$$\begin{array}{lll} \text{(a)} & \mu_j \geq 0\ ? & j \in \mathcal{I} \\ \text{(b)} & g_k(\mathbf{x}) < 0\ ? & k \in \{1, \cdots, m\} \backslash \mathcal{I} \end{array}$$

- If all constraints and assumptions satisfied, output candidate solution
- Otherwise, repeat with other active/inactive assumptions.

# Example

In some cases, Lagrangian multiplier and Karush-Kuhn-Tucker condition is used for finding an optimum point analytically, as shown in the following example. Consider the following problem.

$$\begin{array}{ll} \textbf{Minimize} & f(\mathbf{x}) = \mathbf{c}^{\mathrm{t}}\mathbf{x} \\ & \text{where } \mathbf{c} \neq \mathbf{0} \\ \textbf{subject to} & \mathbf{x}^{\mathrm{t}}\mathbf{x} - 1 \leq 1 \end{array}$$

# Example

$$\begin{aligned} \textbf{Minimize} \qquad & f(\mathbf{x}) = \mathbf{x}^{\mathrm{t}}\mathbf{Q}\mathbf{x} \\ \textbf{subject to} \qquad & \mathbf{x}^{\mathrm{t}}\mathbf{x} - 1 = 0 \end{aligned}$$

where $\mathbf{Q}$ is symmetric and positive definite.

# Class Info

- Midterm exam results: after I finish grading, your score will appear on the LMS.
- Next lecture: Friday, January 27. Duality in Nonlinear Programming. Begin Combinatorial Optimization
- Homework 6. Deadline: Friday, January 27 at 18:00.