I214 System Optimization
Chapter 3: Duality in Linear Programing

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Section “Change of Problem” was moved to slides2b.pdf

Outline

3.1 Dual Problem

3.1.1 Motivation

This section includes the dual program, the weak duality theorem and the strong duality theorem.

"Motivation" Consider an LP, given by:

$$\min \mathbf{c}^{\mathrm{t}}\mathbf{x}, \mathbf{A}\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq 0. \tag{1}$$

then we perform some algebra to show that the best lower bound is the optimal value of:

$$\max \mathbf{b}^{\mathrm{t}}\mathbf{y}, \mathbf{A}^{\mathrm{t}}\mathbf{y} \leq \mathbf{c}, \mathbf{y} \geq \mathbf{0}. \tag{2}$$

This program is a new linear program in $\mathbf{y}$, where the roles of the variables has changed. ★1

3.1.2 Dual of the Standard Form Problem

Let P be a linear program called the *primal linear program*.

The *dual linear program* D is another linear program derived from P in the following way:

- The objective function's c_i in the primal LP becomes the constraints' b_i in the dual LP;
- The constraints b_i in the primal LP becomes part of the objective function in the dual LP;
- The direction of the objective is reversed — maximum in the primal becomes minimum in the dual, or vice versa.

Dual of the Standard Form Problem

In particular, if the primal problem P is given by:

Minimize

$$z = \mathbf{c}^T \mathbf{x}$$

Subject to

$$\begin{array}{rcl} \mathbf{A}\mathbf{x} & = & \mathbf{b} \\ \mathbf{x} & \geq & \mathbf{0} \end{array} \tag{P}$$

Then the dual problem D is given by:

Maximize

$$w = \mathbf{b}^T \mathbf{y}$$

Subject to

$$\begin{array}{rcl} \mathbf{A}^{\mathrm{t}}\mathbf{y} & \leq & \mathbf{c} \end{array} \tag{D}$$

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The Dual of the Dual Is ...

Proposition

The dual of the dual problem is equivalent to the primal problem.

In other words, the dual problem can be transformed back to the primal problem.

3.1.3 Weak Duality Theorem

The weak duality theorem states that the objective value of the dual LP at any feasible solution is always a bound on the objective of the primal LP at any feasible solution.

Proposition

Let $\mathbf{x}$ be any feasible solution of the primal problem (P), and let $\mathbf{y}$ be any feasible solution of the dual problem (D). Then:

$$\mathbf{b}^{\mathrm{t}}\mathbf{y} \leq \mathbf{c}^{\mathrm{t}}\mathbf{x}. \quad \bigstar 3 \tag{3}$$

3.1.4 Strong Duality Theorem

The strong duality theorem states that if the primal problem has an optimal solution then the dual problem has an optimal solution too, and the two optimal solutions are equal.

Proposition

If the primal problem (P) has an optimal solution, then the dual problem (D) also has an optimal solution. Moreover, the minimum value of z in (P) is equal to the maximum value of w in (D).

Relationship Between Solution Types

Proposition

If the primal problem (P) is unbounded, then the dual problem (D) is infeasible. If the dual problem (D) is unbounded, then the primal problem (P) is infeasible.

This table summarizes the relationship between types of solutions of the primal problem and the dual problem.

		Dual		
		optimal exists	unbounded	infeasible
Primal	optimal exists	Yes	No	No
	unbounded	No	No	Yes
	infeasible	No	Yes	Yes

3.2 Generalized Duality

In Section **??** a problem and its dual was given in the following form:

$$\begin{aligned} \min \mathbf{c}^t\mathbf{x} \\ \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned} \qquad\qquad \begin{aligned} \max \mathbf{b}^t\mathbf{y} \\ \mathbf{A}^t\mathbf{y} \leq \mathbf{c} \\ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

In Section **??** a problem and its dual was given in a slightly different form:

$$\begin{aligned} \min \mathbf{c}^t\mathbf{x} \\ \mathbf{A}\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned} \qquad\qquad \begin{aligned} \max \mathbf{b}^t\mathbf{y} \\ \mathbf{A}^t\mathbf{y} \leq \mathbf{c} \\ \mathbf{y} \text{ is free} \end{aligned}$$

Both forms are valid, and these definitions can be generalized.

3.2.1 Form of General Duality

Consider that two problems $P_{\max}$ and $P_{\min}$ are duals, and they have general form:

$$
\begin{aligned}
P_{\max} : \max \mathbf{c}^{\mathrm{t}}\mathbf{x} \\
\mathbf{A}\mathbf{x} \ ? \ \mathbf{b} \\
\mathbf{x} \ ? \ \mathbf{0}
\end{aligned}
\qquad\qquad
\begin{aligned}
P_{\min} : \min \mathbf{b}^{\mathrm{t}}\mathbf{y} \\
\mathbf{A}^{\mathrm{t}}\mathbf{y} \ ? \ \mathbf{c} \\
\mathbf{y} \ ? \ \mathbf{0},
\end{aligned}
$$

where $?$ may be $\leq, =$ or $\geq$ for the constraints and $?$ may be $\leq, \geq$ or "free" for the variable. Table 1 shows possible primal-dual pairs.

3.2.2 Conversion Between Primal-dual pairs

Table 1: Primal-dual pairs of problems. (Guenin, Konemann, Tuncel, page 143) .

	$P_{\max}$		$P_{\min}$	
	$\max \mathbf{c}^t\mathbf{x}$		$\min \mathbf{b}^t\mathbf{y}$	
	$\leq$ constraint	$\leftrightarrow$	≥ 0 variable	
$\mathbf{Ax}\ ?\ \mathbf{b}$	$=$ constraint	$\leftrightarrow$	free variable	$\mathbf{y}\ ?\ 0$
	$\geq$ constraint	$\leftrightarrow$	≤ 0 variable	
	≥ 0 variable	$\leftrightarrow$	$\geq$ constraint	
$\mathbf{x}\ ?\ \mathbf{0}$	free variable	$\leftrightarrow$	$=$ constraint	$\mathbf{A}^t\mathbf{y}\ ?\ \mathbf{c}$
	≤ 0 variable	$\leftrightarrow$	$\leq$ constraint	

There are 9 types of problems — 3 choices for the constraint and 3 choice for the variable. ★4

3.2.3 Strong Duality Example — With Solution

The following is an example of strong duality — the solution to a problem and its dual have the same optimal value, but not the same solution.

Example

Recall an earlier problem:

$$\begin{array}{rllllll}
\textbf{Minimize} & z = -x_1 - x_2 & & & & & \\
\textbf{subject to} & x_1 + 2x_2 + x_3 & & & & = 3 \\
& x_1 & & x_4 & & = 2 \\
& & x_2 & & x_5 & = 1 \\
& x_1\ , & x_2\ , & x_3\ ,\ x_4\ , & x_5 & \geq 0
\end{array} \tag{4}$$

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3.3 Complementary Slackness Condition

The strong duality theorem told us we can get the optimal value of the dual problem w^* from the optimal value of the primal problem z^*.

However, it does not tell us about the the optimal solution $\mathbf{y}^*$.

We can obtain an optimal solution of the dual problem using an optimal solution of the primal problem using the complementary slackness theorem.

3.3.1 Proposition — Complementary Slackness Condition

Proposition

Let $\mathbf{x}$ be a feasible solution of the primal problem (P) and let $\mathbf{y}$ be a feasible solution of the dual problem (D). They are optimal solutions of (P) and (D), respectively, if and only if

$$x_j = 0 \quad \text{or} \quad \mathbf{a}_j^{\mathrm{t}}\mathbf{y} = c_j \quad \text{for } j = 1, 2, \ldots, n \tag{5}$$

$$y_i = 0 \quad \text{or} \quad \mathbf{A}_i\mathbf{x} = b_i \quad \text{for } i = 1, 2, \ldots, m \tag{6}$$

Here $\mathbf{A}_j$ is row j of $\mathbf{A}$ and $\mathbf{a}_i$ is column i of $\mathbf{A}$.

3.3.2 Example — Complementary Slackness Condition

Example

Using the complementary slackness condition, find an optimal solution of the dual problem of Example **??**.

Begin by writing the primal and dual problems in matrix form:

$$\textbf{Minimize} \quad (-1,-1,0,0,0)(x_1,x_2,x_3,x_4,x_5)^{\mathrm{t}}$$

$$\textbf{subject to}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \tag{7}$$

$$\mathbf{x} \geq \mathbf{0}$$

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3.4 Farkas' Lemma

Suppose that you want to prove if a problem is if a given system of equalities is infeasible.

Proposition

Let $\mathbf{A}$ be an $m \times n$ matrix and let $\mathbf{b}$ be an m vector. Then exactly one of these statements are true:

1. $\mathbf{A}\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ has a solution
2. there exists a vector $\mathbf{y}$ such that $\mathbf{A}^{\mathrm{t}}\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^{\mathrm{t}}\mathbf{y} < \mathbf{0}$.

Suppose that there is a vector $\mathbf{y}$ which satisfies (II). By Farkas' lemma, (I) cannot be true, and thus any corresponding linear program does not have a solution.

This $\mathbf{y}$ is a "certificate" or proof that this problem does not have a feasible solution.

Class Info

- No tutorial hours today because Friday follows Monday schedule. Instead, let's discuss HW2 now.
- Homework 2 on LMS. Deadline: December 23 at 18:00.
- Next lecture: Friday, January 6 at 9:00.
- Homework 3 on LMS. Deadline: Friday, January 6 at 18:00.