I214 System Optimization
Chapter 2: Linear Programing (Part B)

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Review — Solve Linear Program (LP) in Standard Form

The standard-form linear program has the form:

$$\begin{array}{ll} \textbf{Maximize} & \mathbf{c}^{\mathrm{t}}\mathbf{x} \\ \textbf{Subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \tag{1}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \text{ and } \mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Review — Basis

The n columns of $\mathbf{A}$ can be partitioned into two sets:

- $\mathcal{B}$ consisting of m columns and
- $\mathcal{N}$ consisting of the other $n - m$ columns.

Then, $\mathbf{A}_{\mathcal{B}}$ is an $m \times m$ matrix, and $\mathbf{A}_{\mathcal{N}}$ is an $(n-m) \times m$ matrix. In what follows, $\mathcal{B}$ will be the *basis* set, and $\mathcal{N}$ will be the *nonbasis* set.

Review — Basic Solution and Canonical Form

Definition

A solution $\mathbf{x}$ is called a *basic solution* if the following two conditions are satisfied:

$$\mathbf{x}_{\mathcal{N}} = \mathbf{0} \text{ and} \tag{2}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{3}$$

By choosing $\mathbf{x}_{\mathcal{N}} = \mathbf{0}$, the second condition is equivalent to $\mathbf{A}_{\mathcal{B}}\mathbf{x}_{\mathcal{B}} = \mathbf{b}$. If $\mathbf{A}_{\mathcal{B}}$ is invertible, then $\mathbf{x}_{\mathcal{B}} = \mathbf{A}_{\mathcal{B}}^{-1}\mathbf{b}$.

Definition

An LP problem with $\mathbf{c}$ in the objective function $\mathbf{c}^{\mathrm{t}}\mathbf{x}$ is in the *canonical form for $\mathcal{B}$* if the equality constraint $\mathbf{A}$ has the form:

$$\mathbf{A}_{\mathcal{B}} = \mathbf{I}_m \text{ and} \tag{4}$$

$$\mathbf{c}_{\mathcal{B}} = 0. \tag{5}$$

where $\mathbf{I}_m$ is the identity matrix.

Review — Canonical Form Example

Example

Minimize $\mathbf{c}^{t}\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ with:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The benefit of the canonical form is that it is easy to find a feasible solution.

Outline

2.5 Simplex Method Using Linear Algebra

2.5.1 Simplex Iteration

1. Start with the problem in canonical form. We easily get a basic solution $\mathbf{x}$ and its value $z(\mathbf{x})$.
2. If all $c_i \geq 0$, then stop and output the basic solution $\mathbf{x}$ as the optimal solution.
3. Choose one of the c_i satisfying $c_i < 0$. The corresponding variable x_i is increased to $t > 0$. Since $c_i < 0$, if we increase x_i, then $c_i x_i$ decreases, and the objective function decreases. Other non-basic variables remain 0. Write $\mathbf{x}_{\mathcal{N}} = (t, 0, \ldots, 0)$.
4. Find the largest value of t that follows the constraints, using the ratio test.
   $\mathbf{Ax} = \mathbf{b}$ can be written as $t\mathbf{A}_i + \mathbf{x}_{\mathcal{B}} = \mathbf{b}$. Apply the ratio test:
   $\mathbf{x}_{\mathcal{B}} = \mathbf{b} - t\mathbf{A}_i \geq \mathbf{0} \Rightarrow \mathbf{b} \geq t\mathbf{A}_i$.
5. Using this value of t, get a new solution $\mathbf{x}'$ and corresponding $z(\mathbf{x}')$.
6. The new $\mathbf{x}'$ is a basic solution because it has m non-zero values. These new m values form new basis $\mathcal{B}'$. Put the problem into canonical form with respect to $\mathcal{B}'$.
7. Go to Step 1.

2.5.2 Simplex Example, First Iteration

Suppose we have a problem in the canonical form, $\min \mathbf{c}^{\mathrm{t}}\mathbf{x}$ with:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

With basis $\mathcal{B} = \{3, 4, 5\}$, the basic solution is $\mathbf{x} = (0, 0, 3, 2, 1)$. The value of the objective function is $z(\mathbf{x}) = \mathbf{c}^{\mathrm{t}}\mathbf{x} = 0$.

Choose One Non-Basic Variable

The current basic solution is:

$$\mathbf{x} = (0, 0, 3, 2, 1). \tag{7}$$

Can we decrease z by increasing the value the nonbasic variables x_1 or x_2? Yes — let us decrease z by increasing x_1 and keeping $x_2 = 0$. In other words, look for a new feasible solution $\mathbf{x}$ with where:

$$x_1 = t. \tag{8}$$

Maximize t While Satisfying Constraints

The new values must satisfy the constraints:

$$\mathbf{b} = \mathbf{A}\mathbf{x} \tag{9}$$

★1

Choose the smallest t which satisfies the non-negative ratios:

$$t = \min\left\{\frac{3}{1}, \frac{2}{1}, -\right\} \tag{10}$$

That is, $t = 2$.

Get a New Solution $\mathbf{x}'$

Apply this value of $t = 2$ to $\mathbf{x}_{\mathcal{B}} = \mathbf{b} - \mathbf{A}_i t$ (★2), to obtain the new solution $\mathbf{x}' = (2, 0, 1, 0, 1)$.

This solution gives objective function $z(\mathbf{x}') = \mathbf{c}^{\mathrm{t}}\mathbf{x}' = -2$, which has improved the solution.

New Basis $\mathcal{B}'$ and New Canonical Form

Since $\mathbf{x}' = (2, 0, 1, 0, 1)$, choose a new basis $\mathcal{B}' = \{1, 3, 5\}$. Using Proposition **??**, the problem can be written in a new canonical form: $\min \mathbf{c}^{\mathrm{t}}\mathbf{x}$, with

$$\mathbf{A}' = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \mathbf{b}' = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{c}' = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Notes:

- The linear term $\mathbf{d}^{\mathrm{t}}\mathbf{b}$ given in Proposition **??** has been dropped.
- We started with $\mathcal{B} = \{3, 4, 5\}$ and we finished with $\mathcal{B}' = \{1, 3, 5\}$ so we say:
  - x_4 *left* the basis, and
  - x_1 *entered* the basis.

2.5.3 Simplex Example, Second Iteration

Now that we have the problem again in canonical form,

$$\mathbf{A}' = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \mathbf{b}' = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{c}' = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

we can attempt again to decrease the objective function.

The nonbasis is $\mathcal{N} = \{2, 4\}$. Note that $c_2 = -1$ and $c_4 = 1$. Since $c_2 < 0$ and $c_4 > 0$, only increasing x_2 will decrease the objective function z, thus we should choose x_2 (and not x_4). ★3

Simplex Example, Second Iteration

Here, x_3 left the basis, and x_2 entered the basis, and the new basis is $\mathcal{B}' = \{1, 2, 5\}$. Using this we put the problem into canonical form:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0.5 & -0.5 & 0 \\ 0 & 0 & -0.5 & 0.5 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 0.5 \\ 0.5 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0.5 \\ 0 \end{bmatrix} \tag{11}$$

2.5.4 Simplex Example, Third Iteration

For the third iteration, we again have the problem in canonical form.

However, we note that that all $c_i \geq 0$, and thus there is no variable to select to decrease $z(\mathbf{x})$. Since all $c_i \geq 0$, this solution is optimal.

The optimal solution is $(2, \frac{1}{2}, 0, 0, 1)$. If we apply this in (6), the minimum of the objective function is $z(\mathbf{x}) = -\frac{5}{2}$.

2.5.5 Condition for Optimality

In the canonical form, if every $c_i \geq 0$, then the basic solution is minimum and optimal.

Proposition

For an LP in canonical form with a feasible basic solution $\mathbf{x}$, if $c_i \geq 0$ for all i, then $\mathbf{x}$ is an optimal feasible solution.

To see this, note that in the canonical form, the basic solution $\mathbf{x}$ gives $\mathbf{c}^{\mathrm{t}}\mathbf{x} = 0$. Since the $c_i \geq 0$ and $x_i \geq 0$, it is not possible to for the objective function $\mathbf{c}^{\mathrm{t}}\mathbf{x}$ to be less than 0.

Since it is possible to generate all basic feasible solutions by repeating pivot operations, we can obtain an optimal solution by this method (if it exists).

2.6 Simplex Method Using Tabular

The tabular simplex method, is a compact way to perform the operations described in the previous section.

Instead of matrix inversion, row and column operations are performed on the tabular.

In what follows, we assume that $\mathrm{rank}(\mathbf{A}) = m$, that is, no redundant equations exist. If they exist, they could be removed.

2.6.1 Simplex Tabular — Example Problem

Minimize z,

$$z + x_1 + x_2 = 0$$

Subject to

$$\begin{array}{cccccccc} x_1 & +2x_2 & +x_3 & & & = & 3 \\ x_1 & & & +x_4 & & = & 2 \\ & x_2 & & & +x_5 & = & 1 \\ x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0 \end{array}$$

We use the following table, called a *simplex tabular*, to represent this problem. ★5

		x_1	x_2	x_3	x_4	x_5
z	0	1	1	0	0	0
x_3	3	1	2	1	0	0
x_4	2	1	0	0	1	0
x_5	1	0	1	0	0	1

How to select the pivot column and the pivot row

Pivot column Find a column with a positive coefficient in z-row. (When there are more than one such rows, we can select any of them, but one with the largest value is often selected.) The nonbasic variable corresponding to the column *enters* the basis.

Pivot row Each value of the basic solution is divided by the coefficient in the pivot column, provided that the coefficient is positive (*computing the ratio*). Select one row with the smallest ratio. The basic variable corresponding to the row *leaves* the basis.

Continuation of Simplex Tabular Example

★6 Continue the example. Iteration 2:

		x_1	x_2	x_3	x_4	x_5
z	-2	0	1	0	-1	0
x_3	1	0	2	1	-1	0
x_1	2	1	0	0	1	0
x_5	1	0	1	0	0	1

Iteration 3:

		x_1	x_2	x_3	x_4	x_5
z	-2.5	0	0	-0.5	-0.5	0
x_2	0.5	0	1	0.5	-0.5	0
x_1	2	1	0	0	1	0
x_5	0.5	0	0	-0.5	0.5	1

Since no coefficient in z-row is positive, the basic solution is optimal.

Generalized Simplex Method

Simplex Method:

1. Find a feasible basic solution.
2. If every coefficient in z-row is nonpositive, then halt. The basic solution is optimal.
3. Otherwise, select one column with a positive coefficient. (In general, select a column with the largest one).
4. Each value of the basic solution is divided by the coefficient in the pivot column, provided that the coefficient is positive (computing the ratio). Select one row with the smallest ratio. If every coefficient is nonpositive, then halt (the problem is unbounded).
5. Apply the pivot operation to the selected row and the column. Go to Step 2.

Unbounded Problems

Any linear program has one of three possible outcomes:

1. It is infeasible — no solution exists
2. It has an optimal solution
3. It is unbounded

A problem is unbounded if for the nonbasic variable in canonical form:

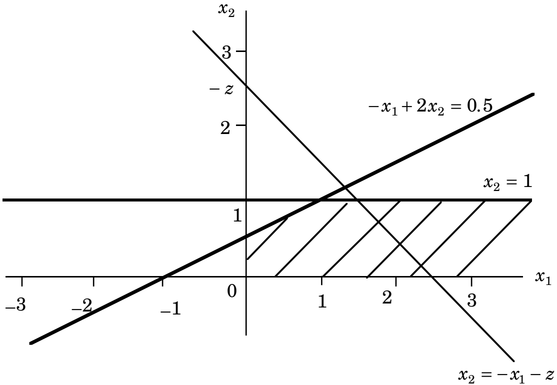
- the coefficient in z-row is positive, and
- all coefficients in the column is nonpositive, i.e., less than or equal to 0,

then the problem is unbounded.

Unbounded Problem — Example

The following simplex tabular is an example of an unbounded problem.

		x_1	x_2	x_3	x_4
z	0	1	1	0	0
x_3	0.5	−1	2	1	0
x_4	1	0	1	0	1



Cycling

Cycling is a situation that the same basis appears more than once in the iteration. This may happen when the improvement of z is 0.

This phenomenon can be avoided by selecting the pivot row and the column using *Bland's Rule*, as follows:

Pivot column
: Find a column with a positive coefficient in z-row. When there are more than one such rows, select one with the smallest index.

Pivot row
: Each value of the basic solution is divided by the coefficient in the pivot column, provided that the coefficient is positive (computing the ratio). Select one row with the smallest value. If there are more than one such rows, then select one for which the corresponding basic variable has the smallest index.

In general, this selection method gives slower convergence.

2.7 Two-Phase Simplex Algorithm

In order to start the simplex method, an initial basic solution is needed.

If all constants in the right side are nonnegative and all inequalities are $\leq$, then the slack variables can be initial basic variable.

However, in general we do not have this form and usually it is not easy to find the initial basic solution.

Basic Solution is Feasible — Two-Phase Method Not Needed

Recall our earlier example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

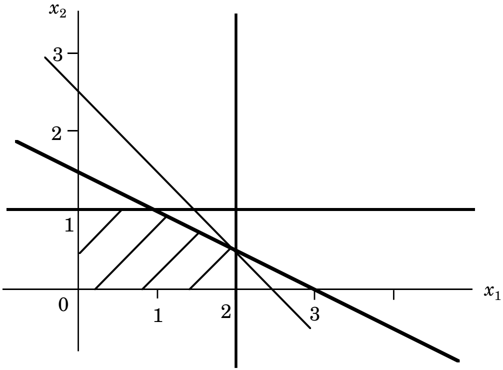


Figure 1: The basic solution $(0, 0)$ is feasible. $(0, 0)$ is inside the shaded feasible region.

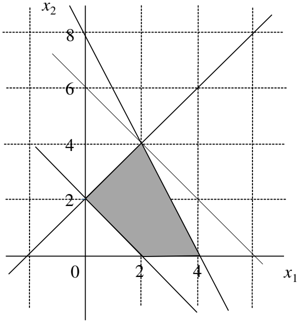
This is canonical form, and the basic solution is $\mathbf{x} = (0, 0, 3, 2, 1)$.

This solution is feasible $\Rightarrow$ Use $\mathbf{x}$ in first step of simplex algorithm.

Basic Solution is Infeasible — Use Two-Phase Method

Maximize $x_1 + x_2$
Subject to

$$\begin{array}{rrcl} -x_1 & +x_2 & \leq & 2 \\ 2x_1 & +x_2 & \leq & 8 \\ x_1 & +x_2 & \geq & 2 \\ x_1, & x_2 & \geq & 0 \end{array}$$



Basic solution is $\mathbf{x} = (0, 0, 2, 8, -2)$. This is infeasible because $x_5 \geq 0$ is not satisfied. Also, $(0, 0)$ is not inside the feasible region.

Need a feasible solution to begin the simplex method.

Phase 1 — Auxiliary Problem

The following method can find an initial basic solution. Consider the problem:

$$\begin{aligned} \textbf{Minimize} \quad & \mathbf{c}^{\mathrm{t}}\mathbf{x} \\ \textbf{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned} \tag{12}$$

Construct the following *auxiliary problem*

$$\begin{aligned} \textbf{Minimize} \quad & y_1 + y_2 + \cdots + y_m \\ \textbf{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{b} \\ & \mathbf{x} \geq 0 \\ & \mathbf{y} \geq 0 \end{aligned} \tag{13}$$

The variables $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ are called *auxiliary variables* or *artificial variables*.

The feasible solution to the auxiliary problem $\mathbf{x} = \mathbf{0}$ is $\mathbf{y} = \mathbf{b}$ is easily obtained.

Two-Phase Method

Proposition

The optimal solution to the auxiliary problem (13) is a feasible solution to the main problem (12).

This proposition gives a method to solve any LP using the Two-Phase Method:

Phase 1 Construct the auxiliary problem. **During iterations, move the all of the artificial variables into the nonbasis** (it is not necessary to completely solve the auxiliary problem).

Phase 2 Now you have a canonical form of the problem with a basis in the original variables. Remove the artificial variables y and proceed with standard simplex method.

Example

Solve the following standard-form LP:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}.$$

This problem only has equality constraints, there are no slack variables.

Phase I: Construct the auxiliary problem and find its solution.

Phase II: Solve the original problem using the basis found in Phase I.

Example

We first solve the following *Phase I* problem:

Minimize $w,$

$$w - y_1 - y_2 = 0$$

Constraint

$$\begin{array}{ccccccccc} z & -3x_1 & -2x_2 & & & & & = & 0 \\ & 2x_1 & +x_2 & -x_3 & & +y_1 & & = & 2 \\ & 4x_1 & +3x_2 & & -x_4 & & +y_2 & = & 6 \\ & x_1, & x_2, & x_3, & x_4, & y_1, & y_2 & \geq & 0 \end{array} \qquad (14)$$

where $y_1,\ y_2$ are called *artificial variables*, that will be selected as basic variables in the first iteration, and w is the sum of them. If $w = 0$ is obtained after the application of the simplex method, then $y_1 = y_2 = 0$ holds. If the optimal value of w is positive, then the original problem is infeasible.

Notice that to have a feasible basic form, the right side of the constraints must be nonnegative before introducing artificial variables.

2.8 Change of Problem

Suppose that we have the optimal canonical form for some problem P.

In some cases, it would be nice to know "nearby" problems P' for which the optimal canonical form is the same.

Even if P and P' have different solutions, it should be easy to find the solution to P' using the optimal canonical form for P.

2.8.1 Change of Constant Terms

Consider changing the constant terms in the constraint from $\mathbf{b}$ to $\mathbf{b} + \Delta\mathbf{b}$. In particularly, for a linear programming problem in a standard form:

$$\begin{aligned} &\textbf{Minimize} \\ &\qquad \mathbf{c}^T\mathbf{x} \\ &\textbf{Constraint} \\ &\qquad \begin{aligned} A\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned} \end{aligned} \tag{15}$$

consider the problem with constant term $\mathbf{b} + \Delta\mathbf{b}$.

$$\begin{aligned} &\textbf{Minimize} \\ &\qquad \mathbf{c}^T\mathbf{x} \\ &\textbf{Subject to} \\ &\qquad \begin{aligned} A\mathbf{x} &= \mathbf{b} + \Delta\mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned} \end{aligned} \tag{16}$$

Change of Constant Terms

Proposition

Problem (15) and problem (16) have the same optimal canonical form if:

$$\overline{\mathbf{b}} + \Delta\overline{\mathbf{b}} \geq \mathbf{0} \tag{17}$$

where

$$\overline{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1}\mathbf{b} \text{ and } \Delta\overline{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1}\Delta\mathbf{b} \tag{18}$$

Example

Minimize z,

$$z + x_1 + x_2 = 0$$

Subject to

$$\begin{array}{rrrrrcl}
-x_1 & +x_2 & +y_1 & & & = & 2 \\
2x_1 & +x_2 & & +y_2 & & = & 8 \\
x_1 & +x_2 & & & -y_3 & = & 2 \\
x_1, & x_2, & y_1, & y_2, & y_3 & \geq & 0
\end{array}$$

... Final tabular

		x_1	x_2	y_1	y_2	y_3
z	-6	0	0	$-\frac{1}{3}$	$-\frac{2}{3}$	0
x_2	4	0	1	$\frac{2}{3}$	$\frac{1}{3}$	0
y_3	4	0	0	$\frac{1}{3}$	$\frac{2}{3}$	1
x_1	2	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	0

Optimal solution: $x_1 = 2,\ x_2 = 4$.

Example

Minimize z,

$$z - \frac{1}{3}y_1 - \frac{2}{3}y_2 = -6$$

Subject to

$$\begin{array}{ccccccc} & x_2 & +\frac{2}{3}y_1 & +\frac{1}{3}y_2 & & = & 4 \\ & & \frac{1}{3}y_1 & +\frac{2}{3}y_2 & +y_3 & = & 4 \\ x_1 & & -\frac{1}{3}y_1 & +\frac{1}{3}y_2 & & = & 2 \\ x_1, & x_2, & y_1, & y_2, & y_3 & \geq & 0 \end{array}$$

Find the range of α so that the above basic form is still optimal in the following modified problem:

Subject to

$$\begin{array}{cccc} -x_1 & +x_2 & \leq & 2+\alpha \\ 2x_1 & +x_2 & \leq & 8 \\ x_1 & +x_2 & \geq & 2 \\ x_1 & x_2 & \geq & 0 \end{array}$$

2.8.2 Change of Coefficients in the Objective Function

Consider changing the objective function from $\min \mathbf{c}^{\mathrm{t}}\mathbf{x}$ to $\min(\mathbf{c}^{\mathrm{t}} + \Delta\mathbf{c}^{\mathrm{t}})\mathbf{x}$. That is, the original problem (15) changes to:

$$
\begin{array}{l}
\textbf{Minimize} \\
\qquad (\mathbf{c}^T + \Delta\mathbf{c}^T)\mathbf{x} \\
\textbf{Subject to} \\
\qquad \begin{array}{rcl} A\mathbf{x} & = & \mathbf{b} \\ \mathbf{x} & \geq & \mathbf{0} \end{array}
\end{array} \tag{19}
$$

Change of Coefficients in the Objective Function

Proposition

Problem (15) and problem (19) have the same optimal canonical form if:

$$p_j + \Delta p_j \leq 0 \tag{20}$$

for $j = 1, 2, \ldots, n$, where

$$p_j = \mathbf{c}_{\mathcal{B}}^{\mathrm{t}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{a}_j - c_j \tag{21}$$

$$\Delta p_j = \Delta \mathbf{c}_{\mathcal{B}}^{\mathrm{t}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{a}_j - \Delta c_j, \tag{22}$$

and $\mathbf{a}_j$ is column j of $\mathbf{A}$.

Example

Example

Continuing the previous example, we try to find the range of β so that the following modified problem has the same optimal solution as the original problem:

Maximize

$$(1+\beta)x_1 + x_2$$

Class Info

- Tutorial Hours: Today at 13:30. Simplex Example Questions on Homework 1?
- Homework 1 on LMS. Deadline: December 16 at 18:00.
- Next lecture: Monday December 19 at 10:50. Linear programming 1 (Part C).
- Homework 2 on LMS. Deadline: December 23 at 18:00.

Tutorial Hours — Problem

Use the simplex method to solve the following problem

Maximize

$$3x_1 + 2x_2 + 4x_3$$

Subject to

$$\begin{array}{llllll} x_1 & +x_2 & +2x_3 & \leq & 4 \\ 2x_1 & & +2x_3 & \leq & 5 \\ 2x_1 & +x_2 & +3x_3 & \leq & 7 \\ x_1, & x_2, & x_3 & \geq & 0 \end{array}$$