Homework 6

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5.1.6.2)

$$A = \begin{bmatrix} 6 & k \\ k & 2 \end{bmatrix}$$

* $\det(A - \lambda I) = \det\begin{bmatrix} 6-\lambda & k \\ k & 2-\lambda \end{bmatrix}$

$= (6-\lambda)(2-\lambda) - k^2$

$\det(A-\lambda I) = 0 \Leftrightarrow (6-\lambda)(2-\lambda) - k^2 = 0$

$\Leftrightarrow 12 - 6\lambda - 2\lambda + \lambda^2 - k^2 = 0$

$\Leftrightarrow \lambda^2 - 8\lambda + (12 - k^2) = 0$

$$\Leftrightarrow \left[\begin{array}{l} \lambda_1 = \dfrac{8 + \sqrt{64 - 4(12-k^2)}}{2} \\ \lambda_2 = \dfrac{8 - \sqrt{64-4(12-k^2)}}{2} \end{array}\right.$$

* A is positive definite $\Leftrightarrow \lambda_1 > 0$ and $\lambda_2 > 0$
  + λ_1 always > 0 if

$64 - 4(12-k^2) \geq 0$

$\Leftrightarrow 12 - k^2 \leq 16$

$\Leftrightarrow k^2 \geq -4$ satisfied $\forall k$

  + $\lambda_2 > 0$ if

$$\frac{8 - \sqrt{64 - 4(12-k^2)}}{2} > 0$$

$\Leftrightarrow 64 - 4(12-k^2) < 64$

$\Leftrightarrow 4(12-k^2) > 0$

$\Leftrightarrow 12 - k^2 > 0$

$\Leftrightarrow k^2 < 12$

$\Leftrightarrow -\sqrt{12} < k < \sqrt{12}$

5.5.1.1)

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$$\text{minimize} \quad -2x_1 x_2^2 \quad : f(x)$$
$$\text{s.t} \quad \frac{1}{2}x_1^2 + x_2^2 - \frac{3}{2} = 0 \quad : h(x)$$

* $L(x, \lambda) = -2x_1 x_2^2 - \lambda\left(\frac{1}{2}x_1^2 + x_2^2 - \frac{3}{2}\right)$

* $\nabla L(x,\lambda) = 0$:

$$\frac{\partial L}{\partial x_1} = -2x_2^2 - \lambda x_1 = 0 \qquad (1)$$

$$\frac{\partial L}{\partial x_2} = -4x_1x_2 - 2\lambda x_2 = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} = -\frac{1}{2}x_1^2 - x_2^2 + \frac{3}{2} = 0 \qquad (3)$$

(1), (2) can be written as

$$\begin{bmatrix} -\lambda & -2x_2 \\ -4x_2 & -2\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Non-zero solution are found when:

$$\det \begin{bmatrix} -\lambda & -2x_2 \\ -4x_2 & -2\lambda \end{bmatrix} = 0$$

$$\Leftrightarrow \quad 2\lambda^2 - 8x_2^2 = 0$$
$$\Leftrightarrow \quad \lambda^2 = 4x_2^2$$
$$\Leftrightarrow \quad \left[\begin{array}{l} \lambda = 2x_2 \\ \lambda = -2x_2 \end{array}\right.$$

* with $\lambda = 2x_2$, (1) (2) (3) become:

$$\begin{cases} -2x_2^2 - 2x_1x_2 = 0 \\ -4x_1x_2 - 4x_2^2 = 0 \\ -\frac{1}{2}x_1^2 - x_2^2 + \frac{3}{2} = 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x_2(x_2 + x_1) = 0 \quad (4) \\ \frac{1}{2}x_1^2 + x_2^2 = \frac{3}{2} \end{cases}$$

$$(4) \Leftrightarrow \left[\begin{array}{l} x_2 = 0 \\ x_2 = -x_1 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x_1 = \pm\sqrt{3} \\ x_1 = \pm 1 \end{array}\right.$$

$$(4) \Leftrightarrow \left[\begin{array}{l} x_2 = 0 \\ x_2 = -x_1 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x_1 = \pm\sqrt{3} \\ x_1 = \pm 1 \end{array}\right.$$

The stationary points are:

$$(-\sqrt{3}, 0),\ (\sqrt{3}, 0),\ (1, -1),\ (-1, 1)$$

* With $\lambda = -2x_2$, (1) (2) (3) become

$$\begin{cases} -2x_2^2 + 2x_1 x_2 = 0 \\ -4x_1 x_2 + 4x_2^2 = 0 \\ \frac{1}{2} x_1^2 + x_2^2 = \frac{3}{2} \end{cases}$$

$$\Leftrightarrow \begin{cases} x_2 (x_1 - x_2) = 0 \quad (5) \\ \frac{1}{2} x_1^2 + x_2^2 = \frac{3}{2} \end{cases}$$

$$(5) \Leftrightarrow \left[\begin{array}{l} x_2 = 0 \\ x_2 = x_1 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x_1 = \pm\sqrt{3} \\ x_1 = \pm 1 \end{array}\right.$$

The stationary points are

$$(-\sqrt{3}, 0),\ (\sqrt{3}, 0),\ (1, 1),\ (-1, -1)$$

* Classify the stationary points:

+ $\nabla_x^2 L(x, \lambda) = \nabla^2 f(x) - \lambda \nabla^2 h(x)$

$$= \begin{bmatrix} 0 & -4x_2 \\ -4x_2 & -4x_1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

+ Let M be the set of all vector orthogonal to the gradient vector of $h(x)$:

$$M = \{ y \in R^2 \mid \nabla h(x)^T y = 0 \} \text{ with } \Delta h(x) = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$$

$$= \{ (y_1, y_2)^T \mid y_1 x_1 + 2x_2 y_2 = 0 \}$$

+ Check optimal condition of stationary points by using the second-order condition:

- $x = (-\sqrt{3}, 0)^T$; $\lambda = 0$

$$M = \{ (y_1, y_2)^T \mid -\sqrt{3}\, y_1 = 0 \} = \{ (0, t)^T \mid t \in R \}$$

$$\Delta_x^2 L(x, \lambda) = \begin{bmatrix} 0 & 0 \\ 0 & 4\sqrt{3} \end{bmatrix}$$

$$y^T \Delta_x^2 L(x, \lambda)\, y = [0\ t] \begin{bmatrix} 0 & 0 \\ 0 & 4\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 \\ t \end{bmatrix}$$

$$y^T \Delta_x^2 L(x,\lambda)\, y = \begin{bmatrix} 0 & t \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 \\ t \end{bmatrix}$$

$$= \begin{bmatrix} 0 & t \end{bmatrix} \begin{bmatrix} 0 \\ 4\sqrt{3}\, t \end{bmatrix}$$

$$= 4\sqrt{3}\, t^2 \geq 0$$

Since $4\sqrt{3}\, t^2 \geq 0$ and $4\sqrt{3}\, t^2 > 0$ for $y \neq 0$, $x = (-\sqrt{3}, 0)$ is a local minimum

- $x = (\sqrt{3}, 0)^T$; $\lambda = 0$

$$M = \{ (0, t)^T \mid t \in R \} \; ; \; \Delta_x^2 L(x,\lambda) = \begin{bmatrix} 0 & 0 \\ 0 & -4\sqrt{3} \end{bmatrix}$$

$$y^T \Delta_x^2 L(x,\lambda)\, y = \begin{bmatrix} 0 & t \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -4\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 \\ t \end{bmatrix}$$

$$= -4\sqrt{3}\, t^2$$

Since $-4\sqrt{3}\, t^2 \leq 0$ and $-4\sqrt{3}\, t^2 < 0$ for $y \neq 0$, $x = (\sqrt{3}, 0)$ is a local maximum

- $x = (-1, -1)$; $\lambda = 2$

$$M = \{ (y_1, y_2)^T \mid -y_1 - 2y_2 = 0 \} = \{ (-2t, t)^T \mid t \in R \}$$

$$\Delta_x^2 L(x,\lambda) = \begin{bmatrix} 0 & 4 \\ 4 & 4 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & 0 \end{bmatrix}$$

$$y^T \Delta_x^2 L(x,\lambda)\, y = \begin{bmatrix} -2t & t \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} -2t \\ t \end{bmatrix}$$

$$= \begin{bmatrix} -2t & t \end{bmatrix} \begin{bmatrix} 8t \\ -8t \end{bmatrix}$$

$$= -24t^2$$

Since $-24t^2 \leq 0$ and $-24t^2 < 0$ for $y \neq 0$, $x = (-1,-1)$ is a local maximum

- $x = (1, 1)$; $\lambda = -2$

$$M = \{ (y_1, y_2)^T \mid y_1 + 2y_2 = 0 \} = \{ (-2t, t)^T \mid t \in R \}$$

$$\Delta_x^2 L(x,\lambda) = \begin{bmatrix} 0 & -4 \\ -4 & -4 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -4 & 0 \end{bmatrix}$$

$$y^T \Delta_x^2 L(x,\lambda)\, y = \begin{bmatrix} -2t & t \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} -2t \\ t \end{bmatrix}$$

$$= 24t^2$$

Since $24t^2 \geq 0$ and $24t^2 > 0$ for $y \neq 0$, $x = (1,1)$ is a local minimum

- $x = (-1, 1)$; $\lambda = 2$

Since $24t^2 \geq 0$ and $24t^2 > 0$ for $y \neq 0$, $x = (1,1)$ is a local minimum

- $x = (-1, 1)$; $\lambda = 2$

$$M = \{ (y_1, y_2)^T \mid -y_1 + 2y_2 = 0 \} = \{ (2t, t)^T \mid t \in \mathbb{R} \}$$

$$\Delta_x^2 L(x, \lambda) = \begin{bmatrix} 0 & -4 \\ -4 & 4 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -4 & 0 \end{bmatrix}$$

$$y^T \Delta_x^2 L(x, \lambda) y = [2t \quad t] \begin{bmatrix} -2 & -4 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} 2t \\ t \end{bmatrix}$$

$$= [2t \quad t] \begin{bmatrix} -8t \\ -8t \end{bmatrix}$$

$$= -24t^2$$

Since $-24t^2 \leq 0$ and $-24t^2 < 0$ for $y \neq 0$, $x = (-1,1)$ is a local maximum

- $x = (1, -1)$; $\lambda = -2$

$$M = \{ (y_1, y_2)^T \mid y_1 - 2y_2 = 0 \} = \{ (2t, t)^T \mid t \in \mathbb{R} \}$$

$$\Delta_x^2 L(x, \lambda) = \begin{bmatrix} 0 & 4 \\ 4 & -4 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 0 \end{bmatrix}$$

$$y^T \Delta_x^2 L(x, \lambda) y = [2t \quad t] \begin{bmatrix} 2 & 4 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 2t \\ t \end{bmatrix}$$

$$= 24t^2$$

Since $24t^2 \geq 0$ and $24t^2 > 0$ for $y \neq 0$, $x = (1,-1)$ is a local minimum