I214 System Optimization
Chapter 4: Network Optimization

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This is the preliminary version of the slides, will be updated just before the lecture.

Outline

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4.1 Graph Theory Notation

- A *graph* is a pair $G = (V, E)$, where V is a set of *vertices* and E is a set of *edges*.
- If all edges have a direction, then G is called a *directed graph*. If no edge has a direction, then G is called an *undirected graph*. An example of a directed graph and an undirected graph are in Fig. 1.

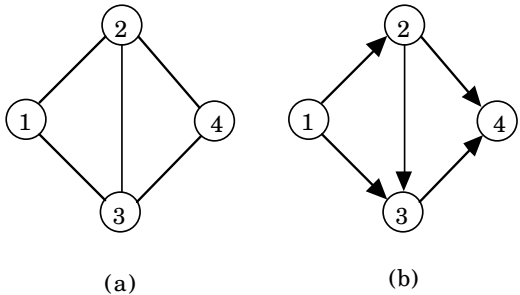

Figure 1: (a) An undirected graph and (b) a directed graph.

Graph Theory Notation

- An edge of a directed graph is an ordered pair (i, j) of vertices, where vertex i is called the *initial vertex* and j is called the *terminal vertex*, for $i \neq j$. An edge of an undirected graph is an unordered pair $\{i, j\}$ for $i \neq j$ where vertices i, j are called *end vertices*.
- A route from some vertex to another vertex following the direction of each edge is called *a walk*. If parallel edges (more than one edges with the same initial and terminal vertices) do not exist, then a walk is specified by a sequence of vertices. In a walk $(v_1, v_2, \cdots, v_n)$, we call v_1 the initial vertex and v_n the terminal vertex.
- If a walk does not go through each edge more than once, it is called *a path*.

Graph Theory Notation

- A *weighted graph* or *network* is a graph where a number (or weight) is assigned to each edge.

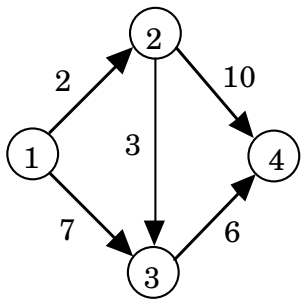

Figure 2: A weighted graph or network.

4.2 Shortest Path Problem

4.2.1 Definition

On a weighted graph or network, the weight of a walk is the sum of the weights on each edge of the walk. Given two vertices on a graph, the *shortest path problem* finds the walk with the minimum weight.

Example

Fig. 3 shows a network of roads that connect seven cities A-G. The number on each edge indicates the distance between two cities. Find a shortest route from A to G.

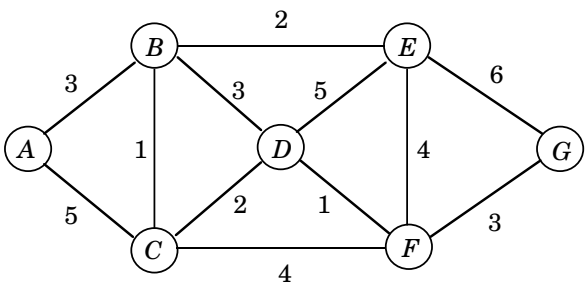

Figure 3: Shortest path problem.

Comments

- Any walk with the smallest length contains no cycles with positive length.
- If a cycle with negative length exists, then we may have a walk with an infinitely small length, i.e., the solution is unbounded. For this reason, we can focus on paths, and not walks.

4.2.2 Dijkstra's Algorithm

Dijkstra's algorithm is a method to find the shortest path in a network. The method is applicable only to networks with non-negative weights.

- Each node i has a label u_i, the label is a number.
- The label represents a distance from the start to that node.
- Initially, the start node has has label $u_1 = 0$. (The distance from the start node to itself is 0).
- For all other nodes, initially, the labels are temporary and initialized with the value ∞.

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- For all other nodes, initially, the labels are temporary and initialized with the value ∞.
- As the algorithm progresses, a temporary label value may change.
- When the label changes from temporary to permanent, then its value will no longer change. A permanent label is denoted u_i^*.
- For the network, the weight on the edge (i, j) is d_{ji}.

Dijkstra's Algorithm

1. *Initialize* The start vertex 1 begins with permanent label $u_1^* = 0$. All other label values are $u_j = \infty$, $j \neq 1$.
2. Let i be a vertex with the smallest temporary label, call i the *current vertex*. For all edges (i, j): if vertex j does not have a permanent label and

   $$u_j > u_i^* + d_{ij},$$

   then update the temporary label with

   $$u_j = u_i^* + d_{ij}. \tag{1}$$

3. For the current vertex i, change the temporary label u_i to a permanent label u_i^*.
4. If all nodes have permanent labels, then output all $u_1^*, u_2^*, \ldots$ as the minimal distances. Otherwise, go to Step 2.

Each permanent label u_i^* indicates the length of a shortest path from vertex 1 to vertex i.

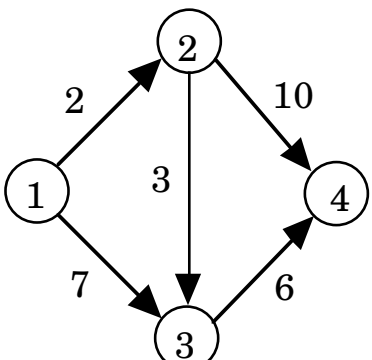
Dijkstra's Algorithm — Find Shortest Path

The shortest path is obtained by adding the following procedure to the algorithm.

- When a temporary label is updated and becomes $u_j = u_i^* + d_{ij}$, the algorithm remembers the link from i to j.
- Denote this link by $\pi(j) = i$. That is, $\pi(j)$ is the neighbor i of j which has the lowest weight.
- Using these π, we can construct a tree from vertex 1 to all other vertices, which indicates a shortest path from vertex 1 to other vertices.

Dijkstra's Algorithm — Example

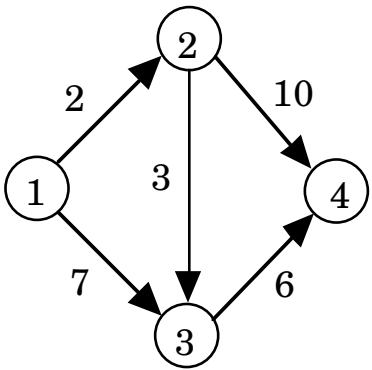
Use Dijkstra's algorithm to find the shortest path from node 1 to node 4.



	u				π			
	1	2	3	4	1	2	3	4
0	0	∞	∞	∞	-			
1								
2								
3								

Dijkstra's Algorithm — Example

Use Dijkstra's algorithm to find the shortest path from node 1 to node 4.



	u				π			
	1	2	3	4	1	2	3	4
0	0	∞	∞	∞	-			
1		$\underline{2}$	7	∞	-	1	1	
2			$\underline{5}$	12	-	1	2	2
3				$\underline{11}$	-	1	2	3

Shortest path has length 11. Shortest path to node 4 is
$4 \leftarrow \pi(4) = 3 \leftarrow \pi(3) = 2 \leftarrow \pi(2) = 1$.

4.2.3 Correctness of Dijkstra's Algorithm

A proof of the correctness of Dijkstra's algorithm is given.

Given that all existing permanent labels are correctly decided, show that for a node i not yet decided, its permanent label u_i^* correctly indicates the length of a shortest path from vertex 1 to vertex i.

4.2.4 Bellman-Ford Algorithm

The Bellman-Ford algorithm also finds the shortest path in a network, but can be applied to networks where some edges have a negative distance.

- Both the Dijkstra algorithm and the Bellman-Ford algorithm maintain a list of labels that indicates the current best distances.
- Dijkstra's algorithm optimizes nodes in serial (i.e. optimizing one at a time) while Bellman-Ford performs all computations in parallel.
- The Bellman-Ford algorithm maintains a list $\mathcal{S}^{(k)}$ of nodes that were updated on iteration k. Since these are the only nodes with new labels, these are used on iteration $k+1$ to reduce the amount of computation needed.

Bellman-Ford Algorithm

1. Label the start vertex 1 $u_1^{(0)} = 0$, and label other vertices $u_j^{(0)} = \infty$, for $j \neq 1$ Let $\mathcal{S}^{(0)} = \{1\}$ and $k = 1$.
2. For each $i \in \mathcal{S}^{(k-1)}$ and each edge (i, j), find the minimum value of

$$u_i^{(k-1)} + d_{ij} \text{ for } i \in S^{k-1},\ (i,j) \in E).$$

   Suppose $u_{i^*}^{(k-1)} + d_{i^*j}$ is the minimum value. If it is less than $u_j^{(k-1)}$, then let $u_j^{(k)} = u_{i^*}^{k-1} + d_{i^*j}$, otherwise let $u_j^{(k)} = u_j^{(k-1)}$. For other j, let $u_j^{(k)} = u_j^{(k-1)}$. The algorithm records the link as $\pi(j) = i^*$.
3. If no labels change, then halt.
4. Let $\mathcal{S}^{(k)}$ be the set of vertices whose labels were updated. Let $k \leftarrow k + 1$ and go to Step 2.

Bellman-Ford Algorithm — Example

Apply the Bellman-Form algorithm to the network in Fig. 5.

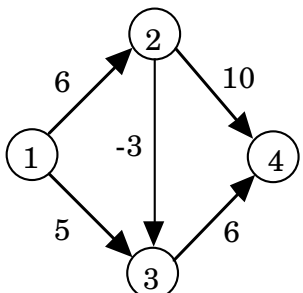


Figure 4: A network with a negative cycle.

	u				π				
k	1	2	3	4	1	2	3	4	S^k
0	0	∞	∞	∞	-				1
1									
2									
3									
4									

Bellman-Ford Algorithm — Example

Apply the Bellman-Form algorithm to the network in Fig. 5.

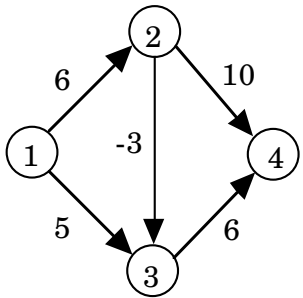


Figure 5: A network with a negative cycle.

	u				π				
k	1	2	3	4	1	2	3	4	S^k
0	0	∞	∞	∞	-				1
1	0	6	5	∞	-	1	1		2, 3
2	0	6	3	11	-	1	2	3	3, 4
3	0	6	3	9	-	1	2	3	4
4	0	6	3	9	-	1	2	3	-

Shortest path has length 9. Shortest path to node 4 is $1 \to 2 \to 3 \to 4$.

Class Info

- Tutorial Hours: Today at 13:30.
- Homework 3 on LMS. Deadline: Friday, January 6 at 18:00.
- Monday, January 9 is Seijin no Hi (Coming of Age Day) — No Class
- Next lecture: Friday, January 9 at 9:00.