Homework 3

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2.1 (d)

$$\text{Maximize} \quad (2+\beta)x_1 + x_2$$
$$\text{s.t} \quad x_1 + 2x_2 \geq 2$$
$$x_1 \leq 3$$
$$x_2 \leq 2$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

* By (c), the original L.P with objective $z = -2x_1 - x_2$ (1) has the following optimal canonical form:

Minimize z

$$z - x_4 - x_6 = -7$$
$$\text{s.t} \quad x_1 + x_4 = 3$$
$$x_3 + x_6 = 2$$
$$x_4 + x_5 - x_6 = 1$$
$$x_2 - x_4 + x_6 = 1$$

with basis $\mathcal{B} = \{x_1, x_3, x_5, x_2\}$;

$$\bar{A} = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

* Consider the following objective

$$\text{Minimize} \quad (-2-\beta)x_1 - x_2 \quad (2)$$

(1) and (2) have the same optimal canonical form if

$$(c_{\mathcal{B}}^T \bar{a}_j - c_j) + (\Delta c_{\mathcal{B}}^T \bar{a}_j - \Delta c_j) \leq 0, \ \forall j = x_1 \dots x_6 \quad (3)$$

Using the basis $\mathcal{B} = \{x_1, x_3, x_5, x_2\}$, we have:

$$c_{\mathcal{B}} = [-2, 0, 0, -1]^T; \quad \Delta c_{\mathcal{B}} = [-\beta, 0, 0, 0]^T$$

+ (3) always equals 0 for $\forall j \in \mathcal{B}$
+ For $j = x_4$,

$$(c_{\mathcal{B}}^T \bar{a}_{x_4} - c_{x_4}) + (\Delta c_{\mathcal{B}}^T \bar{a}_{x_4} - \Delta c_{x_4}) \leq 0$$
$$\Leftrightarrow \quad -1 - \beta \leq 0$$
$$\Leftrightarrow \quad \beta \geq -1$$

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+ For $j = x_6$,

$$(c_B^T \bar{a}_{x_6} - c_{x_6}) + (\Delta c_B^T \bar{a}_{x_6} - \Delta c_{x_6}) \leq 0$$

$$\Leftrightarrow \quad -1 \quad + \quad 0 \quad \leq 0$$

Therefore, if $\beta \geq -1$ then (1) and (2) have the same optimal canonical form

2.2 (d)

$$\begin{aligned} \text{Maximize} \quad & x_1 + x_2 \\ \text{s.t} \quad & -x_1 + 2x_2 \leq 4 \\ & x_1 + 2x_2 \geq 2 \\ & x_1 \leq 4 + \alpha \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

* By (c), the original LP has the following optimal form:

$$\begin{aligned} \text{Minimize } z, \quad & z - x_5 - x_6 = -7 \\ \text{s.t} \quad & x_3 + x_5 - 2x_6 = 2 \\ & x_1 + x_5 = 4 \\ & x_4 + x_5 + 2x_6 = 8 \\ & x_2 + x_6 = 3 \end{aligned}$$

with basis $B = \{x_3, x_1, x_4, x_2\}$; $\bar{b} = [2, 4, 8, 3]^T$

+ Basis matrix:

$$B = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad B^{-1} = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta b = \begin{bmatrix} 0 \\ 0 \\ \alpha \\ 0 \end{bmatrix} ; \quad \Delta \bar{b} = B^{-1} \Delta b = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \\ 0 \end{bmatrix}$$

$$+ \quad \bar{b} + \Delta \bar{b} = \begin{bmatrix} 2 + \alpha \\ 4 + \alpha \\ 8 + \alpha \\ 3 \end{bmatrix} \geq 0 \quad \Leftrightarrow \quad \alpha \geq -2$$

3.4) Minimize $4x_1 - \quad + 2x_3 - x_4$

$$\text{s.t} \quad x_1 + z + 3x_3 + 4x_4 \leq 10$$
$$-x_1 + z - x_3 + x_4 \leq 10$$
$$x_1, x_2, x_3, x_4 \geq 0$$

a) + Standard form:

$$\text{minimize} \quad 4x_1 - 3x_2 + 2x_3 - x_4$$
$$\text{s.t} \quad x_1 + 2x_2 + 3x_3 + 4x_4 + x_5 = 10$$
$$-x_1 + x_2 - x_3 + x_4 + x_6 = 10$$

+ Choose $B = \{x_1, x_6\}$ as basis :

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 10 \\ -1 & 1 & -1 & 1 & 0 & 1 & 10 \end{bmatrix}$$
$$\Downarrow$$
$$\begin{array}{l} \\ r_2 = r_2 + r_1 \end{array} \begin{bmatrix} 1 & 2 & 1 & 4 & 1 & 0 & 10 \\ 0 & 3 & 2 & 5 & 1 & 1 & 20 \end{bmatrix}$$

+ Simplex tabular :

$$z - 4(10 - 2x_2 - x_3 - 4x_4 - x_5) + 3x_2 - 2x_3 + x_4 = 0$$
$$\Leftrightarrow z + 11x_2 + 2x_3 + 17x_4 + 4x_5 = 40$$

		x_1	x_2	x_3	x_4	x_5	x_6
z	40	0	11	2	17	4	0
x_1	10	1	2	1	[4]	1	0
x_6	20	0	3	2	5	1	1

↓

		x_1	x_2	x_3	x_4	x_5	x_6
z	$-5/2$	$-17/4$	$5/2$	$-9/4$	0	$-1/4$	0
x_4	$5/2$	$1/4$	[$1/2$]	$1/4$	1	$1/4$	0
x_6	$15/2$	$-5/4$	$1/2$	$3/4$	0	$-1/4$	1

↓

		x_1	x_2	x_3	x_4	x_5	x_6
z	-15	$-11/2$	0	$-7/2$	-5	$-3/2$	0
x_2	5	$1/2$	1	$1/2$	2	$1/2$	0
x_6	5	$-3/2$	0	$1/2$	-1	$-1/2$	1

Optimal solution : $x^* = [0, 5, 0, 0, 0, 5]^T$, $z^* = -15$

b) Dual problem:

$$\text{maximize} \quad 10y_1 + 10y_2 = w$$
$$\text{s.t} \quad y_1 - y_2 \leq 4$$
$$2y_1 + y_2 \leq -3$$
$$3y_1 - y_2 \leq 2$$
$$4y_1 + y_2 \leq -1$$

$y_1, y_2 \leq 0$

c) The complementary slackness conditions:

$x_1 = 0$ or $[1,-1][y_1, y_2]^T = 4$

$x_2 = 5$ or $[2,1][y_1, y_2]^T = -3$

$x_3 = 0$ or $[3,-1][y_1, y_2]^T = 2$

$x_4 = 0$ or $[4,1][y_1, y_2]^T = -1$

$x_5 = 0$ or $[1,0][y_1, y_2]^T = 0$

$x_6 = 5$ or $[0,1][y_1, y_2]^T = 0$

Therefore, we have

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 0 \end{bmatrix}$$

Optimal solution of the dual problem: $y^* = [-3/2, 0]^T$, $w^* = -15$