I214 System Optimization
Chapter 2: Linear Programing

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Outline

2.1 Formulation of Linear Programming Problems

Show how to formulate LP problems using an example.

Problem

The Stone River Mining Company purchases two kinds of ore.

- The first ore contains iron and copper, and the content is 10% for each.
- The second ore contains iron and lead, and the iron content is 20% and the lead content is 5%.
- The price of the first ore is 20,000 yen per unit, and the price of the second ore is 10,000 yen per unit.

Suppose that 0.8 units of iron, 0.2 units of copper, and 0.05 units of lead are need to be produced. Find the amount of each type of ore that should be purchased.

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Graphical Representation of Feasible Region

A solution that satisfies all the constraints is called a *feasible solution*. The set of all feasible solutions is a called the *feasible region*.

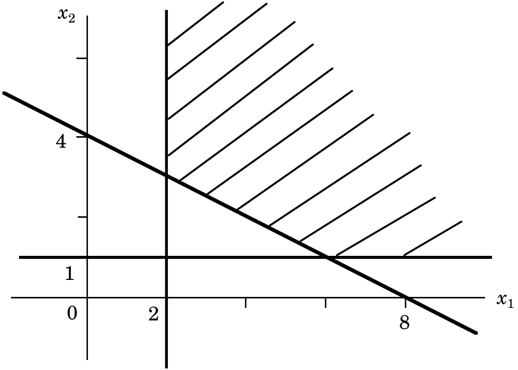

Figure 1: Feasible region corresponding to Problem **??**, after scaling.

Optimal Solution

Since there are infinitely many feasible solutions in general, we need to give some criteria to select the best one. As the criterion, we usually give a function to be maximized (minimized). Such a function is called the *objective function*.

In this example, the objective is to minimize the cost function

$$z(x_1, x_2) = 2x_1 + x_2.$$

Objective Function

Several lines $k = 2x_1 + x_2$ show where the objective function is constant.

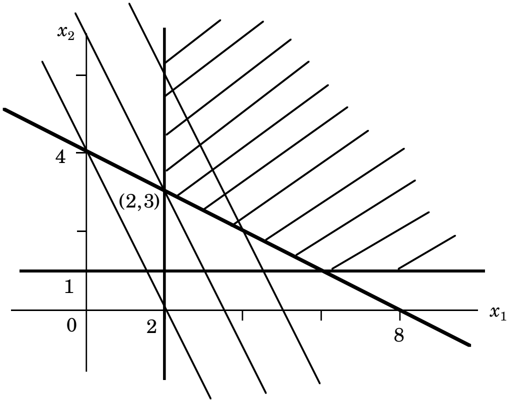

Figure 2: Feasible region, objective function and optimal solution. Several lines $k = 2x_1 + x_2$ show where the objective function is constant.

Not All Solutions Are Unique

- *Multiple optimal solutions* The optimal solution is not always unique, that is, there may be more than one optimal solution.
  For example, infinitely many optimal solutions exist if the objective function is "Minimize $x_1 + 2x_2$".
- *Unbounded problems* Optimal solutions do not always exist, e.g., when "Maximize $2x_1 + x_2$", its value can be increased to the positive infinity.
  Such a problem is called *unbounded*.

In other words, not all problems have a single solution.

2.2 Standard Form of LP

A linear programming problem is said to be in *standard form* if

- Every variable is nonnegative.
- Constraints other than nonnegativity of variables are equalities.
- The objective function is to be minimized (maximized).

$$
\begin{aligned}
&\textbf{Minimize} \\
&\qquad c_1x_1 + \cdots + c_nx_n \\
&\textbf{Subject to} \\
&\qquad \begin{array}{cccc}
a_{11}x_1 & +\cdots & +a_{1n}x_n & = & b_1 \\
\vdots & & \vdots & \vdots & \\
a_{m1}x_1 & +\cdots & +a_{mn}x_n & = & b_m \\
\end{array} \\
&\qquad x_j \geq 0 \quad (j = 1, \cdots, n)
\end{aligned}
\tag{1}
$$

Standard Form — Using Matrix $\mathbf{A}$

As was shown in Chapter 1, this is expressed in matrix form as:

$$\begin{array}{ll} \textbf{minimize} & \mathbf{c}^{\mathrm{t}}\mathbf{x} \\ \textbf{Subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \tag{2}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \text{ and } \mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Conversion to Standard Form

Any linear programming problem can be converted to a standard form problem.

1. **Inequality constraints** are converted to equality constraints by adding *slack variables*.
   Slack variables do not change the problem, but allow us to change inequality constraints into equality constraints.
2. **Free variables** are converted to equality constraints. Suppose that x_j has no sign constraints (for example, $x_j < 0$ is not given).
   This is converted by replacing x_j with two new variables $x_j^+, x_j^- \geq 0$:

$$x_j = x_j^+ - x_j^-, \tag{3}$$

$$x_j^+, x_j^- \geq 0. \tag{4}$$

3. **Maximization** of f is equivalent to minimization of $-f$.
4. **Ignore constants in objective function** Maximization of $f + c$ is equivalent to maximization of f, where c is a constant. Also for minimization.

Example 1

Example

Convert the following LP to standard form:

$$
\begin{aligned}
\textbf{Maximize} \quad & \\
& 4x_1 + 5x_2 \\
\textbf{subject to} \quad & \\
& 2x_1 + 5x_2 \leq 7 \\
& 5x_1 + 6x_2 \leq 9 \\
& 3x_1 + 2x_2 \geq 5 \\
& x_1 \geq 0
\end{aligned}
$$

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Example 2

Example

Formulate the following in standard form:

$$
\begin{aligned}
\textbf{Maximize} \quad & -x_1 - x_2 \\
\textbf{subject to} \quad & x_1 + 2x_2 \leq 3 \\
& x_1 \leq 2 \\
& x_2 \leq 1 \\
& x_1, \ x_2 \geq 0
\end{aligned}
\tag{5}
$$

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2.3 Bases and Canonical Form

This section introduces the basis of the restrictions, and the canonical form of an LP problem. These are key for understanding the simplex method.

In this section, the following running example is considered:

Problem

$$
\begin{array}{rrrrrrl}
\textbf{Minimize} & -x_1 & -\; x_2 & & & & \\
\textbf{subject to} & x_1 & +2x_2 & +x_3 & & & = 3 \\
 & x_1 & & & +x_4 & & = 2 \\
 & & x_2 & & & +x_5 & = 1 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0
\end{array} \tag{6}
$$

Matrix Form of Running Example

The matrix form is:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{7}$$

Indefinite Solution to $\mathbf{Ax} = \mathbf{b}$

Note that the constraints in the running example has 5 variables but only 3 equations. Thus, the solution using $\mathbf{A}$ only is indefinite, and three variables can be seen as dependent on the other two variables. For example, x_3, x_4, x_5 could be seen as dependent on x_1, x_2:

$$\begin{aligned} x_3 &= 3 - x_1 - 2x_2 \\ x_4 &= 2 - x_1 \\ x_5 &= 1 - x_2 \end{aligned}$$

Instead, x_1, x_2, x_4 could be seen as dependent on x_3, x_5:

$$\begin{aligned} x_1 &= 1 - x_3 + 2x_5 \\ x_2 &= 1 - x_5 \\ x_4 &= 1 + x_3 - 2x_5 \end{aligned}$$

That is, there are many choices of the dependent and independent variables.

Bases (Is the Plural of Basis)

Recall the matrix $m \times n$ matrix of constraints $\mathbf{A}$, with $m < n$.

The n columns of $\mathbf{A}$ can be partitioned into two sets:

- $\mathcal{B}$ consisting of m columns and
- $\mathcal{N}$ consisting of the other $n - m$ columns.

Then, $\mathbf{A}_{\mathcal{B}}$ is an $m \times m$ matrix, and $\mathbf{A}_{\mathcal{N}}$ is an $(n - m) \times m$ matrix. In what follows, $\mathcal{B}$ will be the *basis* set, and $\mathcal{N}$ will be the *nonbasis* set.

- The chosen variables $\mathcal{B}$ are called *basic variables*, and other variables $\mathcal{N}$ are called *nonbasic variables*.
- The set of basic variables is called the *basis*, the set of nonbasic variables is called the *nonbasis*.
- Matrix $\mathbf{A}_{\mathcal{B}}$ is called the *basic matrix*, and matrix $\mathbf{A}_{\mathcal{N}}$ is called the *nonbasic matrix*.

Example

Example

(1) Find $\mathbf{A}_\mathcal{B}$ and $\mathbf{A}_\mathcal{N}$ when $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{N} = \{3, 5\}$ for the matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

(2) Instead consider a different choice, $\mathcal{B} = \{2, 3, 5\}$ and $\mathcal{N} = \{1, 4\}$.

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Basic Solution

Try to get a feasible solution by assigning 0 to the nonbasic variables $\mathbf{x}_{\mathcal{N}}$. This is called a basic solution.

Definition

A solution $\mathbf{x}$ is called a *basic solution* if the following two conditions are satisfied:

$$\mathbf{x}_{\mathcal{N}} = \mathbf{0} \text{ and} \tag{8}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{9}$$

By choosing $\mathbf{x}_{\mathcal{N}} = \mathbf{0}$, the second condition is equivalent to $\mathbf{A}_{\mathcal{B}}\mathbf{x}_{\mathcal{B}} = \mathbf{b}$. If $\mathbf{A}_{\mathcal{B}}$ is invertible, then $\mathbf{x}_{\mathcal{B}} = \mathbf{A}_{\mathcal{B}}^{-1}\mathbf{b}$.

Terminology of Basic Solutions

- If a basic solution is feasible, then it is called a *basic feasible solution*.
- But, not all basic solutions are feasible. If at least one of $x_i \leq 0$, then the basic solution is not feasible.
- If a basic feasible solution is optimal, then it is called a *basic optimal solution*.
- We can prove that an optimal solution exists in the set of all basic feasible solutions.

Example — Basic Solution

Continue previous example. Find the basic solution with $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{N} = 3, 5$.
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Canonical Form

The canonical form of an LP is defined with respect to a basis $\mathcal{B}$.

Definition

An LP problem with $\mathbf{c}$ in the objective function $\mathbf{c}^{\mathrm{t}}\mathbf{x}$ is in the *canonical form for* $\mathcal{B}$ if the equality constraint $\mathbf{A}$ has the form:

$$\mathbf{A}_{\mathcal{B}} = \mathbf{I}_m \text{ and} \tag{10}$$

$$\mathbf{c}_{\mathcal{B}} = 0. \tag{11}$$

where $\mathbf{I}_m$ is the identity matrix. ★6

Reason for Canonical Form

An advantage of the canonical form is that it is easy to find a basic solution for a problem with constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$. That is, if $\mathbf{A}$ is in canonical form with respect to $\mathcal{B}$, then $\mathbf{A}_{\mathcal{B}} = \mathbf{I}_m$ and:

$$\mathbf{x}_{\mathcal{B}} = \mathbf{b} \tag{12}$$

$$\mathbf{x}_{\mathcal{N}} = \mathbf{0} \tag{13}$$

is a basic solution.

Conversion to Canonical Form

If an LP is not in canonical form with respect to $\mathcal{B}$, it may be converted to a canonical form if $\mathbf{A}_{\mathcal{B}}$ is nonsingular.

Theorem

Given an LP

$$\begin{aligned} \textbf{Maximize} \quad & z(\mathbf{x}) = \mathbf{c}^{\mathrm{t}}\mathbf{x} \\ \textbf{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{14}$$

and a basis $\mathcal{B}$ with $\mathbf{A}_{\mathcal{B}}$ nonsingular, then the LP:

$$\begin{aligned} \textbf{Maximize} \quad & z(\mathbf{x}) = \mathbf{d}^{\mathrm{t}}\mathbf{b} + (\mathbf{c}^{\mathrm{t}} - \mathbf{d}^{\mathrm{t}}\mathbf{A})\mathbf{x} \\ \textbf{subject to} \quad & \mathbf{A}_{\mathcal{B}}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}_{\mathcal{B}}^{-1}\mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned} \tag{15}$$

is in canonical form, where $\mathbf{d} = (\mathbf{A}_{\mathcal{B}}^{\mathrm{t}})^{-1}\mathbf{c}_{\mathcal{B}}$. ★7

Example — Find Canonical Form

Example

Find the canonical form of (7) with $\mathcal{B} = \{1, 2, 4\}$, we have:

$$\mathbf{A}_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{d} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

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2.4 Geometrical Interpretations

Definition

A set $\mathcal{D} \subseteq \mathbb{R}^n$ is called a convex set if for all $\mathbf{x}, \mathbf{y} \in \mathcal{D}$ and all $0 \leq \lambda \leq 1$:

$$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in \mathcal{D}. \tag{16}$$

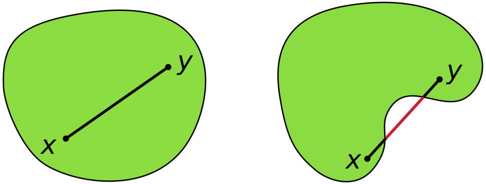

Figure 3: Left: a convex set. Right: a non-convex set. Image credit: Wikipedia/Convex set.

Extreme Point

Definition

Let $\mathcal{S}$ be a convex set. If $\mathbf{z} \in \mathcal{S}$ cannot be represented by a linear combination of two points $\mathbf{x}, \mathbf{y} \in \mathcal{S}$ for $\mathbf{x} \neq \mathbf{y}$:

$$\mathbf{z} = \lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \tag{17}$$

for $0 < \lambda < 1$, then $\mathbf{z}$ is called an *extreme point* of $\mathcal{S}$.

Extreme Point Examples

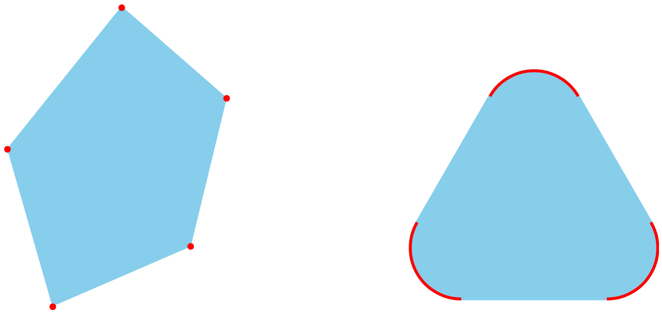

Figure 4: Extreme points in red, for two dimensions. Left: for a polyhedron, the extreme points are the corners. Right: if $\mathcal{S}$ has a curved boundary, points in the curved boundary form are extreme points. Image credit: Wikipedia/Extreme point.

At Least One Solution Is At An Extreme Point

Extreme points are important because at least one solution to the optimization problem will be at an extreme point:

Proposition

For an LP problem over a feasible region $\mathcal{S}$, if the there is a unique solution, then it will occur at an extreme point of $\mathcal{S}$.

If there is only one solution, then it will occur at an extreme point. But since it is possible that there are multiple solutions, not all are guaranteed to be at an extreme point.

Geometrical Interpretation of Running Example

Consider the running example in the following form:

$$
\begin{array}{rl}
\textbf{Minimize} & -x_1 - x_2 \\
\textbf{subject to} & x_1 + 2x_2 \le 3 \\
& x_1 \le 2 \\
& x_2 \le 1 \\
& x_1, \quad x_2 \le 0
\end{array}
$$

The feasible region is the set of points surrounded by the following five lines,

$$
\begin{aligned}
x_1 + 2x_2 &= 3 \\
x_1 &= 2 \\
x_2 &= 1 \\
x_1 &= 0 \\
x_2 &= 0
\end{aligned}
$$

Geometrical Interpretation of Running Example

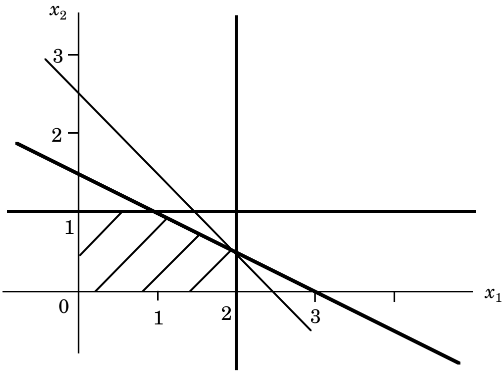

Figure 5: Geometrical interpretation.

The feasible region shown has five extreme points:

$$(0,0),(2,0),(2,0.5),(1,1),(0,1).$$

Observe the following relationship between extreme points and bases. There are $\binom{5}{3} = 10$ ways to choose the basis.

1. Basis $\{x_3, x_4, x_5\}$:

$$
\begin{array}{lllllcl}
z & & & +x_1 & +x_2 & = & 0 \\
x_3 & & & +x_1 & +2x_2 & = & 3 \\
& x_4 & & +x_1 & & = & 2 \\
& & x_5 & & +x_2 & = & 1
\end{array}
$$

Assigning 0 to all nonbasic variables, we obtain $(x_1,\ x_2) = (0,\ 0)$, $z = 0$.

2. Basis $\{x_1,\ x_3,\ x_5\}$:

$$\begin{array}{cccccccc} z & & & +x_2 & -x_4 & = & -2 \\ x_1 & & & & +x_4 & = & 2 \\ & x_3 & & +2x_2 & -x_4 & = & 1 \\ & & x_5 & & +x_2 & = & 1 \end{array}$$

Assigning 0 to all nonbasic variables, we obtain $(x_1,\ x_2) = (2,\ 0)$, $z = -2$.

3. Basis $\{x_1, x_2, x_5\}$:

$$\begin{array}{cccccccc} z & & & -0.5x_3 & +0.5x_4 & = & -2.5 \\ x_1 & & & & +x_4 & = & 2 \\ & x_2 & & +0.5x_3 & -0.5x_4 & = & 0.5 \\ & & x_5 & -0.5x_3 & +0.5x_4 & = & 0.5 \end{array}$$

Assigning 0 to all nonbasic variables, we obtain $(x_1,\ x_2) = (2,\ 0.5)$, $z = -2.5$.

4. Basis $\{x_1,\ x_2,\ x_4\}$:

$$\begin{array}{ccccccc} z & & & -x_3 & +x_5 & = & -2 \\ x_1 & & & +x_3 & -2x_5 & = & 1 \\ & x_2 & & & +x_5 & = & 1 \\ & & x_4 & -x_3 & +2x_5 & = & 1 \end{array}$$

Assigning 0 to all nonbasic variables, we obtain $(x_1,\ x_2) = (1,\ 1)$, $z = -2$.

5. Basis $\{x_2,\ x_3,\ x_4\}$:

$$\begin{array}{ccccccc} z & & & +x_1 & -x_5 & = & -1 \\ x_2 & & & & +x_5 & = & 1 \\ & x_3 & & +x_1 & -2x_5 & = & 1 \\ & & x_4 & +x_1 & & = & 2 \end{array}$$

Assigning 0 to all nonbasic variables, we obtain $(x_1,\ x_2) = (0,\ 1)$, $z = -1$.

6. Basis $\{x_1,\ x_4,\ x_5\}$:

$$\begin{array}{cccccccc}
z & & & -x_2 & -x_3 & = & -3 \\
x_1 & & & +2x_2 & +x_3 & = & 3 \\
& x_4 & & -2x_2 & -x_3 & = & -1 \\
& & x_5 & +x_2 & & = & 1
\end{array}$$

Assigning 0 to all nonbasic variables, we obtain $(x_1,\ x_2) = (3,\ 0)$, $z = -3$. Infeasible.

7. Basis $\{x_1,\ x_2,\ x_3\}$:

$$\begin{array}{cccccccc}
z & & & -x_4 & -x_5 & = & -3 \\
x_1 & & & +x_4 & & = & 2 \\
& x_2 & & & +x_5 & = & 1 \\
& & x_3 & -x_4 & -2x_5 & = & -1
\end{array}$$

Assigning 0 to all nonbasic variables, we obtain $(x_1,\ x_2) = (2,\ 1)$, $z = -3$. Infeasible.

2. Basis $\{x_2,\ x_4,\ x_5\}$:

$$\begin{array}{lllrrcr} z & & & +0.5x_1 & -0.5x_3 & = & -1.5 \\ x_2 & & & +0.5x_1 & +0.5x_3 & = & 1.5 \\ & x_4 & & +x_1 & & = & 2 \\ & & x_5 & -0.5x_1 & -0.5x_3 & = & -0.5 \end{array}$$

   Assigning 0 to all nonbasic variables, we obtain $(x_1,\ x_2) = (0,\ 1.5)$, $z = -1.5$. Infeasible.

3. $\{x_1,\ x_3,\ x_4\}$ cannot be a basis, because $x_2 = 0,\ x_2 = 1$ are parallel lines..
4. $\{x_2,\ x_3,\ x_5\}$ cannot be a basis, because $x_1 = 0,\ x_1 = 2$ are parallel lines.

Complexity of Brute-Force Solution

- There exists at most $\binom{n}{m}$ basic solutions.
- Since the set of basic solutions is finite, the problem is solvable in finite time.
- A brute-force approach is to enumerating all basic solutions, and take the solution with greatest objective function.

Since $\binom{n}{m}$ is exponential:

$$\begin{aligned}\binom{n}{m} &= \frac{n}{m}\frac{n-1}{m-1}\cdots\frac{n-m+1}{1}\\ &\geq (\frac{n}{m})^m.\end{aligned}$$

the brute-force approach is inefficient, as there are an exponential number of candidates.

$\Rightarrow$ More efficient approaches are needed

Class Info

- Homework 1 on LMS. Deadline: December 16 at 18:00.
- Next lecture: Friday, December 16 at 9:00. Continue Linear programming 1.