I214 System Optimization
Chapter 1: Introduction

Brian Kurkoski

Japan Advanced Institute of Science and Technology

2022 December

Outline

1.1 System Optimization

1.2 Example of Linear Programming Problem ★1

1.3 Network Optimization Problems

1.4 Nonlinear Programming

1.5 Combinatorial Optimization

1.6 General Form of Mathematical Programming Problem

1.1 System Optimization

- *Optimization* is widely used engineering, economics and social science.
- *Mathematical programming method* is a methodology for solving optimization problems.
- It formulates a problem in the form of numerical formulas, and then applies a procedure that computes an optimal solution with respect to a given objective function.

System Optimization — Topics

The following topics will be studied in this course:

- Linear programming problems,
- Network optimization problems,
- Nonlinear programming problems,
- Combinatorial optimization problem.

The term "programming" means optimization and not computer programming.

References

This book is a bit technical, but is available in the library:

- Ulrich Faigle, W. Kern, G. Still, "Algorithmic Principles of Mathematical Programming", Springer.

This book is more friendly:

- B. Guenin, J. Könemann, L. Tuncel, "A Gentle Introduction to Optimization," Cambridge University Press, 2014.

In addition, *Lecture Notes* are available on the course website/LMS

Outline

1.1 System Optimization

1.2 Example of Linear Programming Problem ★1

1.3 Network Optimization Problems

1.4 Nonlinear Programming

1.5 Combinatorial Optimization

1.6 General Form of Mathematical Programming Problem

1.2 Example of Linear Programming Problem ★1

Problem [Ref. 4] A company, Stone River Manufacturing, produces three kinds of products **I**, **II**, and **III** from four kinds of materials A, B, C, and D.

- The profit per one unit of production:

I	II	III
$70	$120	$30

- The maximum amount of material available per day (unit):

A	B	C	D
80	50	100	70

- The amount of material necessary for one unit of production (unit):

	I	II	III
A	5	0	6
B	0	2	8
C	7	0	15
D	3	11	0

Find a production plan that maximizes the total profit.

Terminology

- Variables we attempt to find such as x_1, x_2, x_3, are called *decision variables*.
- The function to be maximized (or minimized) is called the *objective function*.
- The restrictions on the variables are called the *constraints*.
- Any value assignment to the variables that satisfies all the constraints is called a *feasible solution*.
- A feasible solution is said to be an *optimal solution* if it maximizes (or minimizes) the objective function.
- The above problem is called a *linear programming problem* because all the constraints are linear equalities or inequalities and the objective function is a linear function of variables.

Linear Program — General Form ★2

The general form of a linear programming problem has n variables $x_1, \ldots, x_n$ and m constraints $b_1, \ldots, b_m$, and is given by:

$$\begin{array}{ll} \textbf{Maximize} & \mathbf{c}^T\mathbf{x} \\ \textbf{Subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \tag{1}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \text{ and } \mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

1.3 Network Optimization Problems

An undirected *graph* consists of nodes, and edges. An edge connects exactly two nodes. Each node is connected to at least one node. A node may also be called a vertex or a point.

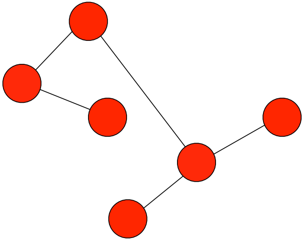


Figure 1: An example of a simple graph — 5 edges connect 6 nodes.

Example of Shortest Path Problem

Fig. **??** shows a network of roads that connect seven cities A-G. The number on each edge indicates the distance between two cities. Find a shortest route from A to G.

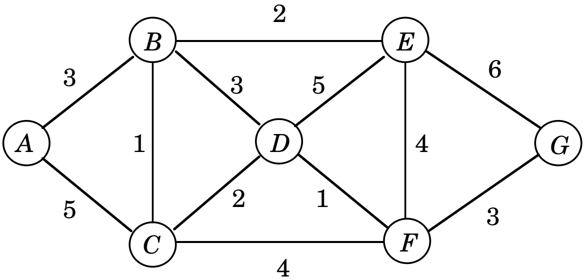


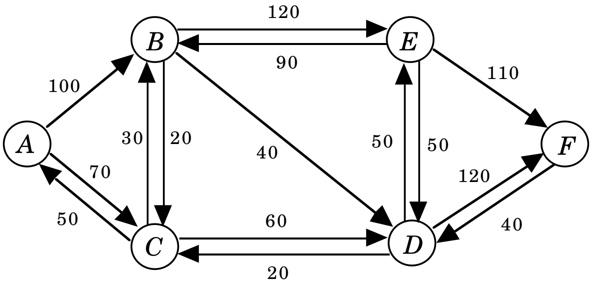
Figure 2: Shortest path problem.

Shortest Path Problem

- This problem is an example of the *shortest path problem* in algorithm theory.
- Since the distance is nonnegative, only routes without any cycles can be an optimal solution.
- Such a route is called a *path*.
- Since the number of paths from A to G is finite, the problem is solvable by enumerating all paths from A to G. However, this procedure is inefficient in general.
- Optimal solution contains optimal solutions to subproblems.

Example of Maximum Flow Problem

Problem [Ref. 4] Consider the Stone River Shipping Company. The figure below shows a network of transportation routes. The number on each edge indicates the maximum amount of goods this company can transport per day. Find the maximum amount of goods this company can transport from city A to city F per day, where we are allowed to use multiple routes in parallel, and goods can be transshipped at each city.



1.4 Nonlinear Programming

- In nonlinear programming problems, either the objective function, or at least one of the constraints, or both, are nonlinear functions.
- In many cases, this nonlinear programming is more challenging than linear programming.

Nonlinear Programming — Example 1 ★4

Problem The price of goods may decrease if the amount of the goods in the market increases. In Stone River Manufacturing's Problem **??**, assume that the profit per unit of production is determined by:

I	II	III
$40 - x_1$	$100 - 3x_2$	$80 - 2x_3$

This is a specific type of nonlinear programming problem called *quadratic programming*, because the objective function is quadratic.

Nonlinear Programming — Example 2

Fig. **??** is a typical curve indicating the time necessary for passing through some road. It is a function of the number of cars x on the road. When x is small, the necessary time can be seen as a constant. However, once x reaches some threshold value, it increases very rapidly.

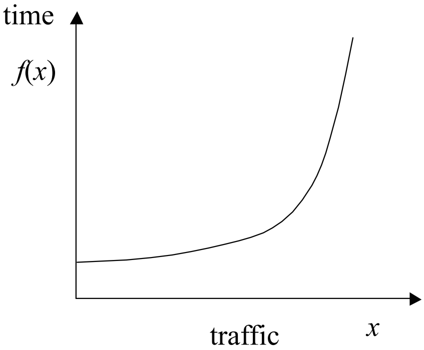


Figure 3: Time required to travel $f(x)$, as a function of the number of cars x.

Nonlinear Programming — Example 2

Problem [Ref. 4] The figure below is a network of roads, where A is a residential area, D is the center of the city, and B, C are junctions. In every morning, w cars move from A to D. Find a best way of selecting routes that seems to be efficient from the perspective of *utilizing the road network*.

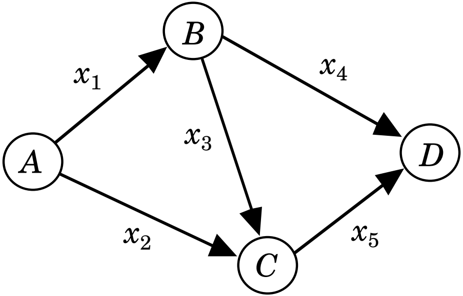


Figure 4: Routes between areas A, B, C and D.

Nonlinear Programming — Example 2 ★5

Let $f_i(x_i)$ denote the function indicating the time necessary for passing through road i. The total time is the number of cars times the time per car:

$$\sum_{i=1}^{5} x_i f_i(x_i).$$

Note that the number of cars entering the network is fixed at w.

1.5 Combinatorial Optimization

- When every feasible solution needs to satisfy some combinatorial constraints (e.g., the sequence of x_i's is a permutation), the problem is called a *combinatorial optimization problem*.
- When every variable needs to be an integer, the problem is called an *integer programming problem*.
- We use *discrete optimization problems* to denote both problems.

Combinatorial Optimization — Knapsack Problem ★6

Problem There are n climbing items $i = 1, \cdots, n$ that will be packed in a knapsack; multiples of any item are allowed. Let a_i (kg) be the weight and let $c_i \in [0, 1]$ be the utility of item i. The total weight must be no more than b (kg). Then, for each item, find the number to be packed so that the total utility is maximized.

Utility c_i is a measure of the importance of the item; items with $c_i = 1$ have the greatest importance, and $c_1 = 0$ means least importance.

Example of Knapsack Problem

Which boxes should be chosen to maximize the amount of money while still keeping the overall weight under or equal to 15 kg?

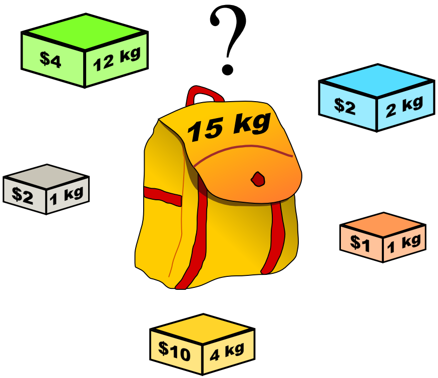


Figure 5: Knapsack Problem

The *one-machine scheduling problem* considers minimizing *tardiness*.

- If the deadline is Wednesday, but the job is completed on the following Friday, the tardiness is two days.
- If the job is completed before Wednesday, the tardiness is always 0.

The *one-machine scheduling problem* considers minimizing *tardiness*.

- If the deadline is Wednesday, but the job is completed on the following Friday, the tardiness is two days.
- If the job is completed before Wednesday, the tardiness is always 0.

Problem There are n jobs to be processed on a machine. For job i, the processing time p_i and the due date d_i are given. The n jobs may be processed in any order. Find the processing order of jobs that minimizes the total tardiness. ★7

1.6 General Form of Mathematical Programming Problem

The general form of mathematical programming problems can be written as:

$$\begin{array}{ll} \textbf{Minimize} & f(\mathbf{x}) \\ \textbf{Subject to} & \mathbf{x} \in \mathcal{S} \end{array} \tag{2}$$

where

- $\mathbf{x}$ is a n-dimensional real vector of variables, and
- the objective function f is defined on the n-dimensional real vector space $\mathbb{R}^n$
- $\mathcal{S} \subset \mathbb{R}^n$ is called the *feasible region*.

Many of the problems in this chapter can be written in this general form.

Class Info

- Tutorial Hours: Today at 13:30. Discuss two formulation problems.
- Homework 1 on LMS. Upload by December 16 at 18:00.
- Next lecture: Monday December 12 at 10:50. Linear programming 1.

Tutorial Hours — Problem 1

Stone River Heating Oil supplies kerosene to customers in the Stone River area. The demand for kerosene is:

Month	1	2	3	4
Demand (liters)	5000	8000	9000	6000

At the beginning of each month, the company purchases oil (kerosene) from a supplier at the following rate:

Month	1	2	3	4
Price ($/liter)	0.75	0.72	0.92	0.9

The company also has a storage tank with capacity 4000 liters. At the beginning of Month 1, it contains 2000 liters. In any month, purchased oil can be delivered directly to the customer; only oil left at the end of the month should be put into storage.

Write an LP that expresses this problem. The solution would express much oil the company should purchase at the beginning of each month, in order to satisfy the customers demand.

Tutorial Hours — Problem 2

The Stone River Hotel wants to rents rooms 1, 2 and 3 on New Year's Eve. There are four possible customers, Alice, Bob, Claire and David. Each customer is willing to pay a fixed amount for each room, as indicated by the following table:

Room	Alice's offer	Bob	Claire	David
1	\$60	\$40	not interested	\$65
2	\$50	\$70	\$55	\$90
3	not interested	\$80	\$75	not interested

The hotel wants to fill Rooms 1, 2, 3 with some potential client A, B, C, D

Each room is to be assigned to exactly one client. Each client should be assigned to *at most* one room (there are more clients than rooms).

1. Formulate this as an integer programming problem to max. hotel's income.
2. Alice and Bob have a history of loud and rude behavior when staying in the hotel. The hotel management decided to not rent rooms to *both* A and B. Modify your answer from part (a) to enforce this restriction.