I217: Functional Programming

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Term 2-1, 2021

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Schedule

10/15	introduction	11/12	interpreters
10/18	algebraic data types I	11/15	compilers
10/22	algebraic data types II	11/19	termination
10/25	program reasoning	11/22	confluence
10/29	applications	11/26	verification
11/1	data structures I	11/29	review
11/5	data structures II	12/6	exam
11/8	computational models		

Evaluation

exam (60) + reports (40)

Termination

Termination

Definition

TRS $\mathcal{R}$ is terminating if there is no infinite rewrite sequence:

$$t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \cdots$$

Question

terminating or not?

$\boxed{1}$ $\{\ \mathsf{f}(\mathsf{f}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{f}(x)))\ \}$

$\boxed{2}$ $\left\{ \begin{array}{rcl} \mathsf{f}(\mathsf{f}(x)) & \to & \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \\ \mathsf{g}(x) & \to & x \end{array} \right\}$

$\boxed{3}$ $\{\ \mathsf{b}(\mathsf{a}(x)) \to \mathsf{a}(\mathsf{b}(x))\ \}$

$\boxed{4}$ $\left\{ \begin{array}{rcl} 0 + y & \to & y \\ x + y & \to & y + x \\ \mathsf{s}(x) + y & \to & \mathsf{s}(x + y) \end{array} \right\}$

Terminating or Not?

$\boxed{5}$ $\{\underline{m} + \underline{n} \to \underline{m+n} \mid m, n \in \mathbb{N}\}$

(e.g. $\underline{1+2}$ stands for $\underline{3}$)

$\boxed{6}$ $\left\{ \begin{array}{rcl} 0 + y & \to & y \\ \mathsf{s}(x) + y & \to & \mathsf{s}(x+y) \end{array} \right\}$

$\boxed{7}$ $\left\{ \begin{array}{ll} \partial(x+y) \to \partial(x) + \partial(y) & \partial(\alpha) \to 1 \\ \partial(x-y) \to \partial(x) - \partial(y) & \partial(\beta) \to 0 \\ \partial(x \times y) \to (\partial(x) \times y) + (x \times \partial(y)) & \\ \partial(x \div y) \to ((\partial(x) \times y) - (x \times \partial(y))) \div (y \times y) & \end{array} \right\}$

Algebras

Definition

$\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ consists of

- carrier A
- interpretations $f_\mathcal{A} \colon A^n \to A$ if $f^{(n)} \in \mathcal{F}$

Definition

- assignment $\alpha \colon \mathcal{V} \to A$
- interpretation function $[\alpha]_\mathcal{A}(\,\cdot\,) \colon \mathcal{T}(\mathcal{F}, \mathcal{V}) \to A$

$$[\alpha]_\mathcal{A}(t) = \begin{cases} \alpha(t) & \text{if } t \in \mathcal{V} \\ f_\mathcal{A}([\alpha]_\mathcal{A}(t_1), \dots, [\alpha]_\mathcal{A}(t_n)) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

Examples of Algebras

consider $\{0, \mathsf{s}, +\}$-algebra $\mathcal{A} = (\mathbb{N}, \{0_\mathcal{A}, \mathsf{s}_\mathcal{A}, +_\mathcal{A}\})$

$$0_\mathcal{A} = 1 \qquad \mathsf{s}_\mathcal{A}(x) = x + 1 \qquad +_\mathcal{A}(x, y) = 2x + y$$

for assignment α with $\alpha(x) = 10$ and $\alpha(y) = 2$

- $[\alpha]_\mathcal{A}(\mathsf{s}(x) + y) = \mathsf{s}_\mathcal{A}(\alpha(x)) +_\mathcal{A} \alpha(y) = 24$
- $[\alpha]_\mathcal{A}(\mathsf{s}(x + y)) = \mathsf{s}_\mathcal{A}(\alpha(x) +_\mathcal{A} \alpha(y)) = 23$

Exercise

does next inequality hold for any $\alpha : \mathcal{V} \to \mathbb{N}$?

$$[\alpha]_\mathcal{A}(\mathsf{s}(x) + y) > [\alpha]_\mathcal{A}(\mathsf{s}(x + y))$$

Termination Criterion

Definition

algebra $\mathcal{A} = (\mathbb{N}, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ (on $\mathbb{N}$) is monotone if

$$f_\mathcal{A}(a_1, \dots, a_i, \dots, a_n) > f_\mathcal{A}(a_1, \dots, b_i, \dots, a_n)$$

for all $f^{(n)} \in \mathcal{F}$ and all $a_1, \dots, a_n, b_i \in \mathbb{N}$ with $a_i > b_i$

Theorem

TRS $\mathcal{R}$ is terminating if there is monotone algebra on $\mathbb{N}$ such that

$$[\alpha]_\mathcal{A}(\ell) > [\alpha]_\mathcal{A}(r)$$

for all rules $\ell \to r \in \mathcal{R}$ and assignments α

Example of Termination Proof

Claim

TRS $\mathcal{R} = \left\{ \begin{array}{c} 0 + y \to y \\ \mathsf{s}(x) + y \to \mathsf{s}(x + y) \end{array} \right\}$ is terminating

Proof.

Consider the algebra $\mathcal{A} = (\mathbb{N}, \{0_\mathcal{A}, \mathsf{s}_\mathcal{A} +_\mathcal{A}\})$ with:

$$0_\mathcal{A} = 1 \qquad \mathsf{s}_\mathcal{A}(x) = x + 1 \qquad +_\mathcal{A}(x, y) = 2x + y$$

$\mathcal{A}$ is **monotone** and for every $x, y \in \mathbb{N}$ we have

$$\begin{aligned} 0_\mathcal{A} +_\mathcal{A} y &= \quad y + 2 > y \quad = y \\ \mathsf{s}_\mathcal{A}(x) +_\mathcal{A} y &= 2x + y + 2 > 2x + y + 1 = \mathsf{s}_\mathcal{A}(x +_\mathcal{A} y) \end{aligned}$$

Hence, $\mathcal{R}$ is terminating. □

Homework 1/2

1 Use a monotone algebra on $\mathbb{N}$ to prove termination, or find an infinite rewrite sequence.

1. $\mathcal{R}_1 = \{\mathsf{f}(\mathsf{f}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{f}(x)))\}$
2. $\mathcal{R}_2 = \{\mathsf{b}(\mathsf{a}(x)) \to \mathsf{a}(\mathsf{b}(x))\}$
3. $\mathcal{R}_3 = \{\mathsf{b}(\mathsf{a}(x)) \to \mathsf{a}(\mathsf{a}(\mathsf{b}(\mathsf{b}(x))))\}$
4. $\mathcal{R}_4 = \{(x + y) + z \to x + (y + z)\}$
5. $\mathcal{R}_5 = \left\{ \begin{array}{ccc} 0 + y & \to & y \\ \mathsf{s}(x) + y & \to & \mathsf{s}(x + y) \\ x + y & \to & y + x \\ (x + y) + z & \to & x + (y + z) \end{array} \right\}$
6. $\mathcal{R}_6 = \left\{ \begin{array}{ccc} [\,] \mathbin{+\!\!+} ys & \to & ys \\ (x : xs) \mathbin{+\!\!+} ys & \to & x : (xs \mathbin{+\!\!+} ys) \end{array} \right\}$

Homework 2/2

2 Hamming's Problem: The set H of natural numbers is inductively defined as follows:

- $1 \in H$, and
- If $n \in H$ then $2n, 3n, 5n \in H$.

The set looks like:

$$H = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, \ldots\}$$

Implement the infinite list h that lists all elements of H in increasing order. (Hint: Use infinite lists.)