I217E: Functional Programming

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Schedule

10/12	introduction	11/9	interpreters
10/14	algebraic data types I	11/11	compilers
10/19	algebraic data types II	11/16	termination
10/21	program reasoning	11/18	confluence
10/26	applications	11/25	verification
10/28	data structures I	11/30	review
11/2	data structures II	12/5	exam
11/4	computational models		

Evaluation

exam (60) + reports (40)

Computational Model

Program as Equational System

Function Definition

$$\begin{aligned} \mathsf{A}_1: &\quad [\,] +\!\!+\, ys = ys \\ \mathsf{A}_2: &\quad (x : xs) +\!\!+\, ys = x : (xs +\!\!+\, ys) \end{aligned}$$

Computation

$$\begin{aligned} & \underline{(1 : (2 : [\,])) +\!\!+ (3 : (4 : [\,]))} & \\ &= 1 : (\underline{(2 : [\,]) +\!\!+ (3 : (4 : [\,]))}) & \text{by } \mathsf{A}_2 \\ &= 1 : (2 : (\underline{[\,] +\!\!+ (3 : (4 : [\,]))})) & \text{by } \mathsf{A}_2 \\ &= 1 : (2 : (3 : (4 : [\,]))) & \text{by } \mathsf{A}_1 \end{aligned}$$

Exercise

Correct next program:

```
-- myProduct [x1,...,xn] = x1 * ... * xn
myProduct []       = 0
myProduct (x : xs) = x * myProduct xs
```


Induction

Theorem

$$\texttt{sum}\ n = \frac{n(n+1)}{2} \textit{ for all } n \in \mathbb{N} \qquad \textit{where} \begin{cases} S_1: & \text{sum } 0 = 0 \\ S_2: & \text{sum } n = n + \text{sum } (n-1) \quad \textit{if } n > 0 \end{cases}$$

Proof.

We show the claim by **mathematical induction on** n.

- If $n = 0$ then $\quad \texttt{sum}\ n = 0 = \frac{0 \cdot (0+1)}{2}$ by S_1
- If $n = n' + 1$ for some $n' \in \mathbb{N}$ then

$$\begin{aligned} \texttt{sum}\ n &= n + \texttt{sum}\ n' && \text{by } S_2 \\ &= n + \frac{n'(n'+1)}{2} && \text{by I.H.} \\ &= \frac{n(n+1)}{2} \end{aligned}$$

$\square$

Recall Definitions of List Operations

$$\begin{aligned} A_1: &\quad [\,] \mathbin{+\!\!+} ys &&= ys \\ A_2: &\quad (x : xs) \mathbin{+\!\!+} ys &&= x : (xs \mathbin{+\!\!+} ys) \\ \\ L_1: &\quad \texttt{length}\ [\,] &&= 0 \\ L_2: &\quad \texttt{length}\ (x : xs) &&= 1 + \texttt{length}\ xs \\ \\ R_1: &\quad \texttt{rev}\ [\,] &&= [\,] \\ R_2: &\quad \texttt{rev}\ (x : xs) &&= \texttt{rev}\ xs \mathbin{+\!\!+} [x] \end{aligned}$$

Theorem

$\texttt{length}\ (xs +\!\!+\, ys) = \texttt{length}\ xs + \texttt{length}\ ys$ *for all lists* xs, ys

Proof.

We show the claim by **structural induction on** xs.

- If $xs = []$ then

$$\begin{aligned} \text{length}\ ([] +\!\!+\, ys) &= \text{length}\ ys && \text{by } A_1 \\ &= \text{length}\ [] + \text{length}\ ys && \text{by } L_1 \end{aligned}$$

- If $xs = x : xs'$ then

$$\begin{aligned} \text{length}\ ((x : xs') +\!\!+\, ys) &= \text{length}\ (x : (xs' +\!\!+\, ys)) && \text{by } A_2 \\ &= 1 + \text{length}\ (xs' +\!\!+\, ys) && \text{by } L_2 \\ &= 1 + \text{length}\ xs' + \text{length}\ ys && \textbf{I.H.} \\ &= \text{length}\ xs + \text{length}\ ys && \text{by } L_2 \end{aligned}$$

□

Theorem

$\text{rev}\ (\text{rev}\ xs) = xs$ *for all lists* xs

Proof.

We show the claim by **structural induction on** xs.

- If $xs = []$ then $\text{rev}\ (\text{rev}\ []) = \text{rev}\ [] = []$ by R_1
- If $xs = x : xs'$ for some x and xs' then

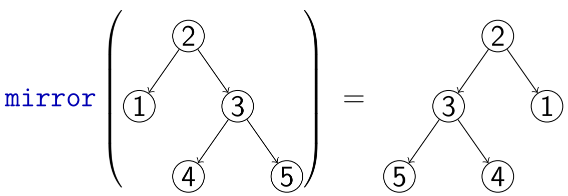
$$\begin{aligned} \text{rev}\ (\text{rev}\ (x : xs')) &= \text{rev}\ ((\text{rev}\ xs') +\!\!+ [x]) && \text{by } R_2 \\ &= x : \text{rev}\ (\text{rev}\ xs') && \textbf{by Lemma} \\ &= x : xs' && \text{I.H.} \end{aligned}$$

□

Lemma

$\text{rev}\ (xs +\!\!+ [x]) = x : \text{rev}\ xs$ for all lists xs

Mirroring



$$\begin{aligned} M_1 &: \text{mirror Leaf} &&= \text{Leaf} \\ M_2 &: \text{mirror}\ (\text{Node}\ \ell\ x\ r) &&= \text{Node}\ (\texttt{mirror}\ r)\ x\ (\texttt{mirror}\ \ell) \end{aligned}$$

Theorem

$\text{mirror}\ (\text{mirror}\ t) = t$ *for all trees* t

Proof.

We show the claim by **structural induction on** t.

- If $t = \text{Leaf}$ then

$$\begin{aligned} \text{mirror}\ (\text{mirror Leaf}) &= \text{mirror Leaf} && \text{by } M_1 \\ &= \text{Leaf} && \text{by } M_1 \end{aligned}$$

- If $t = \text{Node}\ \ell\ x\ r$ for some ℓ, x, and r then

$$\begin{aligned} &\text{mirror}\,(\text{mirror}\ (\text{Node}\ \ell\ x\ r)) \\ &= \text{mirror}\,(\text{Node}\ (\text{mirror}\ r)\ x\ (\text{mirror}\ \ell)) && \text{by } M_2 \\ &= \text{Node}\,(\text{mirror}\,(\text{mirror}\,\ell))\ x\ (\text{mirror}\,(\text{mirror}\,r)) && \text{by } M_2 \\ &= \text{Node}\ \ell\ x\ r && \text{I.H.} \end{aligned}$$

□

Homework

1. Show that $xs +\!\!+ [\,] = xs$ holds for all lists xs.
2. Show that
$$xs +\!\!+ (ys +\!\!+ zs) = (xs +\!\!+ ys) +\!\!+ zs$$
holds for all lists xs, ys, zs
3. Consider the two recursive functions on lists:
$$\begin{aligned}
\texttt{rev}\ [\,] &= [\,] \\
\texttt{rev}\ (x : xs) &= \text{rev}\ xs +\!\!+ [x] \\
\texttt{revapp}\ [\,]\ ys &= ys \\
\texttt{revapp}\ (x : xs)\ ys &= \texttt{revapp}\ xs\ (x : ys)
\end{aligned}$$
Show $\texttt{rev}\ xs = \texttt{revapp}\ xs\ [\,]$ for all lists xs.
Hint: Find a lemma of form revapp $xs\ ys = \cdots$.