Homework 9

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9.2

a) $M \leq \frac{\pi S^2}{\pi N^2} = 100$

b) $M \leq \frac{\mathrm{Vol}(n, \sqrt{P+N})}{\mathrm{Vol}(n, \sqrt{N})} = \frac{\pi^{n/2} (\sqrt{P+N})^n}{\Gamma(\frac{n}{2}+1)} \cdot \frac{\Gamma(\frac{n}{2}+1)}{\pi^{n/2} (\sqrt{N})^n} = \left(1 + \frac{P}{N}\right)^{n/2}$

c) $R = \frac{1}{n} \log M = \frac{1}{2} \log\left(1 + \frac{P}{N}\right)$

9.5

a) Let $K = X + Y$, we have $K \in [-2, 2]$ and

$$p_K(k) = \int_{-1}^{1} p_X(x)\, p_Y(k-x)\, dx$$

$$= \int_{-1}^{1} \frac{1}{2} p_Y(k-x)\, dx$$

As $Y \in [-1, 1]$, in order to have $p_Y(k-x) > 0$, $(k-x)$ must satisfy $-1 \leq k-x \leq 1$ (1)

As $X \in [-1, 1]$ we have $k+1 \geq k-x \geq k-1$ so we have 2 cases:

+ For $k \in [-2, 0]$, we have $k-x \leq k+1 \leq 1$. In order for $k-x \geq -1 \Leftrightarrow x \leq k+1$

$$p_K(k) = \int_{-1}^{k+1} \frac{1}{2} p_Y(k-x)\, dx + \int_{k+1}^{1} \frac{1}{2} p_Y(k-x)\, dx$$

$$= \int_{-1}^{k+1} \frac{1}{4}\, dx + 0$$

$$= \frac{1}{4} |x|_{-1}^{k+1} = \frac{1}{4}(k+2)$$

+ For $k \in [0, 2]$, we have $k-x \geq k-1 \geq -1$. In order for $k-x \leq 1 \Leftrightarrow x \geq k-1$

$$p_K(k) = \int_{-1}^{k-1} \frac{1}{2} p_Y(k-x)\, dx + \int_{k-1}^{1} \frac{1}{2} p_Y(k-x)\, dx$$

$$= 0 + \int_{k-1}^{1} \frac{1}{4}\, dx$$

$$= \frac{1}{4} |x|_{k-1}^{1} = \frac{1}{4}(2-k)$$

So $p_K(k) = \begin{cases} 1/4\,(k+2) & k \in [-2, 0] \\ 1/4\,(2-k) & k \in [0, 2] \\ 0 & \text{otherwise} \end{cases}$

b) $$H(Z) = -\int_{-\frac{1}{c}}^{\frac{1}{c}} p_Z(z) \ln p_Z(z)\, dz$$

$$= -\int_{-\frac{1}{c}}^{0} (c + c^2 z) \ln (c + c^2 z)\, dz - \int_{0}^{\frac{1}{c}} (c - c^2 z) \ln (c - c^2 z)\, dz$$

$$= -2 \int^{0} (c + c^2 z) \ln (c + c^2 z)\, dz$$

$$= -2\int_{-\frac{1}{c}}^{0} (c+c^2z)\ln(c+c^2z)\,dz = -2A$$

$+$ $A = \int_{-\frac{1}{c}}^{0} c\ln(c+c^2z)\,dz + \int_{-\frac{1}{c}}^{0} c^2z\ln(c+c^2z)\,dz$

$+$ $\int_{-\frac{1}{c}}^{0} c\ln(c+c^2z)\,dz = c\left|\frac{(c+c^2z)\ln(c+c^2z)-(c+c^2z)}{c^2}\right|_{-\frac{1}{c}}^{0}$

$$= \frac{c\ln c - c}{c} = \ln c - 1$$

$+$ $\int_{-\frac{1}{c}}^{0} c^2z\ln(c+c^2z)\,dz = \left|c^2z\cdot\frac{(c+c^2z)\ln(c+c^2z)-(c+c^2z)}{c^2}\right|_{-\frac{1}{c}}^{0} - \int_{-\frac{1}{c}}^{0} c^2\,\frac{(c+c^2z)\ln(c+c^2z)-(c+c^2z)}{c^2}\,dz$

$$= 0 - \int_{-\frac{1}{c}}^{0}(c+c^2z)\ln(c+c^2z)\,dz + \int_{-\frac{1}{c}}^{0} c\,dz + c^2\int_{-\frac{1}{c}}^{0} z\,dz$$

$$= -A + c\,|z|_{-\frac{1}{c}}^{0} + c^2\left|\frac{z^2}{2}\right|_{-\frac{1}{c}}^{0} = -A + 1 - \frac{1}{2} = -A + \frac{1}{2}$$

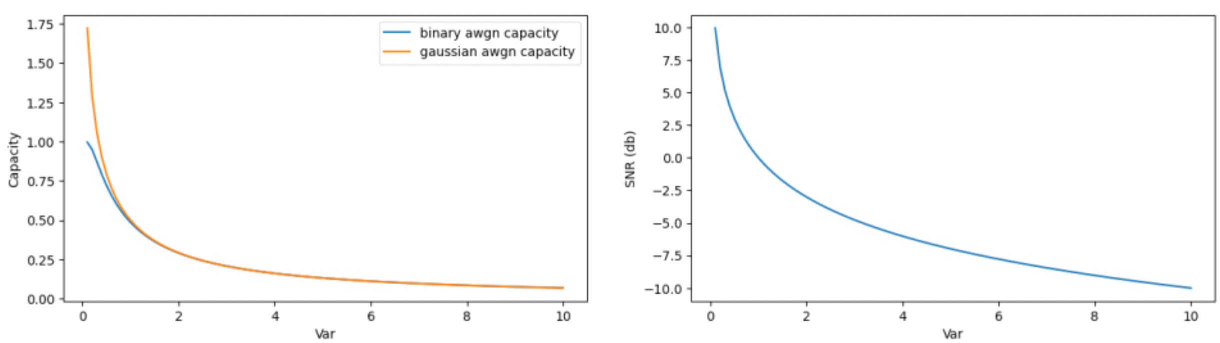
$+$ $A = \ln c - 1 - A + \frac{1}{2} \Leftrightarrow 2A = \ln c - \frac{1}{2} \Leftrightarrow -2A = \frac{1}{2} - \ln c$

$+$ $H(Z) = \frac{1}{2} - \ln c$

c) We have $p_{X+Y}(X+Y) = p_Z(z)$ with $c = \frac{1}{2} \Leftrightarrow H(X+Y) = \frac{1}{2} - \ln\frac{1}{2}$

9.7

The binary-input AWGN has $P=1$. With $Z \sim N(0,\sigma^2)$ we have the plot:



$+$ As $SNR \to 0$, either $\sigma^2 \to \infty$ or $P \to 0$. In either case, $C \to 0$ and as $R < C \Rightarrow R \to 0$

$+$ As $SNR \to \infty$, $\sigma^2 \to 0$ or $P \to \infty$. In either case, $C \to \infty$ and $R \to \infty$

The achievable rates is positively correlated with P and negatively correlated with σ^2

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import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import quad
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plt.rcParams["figure.figsize"] = (16, 4)
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```
def binary_awgn(x, var):
    return (1 / (np.sqrt(2 * np.pi * var))) * np.exp(-(x - 1) ** 2 / (2 * var)) * \
        np.log2(2 / (1 + np.exp(-2 * x / var)))
def gaussian_awgn(var):
    return 0.5 * np.log2(1 + 1 / var)
def snr_db(var):
    return 10 * np.log10(1 / var)

fig, (ax1, ax2) = plt.subplots(1, 2)
ax1.set_xlabel("Var")
ax1.set_ylabel("Capacity")
ax2.set_xlabel("Var")
ax2.set_ylabel("SNR (db)")
var_min, var_max = 0, 10
n_points = 100
var_list = np.linspace(var_min, var_max, num=n_points)
binary_awgn_capacity_list = [quad(binary_awgn, -10, 10, args=(v))[0] for v in var_list]
gaussian_awgn_capacity_list = [gaussian_awgn(v) for v in var_list]
snr_db_list = [snr_db(v) for v in var_list]
ax1.plot(var_list, binary_awgn_capacity_list, label="binary awgn capacity")
ax1.plot(var_list, gaussian_awgn_capacity_list, label="gaussian awgn capacity")
ax2.plot(var_list, snr_db_list)

ax1.legend(loc='upper right')
plt.show()
```