Homework 1

18 April 2023 10:27

1.2

a) Split 9 ingots into 3 stacks of 3 ingots and select 2 stacks for weighting. If those 2 stacks have equal weight then the remaining stack contains the counterfeit, otherwise we select the stack that has lower weight. We split the stack into 3 parts of 1 ingot and repeat the process. The minimum # of uses is 2.

b) Use the above process, for each use of the balance we can have 3 possible outcomes. For n ingots, the minimum # of split and balance usage to guarantee to determine the counterfeit is $\log_3 n$ so we have $t \geq \log_3 n$

1.5

a) $$p_Y(Y=0) = p_X(X=0)\,p_Z(Z=0) + p_X(X=1)\,p_Z(Z=1) = \frac{1}{2}(1-p) + \frac{1}{2}p = \frac{1}{2}$$

$$p_Y(Y=1) = p_X(X=0)\,p_Z(Z=1) + p_X(X=1)\,p_Z(Z=0) = \frac{1}{2}p + \frac{1}{2}(1-p) = \frac{1}{2}$$

$$p_Y(y) = \begin{cases} \frac{1}{2} & y=0 \\ \frac{1}{2} & y=1 \end{cases}$$

b) $$p_{Y|X}(Y=0 \mid X=0) = p_Z(Z=0) = 1-p$$
$$p_{Y|X}(Y=1 \mid X=0) = p_Z(Z=1) = p$$
$$p_{Y|X}(Y=0 \mid X=1) = p_Z(Z=1) = p$$
$$p_{Y|X}(Y=1 \mid X=1) = p_Z(Z=0) = 1-p$$

$p_{Y\|X}(y\|x)$	$y=0$	$y=1$
$x=0$	$1-p$	p
$x=1$	p	$1-p$

c) $$H(Y|X) = H(X \oplus Z \mid X) = H(Z) = -p\log p - (1-p)\log(1-p)$$

d) $$H(Y \oplus X \mid X) = H(X \oplus Z \oplus X \mid X) = H(Z) = -p\log p - (1-p)\log(1-p)$$

1.6

$$h(p) = -p\log p - (1-p)\log(1-p)$$

$$h'(p) = -\log p - \frac{p}{p\ln 2} + \frac{1}{(1-p)\ln 2} + \log(1-p) - \frac{p}{(1-p)\ln 2}$$

$$= -\log p + \log(1-p) + \frac{(1-p)}{(1-p)\ln 2} - \frac{1}{\ln 2}$$

$$= \log \frac{1-p}{p}$$

$(1-p)\ln 2$ $\quad \ln 2$

$$= \log \frac{1-p}{p}$$