Homework 2

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2.6

$$p_X(x) = \begin{cases} 0.25 & x = 1 \\ 0.5 & x = 2 \\ 0.25 & x = 3 \end{cases}$$

a) $E[X] = \sum_x p_X(x) \cdot x = 2$

b) $Var[X] = E[X^2] - (E[X])^2 = 4.5 - 4 = 0.5$

c) + $Var[\bar{X}_n] = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} Var[X_i] = \frac{0.5}{n}$

+ Using the chebyshev inequality, we have:

$$P(|\bar{X}_n - E[\bar{X}_n]| < \varepsilon) \geq 1 - \frac{Var[\bar{X}_n]}{\varepsilon^2}$$

$$\Leftrightarrow P(|\bar{X}_n - E[\bar{X}_n]| < 0.1) \geq 1 - \frac{0.5}{0.01n}$$

+ $P(|\bar{X}_n - 2| < 0.1) \geq 0.999$

$\Leftrightarrow 1 - \frac{0.5}{0.01n} \geq 0.999 \Leftrightarrow 0.01n \geq 500 \Leftrightarrow n \geq 50000$

d) For $\varepsilon = 0.01$, using the Chebyshev inequality we have:

$$P(|X_n - E[\bar{X}_n]| < 0.01) \geq 1 - \frac{0.5}{0.0001n}$$

$\Leftrightarrow 1 - \frac{0.5}{0.0001n} \geq 0.999 \Leftrightarrow n \geq 5000000$

e) 626

f) 826

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import numpy as np
p = [0.25, 0.5, 0.25]
X = [1, 2, 3]
X_mean = np.sum([i * j for i, j in zip(X, p)])
N = 50
n_iter = 1000
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count = 0
for _ in range(n_iter):
    X_bar = np.mean(np.random.choice(X, N, p=p))
    count += 1 if np.abs(X_bar - X_mean) <= 0.1 else 0
print(count)
```

2.9

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Proof:

Let $Y = (X-\mu)^2$, by using the Markov inequality we have

$$P(Y \geq k^2\sigma^2) \leq \frac{E[Y]}{k^2\sigma^2}$$

$$\Leftrightarrow P((X-\mu)^2 \geq k^2\sigma^2) \leq \frac{E[(X-\mu)^2]}{k^2\sigma^2}$$

$$\Leftrightarrow P(|X-\mu| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2} \quad \square$$

2.10

$$\lim_{n\to\infty} P(|\bar{X} - E[X]| < \varepsilon) = 1$$

Proof:

Using the chebyshev inequality, we have

$$P(|\bar{X} - E[\bar{X}]| < \varepsilon) \geq 1 - \frac{Var[\bar{X}]}{\varepsilon^2} \quad (1)$$

As $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, we have

$$+ \ E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = E[X] \quad (2)$$

$$+ \ Var[\bar{X}] = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} Var[X_i] = \frac{Var[X]}{n} \quad (3)$$

Plug (2) (3) into (1), we have

$$P(|\bar{X} - E[X]| < \varepsilon) \geq 1 - \frac{Var[X]}{\ldots}$$

Plug (2) (3) into (1), we have

$$P(|\bar{X} - E[X]| < \varepsilon) \geq 1 - \frac{Var[X]}{n\varepsilon^2}$$

$$\Rightarrow \lim_{n \to \infty} P(|\bar{X} - E[X]| < \varepsilon) \geq 1 - \lim_{n \to \infty} \frac{Var[X]}{n\varepsilon^2} = 1 \quad \square$$