Homework 6

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6.3

+ Let z be the steady-state distribution

We have

$$P = \begin{bmatrix} p & 1-p \\ \frac{1-p}{2} & \frac{1+p}{2} \end{bmatrix}$$

+ Let $Q = P - I_2 = \begin{bmatrix} p-1 & 1-p \\ \frac{1-p}{2} & \frac{p-1}{2} \end{bmatrix}$

$$\tilde{Q} = \begin{bmatrix} 1 & 1-p \\ 1 & \frac{p-1}{2} \end{bmatrix} ; \quad \tilde{Q}^{-1} = \frac{1}{\frac{(p-1)}{2} - (1-p)} \begin{bmatrix} \frac{p-1}{2} & p-1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{-2}{3(p-1)} & \frac{2}{3(p-1)} \end{bmatrix}$$

$$z = [1 \quad 0] \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{-2}{3(p-1)} & \frac{2}{3(p-1)} \end{bmatrix} = \left[\frac{1}{3} \quad \frac{2}{3} \right]$$

+ $H(X) = -\sum_{i=1}^{2} \sum_{j=1}^{2} z_i P_{ij} \log P_{ij}$

$$= -\left[\frac{1}{3}\left(p \log p + (1-p) \log(1-p) \right) + \frac{2}{3}\left(\frac{1-p}{2} \log\left(\frac{1-p}{2}\right) + \frac{1+p}{2} \log\left(\frac{1+p}{2}\right) \right) \right]$$

$$= -\left(\frac{p}{3} \log p + \frac{1-p}{3} \log(1-p) + \frac{1-p}{3} \log(1-p) + \frac{1+p}{2} \log(1+p) - 2 \right)$$

$$= -\frac{p}{3} \log p - \frac{2(1-p)}{3} \log(1-p) - \frac{1+p}{2} \log(1+p) + 2$$

6.4

a)

$$P = \begin{bmatrix} 0.18 & 0.274 & 0.426 & 0.12 \\ 0.171 & 0.367 & 0.274 & 0.188 \\ 0.161 & 0.339 & 0.375 & 0.125 \\ 0.079 & 0.355 & 0.384 & 0.182 \end{bmatrix}$$

+ Let

$$Q = P - I_4 = \begin{bmatrix} -0.82 & 0.274 & 0.426 & 0.12 \\ 0.171 & -0.633 & 0.274 & 0.188 \\ 0.161 & 0.339 & -0.625 & 0.125 \end{bmatrix}$$

$$Q = P - I_4 = \begin{bmatrix} 0.171 & -0.633 & 0.274 & 0.188 \\ 0.161 & 0.339 & -0.625 & 0.125 \\ 0.079 & 0.355 & 0.384 & -0.818 \end{bmatrix}$$

$$\tilde{Q} = \begin{bmatrix} 1 & 0.274 & 0.426 & 0.12 \\ 1 & -0.633 & 0.274 & 0.188 \\ 1 & 0.339 & -0.625 & 0.125 \\ 1 & 0.355 & 0.384 & -0.818 \end{bmatrix}$$

$$\tilde{Q}^{-1} = \begin{bmatrix} 0.155 & 0.341 & 0.349 & 0.155 \\ 1.015 & -1.098 & 0.162 & -0.079 \\ 1.019 & -0.068 & -0.941 & -0.009 \\ 1.108 & -0.092 & 0.056 & -1.072 \end{bmatrix}$$

+ The steady-state distribution is

$$z = [1 \quad 0 \quad 0 \quad 0]\, \tilde{Q}^{-1} = [0.155 \quad 0.341 \quad 0.349 \quad 0.155]$$

b)

$$H(X) = -\sum_{i=1}^{4} z_i \log z_i = 1.895$$

c) $$H(X) = -\sum_{i=1}^{4} \sum_{j=1}^{4} z_i\, p_{ij} \log p_{ij} = 0.286 + 0.659 + 0.649 + 0.278 = 1.872$$

6.6

A Markov chain is a stationary process if the transition probability matrix P remains invariant w.r.t the time shift

$$P(X_n \mid X_{n-1}, \dots, X_1) = P(X_{n+1} \mid X_n, \dots, X_2)$$

or the state distribution remains unchanged as time progresses.

$$z = zP$$