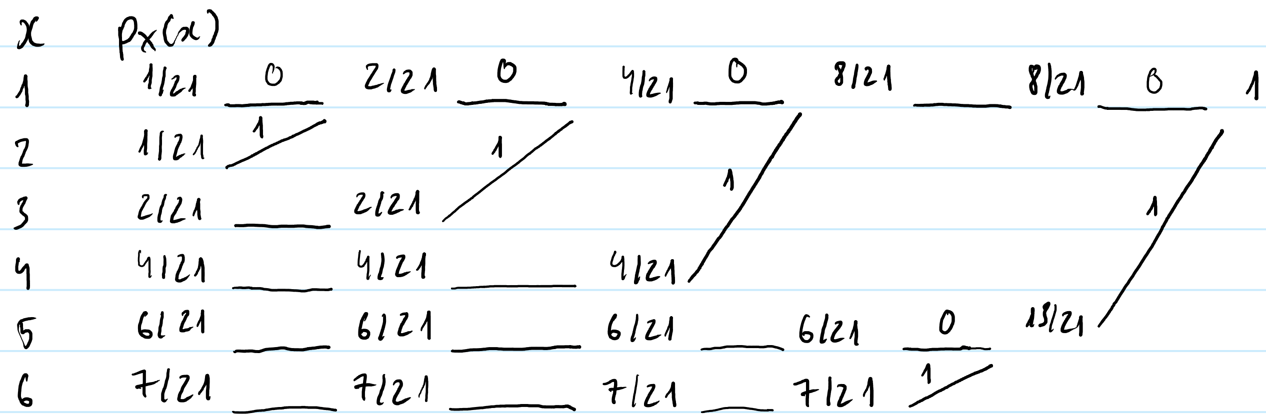
Homework 4

28 April 2023 11:02

4.1

a)



\+ Code book

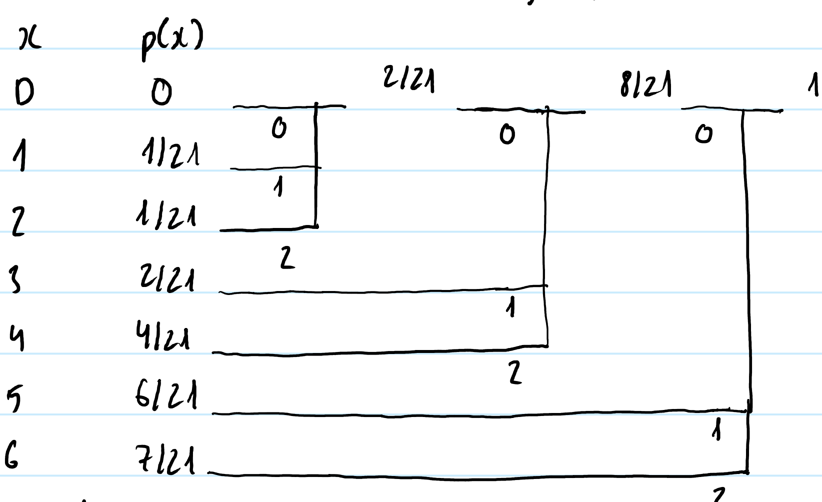
x	$p_X(x)$	$c(x)$	$l(x)$
1	1/21	0000	4
2	1/21	0001	4
3	2/21	001	3
4	4/21	01	2
5	6/21	10	2
6	7/21	11	2

$$L = \sum_{x \in \mathcal{X}} p_X(x)\, l(x)$$

$$= \frac{4}{21} + \frac{4}{21} + \frac{6}{21} + \frac{8}{21} + \frac{12}{21} + \frac{14}{21}$$

$$= \frac{48}{21}$$

b) $D = 3$, $|\mathcal{X}| = 6$, # dummy symbols $= (1 - |\mathcal{X}|) \,\%\, (D-1) = 1$



\+ Code book

x	$p(x)$	$c(x)$	$l(x)$
1	1/21	001	3

$$L = \sum_{x \in |\mathcal{X}|} p_X(x)\, l(x)$$

x	$p(x)$	$c(x)$	$l(x)$
1	1/21	001	3
2	1/21	002	3
3	2/21	01	2
4	4/21	02	2
5	6/21	1	1
6	7/21	2	1

$$L = \sum_{x \in |X|} p(x)\, l(x)$$

$$= \frac{3}{21} + \frac{3}{21} + \frac{4}{21} + \frac{8}{21} + \frac{6}{21} + \frac{7}{21}$$

$$= \frac{31}{21}$$

c) As above

4.3

a) • $h(p) = -\sum_i p_i \log p_i = -\left(\frac{1}{2}\log\frac{1}{2} + \frac{1}{4}\log\frac{1}{4} + \frac{1}{8}\log\frac{1}{8} + \frac{1}{16}\log\frac{1}{16} + \frac{1}{16}\log\frac{1}{16}\right)$

$$= \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4} + \frac{1}{4}$$

$$= \frac{15}{8}$$

• $h(q) = -\sum_i q_i \log q_i = -\left(\frac{1}{2}\log\frac{1}{2} + \frac{1}{8}\log\frac{1}{8} + \frac{1}{8}\log\frac{1}{8} + \frac{1}{8}\log\frac{1}{8} + \frac{1}{8}\log\frac{1}{8}\right)$

$$= 2$$

• $D(p\|q) = \sum_i p_i \log\frac{p_i}{q_i} = \frac{1}{2}\log\frac{1/2}{1/2} + \frac{1}{4}\log\frac{1/4}{1/8} + \frac{1}{8}\log\frac{1/8}{1/8} + \frac{1}{16}\log\frac{1/16}{1/8} + \frac{1}{16}\log\frac{1/16}{1/8}$

$$= 0 + \frac{1}{4} + 0 - \frac{1}{16} - \frac{1}{16}$$

$$= \frac{1}{8}$$

• $D(q\|p) = \sum_i q_i \log\frac{q_i}{p_i} = \frac{1}{2}\log\frac{1/2}{1/2} + \frac{1}{8}\log\frac{1/8}{1/4} + \frac{1}{8}\log\frac{1/8}{1/8} + \frac{1}{8}\log\frac{1/8}{1/16} + \frac{1}{8}\log\frac{1/8}{1/16}$

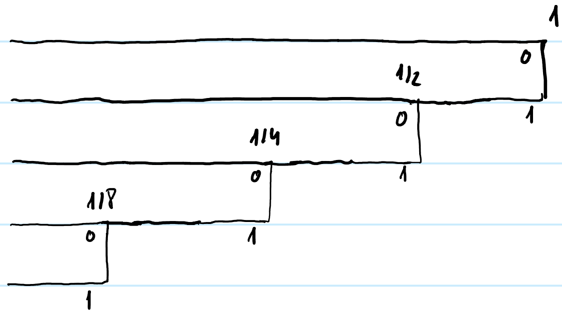
$$= 0 - \frac{1}{8} + 0 + \frac{1}{8} + \frac{1}{8}$$

$$= \frac{1}{8}$$

b) Use a binary Huffman code for p:

b) Use a binary Huffman code for p:

x	p_i
1	1/2
2	1/4
3	1/8
4	1/16
5	1/16



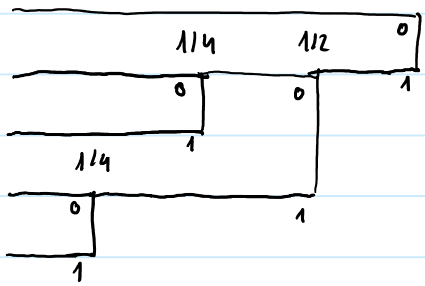
+ Code book:

x	p_i	$C_1(p_i)$	$\ell(p_i)$
1	1/2	0	1
2	1/4	10	2
3	1/8	110	3
4	1/16	1110	4
5	1/16	1111	4

$$L(C_1) = \sum_i p_i \, \ell(p_i) = \frac{1}{2} + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{16}\cdot 4 + \frac{1}{16}\cdot 4 = \frac{15}{8}$$

c) Use a binary Huffman code for q:

x	q_i
1	1/2
2	1/8
3	1/8
4	1/8
5	1/8



+ Codebook:

x	q_i	$C_2(q_i)$	$\ell(q_i)$
1	1/2	0	1
2	1/8	100	3
3	1/8	101	3
4	1/8	110	3
5	1/8	111	3

$$L(C_2) = \sum_i q_i \, \ell(q_i) = \frac{1}{2} + \frac{1}{8}\cdot 3 + \frac{1}{8}\cdot 3 + \frac{1}{8}\cdot 3 + \frac{1}{8}\cdot 3 = 2$$

d) As p and q is 2-adic, we have $L(C_1) = h(p)$ and $L(C_2) = h(q)$ and

$$h(p) \leq L(C_1) \leq h(p) + 1$$
$$h(q) \leq L(C_2) \leq h(q) + 1$$

$$h(p) \leq L(C_1) < h(p)+1$$

$$h(q) \leq L(C_2) \leq h(q)+1$$

satisfied

2) The average length of codewords $\bar{L}$ using C_2 on p distribution:

$$\bar{L} = h(p) + D(p\|q)$$

$$= \frac{15}{8} + \frac{1}{8} = 2$$

$\bar{L}$ exceeds the entropy of p by $D(p\|q) = \frac{1}{8}$

1) When C_1 is used with the q distribution, the average length increase by $D(q\|p) = \frac{1}{8}$

4.4

$$\min L(C) = \sum_{x=1}^{m} p_x l_x$$

$$\text{s.t} \quad \sum_{x=1}^{m} D^{-l_x} \leq 1$$

+ Lagrangian function:

$$L(l, \lambda) = \sum_{x=1}^{m} p_x l_x + \lambda \left[\sum_{x=1}^{m} D^{-l_x} - 1 \right]$$

+ we have the KKT conditions:

$$p_x - \lambda . D^{-l_x} . \ln(D) = 0, \quad x = 1, 2 \dots, m \qquad (1)$$

$$\sum_{x=1}^{m} D^{-l_x} - 1 \leq 0 \qquad (2)$$

$$\lambda \geq 0 \qquad (3)$$

$$\lambda \left[\sum_{x=1}^{m} D^{-l_x} - 1 \right] = 0 \qquad (4)$$

+ By (1), we have

$$p_x = \lambda D^{-l_x} \ln(D), \quad x = 1, 2, \dots, m$$

as $p_x > 0 \ \forall x$, we have $\lambda \neq 0$ and

$$p_x = \lambda D \quad \ln(D) \quad , x = 1, 2, \dots, m$$

as $p_x > 0 \ \forall x$, we have $\lambda \neq 0$ and

$$\log_D p_x = \log_D(\lambda) - l_x + \log_D(\ln D)$$

$$\Leftrightarrow l_x = -\log_D p_x + \log_D(\lambda \ln D)$$

+ Because $\lambda \neq 0$, by (4) we have

$$\sum_{x=1}^{m} D^{-l_x} = 1$$

$$\Leftrightarrow \sum_{x=1}^{m} D^{\log_D p_x} \cdot D^{-\log_D(\lambda \ln D)} = 1$$

$$\Leftrightarrow \sum_{x=1}^{m} p_x \cdot \frac{1}{\lambda \ln D} = 1$$

$$\Leftrightarrow \sum_{x=1}^{m} p_x = \lambda \ln D$$

$$\Leftrightarrow \lambda \ln D = 1$$

$$\Leftrightarrow \lambda = \frac{1}{\ln D}$$

The solution to the optimization problem is

$$\begin{cases} l_x = -\log_D p_x, & x = 1, 2, \dots, m \\ \lambda = \dfrac{1}{\ln D} \end{cases}$$

and the objective junction has the minimum value

$$L(C) = -\sum_{x=1}^{m} p_x \log_D p_x = h(p)$$