JAIST — School of Information Science — I232E Information Theory

Midterm Exam 2022

Instructor: Brian M. Kurkoski and Lei Liu

May 20, 2022 13:30–15:10

Name: ______________________________

Student Number: ______________________________

Write your answers in this test book.
If you need more space, write on the back of the paper.

Exam Policy. This is a *closed book* exam. You may use:

- One page of notes, A4-sized paper, double-sided OK.

You may not use anything else:

- No printed materials, including books, lecture notes and slides
- No notes (except as above)
- No internet-connected devices
- No calculators. Answers such as "log 3" are acceptable.

Question	Points	Score
1	25	
2	20	
3	25	
4	30	
Total:	100	

This exam has 10 pages.

1. What is the correct relationship, $=$, $\geq$, $\leq$ or ? (for unknown) for each pair below.
   (a) (2 points) $I(\mathsf{X};\mathsf{Y})$ ______ 0.
   (b) (2 points) $H(\mathsf{X},\mathsf{Y})$______ $H(\mathsf{X})+H(\mathsf{Y})$
   (c) (2 points) $I(\mathsf{X};\mathsf{Y})+H(\mathsf{X}|\mathsf{Y})$ ______ $H(\mathsf{X})$.
   (d) (2 points) $I(\mathsf{X};\mathsf{X})$______ $H(\mathsf{X})$.
   (e) (2 points) $I(\mathsf{X};\mathsf{Y})$ ______ $H(\mathsf{X})-H(g(\mathsf{Y})|\mathsf{Y})$.
   (f) (2 points) $H(\mathsf{X}|\mathsf{Y})$______ $H(\mathsf{X})+H(\mathsf{Y})$
   (g) (2 points) $H(2\mathsf{X})$ ______ $H(\mathsf{X})$
   (h) (2 points) $H(\mathsf{X}^2)$ ______ $H(\mathsf{X})$
   (i) (2 points) $H(\mathsf{X}_2|\mathsf{X}_1)$______ $H(\mathsf{X}_2|\mathsf{X}_1,\mathsf{X}_0)$
   (j) (3 points) $H(\mathsf{X},\mathsf{Y})+I(\mathsf{X};\mathsf{Y})$______ $H(\mathsf{X})+H(\mathsf{Y})$
   (k) (4 points) $H(\mathsf{X},\mathsf{Y})$______ $H(\mathsf{X}|\mathsf{Y})+H(\mathsf{Y}|\mathsf{X})+I(\mathsf{X};\mathsf{Y})$

Continue your answer on the next page

Continue your answer here.

2. Let X be defined on $\mathcal{X} = \{-1, 0, 1\}$ with $p_{\mathsf{X}}(x) = [\frac{1}{4}, \frac{1}{2}, \frac{1}{4}]$. Let $g(x) = x^2$ and let $\mathsf{Y} = g(\mathsf{X})$, so that $\mathcal{Y} = \{0, 1\}$ and $p_{\mathsf{Y}}(0) = 1/2$ and $p_{\mathsf{Y}}(1) = 1/2$.

   (a) (4 points) Compute $E[\mathsf{X}]$ and $E[g(\mathsf{X})]$.

   (b) (4 points) What is $H(\mathsf{Y}|\mathsf{X})$? It is easily found without computations.

   (c) (4 points) Find $p_{\mathsf{XY}}(x, y)$. Find $p_{\mathsf{X}|\mathsf{Y}}(x|y)$.

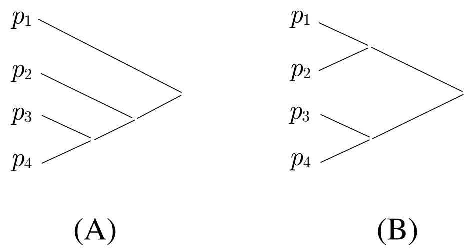
   (d) (4 points) Compute $H(\mathsf{X}|\mathsf{Y} = 0)$ and $H(\mathsf{X}|\mathsf{Y} = 1)$.

   (e) (4 points) Compute $H(\mathsf{X}|\mathsf{Y})$.

Continue your answer on the next page

Continue your answer here.

3. *Huffman code trees* Consider a source with $\mathcal{X} = \{1, 2, 3, 4\}$ and $p_1 > p_2 > p_3 > p_4$ and $p_1 + p_2 + p_3 + p_4 = 1$. There are only two possible binary Huffman codes for this source, with corresponding trees (A) and (B):



(a) (5 points) Let $(p_1, p_2, p_3, p_4) = (0.39, 0.21, 0.2, 0.2)$. Give a binary Huffman code for this source. What is the expected codeword length?

(b) (5 points) Give a ternary Huffman code for the source in part (a). What is the expected codeword length?

(c) (5 points) Give an inequality using p_1, p_3 and p_4 such that tree (A) is always obtained.

(d) (5 points) Show that if $p_1 > \frac{2}{5}$ then $p_3 + p_4 < \frac{2}{5}$.

(e) (5 points) Show that if $p_1 > \frac{2}{5}$, then the length of the corresponding Huffman codeword for "$x = 1$" is 1.

Continue your answer on the next page

Continue your answer here.

4. Consider a two-state Markov chain $\mathbf{X} = [\mathsf{X}_1, \mathsf{X}_2, \cdots]$ with probability transition matrix:

$\mathbf{P}_{\mathsf{X}_n \mid \mathsf{X}_{n-1}}$	$x_{n-1} = 0$	$x_{n-1} = 1$
$x_n = 0$	$4/5$	$1/2$
$x_n = 1$	$1/5$	$1/2$

(a) (3 points) What is the stationary distribution $\mathbf{p}_{\mathsf{X}}$?

(b) (3 points) What is the entropy rate $H(\mathcal{X})$?

(c) (3 points) What is the (single-variable) entropy $\lim_{n\to\infty} H(\mathsf{X}_n)$?

(d) (3 points) Which has lower compression rate, compression using the Markov property, or single-variable compression?

Let $\mathbf{Y} = [\mathsf{Y}_1, \mathsf{Y}_2, \ldots]$ and $\mathsf{Y}_n = [\mathsf{X}_{2n-1}, \mathsf{X}_{2n}]$. Then $\mathsf{Y}_1 = [\mathsf{X}_1, \mathsf{X}_2], \mathsf{Y}_2 = [\mathsf{X}_3, \mathsf{X}_4], \ldots$ is a four-state Markov chain.

(e) (4 points) What is the probability transition matrix $\mathbf{P}_{\mathsf{Y}_n \mid \mathsf{X}_{2n-2}}$? What is the probability transition matrix $\mathbf{P}_{\mathsf{Y}_n \mid \mathsf{Y}_{n-1}}$?

(f) (2 points) What is the stationary distribution $\mathbf{p}_{\mathsf{Y}}$? (Hint: Matrix inverse is not necessary. You can utilize the stationary distribution $\mathbf{p}_{\mathsf{X}}$.)

(g) (3 points) What is the entropy rate $H(\mathcal{Y})$?

(h) (3 points) Which has lower compression rate per symbol, compression using the Markov property of $\mathbf{Y}$, or compression using the Markov property of $\mathbf{X}$?

(i) (3 points) What is the (two-variable) entropy $\lim_{n\to\infty} H(\mathsf{Y}_n)$?

(j) (3 points) Which has lower compression rate per symbol, two-variable compression, or single-variable compression?

Notes: $\log_2 5 \approx 2.3219, \log_2 7 \approx 2.8074$, and you may use the binary entropy function $h(p) \equiv -p\log p - (1-p)\log(1-p)$ for the questions above.

Continue your answer on the next page

Continue your answer here.

Continue your answer on the next page

Continue your answer here.