JAIST — School of Information Science — I232 Information Theory

Final Exam 2021

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June 7, 2021 13:30–15:10

Name: ______________________________

Student Number: ______________________________

Exam policy:

- You may not connect to the internet, except to use the LMS.
- You may not communicate with anyone during the exam, except the instructor.
- This is an open-book exam. You may use only
  - *Information Theory Lecture Notes* by the instructor,
  - *Elements of Information Theory* by Cover and Thomas,
  - anything written in your own hand.
  - You can also use any material on the LMS on your device. Your device screen should not display pages other than the LMS.
- **Not following the above rules could result in failing the course.**

In addition:

- Calculators are not allowed and not needed. Numerical answers such as 5^7 and $1 + 2\log 3$ are acceptable, but simplify as much as reasonably possible.
- You can write on the back of the paper.
- Students should wear a mask during the exam.

Question	Points	Score
1	25	
2	20	
3	25	
4	30	
Total:	100	

This exam has 9 pages.

1. *Information theory concepts* State the key ideas from information theory. (Do not write too much.)

   (a) (5 points) Does compressing over blocks of length n and letting $n \to \infty$ make the "performance" of a source code *strictly* better? If so, state why. If not, show why not.

   (b) (5 points) Your friend has a 16 pixel-by-16 pixel image, in which each pixel is either black or white with probability 0.5, and all pixels are independent. He wants to losslessly compress this to an 8 pixel-by-8 pixel image, again with black and white pixels. If this is possible, find a code or describe the procedure you would use to do so. If this is not possible, state the reason.

   (c) (5 points) If possible, find a prefix code over an alphabet of size $D = 3$ with codeword lengths $(1, 1, 2, 3)$. If not possible, explain the reason.

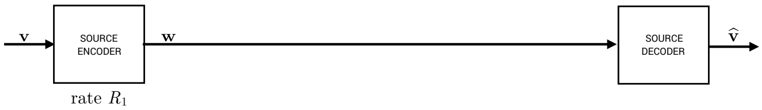
   (d) (5 points) Let X and Y be independent and discrete. Find $H(2\mathsf{X}, -2\mathsf{Y})$ in terms of the entropies $H(\mathsf{X})$, $H(\mathsf{Y})$ and $H(\mathsf{X}, \mathsf{Y})$.

   (e) (5 points) Consider a channel with capacity $C = 0.5$. Suppose a code with length $n = 10^6$ is used for this channel. What can you say about the number of message bits that can be reliably transmitted over this channel?

Continue your answer on the next page

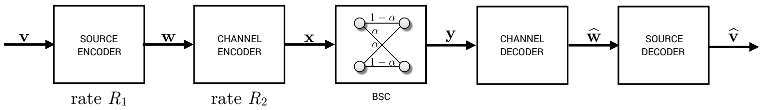
Continue your answer here.

2. *Source coding-channel coding* An information source consists of n_1 bits $\mathsf{V}_1, \ldots, \mathsf{V}_{n_1}$, i.i.d. distributed as $p_{\mathsf{V}}(v) = [1-p, p]$, $v \in \{0, 1\}$. A source encoder of rate $R_1 < 1$ compresses to a **binary** index $\mathbf{w} = (w_1, w_2, \ldots, w_{n_2})$.



(a) (5 points) How many bits n_2 are in the sequence $\mathbf{w}$? In terms of n_1 and p, what is a lower bound on n_2?

Now, to reliably transmit this index $\mathbf{w}$ over a binary symmetric channel with error probability α, a channel encoder of rate $R_2 < 1$ encodes $\mathbf{w}$ to a codeword $\mathbf{x}$.



(b) (5 points) How many bits n_3 are in the sequence $\mathbf{x}$? In terms of n_2 and α, what is a lower bound on n_3?

At the destination, a decoder estimates $\widehat{\mathbf{w}}$, which is decompressed to $\widehat{\mathbf{v}}$.

(c) (10 points) Let the total rate be $R = R_1/R_2$. In terms of p and α, find a condition on R such that the probability of error $\Pr[\mathsf{V} \neq \widehat{\mathsf{V}}]$ can go to 0 as $n \to \infty$.

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Continue your answer here.

3. *Multiple access channel (MAC)* Consider the following multiple access channel (MAC):

$$Y = X_1 \oplus X_2 \oplus N,$$

where $\oplus$ denotes xor, $X_1, X_2 \in \{0, 1\}$ are two binary inputs, and $N \in \{0, 1\}$ is a binary noise with $p_N(0) = p$.

(a) (5 points) For fixed $X_1 = 0$, find the maximum achievable rate between X_2 and Y. Find the optimal distribution of X_2 which achieves this rate.

(b) (10 points) Let R_1 be the achievable rate of X_1 and R_2 be the achievable rate of X_2. Draw the capacity region (R_1, R_2) and label the rates on it.

(c) (10 points) Give a scheme to achieve the whole capacity region in (b)?

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4. *Multiple input and multiple output channel (MIMO) Channel* Consider a MIMO channel:

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}, \tag{1}$$

where $\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma^{\mathbf{2}}\mathbf{I})$ is a 3×1 noise vector, $\mathbf{x} = [x_1, x_2, x_3]^{\mathrm{T}}$ is a message vector with power constraint $\sum_{i=1}^{3} \mathrm{E}\{x_i^2\} = 1$, and $\mathbf{A}$ is a channel matrix given by

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

Assume that $\mathbf{A}$ is known at transmitter. The singular value decomposition (SVD) of $\mathbf{A}$ is given by

$$\mathbf{A} = \underbrace{\begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{\mathbf{V}^{\mathrm{T}}} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}}_{\mathbf{\Lambda}} \underbrace{\begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{\mathbf{V}},$$

where $\mathbf{V}$ is an orthogonal matrix.

(a) (5 points) Find the equivalent parallel AWGN channels for the MIMO channel.

(b) (10 points) Find the optimal power allocation for the parallel AWGN channels in (a).

(c) (10 points) Based on the result in (b), find the capacity of the MIMO channel in (1).

(d) (2 points) Let $\sigma^2 = 6$. Calculate the optimal power allocation in (b) and the capacity in (c).

(e) (3 points) Let $\sigma^2 = 2$. Calculate the optimal power allocation in (b) and the capacity in (c).

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