Homework 3

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3.1

a) $I(X;Y) \geq 0$ because $I(X;Y) = H(X) - H(X|Y)$ (1) and $H(X) \geq H(X|Y)$

b) $H(X,Y) \leq H(X) + H(Y)$ because $H(X,Y) = H(X) + H(Y) - I(X;Y)$

c) $I(X;Y) + H(X|Y) = H(X)$ as (1)

d) $I(X;X) = H(X)$ as (1) and $H(X|X) = 0$

e) $I(X;Y) \leq H(X) - H(g(Y)|Y)$ as (1) and $H(g(Y)|Y) = 0$

f) $H(X|Y) \leq H(X) + H(Y)$ because conditioning reduces entropy

g) $H(2X) = H(X)$ because the probabilities are the same

h) $H(X_2|X_1) \geq H(X_2|X_1, X_0)$ because conditioning reduces entropy

3.4

a) $p_X(x) = \begin{cases} 0.5 & x=0 \\ 0.5 & x=1 \end{cases}$; $p_Y(y) = \begin{cases} 0.5 & y=0 \\ 0.5 & y=1 \end{cases}$;

$p_{X\|Y}(x\|y)$	$y=0$	$y=1$
$x=0$	$2/3$	$1/3$
$x=1$	$1/3$	$2/3$

- $H(X) = -\sum_{x\in[0,1]} p_X(x)\log p_X(x) = 1$

- $H(Y) = -\sum_{y\in[0,1]} p_Y(y)\log p_Y(y) = 1$

- $H(X|Y) = -\sum_{y\in[0,1]} p_Y(y) \sum_{x\in[0,1]} p_{X|Y}(x|y)\log p_{X|Y}(x|y)$

$$= -\left[\frac{1}{2}\left(\frac{2}{3}\log\frac{2}{3} + \frac{1}{3}\log\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\log\frac{1}{3} + \frac{2}{3}\log\frac{2}{3}\right)\right]$$

$= 0.9182$

- $H(X,Y) = H(Y) + H(X|Y) = 1 + 0.9182 = 1.9182$

b) - As $Z = X + Y$, we have

$p_{XYZ}(x=0, y=0, z=0) = p_{XY}(x=0, y=0) = 1/3$

$p_{XYZ}(x=0, y=1, z=1) = p_{XY}(x=0, y=1) = 1/6$

$p_{XYZ}(x=1, y=0, z=1) = p_{XY}(x=1, y=0) = 1/6$

$p_{XYZ}(x=1, y=1, z=2) = p_{XY}(x=1, y=1) = 1/3$

$\begin{cases} 1/3 & x=y=z=0 \\ 1/6 & x=0,\ y=z=1 \end{cases}$

$\begin{cases} 1/3 & z=0 \end{cases}$

$$+\ p_{XYZ}(x,y,z) = \begin{cases} 1/3 & x = y = z = 0 \\ 1/6 & x = 0,\ y = z = 1 \\ 1/6 & x = 1,\ y = 0,\ z = 1 \\ 1/3 & x = y = 1,\ z = 2 \\ 0 & \text{otherwise} \end{cases} \quad ; \quad p_Z(z) = \begin{cases} 1/3 & z = 0 \\ 1/3 & z = 1 \\ 1/3 & z = 2 \end{cases}$$

c) + $I(X,Y;Z) = H(X,Y) + H(Z) - H(X,Y,Z)$

+ $H(Z) = -\sum_{z \in [0,1,2]} p_Z(z) \log p_Z(z) = 1.5849$

+ Because $Z = X + Y$, $H(X,Y,Z) = H(X,Y) = 1.9182$

+ $I(X,Y;Z) = H(Z) = 1.5849$

d) + $I(X;Z) = H(Z) - H(Z|X)$

+ $p_{Z|X}(z=0 | x=0) = \dfrac{p_{ZX}(z=0, x=0)}{p_X(x=0)} = \dfrac{1/3}{1/2} = \dfrac{2}{3}$

$p_{Z|X}(z=1 | x=0) = \dfrac{p_{ZX}(z=1, x=0)}{p_X(x=0)} = \dfrac{1/6}{1/2} = \dfrac{1}{3}$

$p_{Z|X}(z=1 | x=1) = \dfrac{p_{ZX}(z=1, x=1)}{p_X(x=1)} = \dfrac{1/6}{1/2} = \dfrac{1}{3}$

$p_{Z|X}(z=2 | x=1) = \dfrac{p_{ZX}(z=2, x=1)}{p_X(x=1)} = \dfrac{1/3}{1/2} = \dfrac{2}{3}$

+ $H(Z|X) = -\sum_{x \in [0,1]} p_X(x) \sum_{z \in [0,1,2]} p_{Z|X}(z|x) \log p_{Z|X}(z|x)$

$$= -\left[\frac{1}{2}\left(\frac{2}{3} \log \frac{2}{3} + \frac{1}{3} \log \frac{1}{3} \right) + \frac{1}{2}\left(\frac{1}{3} \log \frac{1}{3} + \frac{2}{3} \log \frac{2}{3} \right) \right]$$

$= 0.9182$

+ $I(X;Z) = H(Z) - H(Z|X) = 1.5849 - 0.9182 = 0.6667$

3.8. $H(X|Z) \geq H(X|Y)$

Proof:

+ If $X \to Y \to Z$ form a Markov chain then we have

$I(X;Y,Z) = I(X;Y) + I(X;Z|Y) = I(X;Z) + I(X;Y|Z)$

Because X and Z are conditionally independent given Y, $I(X;Z|Y) = 0$ and

$I(X;Y) = I(X;Z) + I(X;Y|Z)$

$\Leftrightarrow I(X;Y) \geq I(X;Z)$

$\Leftrightarrow \quad H(X) - H(X|Y) \geq H(X) - H(X|Z)$

$\Leftrightarrow \quad H(X|Y) \leq H(X|Z) \quad \square$