Homework 7

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7.3

a) We have:

- # of time pigeons can be sent / hour : $\frac{60}{5} = 12$

- The amount of information carried / release : 8
- Capacity : $12.8 = 96$ bits / hour

b) After get hit, the # of pigeons is $(1-\alpha)$

Capacity : $96(1-\alpha)$ bits / hours

7.7

$$p_{Y|X}(y|x) = \begin{bmatrix} 1-\alpha-\varepsilon & \varepsilon & \alpha \\ \alpha & \varepsilon & 1-\alpha-\varepsilon \end{bmatrix}$$

a) + $p_X^*(x) = [1/2, 1/2]$

+ $$p_{X^*,Y}(x,y) = \begin{bmatrix} \frac{1}{2}(1-\alpha-\varepsilon) & \frac{1}{2}\varepsilon & \frac{1}{2}\alpha \\ \frac{1}{2}\alpha & \frac{1}{2}\varepsilon & \frac{1}{2}(1-\alpha-\varepsilon) \end{bmatrix}$$

+ $$p_Y(y) = \left[\frac{1}{2}(1-\varepsilon) \quad \varepsilon \quad \frac{1}{2}(1-\varepsilon) \right]$$

+ $$C = \max_{p_X(x)} I(X;Y) = H(Y) - H(Y|X^*)$$

$$= -\sum_y p_Y(y) \log p_Y(y) + \sum_x p_{X^*}(x) \sum_y p_{Y|X^*}(y|x) \log p_{Y|X^*}(y|x)$$

$$= -\frac{1}{2}(1-\varepsilon)\log\frac{1}{2}(1-\varepsilon) - \varepsilon\log\varepsilon - \frac{1}{2}(1-\varepsilon)\log\frac{1}{2}(1-\varepsilon)$$

$$+ (1-\alpha-\varepsilon)\log(1-\alpha-\varepsilon) + \varepsilon\log\varepsilon + \alpha\log\alpha$$

$$= (1-\varepsilon)(1-\log(1-\varepsilon)) + (1-\alpha-\varepsilon)\log(1-\alpha-\varepsilon) + \alpha\log\alpha$$

b) If $\varepsilon = 0$, the channel become the BSC with capacity:

$$C = 1 + (1-\alpha)\log(1-\alpha) + \alpha\log\alpha = 1 - h(\alpha)$$

c) If $\alpha = 0$, the channel become the BEC with capacity:

$$C = 1-\varepsilon - (1-\varepsilon)\log(1-\varepsilon) + (1-\varepsilon)\log(1-\varepsilon) = 1-\varepsilon$$

7.9

(a) $I(X;Y) = H(Y) - H(Y|X)$

$$= H(Y) - \sum_x p_X(x)\, H(Y|X=x)$$

$$= H(Y) - \sum_x p_X(x) \log 3$$

$$= H(Y) - \log 3 \le \log 3 - \log 3 = 0$$

So $C = \max_{p_X(x)} I(X;Y) = 0 \quad \forall\, p_X^*(x)$

(b) $I(X;Y) = H(Y) - H(Y|X)$

$$= H(Y) - \sum_x p_X(x)\, H(Y|X=x)$$

$$= H(Y) - \sum_x p_X(x)\, h(1/2)$$

$$= H(Y) - 1 \qquad \le \log 3 - 1$$

So $C = \max_{p_X(x)} I(X;Y) = \log 3 - 1$ when $H(Y)$ is uniform $\Leftrightarrow p_X^*(x) = [1/3, 1/3, 1/3]$

(c) $I(X;Y) = H(Y) - H(Y|X)$

$$= H(Y) - \sum_x p_X(x)\, H(Y|X=x)$$

$$= H(Y) - \left[(p_X(1) + p_X(2))\, h(p) + (p_X(3) + p_X(4))\, h(q) \right] \quad (1)$$

$$\le \log 4 - (p_X(1) + p_X(2))\, h(p) - (p_X(3) + p_X(4))\, h(q)$$

with equality when $H(Y)$ is uniform

$$\Leftrightarrow \begin{cases} p_X(1)\, p + p_X(2)(1-p) = 1/4 \\ p_X(1)(1-p) + p_X(2)\, p = 1/4 \\ p_X(3)\, q + p_X(4)(1-q) = 1/4 \\ p_X(3)(1-q) + p_X(4)\, q = 1/4 \end{cases}$$

$$\Leftrightarrow \begin{cases} p_X(1) + p_X(2) = 1/2 \\ p_X(3) + p_X(4) = 1/2 \end{cases}$$

We have the capacity:

$$C = \max_{p_X(x)} I(X;Y) = 2 - \frac{1}{2}(h(p) + h(q)) \text{ with any } p_X^*(x) \text{ that satisfy: } \begin{cases} p_X(1) + p_X(2) = 1/2 \\ p_X(3) + p_X(4) = 1/2 \end{cases}$$

7.5 c

7.3 c

- Let $\mathcal{X} \in \{0,1\}$ and uniform distribution $X, Y \in \mathcal{X}^8$
- The information each pigeon carry have the following distribution

$$p_{Y|X}(y|x) = \begin{cases} 1-\alpha + \dfrac{\alpha}{256} & y = x \\ \dfrac{\alpha}{256} & \text{otherwise} \end{cases}$$

- $I(X,Y) = H(Y) - H(Y|X)$

$$= \log|Y| + \sum_{x \in \mathcal{X}^n} p_X(x) \sum_y p_{Y|X}(y|x) \log p_{Y|X}(y|x)$$

$$= \log 256 + \sum_{x \in \mathcal{X}^n} \frac{1}{256}\left[\left(1-\alpha + \frac{\alpha}{256}\right) \log\left(1-\alpha+\frac{\alpha}{256}\right) + \frac{255\alpha}{256} \log \frac{\alpha}{256}\right]$$

$$= 8 + \sum_{x \in \mathcal{X}^n} \frac{1}{256}\left[\left(1-\alpha + \frac{\alpha}{256}\right) \log\left(1-\alpha+\frac{\alpha}{256}\right) + \frac{255\alpha}{256} \log \frac{\alpha}{256}\right]$$

- $C = 12 \times I(X,Y)$