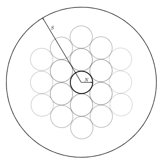
HOMEWORK 9 (2023) — SOLUTIONS

JAIST — SCHOOL OF INFORMATION SCIENCE — I232 INFORMATION THEORY

1. *Packing spheres*
   (a) In two dimensions, consider packing circles of radius $N = 1$ inside a large circle of radius $S = 10$. Find an upper bound on M, the number of circles, by dividing the area of the large circle, by the area of small circle.
   (b) Now consider n-dimensional spheres. The volume of a sphere with radius r in n dimensions is $\text{Vol}(n, r)$:
   $$\text{Vol}(n, r) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} r^n,$$
   where Γ is the gamma function. Let the radius of the large sphere be $\sqrt{P + N}$, and let the radius of the small sphere be $\sqrt{N}$. As before, find an upper bound on the number of spheres M.
   (c) Now let $R = \frac{1}{n} \log M$. Using your answer to part (b), what is an upper bound on R?

Solution:



(a)

$$M \le \frac{\text{Area of large circle}}{\text{Area of small circle}} = \frac{\pi S^2}{\pi N^2} = \frac{100}{1} = 100$$

(b) With $C_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$:

$$\text{Vol}_{\text{big}} = C_n(\sqrt{n(P+N)})^n$$
$$\text{Vol}_{\text{small}} = C_n(\sqrt{n(N)})^n$$

Maximum number of messages M is the maximum number of non-intersecting spheres:

$$M \le \frac{\text{Vol}_{\text{big}}}{\text{Vol}_{\text{small}}} = \frac{C_n(\sqrt{n(P+N)})^n}{C_n(\sqrt{n(N)})^n}$$
$$M \le \left(1 + \frac{P}{N}\right)^{n/2}$$

(c)

$$R = \frac{1}{n} \log M \le \frac{1}{n} \log \left(1 + \frac{P}{N}\right)^{n/2}$$
$$R \le \frac{1}{2} \log \left(1 + \frac{P}{N}\right)$$

2. *Triangular probability distribution* Consider two uniformly distributed random variables X and Y:

$$p_{\mathsf{X}}(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad p_{\mathsf{Y}}(y) = \begin{cases} \frac{1}{2} & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the probability distribution of $\mathsf{X} + \mathsf{Y}$, using convolution.

Solution: Let $\mathsf{Z} = \mathsf{X} + \mathsf{Y}$.

$$\begin{aligned}
p_{\mathsf{Z}}(z) &= \int_{-\infty}^{\infty} p_{\mathsf{X}}(z-w)p_{\mathsf{Y}}(w)dw && -1 \le z-w \le 1, -1 \le w \le 1 \\
&= \int_{\max(z-1,-1)}^{\min(z+1,1)} \frac{1}{2}\frac{1}{2}dw && z-1 \le w \le z+1 \\
&= \frac{1}{4}\Big(\min(z+1,1) - \max(z-1,-1)\Big) \\
&= \begin{cases} \frac{1}{2} + \frac{1}{4}z & -2 \le z \le 0 \\ \frac{1}{2} - \frac{1}{4}z & 0 < z \le 2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

(b) Let Z be distributed as a triangular distribution:

$$p_{\mathsf{Z}}(z) = \begin{cases} c + c^2 z & -\frac{1}{c} \le z \le 0 \\ c - c^2 z & 0 < z \le \frac{1}{c} \\ 0 & \text{otherwise} \end{cases}$$

for $c > 0$. Find differential entropy of Z in nats, using natural log ln.

Solution:

$$\begin{aligned}
H(\mathsf{Z}) &= -\int p_{\mathsf{Z}}(z) \ln p_{\mathsf{Z}}(z) dz \\
&= -\int_{-1/c}^{0} (c + c^2 z) \ln(c + c^2 z) dz - \int_{0}^{1/c} (c - c^2 z) \ln(c - c^2 z) dz
\end{aligned}$$

The two integrals are equal by symmetry. The first integral is:

$$-\int_{-1/c}^{0} (c + c^2 z) \ln(c + c^2 z) dz = \frac{1}{4} + \frac{1}{2} \ln \frac{1}{c}$$

One technique to evaluate the integral is to use Wolfram Alpha or Mathematica:

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Integrate[c(1+cz) Log[ c (1+c z)],{z,-1/c,0}, Assumptions -> c > 0]
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So the differential entropy is twice that:

$$H(\mathsf{Z}) = \frac{1}{2} + \ln \frac{1}{c}$$

(c) What is the differential entropy of $\mathsf{X} + \mathsf{Y}$, in nats?

Solution: Since $\mathsf{X} + \mathsf{Y}$ has a triangular distribution with $c = 1/2$, the differential entropy is $\frac{1}{2} + \ln 2$.

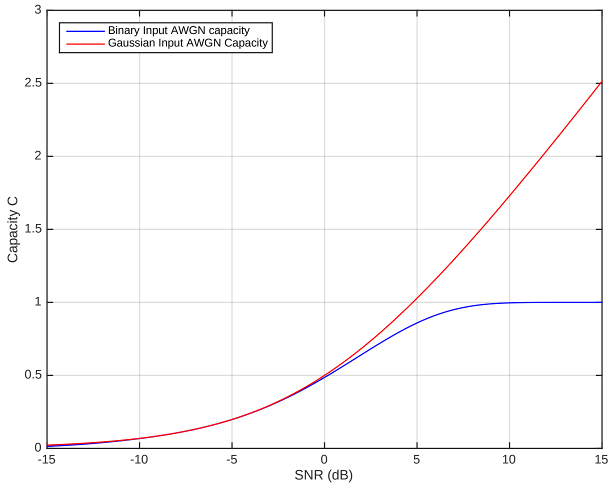
3. *Capacity of the binary-input AWGN channel* The binary-input AWGN channel achieves capacity with $p^*_{\mathsf{X}}(x) = [\frac{1}{2}, \frac{1}{2}]$. The signal-to-noise ratio SNR in decibels (dB) is:

$$\text{SNR in dB} = 10 \log_{10} \frac{1}{\sigma^2}.$$

(a) Make a plot of capacity C versus SNR dB for the binary-input AWGN channel. (b) On the same graph, plot the capacity of the AWGN channel with Gaussian input distribution. (c) What happens to achievable rates as SNR $\to 0$? What happens to achievable rates as SNR $\to \infty$? What can you conclude from this?

Solution: (a), (b) The figure below shows the plot of capacity of the binary-input AWGN channel and the Gaussian-input AWGN channel, and below that is the source code used to generate them.

(c) Note that SNR $\to 0$ means SNR dB $\to -\infty$. As SNR $\to 0$, the capacity goes to 0. As SNR $\to \infty$, the capacity of the binary-input goes to 1, since the input is binary, but the capacity of the Gaussian-input AWGN channel goes to infinity. From these two, we can conclude that as SNR $\to 0$, it is sufficient to use binary inputs, i.e. there is no benefit to using Gaussian-input signaling. But as SNR $\to \infty$, Gaussian inputs are needed to achieve higher capacities.

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clear all
close all

SNRdb = linspace(-15,15);
SNR   = 10.^(SNRdb/10);
var   = 1 ./ SNR;

IXY = zeros(size(var));
for ii = 1:length(var)
    IXY(ii) = biawgnCapacity(var(ii));
end

plot(SNRdb,IXY,'b-');
hold on
C = 0.5 * log2(1 + SNR);
plot(SNRdb,C,'r-');
xlabel('SNR (dB)');
ylabel('Capacity C');
legend('Binary Input AWGN capacity','Gaussian Input AWGN Capacity','Location','
    Northwest');
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```
grid on
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```
function IXY = biawgnCapacity(var)

fun = @(y) (1/sqrt(2*pi*var)) .* exp( - (y-1).^2 / (2*var) ) .* log2( 2  ./ ( 1 + exp( - 2*y / var))  );
IXY = integral(fun,-10,10);
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```
>> var = 0.9578;
>> fun = @(y) (1/sqrt(2*pi*var)) .* exp( - (y-1).^2 / (2*var) ) .* log2( 2  ./ ( 1 + exp( - 2*y / var))  );
>> IXY = integral(fun,-10,10)

IXY =

    0.5000
```