Homework 2 (2023) — SOLUTIONS

JAIST — School of Information Science — I232 Information Theory

1. Let the random variable X be distributed as:

$$p_{\mathsf{X}}(x) = \begin{cases} \frac{1}{4} & \text{if } x = 1 \\ \frac{1}{2} & \text{if } x = 2 \\ \frac{1}{4} & \text{if } x = 3 \end{cases}.$$

Consider the sample mean:

$$\overline{\mathsf{X}}_n = \frac{1}{n}\sum_{i=1}^{n} \mathsf{X}_n.$$

(a) Find $E[\mathsf{X}]$

> **Solution:** $E[\mathsf{X}] = \sum_{x\in\mathcal{X}} x p_{\mathsf{X}}(x) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} = 2$

(b) Find $\mathrm{Var}[\mathsf{X}]$

> **Solution:** $E[\mathsf{X}^2] = \sum_{x\in\mathcal{X}} x^2 p_{\mathsf{X}}(x) = 1^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{2} + 3^2 \cdot \frac{1}{4} = 4.5$
> $\mathrm{Var}\mathsf{X} = E[\mathsf{X}^2] - (E[\mathsf{X}])^2 = 4.5 - 2^2 = 0.5$

(c) Using the Chebyshev inequality, what value of n is needed to guarantee that the probability that $\overline{\mathsf{X}}_n$ is within $\epsilon = 0.1$ of its mean is greater than 0.999?

> **Solution:** From the proof of the law of large numbers:
>
> $$\underbrace{\Pr\left[|\overline{\mathsf{X}}_n - E[\mathsf{X}]| < \epsilon\right]}_{q} \geq 1 - \frac{\mathrm{Var}[\mathsf{X}]}{n\epsilon^2}$$
>
> $$n \leq \frac{\mathrm{Var}[\mathsf{X}]}{(1-q)\epsilon^2}$$
>
> Using $\mathrm{Var}[\mathsf{X}] = 0.5, q = 0.999, \epsilon = 0.1$, we have $n = 50000$ will satisfy the conditions.

(d) Next, it is expected that the mean should be even closer to its mean, within $\epsilon = 0.01$. Now what value of n is needed to guarantee this?

> **Solution:** Changing to $\epsilon = 0.01$, we now need many more samples, $n = 5 \cdot 10^6$ to satisfy the conditions.

You should write a program to for the following parts. Refer to the Information Theory Lecture Notes for an example. For the random variable $\mathbf{X}$, write a program that randomly generates n samples from the distribution $p_{\mathsf{X}}(x)$, and computes $\overline{\mathsf{X}}_n$.

(e) If $n = 50$, perform Monte Carlo experiments to estimate the probability that $\overline{\mathsf{X}}_n$ is within $\varepsilon = 0.1$ of the true mean. You should perform 1000 Monte Carlo experiments, and count the number of times $|\overline{\mathsf{X}}_n - EX| \leq \varepsilon$ is satisfied.

(f) Repeat for $n = 100$.

(g) Repeat for $n = 200$.

Solution: Using 1000 Monte Carlo experiments, the estimated probabilities are:
(e) 0.631
(f) 0.821
(g) 0.949
Clearly, n increases, the sample mean gets closer to the true mean. Below is a Matlab implementation.

```
clear

trueMean = 2;   %true mean of the random variable
n        = 200; %number of samples
epsilon  = 0.1;
numberOfExperiments = 1000;

for ii = 1:numberOfExperiments
    %This generates a random variable with [1/4 1/2 1/4]:
    x = randi(4,1,n);
    x(find(x==4)) = 2;

    sampleMean(ii) = mean(x);
end
numberWithinEpsilon = length(find( abs(sampleMean - trueMean) < epsilon))
probability = numberWithinEpsilon / numberOfExperiments
```

2. Prove the Chebyshev Inequality, for a random variable X with mean μ and variance σ^2:

$$\Pr[|\mathsf{X} - \mu| \geq k\sigma] \leq \frac{1}{k^2}$$

using the Markov inequality.

Solution: The Markov inequality states that for a non-negative random variable Y and constant $a > 0$:

$$\Pr(\mathsf{Y} \geq a) \leq \frac{E[\mathsf{Y}]}{a}.$$

holds. Define $\mathsf{Y} = (\mathsf{X} - \mu)^2$ (note $\mathsf{Y} > 0$), so that $E[\mathsf{Y}] = E[(\mathsf{X} - \mu)^2] = \sigma^2$ then the Markov inequality becomes:

$$\Pr((\mathsf{X} - \mu)^2 \geq a) \leq \frac{\sigma^2}{a}$$

Further, take $a = k^2\sigma^2$ (where $k > 0$), so that:

$$\Pr(|\mathsf{X} - \mu| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2},$$

which is the Chebyshev inequality.

3. Let $\mathsf{X}_1, \mathsf{X}_2, \ldots \mathsf{X}_n$ be a sequence of n independent and identically distributed random variables with expected value $E[\mathsf{X}]$ and variance $\mathrm{Var}[\mathsf{X}]$. The sample mean is:

$$\overline{\mathsf{X}}_n = \frac{1}{n}\sum_{i=1}^{n} \mathsf{X}_i.$$

The law of large numbers states that the sample mean converges in probability towards the expected value:

$$\lim_{n\to\infty} \Pr\left(|\overline{\mathsf{X}}_n - E[\mathsf{X}]| < \epsilon\right) = 1$$

Give a proof of the law of large numbers. Use the Chebyshev inequality in your proof.

Solution: The mean and variance of $\overline{\mathsf{X}}_n$ are:

$$\begin{aligned} E[\overline{\mathsf{X}}_n] &= \frac{1}{n}\sum_{i=1}^{n} E[\mathsf{X}_i] = E[\mathsf{X}] \\ \mathrm{Var}[\overline{\mathsf{X}}_n] &= \mathrm{Var}[\frac{1}{n}\sum_{i=1}^{n} \mathsf{X}_i] = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}[\mathsf{X}_i] \quad \text{by independence} \\ &= \frac{1}{n}\mathrm{Var}[\mathsf{X}] \end{aligned}$$

Apply the Chebyshev inequality to $\overline{\mathsf{X}}_n$:

$$\Pr\left[|\overline{\mathsf{X}}_n - E[\overline{\mathsf{X}}_n]| < \epsilon\right] \geq 1 - \frac{\mathrm{Var}[\overline{\mathsf{X}}_n]}{\epsilon^2}$$

and then:

$$\Pr\left[|\overline{\mathsf{X}}_n - E[\mathsf{X}]| < \epsilon\right] \geq 1 - \frac{\mathrm{Var}[\mathsf{X}]}{n\epsilon^2}$$

Taking the limit of both sides:

$$\lim_{n\to\infty} \Pr\left[|\overline{\mathsf{X}}_n - E[\mathsf{X}]| < \epsilon\right] = 1$$

since $\lim_{n\to\infty} 1 - \frac{\mathrm{Var}[\mathsf{X}]}{n\epsilon^2} = 1$.