Homework 3

23 October 2023 14:22

3.2

- Let $c_1 = [1\ 0\ 1\ p_1\ p_2\ p_3\ p_4]$

$$Hc_1^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{cases} 1 + 1 + p_1 = 0 \\ 1 + 1 + p_2 = 0 \\ 1 + p_2 + p_3 = 0 \\ 1 + p_2 + p_4 = 0 \end{cases} \Leftrightarrow \begin{cases} p_1 = 0 \\ p_2 = 0 \\ p_3 = 1 \\ p_4 = 1 \end{cases}. \quad c_1 = [1\ 0\ 1\ 0\ 0\ 1\ 1]$$

- Let $c_2 = [0\ 1\ 0\ p_1\ p_2\ p_3\ p_4]$

$$Hc_2^T = 0 \Leftrightarrow \begin{cases} 1 + p_1 = 0 \\ p_2 = 0 \\ 1 + p_2 + p_3 = 0 \\ 1 + p_2 + p_4 = 0 \end{cases} \Leftrightarrow \begin{cases} p_1 = 1 \\ p_2 = 0 \\ p_3 = 1 \\ p_4 = 1 \end{cases}. \quad c_2 = [0\ 1\ 0\ 1\ 0\ 1\ 1]$$

3.3

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\downarrow \begin{matrix} R_4 = R_4 + R_1 \\ R_2 = R_2 + R_3 \\ R_3 = R_3 + R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\downarrow R_4 = R_4 + R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\downarrow \begin{matrix} R_4 = R_4 + R_1 \\ R_2 = R_2 + R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

Let $\bar{x} = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7]^T$ and $A\bar{x} = 0$. (1) we have

$$\begin{cases} x_1 + x_5 = 0 \\ x_3 + x_5 + x_6 = 0 \\ x_2 + x_4 + x_6 = 0 \\ x_7 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_5 = x_1 \\ x_6 = x_1 + x_3 \\ x_6 = x_2 + x_4 \\ x_7 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_5 = x_1 \\ x_6 = x_1 + x_3 \\ x_4 = x_1 + x_3 + x_2 \\ x_7 = 0 \end{cases}$$

Let x_1, x_2, x_3 be free variables, we have

$$\bar{x} = [x_1, x_2, x_3, x_1 + x_3 + x_2, x_1, x_1 + x_3, 0]$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad (2)$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We have $G . H^T = 0 \Leftrightarrow G$ form the basis of the null space of H.
From (2), we have

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

3.5

$$G = [I_3 \mid -A^T] = \begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$$

$$\Leftrightarrow A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 2 \end{bmatrix}$$

$$H = [A \mid I_2] = \begin{bmatrix} 2 & 3 & 4 & 1 & 0 \\ 4 & 3 & 2 & 0 & 1 \end{bmatrix}$$

3.8

a) Let $\tilde{G} = \begin{bmatrix} G \\ G' \end{bmatrix}$ be a full rank matrix. We have

$$\tilde{G} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{H} = \tilde{G}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

We have $\tilde{H} = [E \ H^T]$, so

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

b) $\tilde{G} = \begin{bmatrix} G \\ G' \end{bmatrix}$ and $\tilde{H} = [E \ H^T]$ and $\tilde{G} . \tilde{H} = I$

Let u be a information vector and c be the corresponding code word.
We have

$$[u \ \ 0] \begin{bmatrix} G \\ G' \end{bmatrix} = c$$

$$\Leftrightarrow [u \ \ 0] = c \begin{bmatrix} G \\ G' \end{bmatrix}^{-1}$$

$$\Leftrightarrow [u \ \ 0] = c \, [E \ \ H^T]$$

$$\Leftrightarrow [u \quad 0] = c \begin{bmatrix} \\ G' \end{bmatrix}$$

$$\Leftrightarrow [u \quad 0] = c\,[E \quad H^T]$$

$$\Leftrightarrow \begin{cases} u = cE & \text{for any } u \\ 0 = cH^T \end{cases}$$

so E is the inverse encoding matrix and H is a parity check matrix