Lecture 2: Linear Algebra

Yi, Yung (이융)

Mathematics for Machine Learning
https://yung-web.github.io/home/courses/mathml.html
KAIST EE

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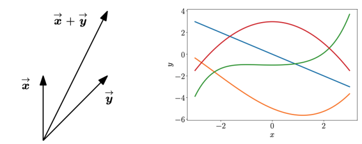
Roadmap

Roadmap

Linear Algebra

- Algebra: a set of objects and a set of rules or operations to manipulate those objects
- Linear algebra
  - Object: vectors $\boldsymbol{v}$
  - Operations: their additions $(\boldsymbol{v} + \boldsymbol{w})$ and scalar multiplication $(k\boldsymbol{v})$
- Examples
  - Geometric vectors
    - High school physics
  - Polynomials
  - Audio signals
  - Elements of $\mathbb{R}^n$



(a) Geometric vectors. (b) Polynomials.

System of Linear Equations

- For unknown variables $(x_1, \cdots, x_n) \in \mathbb{R}^n$,

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

- Three cases of solutions

- No solution	- Unique solution	- Infinitely many solutions
$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 + 3x_3 &= 1 \end{aligned}$	$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= 2 \\ x_2 + 3x_3 &= 1 \end{aligned}$	$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 + 3x_3 &= 5 \end{aligned}$

- Question. Under what conditions, one of the above three cases occur?

Matrix Representation

- A collection of linear equations

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

- Matrix representations:

$$\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \cdots + \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \iff \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}_{\boldsymbol{b}}$$

- Understanding $\boldsymbol{A}$ is the key to answering various questions about this linear system $\boldsymbol{Ax} = \boldsymbol{b}$.

Roadmap

Matrix: Addition and Multiplication

- For two matrices $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{m \times n}$,

$$\boldsymbol{A} + \boldsymbol{B} := \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

- For two matrices $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n \times k}$, the elements c_{ij} of the product $\boldsymbol{C} = \boldsymbol{AB} \in \mathbb{R}^{m \times k}$ is:

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, k.$$

- Example. $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$, compute $\boldsymbol{AB}$ and $\boldsymbol{BA}$.

Identity Matrix and Matrix Properties

- A square matrix[1] $\boldsymbol{I}_n$ with $I_{ii} = 1$ and $I_{ij=0}$ for $i \neq j$, where n is the number of rows and columns. For example,

$$\boldsymbol{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Associativity: For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times p}$, $\boldsymbol{C} \in \mathbb{R}^{p\times q}$, $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$
- Distributivity: For $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times n}$, and $\boldsymbol{C}, \boldsymbol{D} \in \mathbb{R}^{n\times p}$,
  (i) $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$ and (ii) $\boldsymbol{A}(\boldsymbol{C}+\boldsymbol{D}) = \boldsymbol{AC} + \boldsymbol{AD}$
- Multiplication with the identity matrix: For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{I}_m\boldsymbol{A} = \boldsymbol{A}\boldsymbol{I}_n = \boldsymbol{A}$

[1] # of rows = # of cols

Inverse and Transpose

- For a square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}$ is the inverse of A, denoted by $\boldsymbol{A}^{-1}$, if

$$\boldsymbol{AB} = \boldsymbol{I}_n = \boldsymbol{BA}.$$

- Called regular/invertible/nonsingular, if it exists.
- If it exists, it is unique.
- How to compute? For 2×2 matrix,

$$\boldsymbol{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

- For a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times m}$ with $b_{ij} = a_{ji}$ is the transpose of $\boldsymbol{A}$, which we denote by $\boldsymbol{A}^{\mathsf{T}}$.
- Example. For $\boldsymbol{A} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$,

$$\boldsymbol{A}^{\mathsf{T}} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

- If $\boldsymbol{A} = \boldsymbol{A}^{\mathsf{T}}$, $\boldsymbol{A}$ is called symmetric.

Inverse and Transpose: More Properties

- $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I} = \boldsymbol{A}^{-1}\boldsymbol{A}$
- $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
- $(\boldsymbol{A}+\boldsymbol{B})^{-1} \neq \boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}$
- $(\boldsymbol{A}^{\mathsf{T}})^{\mathsf{T}} = \boldsymbol{A}$
- $(\boldsymbol{A}+\boldsymbol{B})^{\mathsf{T}} = \boldsymbol{A}^{\mathsf{T}} + \boldsymbol{B}^{\mathsf{T}}$
- $(\boldsymbol{AB})^{\mathsf{T}} = \boldsymbol{B}^{\mathsf{T}}\boldsymbol{A}^{\mathsf{T}}$
- If $\boldsymbol{A}$ is invertible, so is $\boldsymbol{A}^{\mathsf{T}}$.

Scalar Multiplication

- Multiplication by a scalar $\lambda \in \mathbb{R}$ to $\boldsymbol{A} \in \mathbb{R}^{m\times n}$
- Example. For $\boldsymbol{A} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$, $3 \times \boldsymbol{A} = \begin{pmatrix} 0 & 6 \\ 3 & -3 \\ 0 & 3 \end{pmatrix}$
- Associativity
  - $(\lambda\psi)\boldsymbol{C} = \lambda(\psi\boldsymbol{C})$
  - $\lambda(\boldsymbol{BC}) = (\lambda\boldsymbol{B})\boldsymbol{C} = \boldsymbol{B}(\lambda\boldsymbol{C}) = (\boldsymbol{BC})\lambda$
  - $(\lambda\boldsymbol{C})^{\mathsf{T}} = \boldsymbol{C}^{\mathsf{T}}\lambda^{\mathsf{T}} = \boldsymbol{C}^{\mathsf{T}}\lambda = \lambda\boldsymbol{C}^{\mathsf{T}}$
- Distributivity
  - $(\lambda+\psi)\boldsymbol{C} = \lambda\boldsymbol{C} + \psi\boldsymbol{C}$
  - $\lambda(\boldsymbol{B}+\boldsymbol{C}) = \lambda\boldsymbol{B} + \lambda\boldsymbol{C}$

Roadmap

Example

$$\begin{aligned} -3x \quad\; + 2z &= -1 \\ x - 2y + 2z &= -5/3 \\ -x - 4y + 6z &= -13/3 \end{aligned}$$

- ρ_i: i-th equation
- Express the equation as its augmented matrix.

$$\left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 1 & -2 & 2 & -5/3 \\ -1 & -4 & 6 & -13/3 \end{array}\right) \xrightarrow[-(1/3)\rho_1+\rho_3]{(1/3)\rho_1+\rho_2} \left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 0 & -2 & 8/3 & -2 \\ 0 & -4 & 16/3 & -4 \end{array}\right)$$

$$\xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 0 & -2 & 8/3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The two nonzero rows give $-3x + 2z = -1$ and $-2y + (8/3)z = -2$.

[1]Examples from this slide to the next several slides come from Jim Hefferson's Linear Algebra book.

- Parametrizing $-3x + 2z = -1$ and $-2y + (8/3)z = -2$ gives:

$$\begin{aligned} x &= (1/3) + (2/3)z \\ y &= 1 + (4/3)z \\ z &= z \end{aligned}$$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 4/3 \\ 1 \end{pmatrix} z \mid z \in \mathbb{R} \right\}$$

This helps us understand the set of solutions, e.g., each value of z gives a different solution.

z	0	1	2	$-1/2$
solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$	$\begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 7/3 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 5/3 \\ 11/3 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1/3 \\ -1/2 \end{pmatrix}$

Form of solution sets

- The system $\begin{aligned} x + 2y - \;\; z \quad\;\; &= 2 \\ 2x - \;\; y - 2z + w &= 5 \end{aligned}$ reduces in this way.

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 2 & -1 & -2 & 1 & 5 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & -5 & 0 & 1 & 1 \end{array}\right)$$

- It has solutions of this form.

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 12/5 \\ -1/5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -2/5 \\ 1/5 \\ 0 \\ 1 \end{pmatrix} w \qquad \text{for } z, w \in \Re$$

- Note that taking $z = w = 0$ shows that the first vector is a particular solution of the system.

General $=$ Particular $+$ Homogeneous

- General approach
  1. Find a particular solution to $\boldsymbol{Ax} = \boldsymbol{b}$
  2. Find all solutions to the homogeneous equation $\boldsymbol{Ax} = 0$
     - 0 is a trivial solution
  3. Combine the solutions from steps 1. and 2. to the general solution
- Questions: A formal algorithm that performs the above?
  - Gauss-Jordan method: convert into a "beautiful" form (formally reduced row-echelon form)
  - Elementary transformations: (i) row swapping (ii) multiply by a constant (iii) row addition
- Such a form allows an algorithmic way of solving linear equations

Example: Unique Solution

- Start as usual by getting echelon form.

$$\begin{array}{rcl} x+y-z&=&2\\ 2x-y&=&-1\\ x-2y+2z&=&-1\end{array} \xrightarrow[-1\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{array}{rcl} x+y-z&=&2\\ -3y+2z&=&-5\\ -3y+3z&=&-3\end{array} \xrightarrow{-1\rho_2+\rho_3} \begin{array}{rcl} x+y-z&=&2\\ -3y+2z&=&-5\\ z&=&2\end{array}$$

- Make all the leading entries one.

$$\xrightarrow{(-1/3)\rho_2} \begin{array}{rcl} x+y-z&=&2\\ y-(2/3)z&=&5/3\\ z&=&2\end{array}$$

- Finish by using the leading entries to eliminate upwards, until we can read off the solution.

$$\begin{array}{rcl} x+y-z&=&2\\ y-(2/3)z&=&5/3\\ z&=&2\end{array} \xrightarrow[(2/3)\rho_3+\rho_2]{\rho_3+\rho_1} \begin{array}{rcl} x+y&=&4\\ y&=&3\\ z&=&2\end{array} \xrightarrow{-\rho_2+\rho_1} \begin{array}{rcl} x&=&1\\ y&=&3\\ z&=&2\end{array}$$

Example: Infinite Number of Solutions

$$\begin{array}{rcl} x-y-2w&=&2\\ x+y+3z+w&=&1\\ -y+z-w&=&0\end{array}$$

- Start by getting echelon form and turn the leading entries to 1's.

$$\xrightarrow{-1\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1&-1&0&-2&2\\ 0&2&3&3&-1\\ 0&-1&1&-1&0\end{array}\right)$$

$$\xrightarrow{(1/2)\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1&-1&0&-2&2\\ 0&2&3&3&-1\\ 0&0&5/2&1/2&-1/2\end{array}\right)$$

$$\xrightarrow[(2/5)\rho_3]{(1/2)\rho_2} \left(\begin{array}{cccc|c} 1&-1&0&-2&2\\ 0&1&3/2&3/2&-1/2\\ 0&0&1&1/5&-1/5\end{array}\right)$$

- Eliminate upwards.

$$\xrightarrow{-(3/2)\rho_3+\rho_2} \left(\begin{array}{cccc|c} 1&-1&0&-2&2\\ 0&1&0&6/5&-1/5\\ 0&0&1&1/5&-1/5\end{array}\right)$$

$$\xrightarrow{\rho_2+\rho_1} \left(\begin{array}{cccc|c} 1&0&0&-4/5&9/5\\ 0&1&0&6/5&-1/5\\ 0&0&1&1/5&-1/5\end{array}\right)$$

- The parameterized solution set is:

$$\left\{ \begin{pmatrix} 9/5\\ -1/5\\ -1/5\\ 0\end{pmatrix} + \begin{pmatrix} 4/5\\ -6/5\\ -1/5\\ 1\end{pmatrix} w \mid w \in \mathbb{R} \right\}$$

Cases of Solution Sets

number of solutions of the homogeneous system

particular solution exists?		*one*	*infinitely many*
	yes	unique solution	infinitely many solutions
	no	no solutions	no solutions

Algorithms for Solving System of Linear Equations

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1. Pseudo-inverse

   $\boldsymbol{Ax} = \boldsymbol{b} \iff \boldsymbol{A}^\mathsf{T}\boldsymbol{Ax} = \boldsymbol{A}^\mathsf{T}\boldsymbol{b} \iff \boldsymbol{x} = \left(\boldsymbol{A}^\mathsf{T}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^\mathsf{T}\boldsymbol{b}$

   - $\left(\boldsymbol{A}^\mathsf{T}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^\mathsf{T}$: *Moore-Penrose pseudo-inverse*
   - many computations: matrix product, inverse, etc
2. Gaussian elimination
   - intuitive and constructive way
   - cubic complexity (in terms of # of simultaneous equations)
3. Iterative methods
   - practical ways to solve indirectly

   (a) stationary iterative methods: Richardson method, Jacobi method, Gaus-Seidel method, successive over-relaxation method

   (b) Krylov subspace methods: conjugate gradients, generalized minimal residual, biconjugate gradients

Roadmap

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Group

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- A set $\mathcal{G}$ and an operation $\otimes : \mathcal{G} \times \mathcal{G} \mapsto \mathcal{G}$. $G := (\mathcal{G}, \otimes)$ is called a group, if:
  1. Closure. $\forall x, y \in \mathcal{G},\ x \otimes y \in \mathcal{G}$
  2. Associativity. $\forall x, y, z \in \mathcal{G},\ (x \otimes y) \otimes z = x \otimes (y \otimes z)$
  3. Neutral element. $\exists e \in \mathcal{G},\ \forall x \in \mathcal{G},\ x \otimes e = x$ and $e \otimes x = x$
  4. Inverse element. $\forall x \in \mathcal{G},\ \exists y \in \mathcal{G},\ x \otimes y = e$ and $y \otimes x = e$. We often use $x^{-1} = y$.
- $G = (\mathcal{G}, \otimes)$ is an Abelian group, if the following is additionally met:
  - Communicativity. $\forall x, y \in \mathcal{G},\ x \otimes y = y \otimes x$

Examples

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- $(\mathbb{Z}, +)$ is an Abelian group
- $(\mathbb{N} \cup \{0\}, +)$ is not a group (because inverses are missing)
- $(\mathbb{Z}, \cdot)$ is not a group
- $(\mathbb{R}, \cdot)$ is not a group (because of no inverse for 0)
- $(\mathbb{R}^n, +)$, $(\mathbb{Z}^n, +)$ are Abelian, if $+$ is defined componentwise
- $(\mathbb{R}^{m\times n}, +)$ is Abelian (with componentwise $+$)
- $(\mathbb{R}^{n\times n}, \cdot)$
  - Closure and associativity follow directly
  - Neutral element: $\boldsymbol{I}_n$
  - The inverse $\boldsymbol{A}^{-1}$ may exist or not. So, generally, it is not a group. However, the set of invertible matrices in $\mathbb{R}^{n\times n}$ with matrix multiplication is a group, called general linear group.

Vector Spaces

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Definition. A real-valued vector space $V = (\mathcal{V}, +, \cdot)$ is a set $\mathcal{V}$ with two operations

(a) $+ : \mathcal{V} \times \mathcal{V} \mapsto \mathcal{V}$ (vector addition)

(b) $\cdot : \mathbb{R} \times \mathcal{V} \mapsto \mathcal{V}$ (scalar multiplication),

where

1. $(\mathcal{V}, +)$ is an Abelian group
2. Distributivity.
   - $\forall \lambda \in \mathbb{R}, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}, \ \lambda \cdot (\boldsymbol{x} + \boldsymbol{y}) = \lambda \cdot \boldsymbol{x} + \lambda \boldsymbol{y}$
   - $\forall \lambda, \psi \in \mathbb{R}, \boldsymbol{x} \in \mathcal{V}, \ (\lambda + \psi) \cdot \boldsymbol{x} = \lambda \cdot \boldsymbol{x} + \psi \cdot \boldsymbol{x}$
3. Associativity. $\forall \lambda, \psi \in \mathbb{R}, \boldsymbol{x} \in \boldsymbol{\mathcal{V}}, \ \lambda \cdot (\psi \cdot \boldsymbol{x}) = (\lambda \psi) \cdot \boldsymbol{x}$
4. Neutral element. $\forall \boldsymbol{x} \in \mathcal{V}, \ 1 \cdot \boldsymbol{x} = \boldsymbol{x}$

Example

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- $\mathcal{V} = \mathbb{R}^n$ with
  - Vector addition: $\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \ldots, x_n + y_n)$
  - Scalar multiplication: $\lambda \boldsymbol{x} = (\lambda x_1, \ldots, \lambda x_n)$
- $\mathcal{V} = \mathbb{R}^{m \times n}$ with
  - Vector addition: $\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$
  - Scalar multiplication: $\lambda \boldsymbol{A} = \begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{pmatrix}$

Vector Subspaces

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Definition. Consider a vector space $V = (\mathcal{V}, +, \cdot)$ and $\mathcal{U} \subset \mathcal{V}$. Then, $U = (\mathcal{U}, +, \cdot)$ is called vector subspace (simply linear subspace or subspace) of V if U is a vector space with two operations '$+$' and '$\cdot$' restricted to $\mathcal{U} \times \mathcal{U}$ and $\mathbb{R} \times \mathcal{U}$.

Examples

- For every vector space V, V and $\{0\}$ are the trivial subspaces.
- The solution set of $\boldsymbol{A}\boldsymbol{x} = 0$ is the subspace of $\mathbb{R}^n$.
- The solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ ($\boldsymbol{b} \neq 0$) is not a subspace of $\mathbb{R}^n$.
- The intersection of arbitrarily many subspaces is a subspace itself.

Roadmap

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Linear Independence

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- Definition. For a vector space V and vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in V$, every $\boldsymbol{v} \in V$ of the form $\boldsymbol{v} = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_k \boldsymbol{x}_k$ with $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ is a linear combination of the vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in V$.
- Definition. If there is a non-trivial linear combination such that $0 = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i$ with at least one $\lambda_i \neq 0$, the vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ are linearly dependent. If only the trivial solution exists, i.e., $\lambda_1 = \ldots = \lambda_k = 0$, $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ are linearly independent.
- Meaning. A set of linearly independent vectors consists of vectors that have no redundancy.
- Useful fact. The vectors $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\}$ are linearly dependent, iff (at least) one of them is a linear combination of the others.
  - $x - 2y = 2$ and $2x - 4y = 4$ are linearly dependent.

Checking Linear Independence

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- Gauss elimination to get the row echelon form
- All column vectors are linearly independent iff all columns are pivot columns (why?).
- Example.

$$\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 4 \end{pmatrix}, \quad \boldsymbol{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \quad \boldsymbol{x}_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & -2 \\ -3 & 0 & 1 \\ 4 & 2 & 1 \end{pmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- Every column is a pivot column. Thus, $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ are linearly independent.

Linear Combinations of Linearly Independent Vectors

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- Vector space V with k linearly independent vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_k$
- m linear combinations $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m$. (Q) Are they linearly independent?

$$\begin{aligned} \boldsymbol{x}_1 &= \lambda_{11}\boldsymbol{b}_1 + \lambda_{21}\boldsymbol{b}_2 + \cdots + \lambda_{k1}\boldsymbol{b}_k \\ &\vdots \\ \boldsymbol{x}_m &= \lambda_{1m}\boldsymbol{b}_1 + \lambda_{2m}\boldsymbol{b}_2 + \cdots + \lambda_{km}\boldsymbol{b}_k \end{aligned}$$

$$\boldsymbol{x}_j = \overbrace{\begin{pmatrix} \boldsymbol{b}_1, & \cdots, & \boldsymbol{b}_k \end{pmatrix}}^{\boldsymbol{B}} \overbrace{\begin{pmatrix} \lambda_{1j} \\ \vdots \\ \lambda_{kj} \end{pmatrix}}^{\boldsymbol{\lambda}_j}, \qquad \boldsymbol{x}_j = \boldsymbol{B}\boldsymbol{\lambda}_j$$

- $\sum_{j=1}^{m} \psi_j \boldsymbol{x}_j = \sum_{j=1}^{m} \psi_j \boldsymbol{B}\boldsymbol{\lambda}_j = \boldsymbol{B} \sum_{j=1}^{m} \psi_j \boldsymbol{\lambda}_j$
- $\{\boldsymbol{x}\}$ linearly independent $\Longleftrightarrow$ $\{\boldsymbol{\lambda}\}$ linearly independent

Example

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$$\begin{aligned} \boldsymbol{x}_1 &= \boldsymbol{b}_1 - 2\boldsymbol{b}_2 + \boldsymbol{b}_3 - \boldsymbol{b}_4 \\ \boldsymbol{x}_2 &= -4\boldsymbol{b}_1 - 2\boldsymbol{b}_2 + 4\boldsymbol{b}_4 \\ \boldsymbol{x}_3 &= 2\boldsymbol{b}_1 + 3\boldsymbol{b}_2 - \boldsymbol{b}_3 - 3\boldsymbol{b}_4 \\ \boldsymbol{x}_4 &= 17\boldsymbol{b}_1 - 10\boldsymbol{b}_2 + 11\boldsymbol{b}_3 + \boldsymbol{b}_4 \end{aligned}$$

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{\lambda}_1 & \boldsymbol{\lambda}_2 & \boldsymbol{\lambda}_3 & \boldsymbol{\lambda}_4 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 2 & 17 \\ -2 & -2 & 3 & -10 \\ 1 & 0 & -1 & 11 \\ -1 & -4 & -3 & 1 \end{pmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & -18 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- The last column is not a pivot column. Thus, $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_3$ are linearly dependent.

Roadmap

(1) Systems of Linear Equations

(2) Matrices

(3) Solving Systems of Linear Equations

(4) Vector Spaces

(5) Linear Independence

(6) Basis and Rank

(7) Linear Mappings

(8) Affine Spaces

Generating Set and Basis

- Definition. A vector space $V = (\mathcal{V}, +, \cdot)$ and a set of vectors $\mathcal{A} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subset \mathcal{V}$.
  - If every $v \in \mathcal{V}$ can be expressed as a linear combination of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$, $\mathcal{A}$ is called a generating set of V.
  - The set of all linear combinations of $\mathcal{A}$ is called the span of $\mathcal{A}$.
  - If $\mathcal{A}$ spans the vector space V, we use $V = \text{span}[\mathcal{A}]$ or $V = \text{span}[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k]$
- Definition. The minimal generating set $\mathcal{B}$ of V is called basis of V. We call each element of $\mathcal{B}$ basis vector. The number of basis vectors is called dimension of V.
- Properties
  - $\mathcal{B}$ is a maximally[2] linearly independent set of vectors in V.
  - Every vector $x \in V$ is a linear combination of $\mathcal{B}$, which is unique.

[2] Adding any other vector to this set will make it linearly dependent.

Examples

- Different bases $\mathbb{R}^3$

$$\mathcal{B}_1 = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}, \mathcal{B}_2 = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\},$$

$$\mathcal{B}_3 = \left\{ \begin{pmatrix}0.5\\0.8\\0.4\end{pmatrix}, \begin{pmatrix}1.8\\0.3\\0.3\end{pmatrix}, \begin{pmatrix}-2.2\\-1.3\\3.5\end{pmatrix} \right\}$$

- Linearly independent, but not maximal. Thus, not a basis.

$$\mathcal{A} = \left\{ \begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}2\\-1\\0\\2\end{pmatrix}, \begin{pmatrix}1\\1\\0\\-4\end{pmatrix} \right\}$$

Determining a Basis

- Want to find a basis of a subspace $U = \text{span}[\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m]$
  1. Construct a matrix $\boldsymbol{A} = \begin{pmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_m\end{pmatrix}$
  2. Find the row-echelon form of $\boldsymbol{A}$.
  3. Collect the pivot columns.
- Logic: Collect $\boldsymbol{x}_i$ so that we have only trivial solution. Pivot columns tell us which set of vectors is linearly independent.
- See example 2.17 (pp. 35)

Rank (1)

- Definition. The rank of $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ denoted by $\text{rk}(\boldsymbol{A})$ is # of linearly independent columns
  - Same as the number of linearly independent rows
- $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{pmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

  Thus, $\text{rk}(\boldsymbol{A}) = 2$.
- $\text{rk}(\boldsymbol{A}) = \text{rk}(\boldsymbol{A}^\mathsf{T})$

Rank (2)

- The columns (resp. rows) of $\boldsymbol{A}$ span a subspace U (resp. W) with $\dim(U) = \text{rk}(\boldsymbol{A})$ (resp. $\dim(W) = \text{rk}(\boldsymbol{A})$), and a basis of U (resp. W) can be found by Gauss elimination of $\boldsymbol{A}$ (resp. $\boldsymbol{A}^\mathsf{T}$).
- For all $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\text{rk}(\boldsymbol{A}) = n$, iff $\boldsymbol{A}$ is regular (invertible).
- The linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is solvable, iff $\text{rk}(\boldsymbol{A}) = \text{rk}(\boldsymbol{A}|\boldsymbol{b})$.
- For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, the subspace of solutions for $\boldsymbol{Ax} = 0$ possesses dimension $n - \text{rk}(\boldsymbol{A})$.
- $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ has full rank if its rank equals the largest possible rank for a matrix of the same dimensions. The rank of the full-rank matrix $\boldsymbol{A}$ is min(# of cols, # of rows).

Roadmap

Linear Mapping (1)

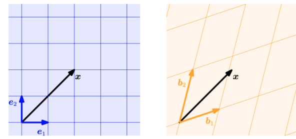
- Interest: A mapping that preserves the structure of the vector space
- Definition. For vector spaces V, W, a mapping $\Phi : V \mapsto W$ is called a linear mapping (or homomorphism/linear transformation), if, for all $\boldsymbol{x}, \boldsymbol{y} \in V$ and all $\lambda \in \mathbb{R}$,
  - $\Phi(\boldsymbol{x} + \boldsymbol{y}) = \Phi(\boldsymbol{x}) + \Phi(\boldsymbol{y})$
  - $\Phi(\lambda \boldsymbol{x}) = \lambda\Phi(\boldsymbol{x})$
- Definition. A mapping $\Phi : \mathcal{V} \mapsto \mathcal{W}$ is called
  - Injective (단사), if $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$, $\Phi(\boldsymbol{x}) = \Phi(\boldsymbol{y}) \implies \boldsymbol{x} = \boldsymbol{y}$
  - Surjective (전사), if $\Phi(\mathcal{V}) = \mathcal{W}$
  - Bijective (전단사), if it is injenctive and surjective.

Linear Mapping (2)

- For bijective mapping, there exists an inverse mapping Φ^{-1}.
- Isomorphism if Ψ is linear and bijective.
- Theorem. Vector spaces V and W are isomorphic, iff $\dim(V) = \dim(W)$.
  - Vector spaces of the same dimension are kind of the same thing.
- Other properties
  - For two linear mappings Φ and Ψ, $\Phi \circ \Psi$ is also a linear mapping.
  - If Φ is an isomorphism, so is Φ^{-1}.
  - For two linear mappings Φ and Ψ, $\Phi + \Psi$ and $\lambda\Psi$ for $\lambda \in \mathbb{R}$ are linear.

Coordinates

- A basis defines a coordinate system.



- Consider an ordered basis $B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n)$ of vector space V. Then, for any $\boldsymbol{x} \in V$, there exists a unique linear combination

  $\boldsymbol{x} = \alpha_1 \boldsymbol{b}_1 + \ldots + \alpha_n \boldsymbol{b}_n$.
- We call $\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ the coordinate of $\boldsymbol{x}$ with respect to $B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n)$.
- Basis change $\Longrightarrow$ Coordinate change

Basis Change

- Consider a vector space V and two coordinate systems defined by $B = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$ and $B' = (\boldsymbol{b}'_1, \ldots, \boldsymbol{b}'_n)$.
- Question. For $(x_1, \ldots, x_n)_B \to (y_1, \ldots, y_n)_{B'}$, what is $(y_1, \ldots, y_n)_{B'}$?
- Theorem. $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}'_1 & \ldots & \boldsymbol{b}'_n \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{b}_1 & \ldots & \boldsymbol{b}_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$
- Regard $\boldsymbol{A}_\Phi = \begin{pmatrix} \boldsymbol{b}'_1 & \ldots & \boldsymbol{b}'_n \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{b}_1 & \ldots & \boldsymbol{b}_n \end{pmatrix}$ as a linear map

Example

- $B = ((1,0),(0,1)$ and $B' = ((2,1),(1,2))$
- $(4,2)_B \to (x,y)_{B'}$?
- Using $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}'_1 & \ldots & \boldsymbol{b}'_n \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{b}_1 & \ldots & \boldsymbol{b}_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Transformation Matrix

- Two vector spaces
  - V with basis $B = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)$ and W with basis $C = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_m)$
- What is the coordinate in C-system for each basis $\boldsymbol{b}_j$? For $j = 1, \dots, n$,

$$\boldsymbol{b}_j = \alpha_{1j}\boldsymbol{c}_1 + \cdots + \alpha_{mj}\boldsymbol{c}_m \Longleftrightarrow \boldsymbol{b}_j = \begin{pmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_m \end{pmatrix} \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{mj} \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_m \end{pmatrix} \overbrace{\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}}^{\boldsymbol{A}_\Phi}$$

- $\hat{x} = \boldsymbol{A}_\Phi \hat{y}$, where $\hat{x}$ is the vector w.r.t B and $\hat{y}$ is the vector w.r.t. C

Basis Change: General Case

- For linear mapping $\Phi : V \mapsto W$, consider bases B, B' of V and C, C' of W
  $B = \begin{pmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n \end{pmatrix},\ B' = \begin{pmatrix} \boldsymbol{b}'_1 & \cdots & \boldsymbol{b}'_n \end{pmatrix} \quad C = \begin{pmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_m \end{pmatrix},\ C' = \begin{pmatrix} \boldsymbol{c}'_1 & \cdots & \boldsymbol{c}'_m \end{pmatrix}.$
- (inter) transformation matrices $\boldsymbol{A}_\Phi$ from B to C and $\boldsymbol{A}'_\Phi$ from B' to C'
- (intra) transformation matrices S from B' to B and T from C' to C
- Theorem. $\boldsymbol{A}'_\Phi = T^{-1}\boldsymbol{A}_\Phi S$

$$\begin{tikzcd}
V \arrow[r, "\Phi"] & W \\
B \arrow[r, "\Phi_{CB}", "\boldsymbol{A}_\Phi"'] & C \\
\tilde{B} \arrow[u, "\Psi_{B\tilde{B}}", "\boldsymbol{S}"'] \arrow[r, "\tilde{\boldsymbol{A}}_\Phi", "\Phi_{\tilde{C}\tilde{B}}"'] & \tilde{C} \arrow[u, "\boldsymbol{T}", "\Xi_{C\tilde{C}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "\Phi"] & W \\
B \arrow[r, "\Phi_{CB}", "\boldsymbol{A}_\Phi"'] & C \arrow[d, "\boldsymbol{T}^{-1}"', "\Xi_{\tilde{C}C} = \Xi_{C\tilde{C}}^{-1}"] \\
\tilde{B} \arrow[u, "\Psi_{B\tilde{B}}", "\boldsymbol{S}"'] \arrow[r, "\tilde{\boldsymbol{A}}_\Phi", "\Phi_{\tilde{C}\tilde{B}}"'] & \tilde{C}
\end{tikzcd}$$

Image and Kernel

- Consider a linear mapping $\Phi : V \mapsto W$. The kernel (or null space) is the set of vectors in V that maps to $0 \in W$ (i.e., neutral element).

  Definition. $\ker(\Phi) := \Phi^{-1}(0_W) = \{\boldsymbol{v} \in V : \Phi(\boldsymbol{v}) = 0_W\}$
- Image/range: set of vectors $w \in W$ that can be reached by Φ from any vector in V
- V: domain, W: codomain

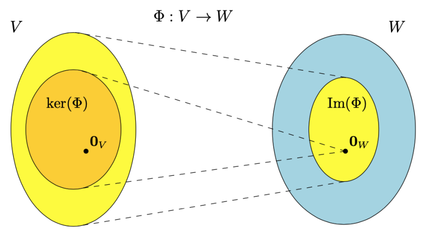


Image and Kernel: Properties

- $0_V \in \ker(\Phi)$ (because $\Phi(0_V) = 0_W$)
- Both $\mathrm{Im}(\Phi)$ and $\ker(\Phi)$ are subspaces of W and V, respectively.
- Φ is one-to-one (injective) $\Longleftrightarrow \ker(\Phi) = \{0\}$ (i.e., only 0 is mapped to 0)
- Since Φ is a linear mapping, there exists $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ such that $\Phi : \boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$. Then, $\mathrm{Im}(\Phi) =$ column space of $\boldsymbol{A}$ which is the span of column vectors of $\boldsymbol{A}$.
- $\mathrm{rk}(\boldsymbol{A}) = \dim(\mathrm{Im}(\Phi))$
- $\ker(\Phi)$ is the solution set of the homogeneous system of linear equations $\boldsymbol{A}\boldsymbol{x} = 0$

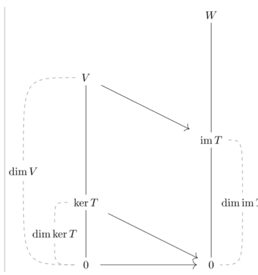
Rank-Nullity Theorem

Theorem.

$$\dim(\ker(\Phi)) + \dim(\mathrm{Im}(\Phi)) = \dim(V)$$

- If $\dim(\mathrm{Im}(\Phi)) < \dim(V)$, the kernel contains more than just 0.
- If $\dim(\mathrm{Im}(\Phi)) < \dim(V)$, $\boldsymbol{A}_\Phi \boldsymbol{x} = 0$ has infinitely many solutions.
- If $\dim(V) = \dim(W)$ (e.g., $V = W = \mathbb{R}^n$), the followings are equivalent: Φ is
  - (1) injective, (2) surjective, (3) bijective,
  - In this case, Φ defines $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{A}$ is regular.
- Simplified version. For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$,

$$\mathrm{rk}(\boldsymbol{A}) + \mathrm{nullity}(\boldsymbol{A}) = n$$

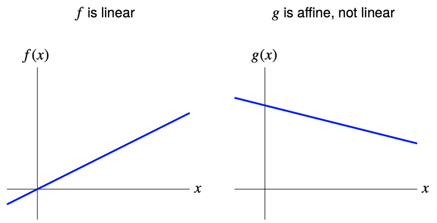


[2] Nullity: the dimension of null space (kernel)

Roadmap

Linear vs. Affine Function

- linear function: $f(x) = ax$
- affine function: $f(x) = ax + b$
- sometimes (ignorant) people refer to affine functions as linear



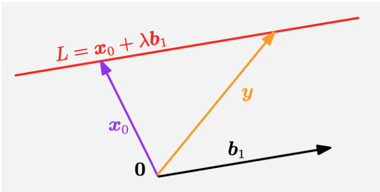
Affine Subspace

- Spaces that are offset from the origin. Not a vector space.
- Definition. Consider a vector space V, $\boldsymbol{x}_0 \in V$ and a subspace $U \subset V$. Then, the subset $L = \boldsymbol{x}_0 + U := \{\boldsymbol{x}_0 + \boldsymbol{u} : \boldsymbol{u} \in U\}$ is called affine subspace or linear manifold of V.
- U is called direction or direction space, and $\boldsymbol{x}_0$ is support point.
- An affine subspace is not a vector subspace of V for $\boldsymbol{x}_0 \notin U$.
- **Parametric equation.** A k-dimensional affine space $L = \boldsymbol{x}_0 + U$. If $(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k)$ is an ordered basis of U, any element $\boldsymbol{x} \in L$ can be uniquely described as

$$\boldsymbol{x} = \boldsymbol{x}_0 + \lambda_1 \boldsymbol{b}_1 + \cdots + \lambda_k \boldsymbol{b}_k, \quad \lambda_1, \ldots, \lambda_k \in \mathbb{R}$$

Example

- In $\mathbb{R}^2$, one-dimensional affine subspace: line. $\boldsymbol{y} = \boldsymbol{x}_0 + \lambda \boldsymbol{b}_1$. $U = \text{span}[\boldsymbol{b}_1]$
- In $\mathbb{R}^3$, two-dimensional affine subspace: plane. $\boldsymbol{y} = \boldsymbol{x}_0 + \lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2$. $U = \text{span}[\boldsymbol{b}_1, \boldsymbol{b}_2]$
- In $\mathbb{R}^n$, $(n-1)$-dimensional affine subspace: hyperplane. $\boldsymbol{y} = \boldsymbol{x}_0 + \sum_{k=1}^{n-1} \lambda_i \boldsymbol{b}_i$. $U = \text{span}[\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n]$



- For a linear mapping $\Phi : V \mapsto W$ and a vector $\boldsymbol{a} \in W$, the mapping $\phi : V \mapsto W$ with $\phi(\boldsymbol{x}) = \boldsymbol{a} + \Phi(\boldsymbol{x})$ is an affine mapping from V to W. The vector $\boldsymbol{a}$ is called the translation vector.

Questions?

Review Questions

1)