

# Number Theory Reading Group

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## 1 Thursday, 9/12/2024, Representation of $\mathfrak{sl}_2(\mathbb{F})$ by Hechi

$$\mathfrak{sl}_2(\mathbb{F}) := \{g \in \mathfrak{gl}_2(\mathbb{F}) \mid \text{Tr}(g) = 0\}$$

We assume  $\text{char}(\mathbb{F}) = 0$  and  $\mathbb{F}$  is algebraically closed.

**Theorem 1.1.**  $\mathfrak{sl}_2(\mathbb{F})$  is semisimple

*Proof.* Direct computation of the Killing Form. □

Recall: if  $\mathfrak{L}$  is semisimple and  $\phi : \mathfrak{L} \rightarrow \mathfrak{gl}(V)$  is a representation.

$\mathfrak{L} \ni x = s + n$  abstract jordan decomposition.

$\implies \phi(x) = \phi(s) + \phi(n)$  is the Jordan decomposition of  $\phi(x)$  in  $\phi(\mathfrak{L})$ .

From now on,  $\mathfrak{L} = \mathfrak{sl}_2(\mathbb{F}) = \mathfrak{sl}(2, \mathbb{F})$ .

$(V, \phi)$  is a representation.

Basis of  $\mathfrak{L}$ :

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus we have  $[h, x] = 2x, [h, y] = -2y, [x, y] = h$ .

Since  $h$  is diagonal,  $h$  is semisimple.

$\implies \phi(h)$  is semisimple and thus diagonalizeable.  $\in \text{End}(V)$ .

We can decompose  $V = \bigoplus_{\lambda} V_{\lambda}$  where  $V_{\lambda} = \{v \in V \mid hv = \lambda v\}$  for all  $\lambda \in \mathbb{F}$ .

We say  $V_{\lambda}$  is a weight space with  $\lambda$  as its weight.

**Lemma 1.2** (7.1). Suppose  $v \in V_{\lambda}$ . Then,

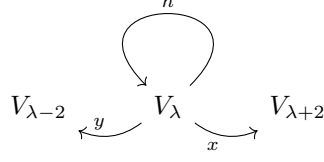
$$1) \quad xv \in V_{\lambda+2}$$

$$2) \quad yv \in V_{\lambda-2}$$

*Proof.* 1)  $h(xv) = [h, x]v + x(hv) = 2xv + \lambda xv = (\lambda + 2)xv$

$$2) \quad h(yv) = [h, y]v + y(hv) = -2yv + \lambda yv = (\lambda - 2)yv$$

□



Note that  $\dim V < \infty$

Thus,  $\exists v \in V$  such that  $x \cdot v = 0$ .

Such a  $v$  is called a maximal vector.

For now, assume  $V$  is irreducible.

Let  $v_0$  be a maximal vector with weight  $\lambda$ .

**Definition.** For  $i > 0$  integer,  $v_i = \frac{y^i \cdot v_0}{i!}$

Also,  $v_{-1} = 0$ .

**Lemma 1.3 (7.2).** 1)  $h \cdot v_i = (\lambda - 2i)v_i$

$$2) \quad y \cdot v_i = (i + 1)v_{i+1}$$

$$3) \quad x \cdot v_i = (\lambda - i + 1)v_{i-1}$$

*Proof.* 1) We use induction. Base case is clear.

Assume it is true for  $i - 1$ .

$$v_{i-1} \in V_{\lambda-2(i-1)}$$

$$\text{Thus, } v_i = \frac{1}{i} \cdot yv_{i-1}$$

Lemma 7.1 implies  $v_i \in V_{\lambda-2i}$ .

$$2) \quad y \cdot v_i = (i + 1)v_{i+1} \text{ by definition of } v_i.$$

$$3) \quad ix \cdot v_i = x(yv_{i-1}) = [x, y]v_{i-1} + yxv_{i-1} = hv_{i-1} + yxv_{i-1} = (\lambda - 2(i - 1))v_{i-1} + (\lambda - i + 2)yv_{i-2} = i(\lambda - i + 1)v_{i-1}$$

□

$\dim V < \infty$  so it must end at some point.

So, at some point, it'll become 0.  $v_0, \dots, v_m \neq 0, v_{m+1} = 0$ .

**Definition.**  $m$  is the integer so that  $v_m \neq 0, v_{m+1} = 0$ .

By Lemma 7.2,

$\text{span}\{v_0, \dots, v_m\}$  is a sub-representation of  $V$ .

Since  $V$  is irreducible,

$$V = \text{span}\{v_0, \dots, v_m\}$$

Note: by 7.2(3),

$$0 = x \cdot v_{m+1} = (\lambda - m)v_m$$

Since  $v_m \neq 0$  we have  $\lambda = m$ .

Thus,  $\dim V = m + 1 = \lambda + 1$

Here  $m$  is the highest weight.

$$V = V_{-m} \oplus V_{-m+2} \oplus \dots \oplus V_{m-2} \oplus V_m$$

**Construction.** Suppose  $L \curvearrowright \mathbb{F}[X, Y]$  [as a  $\mathbb{F}$ -space].

$$\rho(x) = X \frac{\partial}{\partial Y}$$

$$\rho(y) = Y \frac{\partial}{\partial X}$$

$$\rho(h) = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$$

Consider subrepresentations  $\mathbb{F}[X, Y]_m$  [symmetric polynomials of degree  $m$ , dimension  $m + 1$ ].

$$\begin{array}{ccccc} & & \overset{h}{\curvearrowright} & & \\ & & \downarrow & & \\ 0 & \xleftarrow{x} & X^m & \xrightarrow{y} & X^{m-1}Y \quad \dots \end{array}$$

## 2 Thursday, 9/19/2024, Representation of $\mathfrak{sl}_2(\mathbb{F})$ by Hechi

### Root Space Decomposition

Let  $\mathcal{L}$  be a non-zero semisimple lie algebra over  $\mathbb{F}$  with  $\text{char } \mathbb{F} = 0$  and  $\mathbb{F}$  algebraically closed.

**Definition** (Toral Subalgebra). A subalgebra  $\mathcal{H} \subseteq \mathcal{L}$  toral if it consists of semisimple elements.

**Remark.** If every element in  $\mathcal{L}$  is ad-nilpotent, then by Engel's Theorem  $\mathcal{L}$  is nilpotent. Thus it is not semisimple.

So, there exists a non-zero toral subalgebra.

Fix  $\mathcal{H}$  to be the maximal toral subalgebra. A maximal subalgebra exists since  $\mathcal{L}$  is finite dimensional.

**Lemma 2.1** (8.1). A toral subalgebra  $\mathcal{T}$  is abelian.

*Proof.* Suppose  $x \in \mathcal{T}$ . We will prove that  $\text{ad}_{\mathcal{T}} x = 0$  [as a map].

$\text{ad}_{\mathcal{T}} x$  is diagonalizable. Assume some eigenvalue is non-zero. Then, we can find eigenvector  $y \in \mathcal{L}$  with eigenvalue  $a \neq 0$ . So,  $[x, y] = ay$ .

Now,  $\text{ad}_{\mathcal{T}} y(x) = [y, x] = -ay$ . Since  $[y, y] = 0$  we see that  $-ay$  is an eigenvector of  $\text{ad}_{\mathcal{T}} y$  with eigenvalue 0.

$\text{ad}_{\mathcal{T}} y$  is also diagonalizable. Suppose  $v_1, \dots, v_n$  is the eigenbasis of  $\text{ad}_{\mathcal{T}} y$  with eigenvalues  $\lambda_1, \dots, \lambda_n$ . Then  $x = a_1 v_1 + \dots + a_n v_n$  for  $a_i \in \mathbb{F}$ .

WLOG,  $v_1 = y$ .

$$[y, x] = a_1 \lambda_1 v_1 + \dots + a_n \lambda_n v_n = -ay$$

By comparing coefficients,  $a_1 \lambda_1 = -a$ . But  $\lambda_1 = 0$ . This is a contradiction.  $\square$

Now, we fix  $\mathcal{H}$  to be a maximal toral subalgebra. It is not necessarily unique.

Note that  $\text{ad } H$  is a commuting family in  $\text{End}(\mathcal{L})$ . From linear algebra we know that  $\text{ad } H$  is simultaneously diagonalizable.

**Definition** (Root Space Decomposition). Suppose  $\mathcal{H}^*$  is the dual space of  $\mathcal{H}$ . We can write:

$$\begin{aligned} \mathcal{L} &= \bigoplus_{\alpha \in \mathcal{H}^*} \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x \forall h \in H\} \\ &= \mathcal{L}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathcal{L}_\alpha \end{aligned}$$

where  $\Phi = \{\alpha \in \mathcal{H}^* \setminus \{0\} \mid \mathcal{L}_\alpha \neq 0\}$  and  $\mathcal{L}_0 = C_{\mathcal{L}}(\mathcal{H})$  [the centralizer].

This is called the root space decomposition.

**Example.**  $\mathfrak{sl}_2(\mathbb{F})$  has basis:

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then the root space decomposition is:

$$\mathfrak{sl}_2(\mathbb{F}) = \mathcal{H} \oplus \mathcal{L}_{-2} \oplus \mathcal{L}_2$$

$\mathcal{L}_{-2}$  contains the linear form sending  $h$  to  $-2$ .

**Proposition 2.2** (8.1). Let  $\alpha, \beta \in \mathcal{H}^*$ . Then,

$$1) [L_\alpha, L_\beta] \subseteq L_{\alpha+\beta} \text{ [by Jacobi Identity]}$$

2)  $\alpha \neq 0 \implies \forall x \in L_\alpha$  is nilpotent [by 1]

3)  $\alpha + \beta \neq 0 \implies L_\alpha \perp L_\beta$  w.r.t. the Killing Form.

*Proof of 3.* Find  $h \in \mathcal{H}$  such that  $(\alpha + \beta)(h) \neq 0$ . Then,

$$\kappa([h, x], y) = -\kappa([x, h], y) = -\kappa(x, [h, y])$$

$$\implies (\alpha + \beta)(h)\kappa(x, y) = 0$$

□

In particular,  $L_0 \perp L_\alpha$  when  $\alpha \in \Phi$ .

**Corollary 2.3** (8.1). The Killing Form restricted to  $\mathcal{L}_0$ ,  $\kappa|_{\mathcal{L}_0}$  is non-degenerate.

**Proposition 2.4** (8.2).  $\mathcal{H} = \mathcal{L}_0 = C_{\mathcal{L}}(\mathcal{H})$ .

*Proof.* Tedious linear algebra

□

**Corollary 2.5** (8.2). The Killing Form restricted to  $\mathcal{H}$ ,  $\kappa|_{\mathcal{H}}$  is non-degenerate.

This implies, the map  $H \rightarrow H^*$  given by  $x \mapsto \kappa(x, -)$  is an isomorphism.

For each  $\phi \in \mathcal{H}^*$  we can define  $t_\phi \in \mathcal{H}$  to be the pre-image of this isomorphism. So it satisfies

$$\phi(h) = \kappa(t_\phi, h) \quad \forall h \in \mathcal{H}$$

**Proposition 2.6** (8.3). 1)  $\Phi$  spans  $\mathcal{H}^*$

2) If  $\alpha \in \Phi$  then  $-\alpha \in \Phi$

3)  $x \in \mathcal{L}_\alpha, y \in \mathcal{L}_{-\alpha} \implies [x, y] = \kappa(x, y)t_\alpha$

4)  $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$

5)  $\dim[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] = 1$ , spanned by  $t_\alpha$

6) Pick any non-zero  $x_\alpha \in L_\alpha \setminus \{0\}$ . Then there exists  $y_\alpha \in \mathcal{L}_{-\alpha}$  such that  $x_\alpha, y_\alpha, h_\alpha := [x_\alpha, y_\alpha]$  spans a subalgebra isomorphic to  $\mathfrak{sl}_2(\mathbb{F})$ , with the isomorphism  $x_\alpha \mapsto x, y_\alpha \mapsto y, h_\alpha \mapsto h$

7)  $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$ .

If  $V$  is a  $\mathfrak{sl}_2(\mathbb{F})$ -module, recalling that  $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ ,

$$V = \bigoplus_{\lambda \in \mathbb{F}} V_\lambda \text{ eigenspaces of } h$$

Recall that all  $\mathfrak{sl}_2(\mathbb{F})$ -module is of the form:

$$\mathfrak{sl}_2(\mathbb{F}) \curvearrowright \mathbb{F}[X, Y]$$

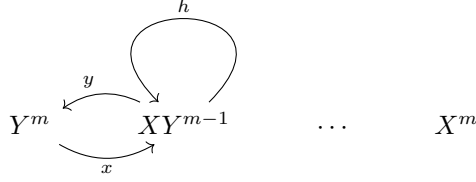
$$\rho(x) = X \frac{d}{dY}, \rho(y) = Y \frac{d}{dX}, \rho(h) = X \frac{d}{dX} - Y \frac{d}{dY}$$

and  $V = \mathbb{F}[X, Y]_m$  [homogeneous polynomials of degree  $m$ ] is irreducible and give us all irreducible representations.

Then we have:

$$V = V_{-m} \oplus V_{-m+2} \oplus \cdots \oplus V_{m-2} \oplus V_m$$

Where  $V_m$  is generated by  $X^m$  and  $V_{-m}$  is generated by  $Y^m$



If  $m$  even,  $0 \neq V_0 \subseteq V$

If  $m$  odd,  $0 \neq V_1 \subseteq V$

**Corollary 2.7.**  $V$  is a  $\mathfrak{sl}_2(\mathbb{F})$ -module. Then  $\dim V_0 + \dim V$  gives the number of summands in the irreducible decomposition of  $V$ .

Consider  $\mathcal{S}_\alpha = \text{span}\{x_\alpha, y_\alpha, h_\alpha\} \cong \mathfrak{sl}_2(\mathbb{F})$  and its adjoint representation ( $\mathcal{L}$  is an  $\mathcal{S}_\alpha$  module).

Fix  $\alpha \in \Phi$  and let  $\mathcal{M} = \mathcal{H} + \sum_{c \in \mathbb{F}^\times} \mathcal{L}_{c\alpha}$ .

By proposition 8.1,  $\mathcal{M}$  is a submodule of  $\mathcal{L}$  [since  $[\mathcal{L}_{c_1\alpha}, \mathcal{L}_{c_2\alpha}] \subseteq \mathcal{L}_{(c_1+c_2)\alpha}$ ].

If  $0 \neq x \in \mathcal{L}_{c\alpha}$  we see that  $[h_\alpha, x] = c\alpha(h_\alpha) \cdot x = 2cx$

$\implies 2c \in \mathbb{Z}$  and a weight of  $h_\alpha$  is 0 or an integer multiple of  $\frac{1}{2}$ .

Then  $\mathcal{M} = \underbrace{\ker \alpha}_{\substack{\text{eigenvalue} \\ \text{vectors of weight 0}}} + \underbrace{\mathbb{F} \cdot h_\alpha}_{\text{weight } 0, \pm 2}$

Therefore,  $\mathcal{M}$  contains vectors of weight only 0 or  $\pm 2$ .

Therefore, if  $\alpha \in \Phi$  we have  $c = \pm 1$ .

$\mathcal{M} = \mathcal{H} + \mathcal{S}_\alpha$ . Suppose  $h_\alpha^c$  is the complement of  $h_\alpha$  in  $\mathcal{H}$ .

Then,  $\mathcal{H} + \mathcal{S}_\alpha = \underbrace{h_\alpha^c}_{\text{abelian}} + \underbrace{\mathcal{S}_\alpha}_{\text{irreducible}}$  has  $\dim \mathcal{H} - 1 + 1 = \dim \mathcal{H} = \dim \mathcal{M} - 2$  irreducible

summands.

On the other hand, the number of irreducible summands of  $\mathcal{M}$  is  $\underbrace{\dim \mathcal{M}_0}_{\dim \mathcal{M} - 2} + \underbrace{\dim \mathcal{M}_1}_0$

Therefore,  $\mathcal{H} + \mathcal{S}_\alpha \subseteq \mathcal{M}$  must be equal.

Therefore,  $\dim \mathcal{L}_\alpha = 1$ .

Now, suppose  $\beta \neq \pm\alpha \in \Phi$ . Then,  $\exists r, q$  such that  $\beta - r\alpha, \beta - (r-1)\alpha, \dots, \beta + q\alpha$  are roots and outside these, i.e.  $\beta - (r+1)\alpha, \beta + (q+1)\alpha$  are not.

To see this, suppose  $K = \sum_{i \in \mathbb{Z}} \mathcal{L}_{\beta+i\alpha} \subseteq \mathcal{L}$  is a  $\mathcal{S}_\alpha$ -submodule. We know that  $\beta + i\alpha \neq 0$ .

Weights:

$$\beta(h_\alpha) + i\alpha(h_\alpha) = \beta(h_\alpha) + 2i$$

So, weights are either all even or all odd.

Therefore,  $K$  is irreducible.

Consider  $\gamma, \delta \in \mathcal{H}^*$ .

Define  $(\gamma, \delta) = \kappa(t_\gamma, t_\delta)$  on  $E_\mathbb{Q} = \text{span}_\mathbb{Q}(\Phi)$  then  $(\cdot, \cdot)$  extends to  $E = E_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{R}$  is positive definite.

Then  $E$  is an Euclidean Space.

$(\Phi, E)$  is called a root system.

### 3 Thursday, 9/26/2024, Root Systems by Zoia

Let  $E$  be an euclidean space. Suppose  $(\alpha, \beta)$  is a symmetric bilinear form on  $E$ .  
 Reflection in  $E$  fixes some hyperplane  $H$ . If  $\alpha$  is perpendicular to  $H$  then the reflection sends  $\alpha$  to  $-\alpha$ .  
 Consider  $\alpha \in E$  and  $P_\alpha = \{\beta \in E \mid (\alpha, \beta) = 0\}$  the hyperplane perpendicular to  $\alpha$ .  
 Suppose  $\sigma_\alpha$  is the reflection w.r.t. this hyperplane. Then,

$$\text{proj}_\alpha(\beta) = \frac{(\beta, \alpha)}{(\alpha, \alpha)} \alpha$$

$$\sigma_\alpha(\beta) = \beta - 2 \text{proj}_\alpha(\beta) = \beta - 2 \frac{(\beta, \alpha)}{(\alpha, \alpha)} \alpha$$

Define:

$$\langle \beta, \alpha \rangle = 2 \frac{(\beta, \alpha)}{(\alpha, \alpha)}$$

Note that  $\langle \beta, \alpha \rangle$  is linear only in  $\beta$ . Then,

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$$

**Lemma 3.1.** Let  $\Phi$  be a finite subset of  $E$  so that  $\Phi$  spans  $E$ . Suppose all reflections  $\sigma_\alpha (\alpha \in \Phi)$  leaves  $\Phi$  invariant. If  $\sigma \in \text{GL}(E)$  fixes hyperplane  $P$  of  $E$  and sends  $0 \neq \alpha \in \Phi$  to  $-\alpha$ , then  $\sigma = \sigma_\alpha$  and  $P = P_\alpha$ .

*Proof.* Suppose  $\tau = \sigma \sigma_\alpha = \sigma \sigma_\alpha^{-1}$ .

Then,  $\tau(\Phi) = \Phi$ ,  $\tau(\alpha) = \alpha$  and  $\tau$  acts as id on  $\mathbb{R} \cdot \alpha$  and  $E/R \cdot \alpha$  eigenvalues are 1.

So we have  $(T - 1)^L$  where  $L = \dim E$ .

$\beta, \tau(\beta), \dots, \tau^k(\beta) \exists k$  that fixes all  $\beta \in \Phi$

$\Phi$  spans  $E$ , so  $\tau^k = 1$ . So  $T^k - 1 = 0$ .

If  $m(T)$  is the minimal polynomial of  $\tau$ , then:

$$m(T) \mid T^k - 1$$

$$m(T) \mid (T - 1)^k$$

Therefore,  $m(T) = T - 1$ .

Therefore,  $\tau = \text{id}$ .

Thus  $\sigma \sigma_\alpha^{-1} = \text{id} \implies \sigma = \sigma_\alpha$  □

**Definition** (Root Systems). A finite subset  $\Phi$  of  $E$  is a root system in  $E$  if:

- 1R)  $\Phi$  spans  $E$ , does not contain 0.
- 2R) If  $\alpha \in \Phi$  then only multiples of  $\alpha$  in  $\Phi$  are  $\pm\alpha$ .
- 3R) If  $\alpha \in \Phi$ , then  $\sigma_\alpha$  leaves  $\Phi$  invariant.  $[\forall \beta \in \Phi, \sigma_\alpha(\beta) \in \Phi]$
- 4R) If  $\alpha, \beta \in \Phi$  then  $\langle \beta, \alpha \rangle \in \mathbb{Z}$ .  $\left[ \langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \right]$

**Definition** (Weyl Group). Let  $\Phi$  be a root system in  $E$ . Denote by  $\mathcal{W}$  the subgroup of  $\text{GL}(E)$  generated by  $\sigma_\alpha (\alpha \in \Phi)$ .

3R  $\implies \mathcal{W}$  is a symmetry group on  $\Phi$ .

**Lemma 3.2.** Let  $\Phi$  be a root system in  $E$  with Weyl group  $\mathcal{W}$ . If  $\sigma \in \text{GL}(E)$  leaves  $\Phi$  invariant, then  $\sigma \sigma_\alpha \sigma^{-1} = \sigma_{\sigma(\alpha)} \forall \alpha \in \Phi$  and  $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle \forall \alpha, \beta \in \Phi$ .

*Proof.*  $\sigma \sigma_\alpha \sigma^{-1}(\sigma(\beta)) = \sigma \sigma_\alpha(\beta) = \sigma(\beta - \langle \beta, \alpha \rangle \alpha) = \sigma(\beta) - \langle \beta, \alpha \rangle \sigma(\alpha)$ .

$\sigma(\beta)$  runs over  $\Phi$ .  $\sigma \sigma_\alpha \sigma^{-1}$  fixes  $\sigma(P_\alpha)$  pointwise and  $\sigma(\alpha) \rightarrow -\sigma(\alpha)$ .

Therefore,  $\sigma \sigma_\alpha \sigma^{-1} = \sigma_{\sigma(\alpha)}$  by the lemma.

$\sigma_{\sigma(\alpha)}(\sigma(\beta)) = \sigma(\beta) - \langle \sigma(\beta), \sigma(\alpha) \rangle \sigma(\alpha)$

Therefore, we must have  $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ . □

**Definition (Isomorphisms).** Suppose  $\Phi, \Phi'$  are root systems with Euclidean spaces  $E, E'$ .

$(\Phi, E) \cong (\Phi', E')$  if there exists map  $\varphi : E \rightarrow E'$  such that  $\varphi$  maps  $\Phi$  to  $\Phi'$  and  $\forall \alpha, \beta \in \Phi$  we have  $\langle \varphi(\beta), \varphi(\alpha) \rangle = \langle \beta, \alpha \rangle$ .

Note that:

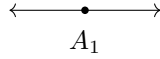
$$\sigma_{\varphi(\alpha)}(\varphi(\beta)) = \varphi(\beta) - \underbrace{\langle \varphi(\beta), \varphi(\alpha) \rangle}_{=\langle \beta, \alpha \rangle} \varphi(\alpha) = \varphi(\beta - \langle \beta, \alpha \rangle \alpha) = \varphi(\sigma_\alpha(\beta))$$

Note that,  $\sigma \mapsto \varphi \sigma \varphi^{-1}$  is an isomorphism of Weyl groups.

Thus,  $\mathcal{W}$  is a subgroup of  $\text{Aut}(\Phi)$ .

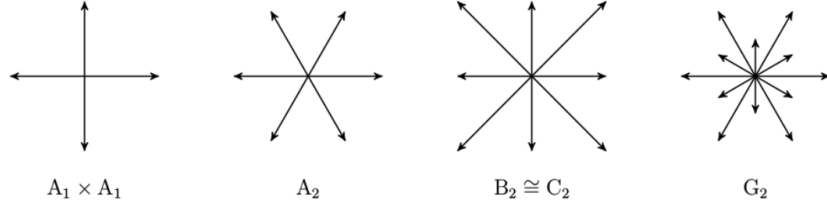
Now we consider root systems of different dimensions. Suppose  $L = \dim E$ .

$L = 1$ : In this case, we have  $\alpha, \alpha \in \Phi$  only. This gives us  $A_1$



$$\mathcal{W}(A_1) = \mathbb{Z}_2$$

$L = 2$ :



$$\mathcal{W}(A_1 \times A_1) = \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$\mathcal{W}(A_2) = S_3$$

$$\mathcal{W}(B_2) = D_4$$

$$\mathcal{W}(G_2) = D_6$$

These are the only possible cases for  $L = 2$ , since:

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{2\|\beta\|\|\alpha\|\cos\theta}{\|\alpha\|\|\alpha\|} = \frac{2\|\beta\|}{\|\alpha\|} \cos\theta \in \mathbb{Z}$$

Similarly,  $\frac{2\|\alpha\|}{\|\beta\|} \cos\theta \in \mathbb{Z}$ . Multiplying,  $4\cos^2\theta \in \mathbb{Z} \implies 4\cos^2\theta = 0, 1, 2, 3, 4$

Thus,  $\cos\theta = 0, \pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2} \implies \theta = \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}$ .

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	$\theta$	$\ \beta\ ^2/\ \alpha\ ^2$
0	0	$\frac{\pi}{2}$	undefined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	2
-1	-2	$\frac{3\pi}{4}$	2
1	3	$\frac{\pi}{6}$	3
-1	-3	$\frac{5\pi}{6}$	3

Table 1: Angle Root System

**Lemma 3.3.** Suppose  $\alpha, \beta$  are non-proportional root.

If  $\langle \alpha, \beta \rangle > 0$  then  $\alpha - \beta$  is a root.

If  $\langle \alpha, \beta \rangle < 0$  then  $\alpha + \beta$  is a root.

*Proof.*  $\langle \alpha, \beta \rangle = 1 \implies \sigma_\beta(\alpha) = \alpha - 1\beta = \alpha - \beta \in \Phi$

If  $\langle \beta, \alpha \rangle = 1$  then  $\sigma_\alpha(\beta) = \beta - 1\alpha = \beta - \alpha \in \Phi$ .

$\sigma_{\beta-\alpha}(\beta - \alpha) = (\beta - \alpha) - \langle \beta - \alpha, \beta - \alpha \rangle(\beta - \alpha) = \alpha - \beta \in \Phi$

□



## 4 Thursday, 10/3/2024, Simple Roots by Zoia

A root system  $\Phi$  of rank  $l$ ,  $E$ -Euclidean Space,  $\mathcal{W}$  is the Weyl Group.

**Definition.** A subset  $\Delta$  of  $\Phi$  is called a base if:

- B1)  $\Delta$  is a basis of  $E$  [ $|\Delta| = l$ ];
- B2)  $\forall \beta \in \Phi, \beta = \sum_{\alpha \in \Delta} k_\alpha \cdot \alpha$ , the expression is unique with  $k_\alpha$  being integers and  $k_\alpha$  are either all non-negative or all non-positive.

**Definition.** The roots from  $\Delta$  are simple roots.

**Definition.** The height of a root  $\beta$  [relative to the base  $\Delta$ ] is:

$$\text{ht}(\beta) = \sum_{\alpha \in \Delta} k_\alpha$$

**Definition.** We have positive roots  $\Phi^+$  and negative roots  $\Phi^-$  from the sign of  $k_\alpha$ . Furthermore  $\Phi^- = -\Phi^+$ .

Also, we define:

$$\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\}$$

**Definition.**  $\gamma \in E$  is regular if:

$$\gamma \in E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$$

Otherwise it is called singular.

Recall that  $P_\alpha = \{\beta \in E \mid (\alpha, \beta) = 0\}$

**Definition.**  $\alpha \in \Phi^+(\gamma)$  is decomposable if  $\alpha = \beta_1 + \beta_2$  with  $\beta_1, \beta_2 \in \Phi^+(\gamma)$ .  $\alpha$  is indecomposable otherwise.

**Definition.** We define  $\Delta(\gamma)$  to be the set of all indecomposable roots in  $\Phi^+(\gamma)$ .

**Theorem 4.1.** Any root system  $\Phi$  has a base. Let  $\gamma \in E$  be a regular.

Then, the set  $\Delta(\gamma)$  of all the indecomposable roots in  $\Phi^+(\gamma)$  is a base of  $\Phi$ .

Conversely, every base of  $\Phi$  is of the form  $\Delta(\gamma)$  for some  $\gamma$ .

*Proof.* We follow the following steps.

Step 1: Each root in  $\Phi^+(\gamma)$  is a non-negative  $\mathbb{Z}$ -linear combination of  $\Delta(\gamma)$ .

Step 2: If  $\alpha, \beta \in \Delta(\gamma)$  then  $(\alpha, \beta) \leq 0$  unless  $\alpha = \beta$ .

Step 3:  $\Delta(\gamma)$  is a linearly independent set.

Step 4:  $\Delta(\gamma)$  is a base of  $\Phi$ .

Step 5: Each base  $\Delta$  of  $\Phi$  has the form  $\Delta(\gamma)$  for some regular  $\gamma \in E$ .

Proof of Step 1: Suppose otherwise. Then  $\exists \alpha \in \Phi^+(\gamma)$  that cannot be expressed as a non-negative  $\mathbb{Z}$  linear combination of  $\Delta(\gamma)$ .

We can have multiple such  $\alpha$ 's. We pick the  $\alpha$  with the smallest  $(\gamma, \alpha)$ .

Note that  $\alpha \notin \Delta(\gamma)$ , since if  $\alpha \in \Delta(\gamma)$  then  $\alpha = 1 \cdot \alpha$ , which violates the assumption.

Thus,  $\alpha$  can be written as sum of two elements in  $\Phi^+(\gamma)$ . Suppose  $\alpha = \beta_1 + \beta_2$  so that  $\beta_1, \beta_2 \in \Phi^+(\gamma)$ . Then,  $(\gamma, \alpha) = (\gamma, \beta_1) + (\gamma, \beta_2)$ . Due to the minimality of  $(\gamma, \alpha)$ , they are both non-negative  $\mathbb{Z}$ -linear combination of  $\Delta(\gamma)$  which means so is  $\alpha$ , a contradiction.

Proof of Step 2: Suppose otherwise. Then,  $(\alpha, \beta) > 0$ .  $\beta$  cannot be  $-\alpha$ , thus  $\alpha - \beta$  is a root. Then either  $\alpha - \beta$  or  $\beta - \alpha$  is in  $\Phi^+(\gamma)$ . WLOG  $\alpha - \beta \in \Phi^+(\gamma)$ . Then  $\alpha = \beta + (\alpha - \beta)$ . Then  $\alpha$  is decomposable, which is a contradiction since  $\Delta(\gamma)$  consists of all indecomposable roots.

Proof of Step 3: Suppose  $\sum_{\alpha \in \Delta(\gamma), r_\alpha \in \mathbb{R}} r_\alpha \cdot \alpha = 0$ .  $r_\alpha$  can be positive or negative. We redistribute so that both sides have positive coefficient:

$$\varepsilon := \sum_{\alpha} s_\alpha \alpha = \sum_{\beta} t_\beta \beta$$

Then,

$$0 \leq (\varepsilon, \varepsilon) = \sum_{\alpha, \beta} \underbrace{s_\alpha t_\beta}_{\geq 0} \underbrace{(\alpha, \beta)}_{\leq 0} \leq 0$$

Thus,  $\varepsilon = 0$ . Now,

$$0 = (\gamma, \varepsilon) = \sum_{\alpha} \underbrace{s_\alpha}_{\geq 0} \underbrace{(\gamma, \alpha)}_{> 0} \geq 0$$

Thus,  $s_\alpha = 0$  for all  $\alpha \in \Delta(\gamma)$ . This implies linear independence.

Proof of Step 4: Note that  $\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma)$ .

B2 is satisfied because of Step 1.

Then  $\Delta(\gamma)$  spans  $E$ . Step 3 implies  $\Delta(\gamma)$  is a basis of  $E$ . Thus we have B1.

Proof of Step 5: Given  $\Delta$ , we select  $\gamma \in E : (\alpha, \gamma) > 0 \forall \alpha \in \Delta$ . B2  $\implies \gamma$  is regular and  $\Phi^+ \subseteq \Phi^+(\gamma)$ . Also,  $\Phi^- \subseteq -\Phi^+(\gamma)$ .

Therefore,  $\Phi^+ = \Phi^+(\gamma)$ .  $\Delta$  consists of indecomposable elements, that is  $\Delta \subseteq \Delta(\gamma)$ .

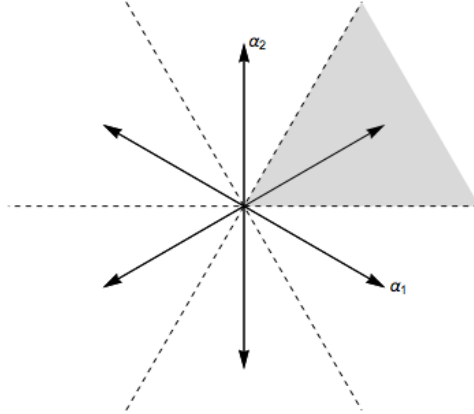
Coordinates are equal, therefore  $\Delta = \Delta(\gamma)$ . □

**Definition** (Weyl Chambers). The connected components of  $E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$  are called the (open) Weyl Chambers of  $E$ .

The fundamental Weyl chamber associated to  $\gamma$  is the open Weyl chamber containing  $\gamma$ . It is denoted by  $\mathcal{C}(\gamma)$ .

Furthermore,  $C(\gamma) = C(\gamma')$  implies  $\gamma$  and  $\gamma'$  are on the same side of each hyperplane  $P_\alpha$ . This also means  $\Delta(\gamma) = \Delta(\gamma')$ , so the Weyl chambers are in 1-1 correspondence with the bases.

For example: here is an open Weyl Chamber for  $A_2$ :



$\mathcal{C}(\Delta)$ -fundamental Weyl chamber relative to the base  $\{\alpha_1, \alpha_2\}$ .

The Weyl group acts on the Weyl chambers by  $\sigma(\mathcal{C}(\gamma)) = \mathcal{C}(\sigma(\gamma))$ .

If  $\sigma \in \mathcal{W}$  and  $\gamma$  is regular.

Also,  $\mathcal{W}$  permutes bases.  $\sigma$  sends  $\Delta$  to  $\sigma(\Delta)$  which is another base.

Since  $\sigma(\Delta(\gamma)) = \Delta(\sigma(\gamma))$  because  $(\sigma\gamma, \sigma\alpha) = (\gamma, \alpha)$ .

## 5 Thursday, 10/17/2024, Weyl Group, Irr. Root System by Zoia

**Lemma 5.1.** Let  $\alpha$  be simple. Then  $\sigma_\alpha$  permutes the positive roots other than  $\alpha$ .

**Corollary 5.2.** Set  $\delta = \frac{1}{2} \sum_{\beta \prec 0} \beta$ . Then,

$$\sigma_\alpha(\delta) = \delta - \alpha \forall \alpha \in \Delta$$

**Lemma 5.3.** Let  $\alpha_1, \dots, \alpha_n \in \Delta$  [not necessarily distinct]. Write  $\sigma_i := \sigma_{\alpha_i}$ . If  $\sigma_1, \dots, \sigma_{t-1}(\alpha_t)$  is negative, then

$$\exists s : 1 \leq s < t : \sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1}$$

**Corollary 5.4.** If  $\sigma = \sigma_1 \cdots \sigma_t$  is an exp for  $\sigma \in \mathcal{W}$ ,  $t$  is as small as possible then  $\sigma(\alpha_t) \prec 0$ .

*Proof.* Suppose  $\sigma(\alpha_t) > 0$ . Then,

$$\underbrace{\sigma_1 \cdots \sigma_t}_{t \text{ factors}} = \underbrace{\sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1}}_{t-2 \text{ factors}}$$

which contradicts minimality.  $\square$

## The Weyl Group

**Definition.**  $\mathcal{W}$  is the subgroup of  $GL(E)$  generated by the reflection  $(\sigma_\alpha)_{\alpha \in \Phi}$ .

**Theorem 5.5.** Let  $\Delta$  be a base of  $\Phi$ .

- a) If  $\gamma \in E$ ,  $\gamma$  is regular,  $\exists \sigma \in \mathcal{W} : (\sigma(\gamma), \alpha) > 0 \forall \alpha \in \Delta$ .
- b) If  $\Delta'$  is another base of  $\Phi$ , then  $\sigma(\Delta') = \Delta$  for some  $\sigma \in \mathcal{W}$ .
- c) If  $\alpha$  is any root  $\implies \exists \sigma \in \mathcal{W} : \sigma(\alpha) \in \Delta$ .
- d)  $\mathcal{W}$  generated by  $\sigma_\alpha (\alpha \in \Delta)$ .
- e) If  $\sigma(\Delta) = \Delta, \sigma \in \mathcal{W}$  then  $\sigma = \text{id}$ .

*Proof.* We consider the subgroup  $\mathcal{W}'$  generated by  $\sigma_\alpha (\alpha \in \Delta)$ .

For a, b, c we prove the theorem for  $\mathcal{W}'$  and for d, e we prove that  $\mathcal{W}' = \mathcal{W}$ .

a)

$$\delta := \frac{1}{2} \sum_{\alpha \prec 0} \alpha$$

Choose  $\sigma \in \mathcal{W}'$  such that  $(\sigma(\gamma), \delta)$  is as big as possible.

If  $\alpha$  is simple then  $\sigma_\alpha \sigma \in \mathcal{W}' \implies (\sigma(\gamma), \delta) \geq (\sigma_\alpha \sigma(\gamma), \delta) = (\sigma(\gamma), \sigma_\alpha(\delta)) = (\sigma(\gamma), \delta - \alpha) = (\sigma(\gamma), \delta) - (\sigma(\gamma), \alpha)$ .

Therefore,  $(\sigma(\gamma), \alpha) \geq 0$ .

Furthermore,  $(\sigma(\gamma), \alpha) \neq 0$  so we have strict inequality. Therefore,

$$\forall \alpha \in \Delta, (\sigma(\gamma), \alpha) > 0$$

Therefore,  $\sigma(\gamma)$  is in the fundamental Weyl chamber of  $\Delta$  and  $\sigma$  sends  $\mathfrak{C}(\gamma)$  to  $\mathfrak{C}(\Delta)$ .

b) Since  $\mathcal{W}'$  permutes the Weyl chambers by  $a$ , it also permutes the bases of  $\Phi$ .

c) Hyperplanes  $P_\beta (\beta \neq \pm \alpha)$  are distinct from hyperplane  $P_\alpha \implies \exists \gamma : \gamma \in P_\alpha, \gamma \notin P_\beta$ . Lets choose  $\gamma'$  so that  $\gamma'$  is close to  $\gamma$  such that  $(\gamma', \alpha) = \varepsilon > 0$  while  $|(\gamma', \beta)| > \varepsilon$  for any  $\beta \neq \pm \alpha$ .

Then  $\alpha \in \Delta(\gamma')$ .

d) We want to show that  $\mathcal{W}' = \mathcal{W}$ . It is enough to show that each reflection  $\sigma_\alpha (\alpha \in \Phi)$  is in  $\mathcal{W}'$ .

Find  $\sigma \in \mathcal{W}'$  such that  $\beta = \sigma(\alpha) \in \Delta$  using c. Then,

$$\sigma_\beta = \sigma_{\sigma(\alpha)} = \sigma \sigma_\alpha \sigma^{-1} \implies \sigma_\alpha = \sigma^{-1} \sigma_\beta \sigma \in \mathcal{W}'$$

e) Let  $\sigma(\Delta) = \Delta$  but  $\sigma \neq \text{id}$ . If  $\sigma$  is written minimally as a product of simple reflections then we have contradiction from corollary 5.4.  $\square$

## Irreducible Root System

$\Phi$  is irreducible if it cannot be partitioned into the union of two proper subsets in the following way: each root in one set is orthogonal to each root in the other subset.

Example:  $A_1, A_2, B_2, G_2$  are irreducible.  $A_1 \times A_1$  is not irreducible.

Claim:  $\Phi$  is irreducible  $\iff \Delta$  cannot be partitioned.

*Proof.*  $\Leftarrow$ : Suppose  $\Phi = \Phi_1 \cup \Phi_2$  with  $(\Phi_1, \Phi_2) = 0$ .

If  $\Delta$  is not wholly contained in  $\Phi_1$  or  $\Phi_2$  then it induces the partition in  $\Delta$ .

Now WLOG suppose  $\Delta \subset \Phi_1$ . Then,  $(\Delta, \Phi_2) = 0$ . Since  $\Delta$  spans  $E$ .

$\implies$ : Let  $\Phi$  be irreducible but suppose  $\Delta = \Delta_1 \cup \Delta_2$  with  $(\Delta_1, \Delta_2) = 0$ .

Each root is conjugate to a simple root (by theorem). Then,

$$\Phi = \Phi_1 \cup \Phi_2$$

where  $\Phi_i$  is the set of roots that are conjugates with those in  $\Delta_i$ .

Since  $\mathcal{W}$  is generated by the  $\sigma_\alpha$  where  $\alpha \in \Delta$ , it follows that each root in  $\Phi_i$  can be obtained from  $\Delta_i$  by  $+$  or  $-$  elements of  $\Delta_i$ .

Therefore,  $\Phi_i$  lies in the subspace  $E_i$  of  $E$  spanned by  $\Delta_i$ .

Then,  $(\Phi_1, \Phi_2) = 0$ .

Since  $\Phi$  is irreducible, it follows that  $\Phi_1 = \emptyset$  or  $\Phi_2 = \emptyset$ .

Therefore,  $\Delta_1 = \emptyset$  or  $\Delta_2 = \emptyset$ . □

**Lemma 5.6.** Let  $\Phi$  be irreducible. Then relative to the partial ordering  $\prec$ , there exists a unique maximal root  $\beta$ .

If  $\beta = \sum_{\alpha} k_{\alpha} \alpha$  ( $\alpha \in \Delta$ ) then all  $k_{\alpha} > 0$ .

**Lemma 5.7.** Let  $\Phi$  be irreducible. Then  $\mathcal{W}$  acts irreducibly on  $E$ . In particular, the  $\mathcal{W}$ -orbit of a root  $\alpha$  spans  $E$ .

**Lemma 5.8.** Let  $\Phi$  be irreducible. Then at most two root lengths occur in  $\Phi$  and all roots of this length are conjugates under  $\mathcal{W}$ .

**Lemma 5.9.** Suppose  $\Phi$  is irreducible with two distinct root lengths. Then the maximal root  $\beta$  of lemma 5.6 is long.

## 6 Thursday, 10/24/2024, Decomposing into Irr. by Hyeonmin

Facts:

- 1) Lemma 9.2 and (R3):  $\forall \alpha, \beta \in \Phi, \forall \sigma \in \mathcal{W}, \langle \beta, \alpha \rangle = \langle \sigma\beta, \sigma\alpha \rangle$ .
- 2) Table 1 in 9.4:  $\forall \alpha, \beta \in E, \alpha \neq \beta, \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0, 1, 2, 3$ .
- 3) Lemma 10.1:  $\forall \alpha, \beta \in \Delta, \alpha \neq \beta, \langle \alpha, \beta \rangle \leq 0$ .
- 4) Theorem 10.3b:  $\mathcal{W}$  acts transitively on the bases. 10.3c:  $\forall \alpha \in \Phi, \exists \sigma \in \mathcal{W}$  such that  $\sigma(\alpha) \in \Delta$ . 10.3d:  $\mathcal{W}$  is generated by  $\sigma_{\alpha} (\alpha \in \Delta)$ .
- 5) Claim 10.4:  $\Phi$  irreducible  $\iff \Delta$  cannot be partitioned into proper  $\Delta_1 \cup \Delta_2$  such that  $(\Delta_1, \Delta_2) = 0$ .

## Classification

Fix  $\Delta \subseteq \Phi$  and let  $l = \dim_{\mathbb{R}} E$ .

**Definition.** Fix an ordering  $(\alpha_1, \dots, \alpha_l)$  of  $\Delta$ .

Then  $(\langle \alpha_i, \alpha_j \rangle)_{ij}$  is called the cartan matrix of  $\Phi$ .

The entries are called the Cartan integers.

**Example.** In  $B_2$  the Cartan matrix is:

$$\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

**Remark.** 1) The Cartan matrix depends on the chosen ordering.

2) The Cartan matrix is independent of the choice of  $\Delta$ . This is because, if  $\Delta'$  is another base, since  $\sigma$  acts transitively, there exists  $\sigma \in \mathcal{W}$  such that  $\sigma\Delta = \Delta'$ . Using the fact  $\langle \alpha_i, \alpha_j \rangle = \langle \sigma\alpha_i, \sigma\alpha_j \rangle$  we see that the matrices are the same.

3) The Cartan Matrix is nonsingular. This is because:

$$(\langle \alpha_i, \alpha_j \rangle)_{i,j} \cdot \text{diag} \left( \frac{(\alpha_i, \alpha_i)}{2}, \dots, \frac{(\alpha_l, \alpha_l)}{2} \right) = \underbrace{((\alpha_i, \alpha_j))_{i,j}}_{\text{nonsingular since inner product of basis}}$$

**Proposition 6.1** (11.1). Let  $(\Phi', E')$  be another root system. Suppose it has base  $\Delta' = \{\alpha'_1, \dots, \alpha'_s\}$  such that  $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$ .

Then the bijection  $\Delta \xrightarrow{\alpha_a \mapsto \alpha'_a} \Delta'$  extends unique to a root system isomorphism  $\phi : (\Phi, E) \rightarrow (\Phi', E')$ .

Therefore, the Cartan matrix of  $\Phi$  determines  $\Phi$  upto isomorphism.

*Proof.*  $\Delta, \Delta'$  are both basis  $\implies \exists$  vector space isomorphism  $\phi : E \rightarrow E'$  such that  $\alpha_a \rightarrow \alpha'_a$ .

$$\forall i, j \quad \sigma_{\phi(\alpha_i)}(\phi(\alpha_j)) = \phi(\alpha_j) - \frac{\langle \phi(\alpha_j), \phi(\alpha_i) \rangle}{\langle \alpha_j, \alpha_i \rangle} \phi(\alpha_i) = \phi(\alpha_j - \langle \alpha_j, \alpha_i \rangle \alpha_i) = \phi(\sigma_{\alpha_i}(\alpha_j))$$

Therefore, we have the commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ \downarrow \sigma_\alpha & & \downarrow \sigma_{\alpha'} \\ E & \xrightarrow{\phi} & E' \end{array} \quad (A)$$

$$\alpha \in \Delta \text{ and } \forall x \in E, \langle x, \alpha \rangle = \langle \phi(x), \phi(\alpha) \rangle \quad (B)$$

And since  $\mathcal{W}$  is generated by  $\sigma_\alpha$  we have for all  $\sigma \in \mathcal{W}$  we have:

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ \downarrow \sigma & & \downarrow \sigma' \\ E & \xrightarrow{\phi} & E' \end{array}$$

Where  $\sigma = \sigma_{\alpha_{l_1}} \dots \sigma_{\alpha_{l_k}}$  and  $\sigma' = \sigma_{\alpha'_{l_1}} \dots \sigma_{\alpha'_{l_k}}$ .

Therefore  $\phi \circ \sigma \circ \phi^{-1} = \sigma' \in \mathcal{W} \quad (D)$ .

Claim 1:  $\phi(\Phi) \subseteq \Phi'$

Let  $\beta \in \Phi \implies \exists \sigma \in \mathcal{W}, \exists \alpha \in \Delta$  such that  $\sigma\alpha = \beta$

$$\text{Also, } \phi(\beta) = \underbrace{(\phi \circ \sigma \circ \phi^{-1})}_{\in \mathcal{W}'} \underbrace{(\phi(\alpha))}_{\in \Phi'} \in \Phi'$$

Claim 2:  $\forall \beta, \gamma \in \Phi, \langle \gamma, \beta \rangle = \langle \phi(\gamma), \phi(\beta) \rangle$ .

Since  $\mathcal{W}$  is generated by reflections  $\exists \sigma \in \mathcal{W}, \exists \alpha \in \Delta$  such that  $\sigma\alpha = \beta$ . Then,

$$\langle \gamma, \beta \rangle = \langle \sigma^{-1}\gamma, \alpha \rangle \stackrel{(B)}{=} \langle \phi \circ \sigma^{-1}(\gamma), \phi(\alpha) \rangle = \langle \sigma' \circ \phi \circ \sigma^{-1}(\gamma), \sigma' \circ \phi(\alpha) \rangle = \langle \phi(\gamma), \phi(\beta) \rangle$$

□

**Definition.** Fix  $\Delta = \langle \alpha_1, \dots, \alpha_l \rangle$  a base. The Coxeter graph of  $\Phi$  consists of  $l$  vertices correlated to  $\Delta$  and  $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$  edges between the  $i$ 'th and  $j$ 'th vertices ( $i \neq j$ ).

**Remark.** This graph is independent of  $\Delta$ . Since  $\Delta'$  another base  $\implies \exists \sigma \in \mathcal{W}$  such that  $\sigma\Delta = \Delta' \implies \langle \alpha_i, \alpha_j \rangle = \langle \sigma\alpha_i, \sigma\alpha_j \rangle$ .  
Also, we have  $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle = 0, 1, 2, 3$ .

e.g. [insert fig table]

We need more information to recover Cartan integer.

We cannot recover  $\langle \alpha_i, \alpha_j \rangle$  when  $\langle \rangle = 2, 3$ .

**Definition.** The Dynkin diagram of  $\Phi$  is the Coxeter graph adding an arrow pointing to the shorter length root.

e.g [insert picture and matrix for B2 fig]

**Proposition 6.2** (11.3).  $\Phi$  decomposes uniquely as the union of irreducible root system  $\Phi_i$  in subspaces  $E_i \subset E$  such that  $E = E_1 \oplus \cdots \oplus E_t$  [orthogonal direct sum].

**Proposition 6.3** (Ex 9.1).  $E' \subset E$ : a subspace. If  $\sigma_\alpha(E') \subseteq E'$  then either  $\alpha \in E'$  or  $E' \subseteq P_\alpha$

*Proof.* Suppose  $E' \subsetneq P_\alpha$ . Then  $\exists \beta \in E'$  such that  $(\alpha, \beta) \neq 0$ .

$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \in E' \implies \alpha \in E'$ . □

*Proof.* (Of 11.3)

$\Delta$  cannot be partitioned into proper  $\Delta_1 \cup \Delta_2$  such that  $(\Delta_1, \Delta_2) = 0 \iff$  the coxeter graph of  $\Phi$  is connected (A).

[insert picture fig]

Assume that the coxeter graph consists of connected components  $C_1, \dots, C_t$ .

Let  $\Delta = \Delta_1 \cup \cdots \cup \Delta_t$  be the partition so that  $\Delta_i$  correlates to  $C_i$ .

$A \implies \Delta_i$  are mutually orthogonal. Define  $E_i = \text{span}_{\mathbb{R}}(\Delta_i)$ .

Therefore,  $E = E_1 \oplus \cdots \oplus E_t$  [orthogonal]. (B)

Define  $\Phi_i = \{\alpha \in \Phi \mid \alpha : \text{a } \mathbb{Z}\text{-linear combination of } \Delta_i\}$ .

Then,  $(\Phi_i, E_i)$  is a root sytem [by checking the axioms].

$\Delta_i \subseteq \Phi_i$  is a base (again checking the axioms).

Therefore,  $\Phi_i$  are irreducible.

Finally, we want to check  $\Phi = \Phi_1 \cup \cdots \cup \Phi_t$ .

The Weyl group  $\mathcal{W}_i$  corresponding to  $\Phi_i$  is the subgroup of  $\mathcal{W}$  generated by  $\sigma_\alpha$  where  $\alpha \in \Delta_i$ .

B  $\implies \sigma_\alpha$  acts trivially on  $E_i$  for any  $\alpha \in \Delta_j, i \neq j$ . (C)

$E_i$  is  $\mathcal{W}_i$  invariant, so  $\sigma_\alpha(E_i) \subseteq E_i \forall \alpha \in \Delta_i \xrightarrow{C} E_i$  is  $\mathcal{W}$ -invariant.

Thus,  $\sigma(E_i) \subseteq E_i$  for any  $\sigma \in \mathcal{W}$

Now we use Exercise 9.1.

$\supseteq$  is trivial by definition.

$\subseteq \alpha \in \Phi \xRightarrow[D]{\text{Ex 9.1}} \sigma_\alpha(E_i) \subseteq E_i \implies \exists 1 \leq j \leq t \text{ such that } \alpha \in E_j \text{ but } \alpha \notin E_i \forall i \neq j$ .

Thus,  $\alpha \in \text{span}_{\mathbb{R}}(\Delta_i) \cap \Phi \implies \alpha$  is a  $\mathbb{Z}$ -linear combination of  $\Delta$  in  $\text{span}(\Delta_i)$ .

Therefore,  $\alpha$  is a  $\mathbb{Z}$ -linear combination of  $\Delta_i$ .

This is the definition of  $\Phi_i$ .

Therefore,  $\alpha \in \Phi_i$ . □

**Theorem 6.4** (Classification Theorem). Let  $\Phi$  be an irreducible root system. Then,  $\Phi$  can only have  $A - G$  type Dynkin diagrams.

## 7 Thursday, 10/31/2024, Irr. rootot System by Hyeonmin

**Theorem 7.1** (11.4). If  $\Phi$  is an irreducible root system, its Dynkin diagram is one of the following:

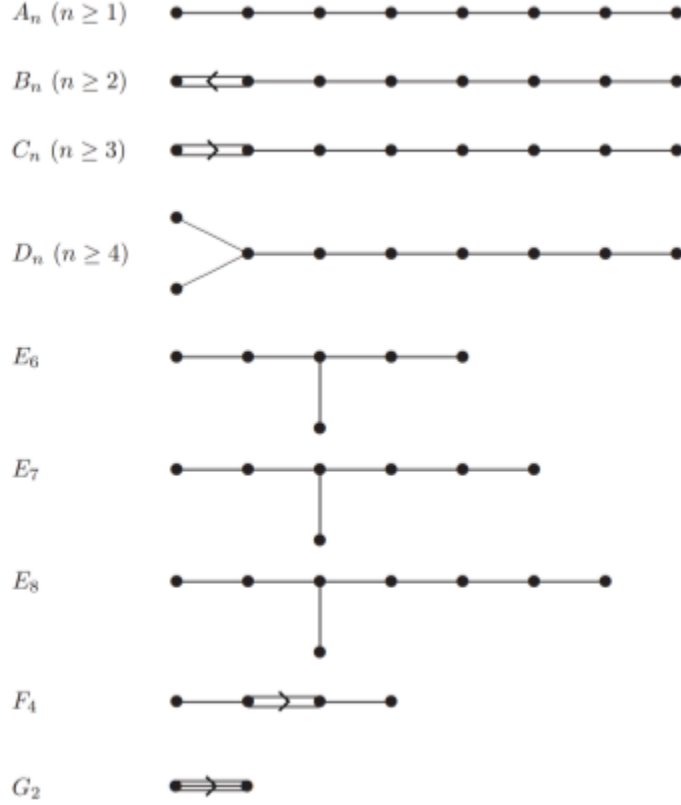


Figure 1: Dynkin Diagrams

*Proof.* Suffices to Show: Coxeter graph is one of the following:

$$A_l, B_l(= C_l), D_l, E_{6,7,8}, F_4, G_2$$

So we can ignore the length of the roots. We work with unit vectors.

**Definition.** Let  $E$  be euclidean space of dimension  $m$  and let  $\mathfrak{A} = \{\varepsilon_1, \dots, \varepsilon_n\} \subseteq E$  be admissible if:

- 1) Vectors are linearly independent.
- 2) Vectors are unit vectors.
- 3)  $(\varepsilon_i, \varepsilon_j) \leq 0$  if  $i \neq j$ .
- 4)  $4(\varepsilon_i, \varepsilon_j)^2 = 0, 1, 2$  or  $3$ .

Then, The graph  $\Gamma$  of  $\mathfrak{A}$ [coxeter graph] has  $n$  vertices and  $4(\varepsilon_i, \varepsilon_j)^2$  as edges.

Existence:

$$\Delta \supseteq \{\alpha_1, \dots, \alpha_n\} \implies \left\{ \frac{\alpha_1}{\sqrt{(\alpha_1, \alpha_1)}}, \dots, \frac{\alpha_n}{\sqrt{(\alpha_n, \alpha_n)}} \right\} := \{\varepsilon_1, \dots, \varepsilon_n\} \text{ admissible.}$$

$$4(\varepsilon_i, \varepsilon_j)^2 = \frac{4(\alpha_i, \alpha_j)^2}{(\alpha_i, \alpha_j)(\alpha_j, \alpha_i)} = \langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$$

- 1) A subset  $\mathfrak{A}' \subseteq \mathfrak{A}$  is still admissible.

- 2)  $\#\{\text{pairs of vertices in } \Gamma \text{ connected by each other}\} < n$ .

Since set  $\varepsilon := \sum_{i=1}^n \varepsilon_i \neq 0$ . Thus  $0 < (\varepsilon, \varepsilon) = \underbrace{n}_{=\sum_i (\varepsilon_i, \varepsilon_i)} + \sum_{i < j} 2(\varepsilon_i, \varepsilon_j)$ .

Suppose  $(\varepsilon_i, \varepsilon_j) < 0$ .

$$4(\varepsilon_i, \varepsilon_j)^2 = 1, 2, 3 \implies 2(\varepsilon_i, \varepsilon_j) \leq -1.$$

- 3)  $\Gamma$  contains no cycles.

Let  $\Gamma' \subseteq \Gamma$  be a vertex on  $k$  vertices.

Then  $\#\{\text{such pairs from (2) in } \Gamma'\} \geq k$ . Contradiction by 2.

- 4)  $\#\{\text{edges originated at a vertex } (\varepsilon) \text{ of } \Gamma\} \leq 3$

Let  $\varepsilon, \eta_1, \dots, \eta_k \in \mathfrak{A}$  be distinct such that  $(\varepsilon, \eta_i) \neq 0$ .

$$(3) \implies (\eta_i, \eta_j) = 0 \forall i \neq j.$$

Set  $\eta'_0 := \varepsilon - \sum_{i=1}^k (\varepsilon, \eta_i) \eta_i$  and  $\eta_0 := \frac{\eta'_0}{(\eta'_0, \eta'_0)}$ . Then,

- $(\eta'_0, \eta_i) = (\varepsilon, \eta_i) - (\varepsilon, \eta_i)(\eta_i, \eta_i) = 0 \implies (\eta_0, \eta_i) = 0$ .
- $\eta_0 \in \text{span}\{\varepsilon, \eta_1, \dots, \eta_k\} \implies (\eta_0, \varepsilon) \neq 0$ .
- $(\eta'_0, \eta'_0) = (\varepsilon, \eta'_0) \implies \sum_{i=0}^k (\varepsilon, \eta_i) \eta_i = (\varepsilon, \eta_0) \eta_0 + \sum_{i=1}^k (\varepsilon, \eta_i) \eta_i$   
 $= \frac{(\varepsilon, \eta'_0)}{(\eta'_0, \eta'_0)} \eta'_0 + \sum_{i=1}^k (\varepsilon, \eta_i) \varepsilon_i = \varepsilon$ .  
 $\implies 1 = (\varepsilon, \varepsilon) = \sum_{i=0}^k (\varepsilon, \eta_i)^2 > \sum_{i=1}^k (\varepsilon, \eta_i)^2 \implies 4 > \sum_{i=1}^k 4(\varepsilon, \eta_i)^2$ .

- 5) The only graph  $\Gamma$  containing a triple edges is  $G_2$ .

- 6) Let  $\{\varepsilon_1, \dots, \varepsilon_k\} \subseteq \mathfrak{A}$  have a subgraph

[insert pic like A1 with veps 1, ..., veps k]

$\mathfrak{A}' = (\mathfrak{A} \setminus \{\varepsilon_1, \dots, \varepsilon_k\}) \cup \{\varepsilon\}$  is admissible where  $\varepsilon = \sum_{i=1}^k \varepsilon_i$ .

Condition 1 is satisfied automatically.

Condition 2:  $\varepsilon$  is a unit:  $2(\varepsilon_i, \varepsilon_{i+1}) = -1 \implies (\varepsilon, \varepsilon) = k + \sum_{i=1}^{k-1} (\varepsilon_i, \varepsilon_{i+1}) = 1$  so it holds.

Condition 3:  $\eta \in \mathfrak{A} \setminus \{\varepsilon_1, \dots, \varepsilon_k\}$  can be connected to at most one of  $\varepsilon_1, \dots, \varepsilon_k$ .

$$3 \implies (\eta, \varepsilon) = 0 \text{ or } (\eta, \varepsilon) = (\eta, \varepsilon_j) \leq 0 \implies (\eta, \varepsilon) \leq 0.$$

Condition 4:  $4(\eta, \varepsilon)^2 = 0$  or  $4(\eta, \varepsilon_i)^2 = 0, 1, 2$  or  $3$

- 7)  $\Gamma$  contains no subgraph of the form:

[insert figure]

- 8) Any connected  $\Gamma$  of an admissible set has one of the following forms:

[insert figure]

- 9) The second type in 8:  $B_l (= C_l), F_4$ .

Set  $\varepsilon := \sum_{i=1}^p i \varepsilon_i, \eta := \sum_{i=1}^q i \eta_i$ .

Then  $2(\varepsilon_i, \varepsilon_{i+1}) = -1 = 2(\eta_j, \eta_{j+1})$ .

Then,  $(\varepsilon, \varepsilon) = \sum_{i=1}^p i^2 - \sum_{i=1}^{p-1} i(i+1) = \frac{p(p+1)}{2}$ , and  $(\eta, \eta) = \frac{q(q+1)}{2}$ . (A)

$$4(\varepsilon_p, \eta_q)^2 = 2 \implies (\varepsilon, \eta)^2 = (p\varepsilon_p, q\eta_q)^2 = \frac{p^2 q^2}{2} \text{ (B)}.$$

The Schwarz inequality of linear independent  $\varepsilon \cdot \eta : (\varepsilon, \eta)^2 < (\varepsilon, \varepsilon)(\eta, \eta)$ .

$$\text{A and B} \implies \frac{2p^2 q^2}{2} < \frac{pq(p+1)(q+1)}{4} \implies pq - p - q - 1 < 0 \implies (p-1)(q-1) < 2$$

$$\implies \underbrace{p = q = 2}_{F_4} \text{ or } \underbrace{p = 1 \text{ or } q = 1}_{B_l}.$$



10) The 3rd type in 8:  $D_l, E_6, E_7, E_8$ .

Set up  $\eta^i := \sum_j \eta_j^i \implies (\eta^i, \eta^j) = 0, (\varepsilon, \eta^i) \neq 0$ .

Applying the method in the proof of (4),

$$\eta^{0'} := \varepsilon - \sum_{i=1}^3 \frac{(\varepsilon, \eta^i)}{(\eta^i, \eta^i)} \eta^i \text{ and } \eta^0 := \frac{\eta^{0'}}{\sqrt{(\eta_0^0, \eta_0^0)}}$$

It is easy to show that:

- $(\eta^0, \eta^i) = 0$
- $(\varepsilon, \eta^0) \neq 0$
- $\varepsilon = (\varepsilon, \eta^0) \eta^0 + \sum_{i=1}^3 \frac{(\varepsilon, \eta^i)}{(\eta^i, \eta^i)} \eta^i$

Then  $1 = (\varepsilon, \varepsilon) > \sum_{i=1}^3 \frac{(\varepsilon, \eta^i)^2}{(\eta^i, \eta^i)}$  by the same reason in (4).

$$(\eta', \eta') = \frac{p(p-1)}{2} \implies \frac{(\varepsilon, \eta')^2}{(\eta', \eta')} = \frac{p-1}{2p} = \frac{1}{2} \left(1 - \frac{1}{p}\right). \text{ Apply } \eta^2, \eta^3.$$

$$\implies 1 > \sum \frac{(\varepsilon, \eta^i)^2}{(\eta^i, \eta^i)} = \frac{1}{2} \left(3 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right) \implies \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1 \text{ (C).}$$

WLOG assume  $\frac{1}{p} \leq \frac{1}{q} \leq \frac{1}{r} \leq \frac{1}{2}$  (D) [if one of them is 1 graph is  $A_l$ ].

$$\text{C, D} \implies \frac{1}{3} < \frac{1}{r} \leq \frac{1}{2} \implies r = 2 \implies \frac{1}{p} + \frac{1}{q} > \frac{1}{2} \text{ (E).}$$

$$\text{D, E} \implies \frac{1}{4} < \frac{1}{q} \leq \frac{1}{2} \implies q = 2, 2 \leq p \implies D_l \text{ or } q = 3, p = 3, 4, 5 \implies E_6, E_7, E_8.$$

Construction:

[matrices for  $A_l, B_l, C_l, D_l$ ]

Let  $E = \mathbb{R}^n$  with the usual inner product and standard basis  $\{\varepsilon_1, \dots, \varepsilon_n\}$ . Let  $I = \{\mathbb{Z}\text{-linear combination of } \{\varepsilon_1, \dots, \varepsilon_n\}\}$ .

$A_l (l \geq 1)E$ : a  $l$ -subspace of  $\mathbb{R}^{l+1}$  orthogonal to  $\varepsilon_1 + \dots + \varepsilon_{l+1}$ .

$I' = I \cap E$ .

$$\Phi = \{\alpha \in I' \mid (\alpha, \alpha) = 2\} = \{(\varepsilon_i - \varepsilon_j), i \neq j\}.$$

$$\text{Root system: } \Delta = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_l - \varepsilon_{l+1}\}$$

$$\text{Weyl Group: } \sigma_{\varepsilon_i - \varepsilon_{i+1}} : \begin{matrix} \varepsilon_i \mapsto \varepsilon_{i+1} \\ \varepsilon_{i+1} \mapsto \varepsilon_i \end{matrix} \leftrightarrow (i, i+1).$$

Thus,  $\mathcal{W} \cong S_{l+1}$ .

$$B_l (l \geq 2)E = \mathbb{R}^l$$

$$\Phi_B = \{\alpha \in I \mid (\alpha, \alpha) = 1 \text{ or } 2\} = \{\pm \varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j), i \neq j\}.$$

$$\Delta = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_l\} \text{ a base.}$$

Weyl Group  $\mathcal{W}$ :

$$\sigma_{\varepsilon_i - \varepsilon_{i+1}} \leftrightarrow (i, i+1)$$

$$\sigma_{\varepsilon_l} \varepsilon_l \mapsto -\varepsilon_l$$

$$\mathcal{W} \cong (\mathbb{Z}/2\mathbb{Z})^l \rtimes S_l.$$

$$D_l (l \geq 4)E = \mathbb{R}^l, \Phi = \{\alpha \in I \mid (\alpha, \alpha) = 2\} = \{\pm(\varepsilon_i \pm \varepsilon_j)\}.$$

$$\Delta = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_{l-1} + \varepsilon_l\}$$

$$\sigma_{\varepsilon_i - \varepsilon_{i+1}} : (i, i+1)$$

$$\sigma_{\varepsilon_{l-1} + \varepsilon_l} : \varepsilon_{l-1} \mapsto -\varepsilon_l, \varepsilon_l \mapsto -\varepsilon_{l-1}.$$

$$\mathcal{W} \cong (\mathbb{Z}/2\mathbb{Z})^{l-1} \rtimes S_l.$$

□

## 7.1 12.2 Automorphism

Claim 1:  $\text{Aut } \Phi \cong \Gamma \rtimes \mathcal{W}$  where  $\Gamma = \{\tau \in \text{Aut } \Phi \mid \tau(\Delta)\Delta\}$ .

Claim 2:  $\Gamma$  may be identified with automorphisms of its dynkin diagram of  $\Phi$ .

## 8 Thursday, 12/5/2024, Cartan Subalgebras (CSA) by Rostyslav

2+2 Definitions.

General Lie Algebra:

- i) In terms of a normalizer
- ii) In terms of Engel subalgebra

Semisimple Lie Algebra:

- i) Maximal toral subalgebras
- ii) Maximal centralizer

If  $x \in \text{End}(V)$ ,  $V = \bigoplus V_a(x)$ ,  $V_a(x) = \ker(x - a \text{id})^m$  where  $m$  is a multiplicity of root of char. poly.

We have  $x|_{V_a(x)} = \underbrace{a}_{\text{scalar}} + \underbrace{n}_{\text{nilpotent}}$ .

Then,  $L = \sqcup_{a \in F} L_a(\text{ad } x) = L_u(\text{ad } x) \oplus L_n(\text{ad } x)$

**Lemma 8.1.** If  $a, b \in F$  then,

$$[L_a(\text{ad } x), L_b(\text{ad } x)] \subset L_{a+b}(\text{ad } x)$$

In particular, when  $L_t(\text{ad } x)$  is a subalgebra of  $L$ ,  $\text{char } F = 0$ ,  $a \neq 0$  then,

$$\forall l \in L_a(\text{ad } x) \text{ } l \text{ is nilpotent}$$

Fact: (simple) adjoints are derivations

$$(\text{ad } x - (a + b))^m[yz] = \sum_{i=0}^m \binom{m}{i} [(\text{ad } x - a)^i(y), (\text{ad } x - b)^{m-i}(z)]$$

For  $m$  sufficiently large all elements on the right will vanish.

**Definition** (Engel Subalgebra). is  $L_0(\text{ad } x)$ .

**Lemma 8.2** (15.2.A). Let  $K \subset L$ -subalgebra. Choose  $z \in K$  such that  $L_0(\text{ad } z)$  is minimal among  $L_0(\text{ad } x)$  for all  $x \in K$ . Suppose  $K \subset L_0(\text{ad } z)$ . Then  $L_0(\text{ad } z) \subset L_0(\text{ad } x) \forall x \in K$ .

Let  $x \in K$  fixed, but arbitrary. Consider a family of endomorphisms of  $L\{\text{ad}(z + cx) \mid c \in F\}$ .  $K_0 = L_0(\text{ad } z)$  is a subalgebra of  $L$  including  $K$  therefore these endomorphisms will stabilize  $K_0 (= L_0(\text{ad } z))$ .

Can induce endomorphism of  $L|_{K_0 (= L_0(\text{ad } z))}$ .

Endomorphism is  $\text{ad}(z + cx)$ . Let  $f, g$  be char. poly in  $K_0$  over  $L/K$  and let  $n = \dim K_0, n - \dim L$ .

$$f(T, c) = T^n + f_1(c)T^{n-1} + \dots + f_n(c)$$

$$g(T, c) = T^{n-r} + g_1(c)T^{n-r-1} + \dots + g_{n-r}(c)$$

$f_i, g_i$  are polynomials.

By definition, eigenvalue 0 appears only in  $K_0$  when  $T = 0$ ,  $g_{n-r}$  is not identically zero on  $F$ . Lets take  $c_1, \dots, c_{r+1}$  not zeros of  $g_{n-r}$ . To say  $g_{n-r}(0) = 0 \iff 0$  is not an eigenvalue of  $\text{ad}(z + cx)$  on the quotient space.

$\implies \forall L_0(\text{ad}(z + cx))$  lie in the subspace of  $K_0$ .

But  $K_0$  is minimal.

$K_0 = L_0(\text{ad } z) = L_0(z + c_i x) \ 1 \leq i \leq r + 1$ .

The only eigenvalues  $\text{ad}(z + cx)$  has is 0.

$\implies f(T, c_i) = T^r, f_i = f(T, c_i).$   
 $\forall f_i$  has  $r + n$  distinct zeros  $\implies \forall f_i$  are identically zero.  
 $L_0(\text{ad}(z + cx)) \supset K_0 \forall c \in F.$   
 Replace  $x$  with  $x - z, c = 1.$   
 $L_0(\text{ad } x) \supset K_0 = L_0(\text{ad } x)$  so we're done.

**Lemma 8.3** (15.2.B). If  $K \subset L$  subalgebra and  $L_0(\text{ad } x) \subset K \implies K$  is self normalizing. In particular, Engel subalgebras are self-normalizing.

$L_0(\text{ad } x) \subset K \implies \text{ad } x$  acts on  $N_L(K)/K$  without an eigenvalue 0 in  $\ker = 0.$   
 $x \in K [N_L(K)x] \subset K \implies \text{ad } x$  acts trivially on  $N_L(K)/K \implies N_L(K) = K.$

**Definition** (Cartan Subalgebra). Cartan Subalgebra is a nilpotent subalgebra of  $L$  that is self-normalizing.

**Theorem 8.4** (15.3). Let  $H$ -subalgebra of  $L$ .  $H$  is a CSA  $\iff H$  is a minimal Engel subalgebra.

$\Leftarrow$  : Assume it is a minimal Engel subalgebra. Then,  $H = L_0(\text{ad } z) \xrightarrow{15.2.B} H$  is self-normalizing.  $\xrightarrow{15.2.A} L_0(\text{ad } z) \subset L_0(\text{ad } x).$

We apply Engel's Theorem which states if  $\forall x \in L$   $\text{ad } x$  is nilpotent then  $L$  is nilpotent.  $\forall x \in H$  in particular  $\text{ad}_H x$  is nilpotent so  $H$  is nilpotent.

$\implies$  : Let  $H$  be CSA,  $H$  is nilpotent by definition.  $H \subset L_0(\text{ad } x)$ . We want to prove  $\exists z$  such that  $H = L_0(\text{ad } z)$ .

Lets assume that is not the case.

Take  $L_0(\text{ad } z)$  smallest,  $\xrightarrow{15.2.A} L_0(\text{ad } z) \subset L_0(\text{ad } x).$

$L_0(\text{ad } z)/H$  here  $x \in H$  will act as a nilpotent.

$H$  annihilates some  $y + H$  where  $y \neq 0.$

$\exists y \in H$  such that  $[Hy] \subset H$ . But  $H$  is self normalizing. Contradiction! So we're done.