Number Theory Reading Group

Thanic Nur Samin

Contents

1 Thursday, 9/12/2024, Representation of $\mathfrak{sl}_2(\mathbb{F})$ by Hechi

$$\mathfrak{sl}_2(\mathbb{F}) \coloneqq \{g \in \mathfrak{gl}_2(\mathbb{F}) \mid \mathrm{Tr}(g) = 0\}$$

We assume $\mathrm{char}(\mathbb{F}) = 0$ and $\mathbb{F}$ is algebraically closed.

Theorem 1.1. $\mathfrak{sl}_2(\mathbb{F})$ is semisimple

Proof. Direct computation of the Killing Form. □

Recall: if $\mathfrak{L}$ is semisimple and $\phi : \mathfrak{L} \to \mathfrak{gl}(V)$ is a representation.
$\mathfrak{L} \ni x = s + n$ abstract jordan decomposition.
$\implies \phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$ in $\phi(\mathfrak{L})$.
From now on, $\mathfrak{L} = \mathfrak{sl}_2(\mathbb{F}) = \mathfrak{sl}(2, \mathbb{F})$.
(V, ϕ) is a representation.
Basis of $\mathfrak{L}$:

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus we have $[h, x] = 2x, [h, y] = -2y, [x, y] = h$.
Since h is diagonal, h is semisimple.
$\implies \phi(h)$ is semisimple and thus diagonalizeable. $\in \mathrm{End}(V)$.
We can decompose $V = \bigoplus_\lambda V_\lambda$ where $V_\lambda = \{v \in V \mid hv = \lambda v\}$ for all $\lambda \in \mathbb{F}$.
We say V_λ is a <u>weight space</u> with λ as its weight.

Lemma 1.2 (7.1)**.** Suppose $v \in V_\lambda$. Then,

1) $xv \in V_{\lambda+2}$
2) $yv \in V_{\lambda-2}$

Proof. 1) $h(xv) = [h, x]v + x(hv) = 2xv + \lambda xv = (\lambda + 2)xv$

2) $h(yv) = [h, y]v + y(hv) = -2yv + \lambda yv = (\lambda - 2)yv$

□

$$\begin{tikzcd}
V_{\lambda-2} & V_\lambda \arrow[l, bend left, "y"] \arrow[r, bend right, "x"'] \arrow["h", loop above] & V_{\lambda+2}
\end{tikzcd}$$

Note that $\div V < \infty$
Thus, $\exists v \in V$ such that $x \cdot v = 0$.
Such a v is called a <u>maximal vector</u>.
For now, assume V is irreducible.
Let v_0 be a maximal vector with weight λ.

Definition. For $i > 0$ integer, $v_i = \frac{y^i \cdot v_0}{i!}$
Also, $v_{-1} = 0$.

Lemma 1.3 (7.2)**.** 1) $h \cdot v_i = (\lambda - 2i)v_i$

2) $y \cdot v_i = (i+1)v_{i+1}$

3) $x \cdot v_i = (\lambda - i + 1)v_{i-1}$

Proof. 1) We use induction. Base case is clear.

Assume it is true for $i-1$.

$v_{i-1} \in V_{\lambda-2(i-1)}$

Thus, $v_i = \frac{1}{i} \cdot y v_{i-1}$

Lemma 7.1 implies $v_i \in V_{\lambda-2i}$.

2) $y \cdot v_i = (i+1)v_{i+1}$ by definition of v_i.

3) $ix \cdot v_i = x(yv_{i-1}) = [x,y]v_{i-1} + yxv_{i-1} = hv_{i-1} + yxv_{i-1} = (\lambda - 2(i-1))v_{i-1} + (\lambda - i + 2)yv_{i-2} = i(\lambda - i + 1)v_{i-1}$

$\square$

$\dim V < \infty$ so it must end at some point.
So, at some point, it'll become 0. $v_0, \cdots, v_m \neq 0, v_{m+1} = 0$.

Definition. m is the integer so that $v_m \neq 0, v_{m+1} = 0$.

By Lemma 7.2,
$\operatorname{span}\{v_0, \cdots, v_m\}$ is a sub-representation of V.
Since V is irreducible,
$V = \operatorname{span}\{v_0, \cdots, v_m\}$
<u>Note</u>: by 7.2(3),
$0 = x \cdot v_{m+1} = (\lambda - m)v_m$
Since $v_m \neq 0$ we have $\lambda = m$.
Thus, $\dim V = m + 1 = \lambda + 1$
Here m is the <u>highest weight</u>.

$$V = V_{-m} \oplus V_{-m+2} \oplus \cdots \oplus V_{m-2} \oplus V_m$$

Construction. Suppose $L \curvearrowright \mathbb{F}[X,Y]$ [as a $\mathbb{F}$-space].
$\rho(x) = X\frac{\partial}{\partial Y}$
$\rho(y) = Y\frac{\partial}{\partial X}$
$\rho(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$
Consider subrepresentations $\mathbb{F}[X,Y]_m$ [symmetric polynomials of degree m, dimension $m+1$].

$$\begin{tikzcd}
0 & X^m \arrow[l, bend left, "x"] \arrow[r, bend right, "y"'] \arrow["h", loop above] & X^{m-1}Y & \cdots
\end{tikzcd}$$

2 Thursday, 9/19/2024, Representation of $\mathfrak{sl}_2(\mathbb{F})$ by Hechi

Root Space Decomposition

Let $\mathcal{L}$ be a non-zero semisimple lie algebra over $\mathbb{F}$ with $\operatorname{char}\mathbb{F} = 0$ and $\mathbb{F}$ algebraically closed.

Definition (Toral Subalgebra)**.** A subalgebra $\mathcal{H} \subseteq \mathcal{L}$ toral if it consists of semisimple elements.

Remark. If every element in $\mathcal{L}$ is ad-nilpotent, then by Engel's Theorem $\mathcal{L}$ is nilpotent. Thus it is not semisimple.
So, there exists a non-zero toral subalgebra.

Fix $\mathcal{H}$ to be the maximal toral subalgebra. A maximal subalgebra exists since $\mathcal{L}$ is finite dimensional.

Lemma 2.1 (8.1)**.** A toral subalgebra $\mathcal{T}$ is abelian.

Proof. Suppose $x \in \mathcal{T}$. We will prove that $\operatorname{ad}_{\mathcal{T}} x = 0$ [as a map].
$\operatorname{ad}_T x$ is diagonalizeable. Assume some eigenvalue is non-zero. Then, we can find eigenvactor $y \in T$ with eigenvalue $a \neq 0$. So, $[x, y] = ay$.
Now, $\operatorname{ad}_T y(x) = [y, x] = -ay$. Since $[y, y] = 0$ we see that $-ay$ is an eigenvector of $\operatorname{ad}_T y$ with eigenvalue 0.
$\operatorname{ad}_T y$ is also diagonalizeable. Suppose $v_1, \cdots, v_n$ is the eigenbasis of $\operatorname{ad}_T y$ with eigenvalues $\lambda_1, \cdots, \lambda_n$. Then $x = a_1 v_1 + \cdots + a_n v_n$ for $a_i \in \mathbb{F}$.
WLOG, $v_1 = y$.

$$[y, x] = a_1\lambda_1 v_1 + \cdots + a_n \lambda_n v_n = -ay$$

By comparing coefficients, $a_1\lambda_1 = -a$. But $\lambda_1 = 0$. This is a contradiction.

□

Now, we fix $\mathcal{H}$ to be a maximal toral subalgebra. It is not necessarily unique.
Note that $\operatorname{ad} H$ is a commuting family in $\operatorname{End}(\mathcal{L})$. From linear algebra we know that $\operatorname{ad} H$ is simultaneously diagonalizeable.

Definition (Root Space Decomposition)**.** Suppose $\mathcal{H}^*$ is the dual space of $\mathcal{H}$. We can write:

$$\mathcal{L} = \bigoplus_{\alpha \in H^*} \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x \forall h \in H\}$$

$$= \mathcal{L}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathcal{L}\alpha$$

where $\Phi = \{\alpha \in H^* \setminus \{0\} \mid \mathcal{L}\alpha \neq 0\}$ and $\mathcal{L}_0 = C_{\mathcal{L}}(\mathcal{H})$ [the centralizer].
This is called the root space decomposition.

Example. $\mathfrak{sl}_2(\mathbb{F})$ has basis:

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then the root space decomposition is:

$$\mathfrak{sl}_2(\mathbb{F}) = \mathcal{H} \oplus \mathcal{L}_{-2} \oplus \mathcal{L}_2$$

$\mathcal{L}_{-2}$ contains the linear form sending h to -2.

Proposition 2.2 (8.1)**.** Let $\alpha, \beta \in \mathcal{H}^*$. Then,

1) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ [by Jacobi Identity]

2) $\alpha \neq 0 \implies \forall x \in L_\alpha$ is nilpotent [by 1]

3) $\alpha + \beta \neq 0 \implies L_\alpha \perp L_\beta$ w.r.t. the Killing Form.

Proof of 3. Find $h \in \mathcal{H}$ such that $(\alpha+\beta)(h) \neq 0$. Then,

$$\kappa([h,x],y) = -\kappa([x,h],y) = -\kappa(x,[h,y])$$

$$\implies (\alpha+\beta)(h)\kappa(x,y) = 0$$

□

In particular, $L_0 \perp L_\alpha$ when $\alpha \in \Phi$.

Corollary 2.3 (8.1)**.** The Killing Form restricted to $\mathcal{L}_0$, $\kappa|_{\mathcal{L}_0}$ is non-degenerate.

Proposition 2.4 (8.2)**.** $\mathcal{H} = \mathcal{L}_0 = C_{\mathcal{L}}(\mathcal{H})$.

Proof. Tedious linear algebra □

Corollary 2.5 (8.2)**.** The Killing Form restricted to $\mathcal{H}$, $\kappa|_{\mathcal{H}}$ is non-degenerate.

This implies, the map $H \to H^*$ given by $x \mapsto \kappa(x,-)$ is an isomorphism.
For each $\phi \in \mathcal{H}^*$ we can define $t_\phi \in \mathcal{H}$ to be the pre-image of this isomorphism. So it satisfies

$$\phi(h) = \kappa(t_\phi, h) \quad \forall h \in \mathcal{H}$$

Proposition 2.6 (8.3)**.** 1) Φ spans $\mathcal{H}^*$

2) If $\alpha \in \Phi$ then $-\alpha \in \Phi$

3) $x \in \mathcal{L}_\alpha, y \in \mathcal{L}_{-\alpha} \implies [x,y] = \kappa(x,y)t_\alpha$

4) $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$

5) $\dim[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] = 1$, spanned by t_α

6) Pick any non-zero $x_\alpha \in L_\alpha \setminus \{0\}$. Then there exists $y_\alpha \in \mathcal{L}_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha \coloneqq [x_\alpha, y_\alpha]$ spans a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{F})$, with the isomorphism $x_\alpha \mapsto x, y_\alpha \mapsto y, h_\alpha \mapsto h$

7) $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$.

If V is a $\mathfrak{sl}_2(\mathbb{F})$-module, recalling that $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$$V = \bigoplus_{\lambda \in \mathbb{F}} V_\lambda \text{ eigenspaces of } h$$

Recall that all $\mathfrak{sl}_2(\mathbb{F})$-module is of the form:

$$\mathfrak{sl}_2(\mathbb{F}) \curvearrowright \mathbb{F}[X,Y]$$

$$\rho(x) = X\frac{\mathrm{d}}{\mathrm{d}Y}, \rho(y) = Y\frac{\mathrm{d}}{\mathrm{d}X}, \rho(h) = X\frac{\mathrm{d}}{\mathrm{d}X} - Y\frac{\mathrm{d}}{\mathrm{d}Y}$$

and $V = \mathbb{F}[X,Y]_m$ [homogeneous polynomials of degree m] is irreducible and give us all irreducible representations.
Then we have:

$$V = V_{-m} \oplus V_{-m+2} \oplus \cdots \oplus V_{m-2} \oplus V_m$$

Where V_m is generated by X^m and V_{-m} is generated by Y^m

$$\begin{tikzcd}
Y^m \arrow[r, bend right, "x"'] & XY^{m-1} \arrow[l, bend right, "y"'] \arrow[loop above, "h"] & \cdots & X^m
\end{tikzcd}$$

If m even, $0 \neq V_0 \subseteq V$
If m odd, $0 \neq V_1 \subseteq V$

Corollary 2.7. V is a $\mathfrak{sl}_2(\mathbb{F})$-module. Then $\dim V_0 + \dim V$ gives the number of summands in the irreducible decomposition of V.

Consider $\mathcal{S}_\alpha = \text{span}\{x_\alpha, y_\alpha, h_\alpha\} \cong \mathfrak{sl}_2(\mathbb{F})$ and its adjoint representation ($\mathcal{L}$ is an $\mathcal{S}_\alpha$ module).
Fix $\alpha \in \Phi$ and let $\mathcal{M} = \mathcal{H} + \sum_{c \in \mathbb{F}^\times} \mathcal{L}_{c\alpha}$.
By proposition 8.1, $\mathcal{M}$ is a submodule of $\mathcal{L}$ [since $[\mathcal{L}_{c_1\alpha}, \mathcal{L}_{c_2\alpha}] \subseteq \mathcal{L}_{(c_1+c_2)\alpha}$].
If $0 \neq x \in \mathcal{L}_{c\alpha}$ we see that $[h_\alpha, x] = c\alpha(h_\alpha) \cdot x = 2cx$
$\implies 2c \in \mathbb{Z}$ and a $\underbrace{\text{weight}}_{\text{eigenvalue}}$ of h_α is 0 or an integer multiple of $\frac{1}{2}$.

Then $\mathcal{M} = \underbrace{\ker \alpha}_{\text{vectors of weight 0}} + \underbrace{\mathbb{F} \cdot h_\alpha}_{\text{weight } 0, \pm 2}$
Therefore, $\mathcal{M}$ contains vectors of weight only 0 or ± 2.
Therefore, if $\alpha \in \Phi$ we have $c = \pm 1$.
$\mathcal{M} = \mathcal{H} + \mathcal{S}_\alpha$. Suppose h_α^c is the complement of h_α in $\mathcal{H}$.
Then, $\mathcal{H} + \mathcal{S}_\alpha = \underbrace{h_\alpha^c}_{\text{abelian}} + \underbrace{\mathcal{S}_\alpha}_{\text{irreducible}}$ has $\dim \mathcal{H} - 1 + 1 = \dim \mathcal{H} = \dim \mathcal{M} - 2$ irreducible summands.
On the other hand, the number of irreducible summands of $\mathcal{M}$ is $\underbrace{\dim \mathcal{M}_0}_{\dim \mathcal{M} - 2} + \underbrace{\dim \mathcal{M}_1}_{0}$
Therefore, $\mathcal{H} + \mathcal{S}_\alpha \subseteq \mathcal{M}$ must be <u>equal</u>.
Therefore, $\dim \mathcal{L}_\alpha = 1$.
Now, suppose $\beta \neq \pm\alpha \in \Phi$. Then, $\exists r, q$ such that $\beta - r\alpha, \beta - (r-1)\alpha, \cdots, \beta + q\alpha$ are roots and outside outside these, i.e. $\beta - (r+1)\alpha, \beta + (q+1)\alpha$ are not.
To see this, suppose $K = \sum_{i \in \mathbb{Z}} \mathcal{L}_{\beta + i\alpha} \subseteq \mathcal{L}$ is a $\mathcal{S}_\alpha$-submodule. We know that $\beta + i\alpha \neq 0$.
Weights:

$$\beta(h_\alpha) + i\alpha(h_\alpha) = \beta(h_\alpha) + 2i$$

So, weights are either all even or all odd.
Therefore, K is irreducible.
Consider $\gamma, \delta \in \mathcal{H}^*$.
Define $(\gamma, \delta) = \kappa(t_\gamma, t_\delta)$ on $E_\mathbb{Q} = \text{span}_\mathbb{Q}(\Phi)$ then $(\cdot, \cdot)$ extends to $E = E_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{R}$ is positive definite.
Then E is an Euclidean Space.

(Φ, E) is called a <u>root system</u>.

3 Thursday, 9/26/2024, Root Systems by Zoia

Let E be an euclidean space. Suppose (α, β) is a symmetric bilinear form on E. Reflection in E fixes some hyperplane H. If α is perpendicular to H then the reflection sends α to $-\alpha$

Consider $\alpha \in E$ and $P_\alpha = \{\beta \in E \mid (\alpha, \beta) = 0\}$ the hyperplane perpendicular to α. Suppose σ_α is the reflection w.r.t. this hyperplane. Then,

$$\mathrm{proj}_\alpha(\beta) = \frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

$$\sigma_\alpha(\beta) = \beta - 2\,\mathrm{proj}_\alpha(\beta) = \beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

Define:

$$\langle \beta, \alpha \rangle = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}$$

Note that $\langle \beta, \alpha \rangle$ is linear only in β. Then,

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$$

Lemma 3.1. Let Φ be a finite subset of E so that Φ spans E. Suppose all reflections $\sigma_\alpha (\alpha \in \Phi)$ leaves Φ invariant. If $\sigma \in \mathrm{GL}(E)$ fixes hyperplane P of E and sends $0 \neq \alpha \in \Phi$ to $-\alpha$, then $\sigma = \sigma_\alpha$ and $P = P_\alpha$.

Proof. Suppose $\tau = \sigma\sigma_\alpha = \sigma\sigma_\alpha^{-1}$.
Then, $\tau(\Phi) = \Phi, \tau(\alpha) = \alpha$ and τ acts as id on $\mathbb{R} \cdot \alpha$ and $E/R \cdot \alpha$ eigenvalues are 1.
So we have $(T-1)^L$ where $L = \dim E$.
$\beta, \tau(\beta), \ldots \tau^k(\beta)$ $\exists k$ that fixes all $\beta \in \Phi$
Φ spans E, so $\tau^k = 1$. So $T^k - 1 = 0$.
If $m(T)$ is the minimal polynomial of τ, then:

$$m(T) \mid T^k - 1$$

$$m(T) \mid (T-1)^k$$

Therefore, $m(T) = T - 1$.
Therefore, $\tau = \mathrm{id}$.
Thus $\sigma\sigma_\alpha^{-1} = \mathrm{id} \implies \sigma = \sigma_\alpha$ ∎

Definition (Root Systems)**.** A finite subset Φ of E is a root system in E if:

1R) Φ spans E, does not contain 0.

2R) If $\alpha \in \Phi$ then only multiples of α in Φ are $\pm\alpha$.

3R) If $\alpha \in \Phi$, then σ_α leaves Φ invariant. $[\forall \beta \in \Phi, \sigma_\alpha(\beta) \in \Phi]$

4R) If $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in \mathbb{Z}$. $\left[\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\right]$

Definition (Weyl Group)**.** Let Φ be a root system in E. Denote by $\mathcal{W}$ the subgroup of $\mathrm{GL}(E)$ generaed by $\sigma_\alpha(\alpha \in \Phi)$.
3R $\implies$ $\mathcal{W}$ is a symmetry group on Φ.

Lemma 3.2. Let Φ be a root system in E with Weyl group $\mathcal{W}$. If $\sigma \in \mathrm{GL}(E)$ leaves Φ invariant, then $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)} \forall \alpha \in \Phi$ and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle \forall \alpha, \beta \in \Phi$.

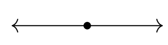
Proof. $\sigma\sigma_\alpha\sigma^{-1}(\sigma(\beta)) = \sigma\sigma_\alpha(\beta) = \sigma(\beta - \langle \beta, \alpha \rangle \alpha) = \sigma(\beta) - \langle \beta, \alpha \rangle \sigma(\alpha)$.
$\sigma(\beta)$ runs over Φ. $\sigma\sigma_\alpha\sigma^{-1}$ fixes $\sigma(P_\alpha)$ pointwise and $\sigma(\alpha) \to -\sigma(\alpha)$.
Therefore, $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ by the lemma.
$\sigma_{\sigma(\alpha)}(\sigma(\beta)) = \sigma(\beta) - \langle \sigma(\beta), \sigma(\alpha) \rangle \sigma(\alpha)$
Therefore, we must have $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$. ∎

Definition (Isomorphisms)**.** Suppose Φ, Φ' are root systems with Euclidean spaces E, E'.
$(\Phi, E) \cong (\Phi', E')$ if there exists map $\varphi : E \to E'$ such that φ maps Φ to Φ' and $\forall \alpha, \beta \in \Phi$ we have $\langle \varphi(\beta), \varphi(\alpha)\rangle = \langle \beta, \alpha \rangle$.

Note that:

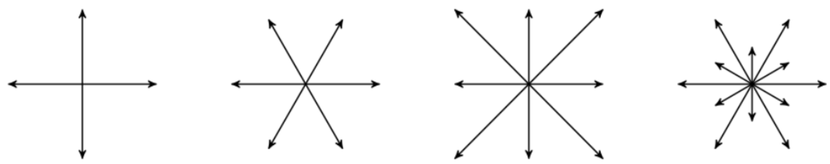
$$\sigma_{\varphi(\alpha)}(\varphi(\beta)) = \varphi(\beta) - \underbrace{\langle \varphi(\beta), \varphi(\alpha)\rangle}_{=\langle \beta,\alpha\rangle} \varphi(\alpha) = \varphi(\beta - \langle \beta, \alpha\rangle \alpha) = \varphi(\sigma_\alpha(\beta))$$

Note that, $\sigma \mapsto \varphi\sigma\varphi^{-1}$ is an isomorphism of Weyl groups.
Thus, $\mathcal{W}$ is a subgroup of $\mathrm{Aut}(\Phi)$.
Now we consider root systems of different dimensions. Suppose $L = \dim E$.
$\underline{L = 1}$: In this case, we have $\alpha, \alpha \in \Phi$ only. This gives us A_1



A_1

$\mathcal{W}(A_1) = \mathbb{Z}_2$
$\underline{L = 2}$:



$A_1 \times A_1$ A_2 $B_2 \cong C_2$ G_2

$\mathcal{W}(A_1 \times A_1) = \mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathcal{W}(A_2) = S_3$
$\mathcal{W}(B_2) = D_4$
$\mathcal{W}(G_2) = D_6$
These are the only possible cases for $L = 2$, since:

$$\langle \beta, \alpha\rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = \frac{2\|\beta\|\|\alpha\|\cos\theta}{\|a\|\|a\|} = \frac{2\|\beta\|}{\|\alpha\|}\cos\theta \in \mathbb{Z}$$

Similarly, $\frac{2\|\alpha\|}{\|\beta\|}\cos\theta \in \mathbb{Z}$. Multiplying, $4\cos^2\theta \in \mathbb{Z} \implies 4\cos^2\theta = 0, 1, 2, 3, 4$
Thus, $\cos\theta = 0, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2} \implies \theta = \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}$.

$\langle \alpha, \beta\rangle$	$\langle \beta, \alpha\rangle$	θ	$\|\beta\|^2/\|\alpha\|^2$
0	0	$\frac{\pi}{2}$	undefined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	2
-1	-2	$\frac{3\pi}{4}$	2
1	3	$\frac{\pi}{6}$	3
-1	-3	$\frac{5\pi}{6}$	3

Table 1: Angle Root System

Lemma 3.3. Suppose α, β are non-proportional root.
If $(\alpha, \beta) > 0$ then $\alpha - \beta$ is a root.
If $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root.

Proof. $\langle \alpha, \beta\rangle = 1 \implies \sigma_\beta(\alpha) = \alpha - 1\beta = \alpha - \beta \in \Phi$
If $\langle \beta, \alpha\rangle = 1$ then $\sigma_\alpha(\beta) = \beta - 1\alpha = \beta - \alpha \in \Phi$.
$\sigma_{\beta-\alpha}(\beta - \alpha) = (\beta - \alpha) - \langle \beta - \alpha, \beta - \alpha\rangle(\beta - \alpha) = \alpha - \beta \in \Phi$ $\square$

4 Thursday, 10/3/2024, Simple Roots by Zoia

A root system Φ of rank l, E-Euclidean Space, $\mathcal{W}$ is the Weyl Group.

Definition. A subset Δ of Φ is called a base if:

B1) Δ is a basis of E [$|\Delta| = l$];

B2) $\forall \beta \in \Phi, \beta = \sum_{\alpha\in\Delta} k_\alpha \cdot \alpha$, the expression is unique with k_α being integers and k_α are either all non-negative or all non-positive.

Definition. The roots from Δ are simple roots.

Definition. The height of a root β [relative to the base Δ] is:

$$\mathrm{ht}(\beta) = \sum_{\alpha\in\Delta} k_\alpha$$

Definition. We have positive roots Φ^+ and negative roots Φ^- from the sign of k_α. Furthermore $\Phi^- = -\Phi^+$.
Also, we define:

$$\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\}$$

Definition. $\gamma \in E$ is regular if:

$$\gamma \in E \setminus \bigcup_{\alpha\in\Phi} P_\alpha$$

Otherwise it is called singular.
Recall that $P_\alpha = \{\beta \in E \mid (\alpha, \beta) = 0\}$

Definition. $\alpha \in \Phi^+(\gamma)$ is decomposable if $\alpha = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in \Phi^+(\gamma)$.
α is indecomposable otherwise.

Definition. We define $\Delta(\gamma)$ to be the set of all indecomposable roots in $\Phi^+(\gamma)$.

Theorem 4.1. Any root system Φ has a base. Let $\gamma \in E$ be a regular.
Then, the set $\Delta(\gamma)$ of all the indecomposable roots in $\Phi^+(\gamma)$ is a base of Φ.
Conversely, every base of Φ is of the form $\Delta(\gamma)$ for some γ.

Proof. We follow the following steps.
Step 1: Each root in $\Phi^+(\gamma)$ is a non-negative $\mathbb{Z}$-linear combination of $\Delta(\gamma)$.
Step 2: If $\alpha, \beta \in \Delta(\gamma)$ then $(\alpha, \beta) \leq 0$ unless $\alpha = \beta$.
Step 3: $\Delta(\gamma)$ is a linearly independent set.
Step 4: $\Delta(\gamma)$ is a base of Φ.
Step 5: Each base Δ of Φ has the form $\Delta(\gamma)$ for some regular $\gamma \in E$.
Proof of Step 1: Suppose otherwise. Then $\exists \alpha \in \Phi^+(\gamma)$ that cannot be expessed as a non-negative $\mathbb{Z}$ linear combination of $\Delta(\gamma)$.
We can have multiple such α's. We pick the α with the smallest (γ, α).
Note that $\alpha \notin \Delta(\gamma)$, since if $\alpha \in \Delta(\gamma)$ then $\alpha = 1 \cdot \alpha$, which violates the assumption.
Thus, α can be written as sum of two elements in $\Phi^+(\gamma)$. Suppose $\alpha = \beta_1 + \beta_2$ so that $\beta_1, \beta_2 \in \Phi^+(\gamma)$. Then, $(\gamma, \alpha) = (\gamma, \beta_1) + (\gamma, \beta_2)$. Due to the minimality of (γ, α), they are both non-negative $\mathbb{Z}$-linear conbination of $\Delta(\gamma)$ which means so is α, a contradiction.
Proof of Step 2: Suppose otherwise. Then, $(\alpha, \beta) > 0$. β cannot be $-\alpha$, thus $\alpha - \beta$ is a root. Then either $\alpha - \beta$ or $\beta - \alpha$ is in $\Phi^+(\gamma)$. WLOG $\alpha - \beta \in \Phi^+(\gamma)$. Then $\alpha = \beta + (\alpha - \beta)$. Then α is decomposable, which is a contradiction since $\Delta(\gamma)$ consists of all indecomposable roots.
Proof of Step 3: Suppose $\sum_{\alpha\in\Delta(\gamma), r_\alpha\in\mathbb{R}} r_\alpha \cdot \alpha = 0$. r_α can be positive or negative. We redistribute so that both sides have positive coefficient:

$$\varepsilon := \sum_\alpha s_\alpha \alpha = \sum_\beta t_\beta \beta$$

Then,

$$0 \leq (\varepsilon, \varepsilon) = \sum_{\alpha,\beta} \underbrace{s_\alpha t_\beta}_{\geq 0} \underbrace{(\alpha, \beta)}_{\leq 0} \leq 0$$

Thus, $\varepsilon = 0$. Now,

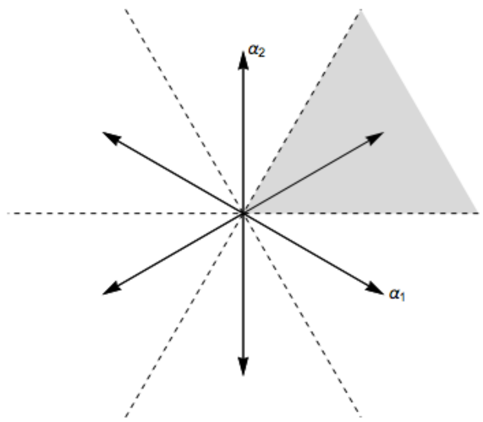
$$0 = (\gamma, \varepsilon) = \sum_{\alpha} \underbrace{s_\alpha}_{\geq 0} \underbrace{(\gamma, \alpha)}_{>0} \geq 0$$

Thus, $s_\alpha = 0$ for all $\alpha \in \Delta(\gamma)$. This implies linear independence.
Proof of Step 4: Note that $\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma)$.
B2 is satisfied because of Step 1.
Then $\Delta(\gamma)$ spans E. Step 3 implies $\Delta(\gamma)$ is a basis of E. Thus we have B1.
Proof of Step 5: Given Δ, we select $\gamma \in E : (\alpha, \gamma) > 0 \forall \alpha \in \Delta$. B2 $\implies \gamma$ is regular and $\Phi^+ \subseteq \Phi^+(\gamma)$. Also, $\Phi^- \subseteq -\Phi^+(\gamma)$.
Therefore, $\Phi^+ = \Phi^+(\gamma)$. Δ consists of indecomposable elements, that is $\Delta \subseteq \Delta(\gamma)$.
Coordinates are equal, therefore $\Delta = \Delta(\gamma)$.

$\square$

Definition (Weyl Chambers)**.** The connected components of $E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$ are called the (open) Weyl Chambers of E.
The fundamental Weyl chamber associated to γ is the open Weyl chamber containing γ. It is denoted by $\mathcal{C}(\gamma)$.
Furthermore, $C(\gamma) = C(\gamma')$ implies γ and γ' are on the same side of each hyperplane P_α. This also means $\Delta(\gamma) = \Delta(\gamma')$, so the Weyl chambers are in 1-1 correspondence with the bases.

For example: here is an open Weyl Chamber for A_2:



$\mathcal{C}(\Delta)$-fundamental Weyl chamber relative to the base $\{\alpha_1, \alpha_2\}$.

The Weyl group acts on the Weyl chambers by $\sigma(\mathcal{C}(\gamma)) = \mathcal{C}(\sigma(\gamma))$.
If $\sigma \in \mathcal{W}$ and γ is regular.
Also, $\mathcal{W}$ permutes bases. σ sends Δ to $\sigma(\Delta)$ which is another base.
Since $\sigma(\Delta(\gamma)) = \Delta(\sigma(\gamma))$ because $(\sigma\gamma, \sigma\alpha) = (\gamma, \alpha)$.

5 Thursday, 10/17/2024, Weyl Group by Zoia

Lemma 5.1. Let α be simple. Then σ_α permutes the positive roots other than α.

Corollary 5.2. Set $\delta = \frac{1}{2} \sum_{\beta \prec 0} \beta$. Then,

$$\sigma_\alpha(\delta) = \delta - \alpha \, \forall \alpha \in \Delta$$

Lemma 5.3. Let $\alpha_1, \cdots, \alpha_n \in \Delta$ [not necessarily distinct]. Write $\sigma_i := \sigma_{\alpha_i}$. If $\sigma_1, \cdots, \sigma_{t-1}(\alpha_t)$ is negative, then

$$\exists s : 1 \le s < t : \sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1}\sigma_{s+1} \cdots \sigma_{t-1}$$

Corollary 5.4. If $\sigma = \sigma_1 \cdots \sigma_t$ is an exp for $\sigma \in \mathcal{W}$, t is as small as possible theen $\sigma(\alpha_t) \prec 0$.

Proof. Suppose $\sigma(\alpha_t) > 0$. Then,

$$\underbrace{\sigma_1 \cdots \sigma_t}_{t \text{ factors}} = \underbrace{\sigma_1 \cdots \sigma_{s-1}\sigma_{s+1} \cdots \sigma_{t-1}}_{t-2 \text{ factors}}$$

which contradicts minimality. □

The Weyl Group

Definition. $\mathcal{W}$ is the subgroup of $GL(E)$ generated by the reflection $(\sigma_\alpha)_{\alpha \in \Phi}$.

Theorem 5.5. Let Δ be a base of Φ.

a) If $\gamma \in E$, $\underline{\gamma \text{ is regular}}$, $\exists \sigma \in \mathcal{W} : (\sigma(\gamma), \alpha) > 0 \, \forall \alpha \in \Delta$.

b) If Δ' is another base of Φ, then $\sigma(\Delta') = \Delta$ for some $\sigma \in \mathcal{W}$.

c) If α is any root $\implies \exists \sigma \in \mathcal{W} : \sigma(\alpha) \in \Delta$.

d) $\mathcal{W}$ generated by σ_α $(\alpha \in \Delta)$.

e) If $\sigma(\Delta) = \Delta, \sigma \in \mathcal{W}$ then $\sigma = \text{id}$.

Proof. We consider the subgroup $\mathcal{W}'$ generated by $\sigma_\alpha(\alpha \in \Delta)$.
For a, b, c we prove the theorem for $\mathcal{W}'$ and for d, e we prove that $\mathcal{W}' = \mathcal{W}$.
a)

$$\delta := \frac{1}{2} \sum_{\alpha \prec 0} \alpha$$

Choose $\sigma \in \mathcal{W}'$ such that $(\sigma(\gamma), \delta)$ is as big as possible.
If α is simple then $\sigma_\alpha \sigma \in \mathcal{W}' \implies (\sigma(\gamma), \delta) \ge (\sigma_\alpha\sigma(\gamma), \delta) = (\sigma(\gamma), \sigma_\alpha(\delta)) = (\sigma(\gamma), \delta - \alpha) = (\sigma(\gamma), \delta) - (\sigma(\gamma), \alpha)$.
Therefore, $(\sigma(\gamma), \alpha) \ge 0$.
Furthermore, $(\sigma(\gamma), \alpha) \ne 0$ so we have strict inequality. Therefore,

$$\forall \alpha \in \Delta, (\sigma(\gamma), \alpha) > 0$$

Therefore, $\sigma(\gamma)$ is in the fundamental Weyl chamber of Δ and σ sends $\mathfrak{C}(\gamma)$ to $\mathfrak{C}(\Delta)$.
b) Since $\mathcal{W}'$ permutes the Weyl chambers by a, it also permutes the bases of Φ.
c) Hyperplanes P_β $(\beta \ne \pm\alpha)$ are distinct from hyperplane $P_\alpha \implies \exists \gamma : \gamma \in P_\alpha, \gamma \notin P_\beta$. Lets choose γ' so that γ' is close to γ such that $(\gamma', \alpha) = \varepsilon > 0$ while $|(\gamma', \beta)| > \varepsilon$ for any $\beta \ne \pm\alpha$.
Then $\alpha \in \Delta(\gamma')$.
d) We want to show that $\mathcal{W}' = \mathcal{W}$. It is enough to show that each reflection $\sigma_\alpha (\alpha \in \Phi)$ is in $\mathcal{W}'$.
Find $\sigma \in \mathcal{W}'$ such that $\beta = \sigma(\alpha) \in \Delta$ using c. Then,

$$\sigma_\beta = \sigma_{\sigma(\alpha)} = \sigma\sigma_\alpha\sigma^{-1} \implies \sigma_\alpha = \sigma^{-1}\sigma_\beta\sigma \in \mathcal{W}'$$

□