M702 ANT

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Abstract

Chapter 1: Local Class Field Theory (LCFT).
Chapter 2: p-divisible groups (eg LT formal groups) and associated Galois representations V and the Hodge-Tate Decomposition of $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ and also the diagonal action of $\mathscr{G}_K$.
Tate: p-divisible groups.
Chapter 3: Sen theory, Fontaine's period rings (φ, Γ)-modules.

1 Local Class Field Theory (LCFT)

1.1 Lubin Tate Theory

[N] Neukirch, Alg. NT
[S] Serre, Local Class Field Theory (Cassels-Frohlich)
[LT] Lubin, Tate Formal complex multiplication
K = non-archimedean local field (locally compact) $\supset \mathcal{O} = \mathcal{O}_K$ = valuation ring $\supset P_K$ = valuation ideal.
Residue Field $k = \mathcal{O}/P_K$, $\text{char}(k) = p > 0, q := |k| = p^f$.
Normalized Valuation $v = v_K : K \twoheadrightarrow \mathbb{Z} \cup \{\infty\}, |a| = q^{-v(a)}$.
$U_K = \mathcal{O}_K^\times$.

Definition. $e(x) \in \mathcal{O}[[x]]$ (a formal power series) is called a Lubin-Tate (LT) series for the uniformizer π (fixed) if the following conditions are fulfilled:

- $e(x) \equiv \pi x \mod \deg 2$.
- $e(x) \equiv x^q \mod \pi$.

Set $\mathscr{E}_\pi$ = set of LT series for the uniformizer π.
Recall: Let R be any $\mathcal{O}$-algebra ($i : \mathcal{O} \to R$ ring homomorphism).
A formal $\mathcal{O}$-module over R is a 1-dimensional commutative formal group $F(x, y) \in R[[x, y]]$ over R (some people call it a formal group law) together with a unital (sending 1 to 1) ring homomorphism:

$$[\cdot]_F : \mathcal{O} \to \text{End}_R(F) = \{f(x) \in R[[x]] \mid f(0) = 0, f(F(x, y)) = F(f(x), f(y))\}$$

such that $\forall a \in \mathcal{O} : [a]_F(x) = i(a)x \mod \deg 2$.
We have the following properties:
$F(x, y) = x + y +$ higher order terms
Associativity: $F(x, F(y, z)) = F(F(x, y), z)$
Commutativity: $F(x, y) = F(y, x)$.
$\implies \exists! \iota(x) \in R[[x]] : F(x, \iota(x)) = 0$. Also, $\iota(x) = -x +$ higher order terms.
If R is a local $\mathcal{O}$-algebra with maximal ideal M ($i^{-1}(M) = P_K, k = \mathcal{O}/P_K \to R/M$) then a formal $\mathcal{O}$-module F over R is callled a LT $\mathcal{O}$-module over R if in addition it is a formal $\mathcal{O}$-module and for any uniformizer π of K: $[\pi]_F(x) \equiv x^q \mod M$.

Remark. If F is a LT $\mathcal{O}$-module over $\mathcal{O}$ [$i : \mathcal{O} \xrightarrow{\text{id}} \mathcal{O}$] then $[\pi]_F(x) \in \mathscr{E}_\pi$ [meaning it is a Lubin Tate series] for any uniformizer π.

Example. 1) $K = \mathbb{Q}_p, F = \widehat{\mathbb{G}}_m, \widehat{\mathbb{G}}_m(x,y) = x + y + xy = (1+x)(1+y) - 1$.

Then, $[\cdot] : \mathbb{Z}_p \to \mathrm{End}_{\mathbb{Z}_p}(\widehat{\mathbb{G}}_m), [a](x) = (1+x)^a - 1 := \sum_{n=1}^{\infty} \binom{a}{n} x^n, \binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!} \in \mathbb{Z}_p$ for any $a \in \mathbb{Z}_p, n \geq 1$.

Exercise. 1) $\forall a \in \mathbb{Z}_p \forall n \geq 0, \binom{a}{n}$ as defined above is in $\mathbb{Z}_p$.

2) If K is a proper extension of $\mathbb{Q}_p$ then $\binom{a}{n} \notin \mathcal{O}_K$ for infinitely many $a \in \mathcal{O}_K$.

2) $K = \mathbb{F}_q((t)), F = \widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a(x,y) \equiv x + y$. Set $[t](x) = tx + x^q$. Then,

$$\Bigg[\underbrace{\sum_{\nu=0}^{\infty} \alpha_\nu t^\nu}_{a}\Bigg](x) := \sum_{\nu=0}^{\infty} \alpha_\nu [t]^{\circ\nu}(x) = \sum_{n=1}^{\infty} a_n x^n \text{ where } a_1 = a$$

gives $F = \widehat{\mathbb{G}}_a$ the structure of a LT $\mathcal{O}$-module over $\mathcal{O}$.

Theorem 1.1.1. i) For all uniformizer π of K and any $e \in \mathscr{E}_\pi$ there exists unique LT $\mathcal{O}$-module F_e over $\mathcal{O}$ such that:

$$[\pi]_{F_e}(x) = e(x)$$

ii) $\forall e, e' \in \mathscr{E}_\pi$ there is an isomorphism of formal $\mathcal{O}$-modules $f : F_e \to F_{e'}$ ($f \in x\mathcal{O}[[x]], f(F_e(x,y)) = F_{e'}(f(x), f(y))$.

$\forall a \in \mathcal{O} : f([a]_{F_e}(x)) = [a]_{F_{e'}}(f(x))$.

$f'(0) \in \mathcal{O}^\times$.

iii) Let K^{nr} be the maximal unramified extension of K (inside some fixed algebraic closure $\overline{K}$) and let $K_{nr} := \widehat{K^{nr}}$ be the completion of K^{nr} and let $\mathcal{O}_{K_{nr}}$ be its valuation ring. Then for any two uniformizers π, π' of K and LT series $e \in \mathscr{E}_\pi$ and $e' \in \mathscr{E}_{\pi'}$, $\exists$ an isomorphism of formal $\mathcal{O}$-modules $F_e \to F_{e'}$ over $\mathcal{O}_{K_{nr}}$.

Formal Complex Multiplication

Let $\overline{K}$ be the fixed algebraic closure of $K \supset \mathcal{O}_{\overline{K}} \supset P_{\overline{K}}$. Let $\pi =$ the fixed uniformizer, $e \in \mathscr{E}_\pi, F_e =$ LT $\mathcal{O}$-module over $\mathcal{O}$.

Set $F[\pi^m] = \left\{ \alpha \in P_{\overline{K}} \mid \underbrace{[\pi^m]_{F_e}}_{e^{\circ m}(x)}(\alpha) = 0 \right\}$. This can be shown to be finite from Theorem 1.1.1.ii by setting $e'(x) = \pi x + x^q$. Then the isomorphism will provide a bijection to $F_{e'}[\pi^m, e' \in \mathscr{E}_\pi]$. Then the zeros of the power series are the zeros of the iteration of the polynomial. Hence the set is finite.

$L_{\pi,m} := K(F_e(\pi^m))$ called the field of π^m-torsion points of F_e. It doesn't depend on e, though it does depend on π.

Example. if $K = \mathbb{Q}_p$ and $e(x) = (1+x)^p - 1$ then, $L_{p,m} = \mathbb{Q}_p(\zeta - 1 \mid \zeta^{p^m} = 1) = \mathbb{Q}_p(\mu_{p^m})$.

If we take $e'(x) = px + x^p$, the power series and the torsion points $F_e[p^m]$ and $F_{e'}[p^m]$ are different but the fields $\mathbb{Q}_p(F_{e[p^m]})$ and $\mathbb{Q}_p(F_{e'}[p^m])$ has to be the same!

Theorem 1.1.2. i) $F_e[\pi^m]$ is a free $\mathcal{O}/(\pi^m)$ module of rank 1 [note that $[\pi^m]$ annihilates $[a](x)$ since $[\pi^m]_{F_e}(\alpha) = 0$].

ii) $\forall m \geq 1$ the maps $\mathcal{O}/(\pi^m) \to \mathrm{End}_{\mathcal{O}}(F_e[\pi^m]), a \mod \pi^m \mapsto [\alpha \mapsto [a](\alpha)]$.

Also, $\mathcal{O}^\times/(1 + (\pi^m)) \to \mathrm{Aut}_{\mathcal{O}}(F_e[\pi^m])$, same formula are isomorphism (of finite groups).

iii) $L_{\pi,m}$ does not depend on $e \in \mathscr{E}_\pi$ but depends on π. In particular, if $e'(x) = \pi x + x^q$ then $L_{\pi,m} = K(F_{e'}[\pi^m])$.

iv) $L_{\pi,m}$ is a finite purely ramified Galois extension (so it does not contain a proper unramified extension) of K of degree $(q-1)q^{m-1}$.

The map $G(L_{\pi,m}/K) \to \mathrm{Aut}_{\mathcal{O}}(F_e[\pi^m]) \overset{\text{ii, canonical}}{\cong} \mathcal{O}^\times/(1+(\pi^m))$ given by $\sigma \mapsto a \mod (1+(\pi^m))$.

If $\forall \alpha \in F_e[\pi^m]\colon \sigma(\alpha) = [a]_{F_e}(\alpha)$, is an isomorphism.

v) If $L_\pi = \bigcup_{m\geq 1} L_{\pi,m}$, then the maps in iv induce an isomorphism:

$$G(L_\pi/K) = \varprojlim_m G(L_{\pi,m}/K) \xrightarrow{\cong} \varprojlim \mathcal{O}^\times/(1+(\pi^m)) \cong \mathcal{O}^\times$$

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Recall: we fixed an algebraic closure $\overline{K}$. Residue field of $\overline{K} = \overline{k} =$ algebraic closure of $k = \mathbb{F}_q$.

Theorem 1.1.3. If L/K is abelian, $L_\pi \subset L$, and L/L_π is purely rammified, then $L_\pi = L$.

Proof. Proof uses the Hasse-Arf theorem, which says that the jumps (or breaks) of the upper ramification filtration $(G(L/K)^t, t \geq -1)$ are integers. $\square$

Remark. $G(L_\pi/K)^m = \mathrm{Gal}(L_\pi/L_{\pi,m}), m \geq 0$.
$L_{\pi,0} \coloneqq K$.