Number Theory Reading Group

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1 Thursday, 9/12/2024, Representation of $\mathfrak{sl}_2(\mathbb{F})$ by Hechi

$$\mathfrak{sl}_2(\mathbb{F}) \coloneqq \{g \in \mathfrak{gl}_2(\mathbb{F}) \mid \mathrm{Tr}(g) = 0\}$$

We assume $\mathrm{char}(\mathbb{F}) = 0$ and $\mathbb{F}$ is algebraically closed.

Theorem 1.1. $\mathfrak{sl}_2(\mathbb{F})$ is semisimple

Proof. Direct computation of the Killing Form. □

Recall: if $\mathfrak{L}$ is semisimple and $\phi : \mathfrak{L} \to \mathfrak{gl}(V)$ is a representation.
$\mathfrak{L} \ni x = s + n$ abstract jordan decomposition.
$\implies \phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$ in $\phi(\mathfrak{L})$.
From now on, $\mathfrak{L} = \mathfrak{sl}_2(\mathbb{F}) = \mathfrak{sl}(2, \mathbb{F})$.
(V, ϕ) is a representation.
Basis of $\mathfrak{L}$:

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus we have $[h, x] = 2x, [h, y] = -2y, [x, y] = h$.
Since h is diagonal, h is semisimple.
$\implies \phi(h)$ is semisimple and thus diagonalizeable. $\in \mathrm{End}(V)$.
We can decompose $V = \bigoplus_\lambda V_\lambda$ where $V_\lambda = \{v \in V \mid hv = \lambda v\}$ for all $\lambda \in \mathbb{F}$.
We say V_λ is a <u>weight space</u> with λ as its weight.

Lemma 1.2 (7.1)**.** Suppose $v \in V_\lambda$. Then,

1) $xv \in V_{\lambda+2}$
2) $yv \in V_{\lambda-2}$

Proof. 1) $h(xv) = [h, x]v + x(hv) = 2xv + \lambda xv = (\lambda + 2)xv$

2) $h(yv) = [h, y]v + y(hv) = -2yv + \lambda yv = (\lambda - 2)yv$

□

$$\begin{tikzcd} V_{\lambda-2} & V_\lambda \arrow[l, bend left, "y"'] \arrow[r, bend right, "x"'] \arrow[loop above, "h"] & V_{\lambda+2} \end{tikzcd}$$

Note that $\div V < \infty$
Thus, $\exists v \in V$ such that $x \cdot v = 0$.
Such a v is called a <u>maximal vector</u>.
For now, assume V is irreducible.
Let v_0 be a maximal vector with weight λ.

Definition. For $i > 0$ integer, $v_i = \frac{y^i \cdot v_0}{i!}$
Also, $v_{-1} = 0$.

Lemma 1.3 (7.2)**.** 1) $h \cdot v_i = (\lambda - 2i)v_i$

2) $y \cdot v_i = (i+1)v_{i+1}$

3) $x \cdot v_i = (\lambda - i + 1)v_{i-1}$

Proof. 1) We use induction. Base case is clear.
Assume it is true for $i-1$.
$v_{i-1} \in V_{\lambda - 2(i-1)}$
Thus, $v_i = \frac{1}{i} \cdot yv_{i-1}$
Lemma 7.1 implies $v_i \in V_{\lambda - 2i}$.

2) $y \cdot v_i = (i+1)v_{i+1}$ by definition of v_i.

3) $ix \cdot v_i = x(yv_{i-1}) = [x,y]v_{i-1} + yxv_{i-1} = hv_{i-1} + yxv_{i-1} = (\lambda - 2(i-1))v_{i-1} + (\lambda - i + 2)yv_{i-2} = i(\lambda - i + 1)v_{i-1}$

□

$\dim V < \infty$ so it must end at some point.
So, at some point, it'll become 0. $v_0, \cdots, v_m \neq 0, v_{m+1} = 0$.

Definition. m is the integer so that $v_m \neq 0, v_{m+1} = 0$.

By Lemma 7.2,
$\text{span}\{v_0, \cdots, v_m\}$ is a sub-representation of V.
Since V is irreducible,
$V = \text{span}\{v_0, \cdots, v_m\}$
Note: by 7.2(3),
$0 = x \cdot v_{m+1} = (\lambda - m)v_m$
Since $v_m \neq 0$ we have $\lambda = m$.
Thus, $\dim V = m + 1 = \lambda + 1$
Here m is the highest weight.

$$V = V_{-m} \oplus V_{-m+2} \oplus \cdots \oplus V_{m-2} \oplus V_m$$

Construction. Suppose $L \curvearrowright \mathbb{F}[X,Y]$ [as a $\mathbb{F}$-space].
$\rho(x) = X\frac{\partial}{\partial Y}$
$\rho(y) = Y\frac{\partial}{\partial X}$
$\rho(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$
Consider subrepresentations $\mathbb{F}[X,Y]_m$ [symmetric polynomials of degree m, dimension $m+1$].

$$\begin{tikzcd} 0 & X^m \arrow[l, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[loop above, "h"] & X^{m-1}Y & \cdots \end{tikzcd}$$