

# Group Representations MATH 607

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Texts: Lang, Algebra, Revised Third Edition, Chapter 17 (sections 1-5) and 18 (sections 1-8)

Serre, Linear Representations of Finite Groups, Parts II and III

**Monday, 8/26/2024**

Today:

History

Modular

Quotients

Matrices

Lang XVII, Section 1

## (Fake) History

History of Groups

Most notions (let's say what is a vector space, what is a group) were vague.

Originally, groups were seen as:

- Symmetry Groups  $S_n$
- $GL_n(\mathbb{R})$  aka  $n \times n$  invertible matrices
- Subgroups of the above
- Representations of the above

For representation, consider  $G$  and a homomorphism  $G \rightarrow S_n$  [which is a group action  $G \curvearrowright \{1, 2, \dots, n\}$ ] or a homomorphism  $G \rightarrow GL_n$  [which is a group action on vector space].

Part I of this course will be Ring Theory.

## Part I: Ring Theory

### Module

Convention:  $R$  = Ring with unity

**Definition** (Left Module). Left Module is an abelian group  $M$  with a function  $R \times M \rightarrow M$  so that  $(r, m) \mapsto rm$  such that  $R \times M \rightarrow M$  is  $\mathbb{Z}$ -bilinear.

Meaning, we have:

$$(r + r')m = rm + r'm$$

$$r(m + m') = rm + rm'$$

$$\text{Also } (rr')m = r(r'm)$$

$$\text{And finally } 1m = m$$

By default, module = left module (since Jim doesn't want Trump to get reelected, he prefers left module)

module / field [module over field] = vector space

We can have submodules  $M' \triangleleft M$

We have quotients  $M/M'$

We have the short exact sequence:

$$0 \rightarrow M' \rightarrow M \rightarrow M/M' \rightarrow 0$$

which means in each homomorphism,  $\text{im} = \ker$   
 So,  $M' \rightarrow M$  is injective and  $M \rightarrow M/M'$  is surjective.  
 Also, kernel of  $M \rightarrow M/M'$  is  $M'$

**Remark.** Note that  $R$  is itself an  $R$ -module.

Convention: Submodule  $M$  of  $R$  = left ideal of  $R$ .

Left ideals are not enough to take quotients (like how we need normal subgroup for group quotients).

So we need two sided ideals.

**Definition** (Two Sided Ideals).  $I \subset R$  is 2-sided ideal if  $I$  is abelian subgroup and  $ri \in I, ir \in I$  aka “closed”.

**Example.** Consider a homomorphism  $f : R \rightarrow R'$ . Then  $\ker f$  is a 2-sided ideal of  $R$ .

For ring homomorphism we need:

$$f(r + r') = f(r) + f(r')$$

$$f(rr') = f(r)f(r')$$

$$f(1) = 1$$

If  $I \subset R$  is 2-sided then  $R/I$  is a quotient ring.

For example,  $M_2(\mathbb{R})$  has no proper 2-sided ideal. But there exists left ideals!

$$\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \text{ is a left ideal}$$

Matrices are a good ‘source’ of non-commutative rings.

Given any ring  $R$  we can consider ring  $M_n(R)$  of  $n \times n$  matrices.

Given  $R$ -module  $M$  we can get  $\text{End}_R(M) = \{f : M \rightarrow M, f \text{ is } R\text{-module map}\}$

We have  $(f + g)m = f(m) + g(m), (fg)m = f(g(m))$ .

This is a ‘coordinate free approach’ to matrices.

**Remark.**  $M_n(R)$  and  $\text{End}_R(R^n)$  often looks the same, but in general  $M_n(R) \not\cong \text{End}_R(R^n)$ .

Let’s first take  $n = 1$ . Let  $r_0 \in R$ .

Consider  $R \rightarrow R$  map  $r \mapsto r_0 r$

We don’t like this because this is not a left module map!!!

So this is not even in  $\text{End}_R(R)$

What if we consider  $r \mapsto r r_0$ ?

This is a left module map, aka  $\in \text{End}_R(R)$

But  $R \rightarrow \text{End}_R(R)$  is not a ring homomorphism.

So we are going to take the opposite ring.

Fix 1:

Given ring  $R$ , we can look into the mirror and find opposite ring  $R^{op}$

Elements of  $R^{op}$  = elements of  $R$ .

0, 1, + remain the same

But multiplication is reversed: define  $r \cdot_{op} r' = r' r$

Alternate notation, we write  $op$  on elements.

$$\text{Then } r^{op}(r')^{op} = (r' r)^{op}$$

Then we have isomorphism  $R^{op} \cong \text{End}_R(R)$  which is a ring homomorphism!

**Exercise.** 1)  $R \cong R^{op} \iff \exists$  antiautomorphism  $\alpha : R \rightarrow R$

Antiautomorphism means  $\alpha$  preserves 0, 1, + but reverses multiplication

2)  $R$  commutative, then  $(M_n R) \cong (M_n R)^{op}$

3) Real quaternions  $\mathbb{H} \cong \mathbb{H}^{op}$

**Remark.** If you take right modules, you don’t need  $op$ .

There is a contravariant endofunctor in the category of rings which takes objects of rings to their opposite.

$\text{Ring}^{op} \rightarrow \text{Ring}$  [opposite category, not the same thing]

$R \mapsto R^{op}$

Fix 2: [From Lang]

Suppose we have module homomorphism  $\phi : E = E_1 \oplus \cdots \oplus E_n \rightarrow F_1 \oplus \cdots \oplus F_m = F$

Then we have  $E_j \rightarrow E \xrightarrow{\phi} F \rightarrow F_i$  which we define to be  $E_j \xrightarrow{\phi_{ij}} F_i$

Then we have a matrix  $M(\phi)$  so that  $M(\phi) = (\phi)_{ij}$

Then for  $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in E_1 \oplus \cdots \oplus E_n$

Then  $\phi(x) = (\phi_{ij}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

So, if we have  $E^n = E \oplus \cdots \oplus E$  [n times]

Lang says, there is a ring isomorphism

$$\text{End}_R(E^n) \xrightarrow{\cong} M_n(\text{End}_R E)$$

$$\phi \mapsto (\phi_{ij})$$

If  $E = R$  as left module, then  $\text{End}_R R \cong R^{op}$

By combining these,  $\text{End}_R(R^n) \cong M_n(R^{op})$

## Wednesday, 8/28/2024

Today:

Group ring

Category

Simple modules

Question: The course is about ‘group representations’. So why study rings?

Answer: A group representation [homomorphism  $G \rightarrow GL_n(\mathbb{R})$ ] is exactly the same as a module over the ring  $\mathbb{R}G$ .

So knowing everything about modules would tell us everything about representation.

Abelian Category!

Suppose we have a ring  $R$  and a group  $G$ . We can get a ring out of  $G$

**Definition** (Group Ring  $RG$ ). As an abelian group, this is the free  $R$ -module with basis the elements of  $G$ .

Elements are symbols of the form  $r_1 g_1 + \cdots + r_n g_n$  [finite linear combination].

0 is the trivial linear combination. So  $0 = 0$

$1 = 1e = 1_R e_G$

Multiplication is defined in the obvious way.

$$(\sum_i r_i g_i)(\sum_j r'_j g'_j) = \sum_{i,j} r_i r'_j g_i g'_j$$

Suppose  $V$  is a  $R$ -module.

Then a homomorphism  $\rho : G \rightarrow \text{Aut}_R(V) \leftrightarrow V$  is  $RG$ -module.

$$\rho \mapsto (\sum_i r_i g_i)v := \sum_i r_i \rho(g_i)v$$

$g \mapsto (v \mapsto gv) \leftarrow V$   $RG$  module.

**Example.**  $C_2 = \{1, t\}$

Then we have  $\mathbb{Z}C_2 = \{a + bt \mid a, b \in \mathbb{Z}, t^2 = 0\} = \mathbb{Z}[t]/(t^2)$

Note that  $(1+t)(1-t) = 1 - t^2 = 0$  so we have zero divisors.

Take  $C_\infty = \langle t \rangle$

Then  $\mathbb{Z}C_\infty = \mathbb{Z}[t, t^{-1}]$  the laurent polynomial ring.

$\mathbb{Q}C_\infty = \mathbb{Q}[t, t^{-1}]$  is a PID [since it is a euclidean ring]

Now we see categories.

If we fix  $R$  then we have a functor  $\text{Group} \rightarrow \text{Ring}$  given by  $G \mapsto RG$

Or we could say we have a functor  $\text{Ring} \times \text{Group} \rightarrow \text{Ring}$  given by  $(R, G) \mapsto RG$

**Definition.** A category  $\mathcal{C}$  consists of:

- objects  $\text{Ob } \mathcal{C}$
- morphism  $C(X, Y)$  for  $X, Y \in \text{Ob } \mathcal{C}$
- compositions  $C(X, Y) \times C(Y, Z) \rightarrow C(X, Z)$  given by  $(g, f) \mapsto f \circ g$
- identity  $\text{Id}_X \in C(X, X) \forall X \in \text{Ob } \mathcal{C}$

Such that we have:

- associativity:  $(f \circ g) \circ h = f \circ (g \circ h)$
- composition with identity:  $\text{Id}_Y \circ f = f = f \circ \text{Id}_X$  for  $f \in C(X, Y)$

For example in the category of groups, we have objects groups and morphisms homomorphism.

Morphism notations:  $f : X \rightarrow Y$  or  $X \xrightarrow{f} Y$  for  $f \in C(X, Y)$

**Definition.**  $f : X \rightarrow Y$  is isomorphism if  $\exists g : Y \rightarrow X$  such that  $f \circ g = \text{Id}$ ,  $g \circ f = \text{Id}$ . Then we say  $X$  and  $Y$  are isomorphic and write  $X \cong Y$ .

**Example.** Example of Categories:

- Set
- Ring
- Group
- Ab (Abelian Groups)
- $R$ -modules (objects are modules, morphisms are homomorphisms  $h(rm) = rh(m)$ )
- Given a group  $G$  we can get a category  $BG$  such that:  
 $\text{Ob } BG = \{*\}$  and  $BG(*, *) = G$

In this category, there is only one object  $*$ . The elements of the group are morphisms.

**Definition.** Functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is  $F : \text{Ob } \mathcal{C} \rightarrow \text{Ob } \mathcal{D}$  given by  $X \mapsto F(X)$

And  $F : C(X, Y) \rightarrow \mathcal{D}(F(X), F(Y))$  such that

$X \xrightarrow{f} Y$  gives us  $F(X) \xrightarrow{F(f)} F(Y)$

such that  $F(f \circ g) = F(f) \circ F(g)$  and  $F(\text{Id}_X) = \text{Id}_{F(X)}$

**Example.** Unit Functor  $\text{Ring} \rightarrow \text{Group}$  given by  $R \mapsto R^\times = \{r \in R \mid \exists s \in R, rs = 1 = sr\}$

For example,  $\mathbb{Q}^\times \cong C_2 \oplus \mathbb{Z}^\infty [= \pm p_1^{e_1} p_2^{e_2} \cdots]$

$\mathbb{Z}^\times \cong \{\pm 1\} = C_2$

$(\mathbb{Z}C_2)^\times \cong \{\pm 1, \pm t\} \cong C_2 \times C_2$

**Definition.**  $R$  is a division ring (= skew field) if  $1 \neq 0$  and  $R^\times = R - 0$ .

**Definition.** Quaternions

$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d, \in \mathbb{R}\}$

Where  $i^2 = j^2 = k^2 = -1$

$ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$

This is a division ring since we can write down inverses.

$\alpha = a + bi + cj + dk$  gives us  $\bar{\alpha} = a - bi - cj - dk$

So,  $\text{norm}(\alpha) = \alpha \bar{\alpha} = a^2 + b^2 + c^2 + d^2$

So,  $\alpha^{-1} = \frac{\bar{\alpha}}{\text{norm}(\alpha)}$

**Remark.** Note that the quaternion group  $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$  is a subgroup of  $\mathbb{H}^\times = GL_1(\mathbb{H})$ .  
So,  $\mathbb{H}$  is a  $\mathbb{R}Q_8$  module.

**Theorem 1** (Weddenburn's Little Theorem). a. A finite commutative domain is a field [easy]  
b. A finite skew field is a field [aka commutative]

a is easy: suppose  $F$  is finite commutative domain. For  $0 \neq f \in F$ , consider multiplication by  $f$  as a map  $F \rightarrow F$ . It is injective, and finiteness implies surjective. So, it is bijective, and there exists inverse.  
eg  $\mathbb{Z}/p$  is a field.

## Simple Modules

These are like primes. We also have some analogue of prime factorization.

**Definition.**  $R$ -module  $E$  is simple if:  
 $E \neq 0$

No proper submodules, aka  $M \triangleleft E \implies M = 0$  or  $E$

In other words,  $E$  is a simple module if it only has two submodules: 0 and  $E$ .

eg simple  $\mathbb{R}$ -modules are 1 dim vector spaces, aka  $\mathbb{R}$

**Exercise.** a)  $\mathbb{R}^2$  is a simple  $M_2(\mathbb{R})$ -module

b) Express  $M_2(\mathbb{R})$  as direct sum of simple modules.

## Friday, 8/30/2024

**Exercise.** Suppose finite  $G \neq 1$  and  $R \neq 0$  Prove that  $RG$  has zero divisors.

**Definition.** Direct product of rings  $R \times S$ , addition and multiplication is done componentwise.

It is a product in the category of rings. aka:

$$\begin{array}{ccccc} & & T & & \\ & f_1 \swarrow & \vdots f \downarrow & \searrow f_2 & \\ R & \xleftarrow{\pi_1} & R \times S & \xrightarrow{\pi_2} & S \end{array}$$

for any pair of ring homomorphisms  $T \xrightarrow{f_1} R$  and  $T \xrightarrow{f_2} S$  we have a unique ring homomorphism  $f : T \xrightarrow{f} R \times S$  so that the diagram commutes.

**Definition.**  $e \in R$  is an idempotent if  $e^2 = e$ .

0, 1 are trivial idempotents.

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  is an idempotent in  $M_2(\mathbb{R})$

$(0, 1)$  is an idempotent in  $\mathbb{R} \times \mathbb{R}$

If  $e$  is an idempotent so is  $1 - e$

**Definition.** Idempotent  $e \in R$  is central if  $\forall r$  we have  $er = re$

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  is not central, but  $(0, 1)$  is.

**Exercise.** A ring can be written as a product ring, aka  $R \cong R_1 \times R_2$  with  $R_i \neq 0$  if and only if there exists a nontrivial central idempotent.

## Semisimple Modules

**Definition.**  $E$  is a simple  $R$ -module if it doesn't have any nontrivial submodules.  
If  $E \neq 0$  and  $M \triangleleft E$  then  $M \neq 0$  or  $M = E$

**Example.**  $R^2$  is a simple  $M_2\mathbb{R}$ -module.  
 $\mathbb{R} \times 0$  is a simple  $\mathbb{R} \times \mathbb{R}$  module.  
 $\mathbb{Z}/p\mathbb{Z}$  is a simple  $\mathbb{Z}$ -module

**Lemma 2.** [Schur's Lemma]: Let  $E, F$  be simple  $R$ -modules. Then any nonzero homomorphism  $f : E \rightarrow F$  is an isomorphism.

*Proof.*  $f \neq 0$  means  $\ker f \neq E$  and  $\text{im } f \neq 0$ .

Since they are submodules,  $\ker f = 0$  and  $\text{im } f = F$

So  $f$  is bijective. □

**Corollary 3.** If  $E$  is simple, then  $\text{End}_R E$  is a skew field [any non-zero element is invertible]

**Example.** Commutative example:  $\text{End}_{M_2\mathbb{R}}(\mathbb{R}^2)$  is a skew field.  
In fact,  $\text{End}_{M_2\mathbb{R}}(\mathbb{R}^2) \cong \mathbb{R}$

**Definition** (Direct Sum). Suppose  $M_i \triangleleft M$  for  $i \in I$   
Then,  $M = \bigoplus_{i \in I} M_i$  means,  $\forall m \in M_i$  we have  $m = \sum_{i \in I} m_i$  with  $m_i \in M_i$  uniquely.  
There are notions of internal and external direct sums. The above is an internal direct sum.

External direct sum: given  $\{M_i\}_{i \in I}$  we can construct  $\bigoplus_{i \in I} M_i$

**Proposition 4** (Universal Property). Given a collection of homomorphisms  $\{t_i : M_i \rightarrow N\}_{i \in I}$ , it extends directly to a homomorphism  $\bigoplus M_i \rightarrow N$ . We denote this by  $\bigoplus f_i$

**Remark.** Note: Maps to product are easy, maps from direct sum are easy.

**Proposition 5** (1.2, Lang XVII). Suppose we have isomorphism  $E_1^{n_1} \oplus \dots \oplus E_r^{n_r} \xrightarrow{\cong} F_1^{m_1} \oplus \dots \oplus F_s^{m_s}$  with  $E_i$  and  $F_j$  simple and non-isomorphic [ie for all  $k \neq i, E_k \not\cong E_i$  and  $k \neq j, F_k \not\cong F_j$  ]

Then  $r = s$  and there exists a permutation  $\sigma \in S_r$  so that  $E_j \cong F_{\sigma(j)}$  and  $n_j = m_{\sigma(j)}$

Corollary: If  $E$  is a finite direct sum of simple modules, then the isomorphism class of simple components of  $E$  and multiplicities are well-defined.

*Proof.* We use Schur's Lemma.

We write  $\phi$  as a matrix  $(\phi_{ji} : E_i^{n_i} \rightarrow F_j^{m_j})$

Since  $\phi$  is injective, for all  $i$  there exists a  $j$  such that  $\phi_{ji} \neq 0$

Then,  $E_i \cong F_j$  by Schur's Lemma

Note that  $F_j$  are isomorphic. So, for all  $i$ , the  $j$  such that  $\phi_{ji} \neq 0$  is unique!

We also get  $\sigma : \{1, \dots, r\} \rightarrow \{1, \dots, s\}$  so that  $\sigma(i) = j$

Since  $\sigma^{-1}$  exists  $\sigma^{-1}$  exists, and thus  $r = s$

Since  $\phi$  is an isomorphism, individual  $\phi_{ji} : E_i^{n_i} \rightarrow F_{\sigma(i)}^{m_{\sigma(i)}}$  are isomorphisms.

To complete the proof, we need a lemma

Lemma: Let  $E$  be simple. If  $E^n \cong E^m$  then  $n = m$

Proof of lemma; Let  $D = \text{End}_R E$ . By Schur's Lemma,  $D$  is a division ring.

Since  $E^n \cong E^m$ , we have  $\text{End}_R(E^n) \cong \text{End}_R(E^m)$

So,  $M_n(D) \cong M_m(D)$

Also, isomorphism not just as rings, but also as  $D$ -modules.

Every module over a skew field is free, and the number of dimensions is the same.

So,  $n^2 = m^2 \implies n = m$

This finishes the proof. □

## Lang XVII section 2

**Theorem 6.** Let  $E$  be an  $R$ -module. Then TFAE:

SS1:  $E$  is a sum of simple modules [so, we can write  $m \in E$  as sum of  $m_i$  but it is not unique]

SS2:  $E$  is a direct sum of simple modules [we can write as a sum, and it's unique]

SS3: Every submodule of  $E$  is a summand.

$F \triangleleft E \iff$  we can find  $F'$  so that  $E = F \oplus F'$

SS3' : any monomorphism  $F \rightarrow E$  'splits'

SS3'' Short exact sequence

$$0 \rightarrow F \rightarrow E \rightarrow H \rightarrow 0$$

splits.

This leads us to:

**Definition.**  $E$  is semisimple if it satisfies one of the above.

Davies: SS2 is best

eg:  $R = \mathbb{R} \times \mathbb{R}$

$E = \mathbb{R} \times \mathbb{R}$  is semisimple but not simple.

Because:  $E = \mathbb{R} \times 0 \oplus 0 \times \mathbb{R}$

## Wednesday, 9/4/2024

Recap: Semisimple modules.

**Lemma 7.** If  $E = \sum_{i \in I} E_i$  with  $E_i$  simple. Then,  $\exists J \subset I$  such that  $E = \bigoplus_{j \in J} E_j$

**Corollary 8.** SS1  $\implies$  SS2

*Proof.* Let  $J \subset I$  be maximal such that  $\sum_{j \in J} E_j = \bigoplus_{j \in J} E_j$

This exists by Zorn's lemma.

$\forall i \in I - J$ , we have  $E_i \cap \bigoplus_{j \in J} E_j \neq \emptyset$  by maximality.

Since  $E_i$  is simple,  $E_i \subset \bigoplus_{j \in J} E_j$ . Therefore,  $E = \bigoplus_{j \in J} E_j$ . □

True or False? Every module has a maximal proper submodule.

False!!! Exercise.

**Exercise.** a) If  $M \triangleleft F$  proper and  $M$  maximal, then  $F/M$  is simple.

b) Find a ring  $R$ , module  $M$  which does not have proper maximal submodules.

c) If  $F$  is a finitely generated  $R$ -module, then it is contained in a proper maximal submodule.

*Proof of SS2  $\implies$  SS3.* Suppose  $F \triangleleft E = \bigoplus_{i \in I} E_i$  with  $E_i$  simple. Let  $J \subset I$  be maximal such that:

$$F + \bigoplus_{j \in J} E_j = F \oplus \bigoplus_{j \in J} E_j$$

Take any  $i \in I - J$ . Then,  $E_i \cap \left[ F \oplus \bigoplus_{j \in J} E_j \right] \neq 0$  by maximality of  $J$ .

Since  $E_i$  is simple,  $E_i \subset F \oplus \bigoplus_{j \in J} E_j$ .

Therefore,  $E = F \oplus \underbrace{\bigoplus_{j \in J} E_j}_{F'}$ .

We have found  $F'$ , which proves SS3. □

*Proof of SS3  $\implies$  SS1.*

**Lemma 9.**  $0 \neq F \triangleleft E$  and  $E$  satisfies SS3. Then, there exists simple finitely generated  $S \triangleleft F$ .

Plan:  $M \triangleleft F_0 \triangleleft F \triangleleft E$ .  
 $\neq$  f.g.

Then, choose  $0 \neq v \in F$ . Let  $F_0 = Rv$ .

**Exercise.**  $M$  exists. [Zorn's Lemma]

Let  $E = \sum_{\text{simple } S \triangleleft E} S$ .

Then, by SS3,  $E = E_0 \oplus E'_0$ .

Lemma and definition of  $E_0$  implies:  $E'_0 = 0$ . So,  $E$  is indeed a sum of simple  $R$ -modules. We're done! □

**Proposition 10 (2.2).** Every quotient module and submodule of a semisimple module is semisimple.

*Proof.* Quotients: Suppose  $M = E/N$ . We have surjective  $f : E \rightarrow M$  with  $E$  semisimple.

SS1 implies  $E = \sum_{i \in I} S_i$  with  $S_i$  simple.

Then,  $M = \sum_{i \in I} f(S_i)$

Schur's lemma implies  $f(S_i)$  is either 0 or simple, so  $M$  satisfies SS1.

Submodules: Suppose  $F \triangleleft E$  with  $E$  semisimple. SS3 implies  $E = F \oplus F'$ . Thus  $E \cong E/F'$ , so it is semisimple by the quotient result. □

Preview:

**Definition.** A ring  $R$  is semisimple if and only if all  $R$ -modules are semisimple.

Lang defines semisimple differently: A ring  $R$  is semisimple if it is semisimple as an  $R$ -module.

**Theorem 11** (Artin-Weddenburn Theorem). A ring is semisimple if and only if it is isomorphic to a finite product of matrix rings over division algebras:

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$$

$\mathbb{C}G, \mathbb{R}G$  are semisimple. We also have the result:

**Theorem 12** (Maschke's Theorem). The group ring  $kG$  is semisimple if  $G$  is finite and  $k$  is a field of characteristic prime to  $G$ .

This also works with  $\text{char } k = 0$ . It is in fact an if and only if.

So  $\mathbb{F}_p G$  is also semisimple given  $p \nmid |G|$

*Proof.* Outline: let  $|G| = n$ . We will verify SS3.

Let  $F \triangleleft E$  be  $kG$  modules.

$k$  is a field, so there exists a  $k$ -linear projection  $\pi : E \rightarrow F$  such that  $\pi(f) = f$  for  $f \in F$  [take a basis of  $F$  as a  $k$ -vector space, complete it to a basis of  $E$ ].

Now, define an 'average'.

$$\pi'(e) = \frac{\sum_{g \in G} g\pi(g^{-1}e)}{n}$$

Then,  $\pi' : E \rightarrow F$  is a  $kG$ -linear projection, meaning  $\pi'(ge) = g\pi'(e)$ .

Then  $E = \text{im } \pi' \oplus \ker \pi'$

□



Friday, 9/6/2024

### Lang XVII, Section 3

“Density Theorem”

Suppose  $R$  is a ring and  $E$  is a  $R$ -module. Then we have maps  $R \times E \rightarrow E$  by multiplication on the left.

**Definition** (Commutant).  $R' = R'(E) = \text{End}_R(E)$  is a ring.

$\phi \in R' \iff \phi : E \rightarrow E$  such that  $\phi(re) = r\phi(e)$ . It ‘commutes with  $E$ ’.

Note that  $E$  is also an  $R'$ -module, with  $R' \times E \rightarrow E$  given by  $(\phi, e) \mapsto \phi(e)$ .

**Definition** (Double Commutant). We can iterate on the previous definition.

$$R'' = R'(R'E) = \text{End}_{R'}(E)$$

Therefore,

$$R'' = \text{End}_{R'}(E) = \text{End}_{\text{End}_R(E)}(E)$$

This means,  $f \in R'' \iff f : E \rightarrow E, \forall \phi \in R', f \circ \phi = \phi \circ f$ . So, things in  $R''$ :

commute with things which commute with  $r \in R$ .

**Example.** Suppose  $R = \mathbb{R}$  and  $E = \mathbb{R}^n$ . Then,

$$R' = \text{End}_{\mathbb{R}}(\mathbb{R}^n) = M_n(\mathbb{R})$$

$$R'' = \text{End}_{M_n(\mathbb{R})}(\mathbb{R}^n) \xleftarrow[r]{rI} \mathbb{R}$$

Suppose  $V =$  vector space.

$V^* = \text{Hom}(V, \mathbb{R})$

Then we have evaluation map  $ev : V \rightarrow V^*$  given by  $v \mapsto (\phi \mapsto \phi(v))$ .

$ev$  is 1-1.

$ev$  is onto iff  $\dim V < \infty$ .

With inspiration from this, we define,

**Definition** (Evaluation map).  $ev : R \rightarrow R''$  given by  $r \mapsto (e \mapsto re)$

We define  $f_r : E \rightarrow E$  given by  $f_r = ev(r)$

**Proposition 13.** a)  $f_r \in R''$

b)  $ev$  is a ring homomorphism.

*Proof.* a)  $f_r(\phi(e)) = r\phi(e) = \phi(re)\phi(f_r(e))$

b)  $ev(r + r') = ev(r) + ev(r'), ev(1) = 1$ .

$$(ev(r))(ev(r'))e = ev(r)(r'e) = rr'e = ev(rr')e$$

□

**Lemma 14** (3.1). Suppose  $E$  is semisimple over  $R$ ,  $e \in E$  and  $f \in R''$

Then  $\exists r \in R$  such that  $re = f(e)$  [i.e.  $f(e) = ev(r)(e)$ ]

*Proof.*  $E$  is semisimple, and  $Re$  is a submodule. Therefore, we can write  $E = Re \oplus F$ .

Define  $\pi : E \rightarrow E$  be projection to  $Re$ .

Then  $\pi \in E' \implies f \circ \pi = \pi \circ f \implies f(e) = f(\pi(e)) = \pi(f(e)) = re$  for some  $r \in R$ . □

We will prove a stronger version of this lemma called the Jacobson Density Theorem.

**Theorem 15** (3.2, Jacobson Density Theorem). Suppose  $E$  is semisimple over  $R$

$e_1, \dots, e_n \in E$

$f \in R''$

Then,  $\exists r \in R$  such that  $re_i = f(e_i) \forall i$ .

Therefore, if  $E$  is finitely generated over  $R'$ , then  $R \rightarrow R''$  is onto.

*Proof.* We use a diagonal trick.

Special Case:  $E$  is simple.

Idea: Apply the lemma on  $E$  with  $\underline{e} = (e_1, \dots, e_n)$  and  $f^n : E^n \rightarrow E^n$  such that  $f(y_1, \dots, y_n) = (f(y_1), \dots, f(y_n))$ .

We need to check that  $f \in R'(R'(E))$  to apply it.

This would imply that  $f^n \in R'(M_n R) \underset{E \text{ simple}}{=} R'(R'(E^n))$

Therefore,  $\exists r$  such that  $r\underline{e} = f^n(\underline{e})$ . This finishes the proof.

For  $E$  semisimple, key idea is  $f^n \in R'(R'(E))$  as above. [Complicated for infinite sums. We avoid.] □

Application:

**Theorem 16** (Burnside's Theorem). Suppose  $k$  is an algebraically closed field.

Take subring  $R$  such that  $k \subset R \subset M_n(k)$

If  $k^n (= E)$  is a simple  $R$ -module, then prove that:

$$R = M_n(k)$$

**Exercise.** Suppose  $D_{2n}$  is the dihedral group of order  $2n$ , aka

$$D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, sr s^{-1} = r^{-1} \rangle$$

Let  $\zeta_n = e^{2\pi i/n} \in \mathbb{C}$

Then we can define a homomorphism  $D_{2n} \rightarrow GL_2(\mathbb{C})$  given by:

$$\begin{aligned} r &\mapsto \begin{bmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{bmatrix} \\ s &\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

This gives us a ring map  $\pi : \mathbb{C}D_{2n} \rightarrow M_2\mathbb{C}$

Prove the following:

- Prove that  $\mathbb{C}^2$  is a simple  $\mathbb{C}D_{2n}$  module [can be done without technology]
- Use Burnside's theorem to show that  $\pi$  is onto.

Note that Burnside's theorem doesn't work if  $k$  is not algebraically closed.

We have:

$$\mathbb{R} \subset \mathbb{C} \subset M_2\mathbb{R}$$

since we can embed  $\mathbb{C}$  into  $M_2\mathbb{R}$ .

$\mathbb{C}$  is a simple  $R$  module, but  $\mathbb{C} \neq M_2\mathbb{R}$

*Proof of Burnside's Theorem.* Step 1: We show that  $\text{End}_R(E) = k$

Note that,  $k \subset \underset{\text{central}}{\text{End}_R(E)} \subset \underset{\text{finite dim}/k}{\overline{\text{End}_k(E)}} \subset \underset{\text{skew field}}{\text{End}_k(E)}$

$\forall \alpha \in \text{End}_R(E), k(\alpha)$  is a field and finite dimensional  $/k$ .

Therefore,  $k(\alpha) = k$  since  $k$  is algebraically closed.

Thus,  $\alpha \in k$ . This finishes Step 1.

Step 2: We show that  $R = \text{End}_k(E)$ .

$\overline{R} \subset \overline{\text{End}_k(E)}$  by hypothesis.

Suppose  $A \in \text{End}_k(E)$ . Let  $e_1, \dots, e_n$  be a  $k$ -basis for  $E = k^n$ .

Density theorem implies:  $\exists r \in R$  such that  $Ae_i = re_i$  for all  $i$ .

Therefore,  $A = r \in R$ . □

**Monday, 9/9/2024**

Today:

Density Theorem

Characters determine representation

Artin-Wedderburn Theorem

Homework due Monday 9/16, Exercises 1-7

Recall Jacobson Density Theorem:

If  $E$  is semisimple over  $R$ ,  $e_1, \dots, e_n \in E$  and  $f \in R''$  then,

$$\exists r \in R \text{ s.t. } f(e_i) = re_i \forall i$$

Recall that  $R''$  is defined as follows:

$$f \in R'' \iff f : E \rightarrow E \text{ s.t. } \forall \phi \in R' = \text{End}_R E, f \circ \phi = \phi \circ f$$

Also recall Burnside's Theorem:

Suppose  $k$  is an algebraically closed field, and  $k \subset R \subset M_n(k)$  are subrings

If  $k^n$  is a simple  $R$ -module, then

$$R = M_n(k)$$

### 3.7 Existence of Projection Operators

**Theorem 17.** Suppose  $E = V_1 \oplus \dots \oplus V_m$ , simple non-isomorphic  $R$ -modules. Then, for any  $i$ , there exists  $r_i \in R$  such that,

$$r_i v = \begin{cases} v, & \text{if } v \in V_i; \\ 0, & \text{if } v \in V_j, i \neq j \end{cases}$$

So, each projection map is just multiplication.

*Proof.* This is a consequence of the density theorem.

Choose nonzero  $e_k \in V_k$ .

Let  $f = \pi_i : E \rightarrow E$  which is a projection on  $V_i$ .

Note that  $f \in R''$  since for all  $\phi \in R', \phi(V_k) \subset V_k$  [Schur's Lemma, non-isomorphic].

Density theorem  $\implies \exists r_i \in R$  such that  $r_i e_k = \pi_i(e_k)$ .

Note that  $V_k = R e_k$  so  $\forall v \in V_k, v = r e_k$ .

So,  $r_i v = r_i r e_k = r \pi_i(e_k) = \pi_i(r e_k) = \pi_i(v)$

Which is what we wanted. □

#### Correction to the Existence of Projection Operators

Suppose  $k$  is a field,  $R$  is a  $k$ -algebra so that  $R$  is semisimple. Suppose  $R$ -module  $E = V \oplus V', \dim_k E < \infty$ .

For all simple  $L \triangleleft V, \forall L' \triangleleft V'$  then  $L \cong L'$

Then,  $\exists r \in R$  such that for all  $e \in E$ ,

$$r e = \begin{cases} e, & \text{if } e \in V; \\ 0, & \text{if } e \in V'; \end{cases}$$

*Proof.* We apply density theorem. Since we have finite dimension, we have:

$$\{e_1, \dots, e_n\} = (k\text{-basis of } V) \cup (k\text{-basis of } V')$$

Let  $\pi_V : E \rightarrow E$  be the projection on  $V$ .

Then,  $\pi_V \in R''$  [the second commutant] since  $\forall \phi \in R', \phi(V) \subset V, \phi(V') \subset V'$ .

Density theorem implies  $\exists r$  such that  $r e_i = \pi_V(e_i)$ .

Then  $\forall a \in k \subset \text{center } R$ ,

$$r(a e_k) = a(r e_k) = a \pi_V(e_k) = \pi_V(a e_k)$$

Therefore,  $r e = \pi_V(r e)$ . □

Question: What is a  $k$ -algebra?

Following Atiyah-McDonald, let  $k$  be a commutative ring [often but not always a field]. Then,

$$R \text{ is a } k\text{-algebra} \stackrel{\text{def}}{\iff} \text{homomorphism } h : k \rightarrow R, h(k) \subset \text{center}(R)$$

**Example.** Any ring is a  $\mathbb{Z}$ -algebra, homomorphism sends  $n$  to  $1 + 1 + \cdots + 1$

$k$  field,  $R \neq 0 \implies k \hookrightarrow R$

$k$ -algebra  $\iff k \subset \text{center}(R)$

**Corollary 18** (3.8). Suppose  $\text{char } k = 0$ ,  $R$  is a  $k$ -algebra,  $E, F$  semisimple over  $R$ , finite dimensional over  $k$ .

For  $r \in R$ , let:

$f_r^E : E \rightarrow E$  be  $f_r^E(e) = re$

$f_r^F : F \rightarrow F$  be  $f_r^F(f) = rf$

If  $\text{Tr}(f_r^E) = \text{Tr}(f_r^F)$  for all  $r \in R$ ,

Then  $E \cong F$  as  $R$ -modules.

*Proof.* Let  $V$  be a simple  $R$ -module.

Suppose  $E = V^n \oplus$  direct sum of simple  $R$ -modules not isomorphic to  $V$

$F = V^m \oplus$  direct sum of simple  $R$ -modules not isomorphic to  $V$

We want to show  $n = m$

Let  $r_v \in R$  be the projection operation from 3.7.

Then,  $\text{Tr}(f_{r_v}^E) = \text{Tr}(r_v \cdot : E \rightarrow E) = \dim_k V^n = n \dim_k V$

Similarly,  $\text{Tr}(f_{r_v}^F) = m \dim_k V \implies n = m$

□

**Corollary 19** (Characters determine representations). Suppose  $k$  is a field and  $\text{char } k = 0$ . Let  $G$  be a finite group. Suppose:

$\rho : G \rightarrow GL_n(k)$

$\rho' : G \rightarrow GL_m(k)$

with  $kG$ -modules  $E = k^n$  over  $\rho$  and  $F = k^m$  over  $\rho'$

If  $\text{Tr}(\rho(g)) = \text{Tr}(\rho'(g))$  for all  $g$ ,

Then  $E \cong F$  as  $kG$ -modules.

Note that, substituting  $g = 1$  gives us:

$\text{Tr}(\rho(1)) = \text{Tr}(\rho'(1)) \implies \text{Tr}(I) = \text{Tr}(I) \implies n = m$ .

**Definition** ((semi)simple rings). Note that if  $R$  is a ring, then  $R$  is a left module as well. We write  ${}_R R$  when we're considering it as a left module, and  ${}_R R_R$  when we are considering a two sided ideal.

$R$  is called a semisimple ring if  ${}_R R$  is a semisimple  $R$ -module.

$R$  is called a simple ring if  $R$  is a semisimple ring, and for all simple  $L, L' \triangleleft_R R \implies L \cong L'$

This means,  ${}_R R = \oplus_{i \in I} L_i$  where  $L_i$  are simple (left) ideals such that  $L_i \cong L_j$  for all  $i, j$ .

Recall that an ideal is simple if it has no proper sub-ideals.

**Example.**  $M_2(\mathbb{H})$  is a simple ring. We can write it as direct sum of two ideals

$$\begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$$

**Example.**  $M_2(\mathbb{H}) \times \mathbb{R}$  is semisimple.

$$\begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix} \times 0 \oplus \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix} \times 0 \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \times \mathbb{R}$$

Artin-Wedderburn generalizes this.

**Theorem 20** (Artin-Wedderburn Theorem). i)  $R$  simple  $\iff R \cong M_n(D)$   
where  $D$  is a skew-field.

ii)  $R$  semisimple  $\iff R \cong R_1 \times \cdots \times R_s$  simple rings.

## Wednesday, 9/11/2024

Today we discuss the Artin-Wedderburn Theorem.

Exercise:  $C_2 = \{1, g\}$ , prove that  $\mathbb{Q}C_2$  is a semisimple ring.

$\mathbb{Q}C_2 = B_1 \oplus B_2$  2-sided ideals

$\mathbb{Q}C_2 \cong \mathbb{Q} \times \mathbb{Q}$ .

**Lemma 21.** Suppose we have a ring  $R$  which is decomposed as a sum of (left) ideals:

$${}_R R = \bigoplus_{i \in I} L_i \quad \text{with } L_i \neq 0$$

Then  $|I| < \infty$ .

*Proof.* Suppose  ${}_R R = \bigoplus_{j \in J} L_j$  where  $L_j$  are ideals. We want to prove that only finitely many are non-zero.

Note that,  $1 = \sum_{j \in J} e_j$ . We use only finitely many elements here, so  $1 = \sum_{i \in I} e_i$  where  $e_i \neq 0, I \subset J, |I| < \infty$ .

For all  $r \in R$  we have  $r = r \cdot 1 = r \sum_{i \in I} e_i = \sum_{i \in I} r e_i \in \sum_{i \in I} L_i$ .

Therefore,  ${}_R R = \bigoplus_{i \in I} L_i$  a finite sum!  $\square$

Now we go to the theorem.

*Proof of Artin-Wedderburn Theorem Part I.* We want to prove:  $R$  simple ring  $\iff R \cong M_n D$  where  $D$  is a skew field.

First, note that  ${}_R R \cong L^n$  where  $L$  is a simple ideal [so no proper sub-ideals]. Therefore,

$$R^{op} \cong \text{End}_R({}_R R) \cong \text{End}_R(L^n) \cong M_n(\underbrace{\text{End}_R L}_{\text{division ring}})$$

Taking transpose,

$$R \cong M_n(\text{End}_R L)^{op} \cong M_n((\text{End}_R L)^{op}) = M_n(D)$$

So we are done with one direction!

The other direction is an exercise. Here are the steps:

$$\text{Step 1: } M_n D = \begin{bmatrix} * & 0 & \cdots & 0 \\ * & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 0 & \cdots & * \\ 0 & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{bmatrix}$$

$$\text{Step 2: Each summand is isomorphic to } D^n = \begin{bmatrix} * \\ * \\ \vdots \\ * \end{bmatrix}$$

Step 3:  $D^n$  is a simple module.  $\square$

**Remark.**  $R$  simple  $\iff R$  artinian,  $R$  has no proper 2-sided ideals. Some definitions forgo the artinian condition, in this case these are called artinian simple rings.

**Lemma 22** (4.2). Suppose  $L$  is a simple ideal and  $M$  is a simple module so that  $L \not\cong M$ . Then  $LM = 0$ .

*Proof.* This is a direct consequence of Schur's lemma. Consider the map  $\phi_m : L \rightarrow M$  given by  $l \mapsto lm$  for  $m \in M$ . Since this can't be an isomorphism, it must be the zero map. Thus,  $lm = 0$ .  $\square$

*Proof of Artin-Wedderburn Theorem Part II. Idea:* Decompose  $R$  as direct sum of simple ideals. Partition the set of simple ideals so that members of a partition are isomorphic to each other, members of a partition are not isomorphic to members of another partition. Direct sum of each partition gives us one  $R_j$ .

Suppose  $R$  is semisimple.

Let  $L_1, \dots, L_s$  be a set of pairwise non-isomorphic simple ideals [meaning  $L_i \not\cong L_j$ ]

So that, for all simple  $L <_R R$ ,  $L \cong L_i$  for some  $i$ .

Let  $B_i = \sum_{L \cong L_i} L$ .

Claim:  $B_i$  is a 2-sided ideal.

Proof of Claim:

$$B_i R \underset{4.2}{=} B_i B_i \subset R B_i \underset{B_i \text{ is a left ideal}}{=} B_i$$

Thus the claim is proven.

Claim: We have a ‘block decomposition of  $R$ ’, meaning,

Proof of Claim:

$${}_R R_R = B_1 \oplus \dots \oplus B_s$$

Subclaim:  $B_i \cap \sum_{j \neq i} B_j = 0$

Proof of Subclaim: Every  $r \in R$ , we have that  $r \in L$  where  $L$  is simple.  $L \subset B_i \implies L \cong L_i$ .  $L \subset \sum_{j \neq i} B_j \implies L \cong B_j$  for some  $j \neq i$  which is not possible.

Now, we go back to the main proof.

We can write  $1 = e_1 + \dots + e_s$ .

Then,  $R_i := (B_i, e_i)$  is a ring!

We have  $R \cong (R_1, e_1) \times \dots \times (R_s, e_s)$ , so we’re done.

The other direction is an exercise. □

## Friday, 9/13/2024

Key idea:

$${}_R R = L^n \implies \text{End}_R R \cong M_n(\text{End}_R L)$$

Note that  $R^{op} \cong \text{End}_R R$  [function composition is written in the opposite direction].

Suppose  $L_1, \dots, L_s$  are non-isomorphic simple  $R$ -ideals.

$L$  simple  $\implies L \cong L_i$ .

Define  $B = \sum_{\text{simple } L \cong L_i} L \triangleleft_R R_R$ .

We can prove that it is a two sided ideals.

Then we can write  $R \cong R_1 \times \dots \times R_s$  simple, where

$R_i = (B_i, e_i)$  [ $e_i$  is the identity in  $B_i$ ].

**Theorem 23** (4.4). Suppose  $E$  is a  $R$ -module.

$$E_i := \sum_{\substack{\text{simple } M \triangleleft E \\ M \cong L_i}} M$$

Then,  $E = \bigoplus_{i=1}^s E_i$

$E_i = e_i E = B_i M$ .

**Corollary 24** (4.5). If  $R$  is semisimple,  $M$  a simple  $R$ -module, then  $M \cong L_i$  for some  $i$ .

**Corollary 25** (4.6). All simple modules of a simple ring are isomorphic.

$$M \cong \oplus L$$

## External Product vs. Internal Product

**Definition** (External Product). If we have [finite] rings  $R_1, \dots, R_s$  we can construct the ring:

$$R_1 \times R_2 \times \dots \times R_s$$

**Definition** (Internal Product). ‘Block Decomposition’: If we have a ring  $R$  and we can write it as sum of 2 sided ideals:

$${}_R R_R = B_1 \oplus \dots \oplus B_s$$

Then we have  $e_j \in B_j$  so that:

$$1 = e_1 + \dots + e_s$$

Then, each  $B_j$  has a ring structure with  $e_j$  as identity. Then,

$$R \cong (B_1, e_1) \times \dots \times (B_s, e_s)$$

Just for clarity:

**Definition** (Direct Sum of Ideals).

$${}_R R_R = B_1 \oplus \dots \oplus B_s$$

If and only if for every  $r \in R$ ,

$$r = b_1 + \dots + b_s$$

where  $b_j \in B_j$  and the expression is unique.

Jim’s Rant: A subring has to have the same identity. So,  $(B_j, e_j)$  is not a subring.

Block Decomposition is not a direct sum of rings!

This is because in category theory, sum refers to the co-product.

**Lemma 26.** Let  $k$  be a field, and let  $D$  be a skew-field which is a  $k$ -algebra such that  $\dim_k D < \infty$ . Then,

- a)  $\forall \alpha \in D$  we have  $k[\alpha]$  is a field.
- b)  $k$  algebraically closed  $\implies D = k$ .

**Example.** If  $k \in \mathbb{R}, D = \mathbb{H}, \alpha \in \mathbb{H} - \mathbb{R}$  then  $k[\alpha] \cong \mathbb{C}$ .

It is not completely obvious since  $k[i + j] \cong \mathbb{C}$  as well.

*Proof.* a)  $D$  is a  $k$ -algebra. Therefore,  $k[\alpha]$  is commutative. We just need to find inverse.

Let  $0 \neq \beta \in k[\alpha]$ . It is enough to prove that for  $\beta \in k[\alpha]$ , multiplication map  $\cdot\beta : k[\alpha] \rightarrow k[\alpha]$  is bijective.

$\cdot\beta$  is a finite dimensional linear transformation so those are true.

- b) For all  $\alpha \in D$  we have:  $k[\alpha] = k$  since  $k$  is closed. So,  $\alpha \in K$ . Thus  $D = k$ . □

**Corollary 27.** Suppose  $G$  is finite. Then,

$$\mathbb{C}G \cong \prod_{i=1}^s M_{n_i}(\mathbb{C})$$

*Proof.* Artin-Wedderburn Theorem plus the previous lemma. □

**Example.** Suppose  $C_n = \langle g \rangle$  cyclic and  $\zeta_n = e^{2\pi i/n}$ . Then,  
 $\mathbb{Q}C_2 \cong \mathbb{Q}_+ \times \mathbb{Q}_-$  where  $g \mapsto (1, -1)$ .  
If  $p$  is prime we can write:  
 $\mathbb{Q}(C_p) \cong \mathbb{Q} \times \mathbb{Q}(\zeta_p)$  where  $g \mapsto (1, \zeta_p)$ .  
 $\mathbb{C}[C_n] \cong \mathbb{C}^n$  where:  
 $g \mapsto (1, \zeta_n, \dots, \zeta_n^{n-1})$   
 $\mathbb{Q}[C_2 \times C_2] \cong \mathbb{Q}^4$  where:

$$(1, g) \mapsto (1, 1, -1, -1)$$

$$(g, 1) \mapsto (1, -1, 1, -1)$$

$\mathbb{R}[Q_8] \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{H}$  where  $\mathbb{R}[Q_8] \twoheadrightarrow \mathbb{R}[C_2 \times C_2]$   
Some other examples:  $\mathbb{Q}[C_n], \mathbb{C}[Q_8], \mathbb{Q}[D_{2n}], \mathbb{R}[D_{2n}], \mathbb{C}[D_{2n}]$

## Representation Theory

Here,  $G$  is a finite group and  $k$  is a field.

Representations	Modules over $kG$	Characters
$\rho : G \rightarrow GL(V)$ where $V$ is a finite dimensional vector space	$V$ is a $kG$ module	$\chi : G \rightarrow k, \chi_\rho(g) = \text{Tr } \rho(g)$

Table 1: Representations, Modules and Characters

## Monday, 9/16/2024

We have:

$$\text{representation} \iff \text{modules over } kG \implies [\iff \text{only if } \text{char } k = 0] \text{ characters.}$$

rep  $\rightarrow kG$ -module

$$\rho \mapsto V_\rho \text{ by } (\sum_g a_g g)v := \sum_g a_g \rho(g)v$$

$$\rho_v \leftarrow V$$

$$\rho_V(g)v := gv$$

Recall the definition of character:

We have the trace map:

$$\text{Tr} : M_n k \rightarrow k$$

Where  $\text{Tr}(a_{ij}) = \sum_j a_{jj}$  [or the sum of eigenvalues]

We have  $\text{Tr}(AB) = \text{Tr}(BA)$  which implies  $\text{Tr}(PAP^{-1}) = \text{Tr}(A)$ .

So, Tr is basis independent. Thus,

$$\text{Tr} : \text{End}_k V \rightarrow k$$

**Definition** (character). Trace is an endomorphism map. This gives us:

$$G \xrightarrow{\rho} GL(V) \xrightarrow{\text{Tr}} k$$

$\chi_\rho$

This is called the character of  $\rho$



There's a correspondence between  $kG$  modules and Representations concepts:

Representations	Modules over $kG$
irreducible	simple
	isomorphism
	direct sum
	Hom
	dual
	tensor product

Table 2: Rep and  $kG$ -mod

#### Irreducible vs Simple

We say irreducible representation, when we on the other hand say simple modules.

Same concept!

#### Isomorphism

Suppose we have two representations:

$$\rho : G \rightarrow GL(V)$$

$$\rho' : G \rightarrow GL(V')$$

We say two representations are isomorphic when:

$$\rho \cong \rho' \stackrel{\text{def}}{\iff} V_\rho \stackrel{\phi}{\cong} V_\rho \stackrel{\phi}{\cong} V_{\rho'} \iff \exists k \text{ isomorphism s.t.} \\ \phi(gv) = g\phi(v)$$

$\phi : V \rightarrow V'$  s.t.  $\forall g \in G$  we have the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ \downarrow \phi & & \downarrow \phi \\ V' & \xrightarrow{\rho'(g)} & V' \end{array}$$

$\phi$  is called the intertwining map.

**Corollary 28.**  $\rho \cong \rho' \implies \chi_\rho = \chi_{\rho'}$

#### Direct Sum

Suppose  $V \oplus W$  is a  $kG$ -module.

$$\rho_{V \oplus W} : G \rightarrow GL(V \oplus W)$$

is given by:

$$\rho_{V \oplus W} = \begin{bmatrix} \rho_V & 0 \\ 0 & \rho_W \end{bmatrix}$$

We also have  $\chi_{V \oplus W} = \chi_V + \chi_W$ .

#### Two Representations

**Definition** (Trivial Representations).

$$\rho : G \rightarrow GL(k)$$

$$g \mapsto 1$$

Is the trivial representation. Also,  $\chi_\rho \equiv 1$ .

**Definition** (Regular Representation). Consider the  $kG$ -module  ${}_kGkG$ . We have:

$$\rho_{kG} : G \rightarrow GL(kG)$$

This is injective.

Note that  $G \curvearrowright G$  by multiplication, this is a free action. For finite group  $G$  with  $|G| = n$ ,  
 $G \hookrightarrow \text{Bijection}(G, G)$  so  $G$  is a subgroup of  $S_n$ . So we have:

$$\begin{array}{c} \text{regular rep.} \\ \curvearrowright \\ G \longrightarrow S_n \longrightarrow GL(k^n) \end{array}$$

With the action of ‘permuting the standard basis’.

Exercise: Compute character of Regular Representation.

We have, in line of the previous theorem:

**Theorem 29** (Maschke’s Theorem). If  $V \subset W$  as  $kG$ -modules and  $\text{char } k \nmid |G|$  then  $\exists V'$  such that  $W = V \oplus V'$

*Proof.* First, find a  $k$ -linear map  $\pi : W \rightarrow V$  such that  $\pi(v) = v$  for all  $v \in V$ .

We average it to make it  $kG$ -linear:

$\pi' : W \rightarrow V$  given by:

$$\pi'(w) := \frac{\sum_g g\pi(g^{-1}w)}{|G|}$$

We have:  $\pi'$  is  $kG$ -linear and  $\pi'(v) = v$

We can take  $V' := \ker \pi$

□

Thus, for  $w \in W$  we can write  $w = \pi'(w) + (w - \pi'(w))$ .

Note that Maschke’s theorem implies  $kG$  is semisimple. Artin Wedderburn implies semisimple  $kG$  module is a direct sum of irreducible modules.

$$\begin{aligned} V &\cong \bigoplus_i n_i V_i \\ \chi_V &= \sum_i n_i \chi_i \end{aligned}$$

Homomorphisms:

Suppose  $V, W$  are  $kG$ -modules, “representations”. Then,

$\text{Hom}_{kG}(V, W)$  is a  $k$ -vector space.

$\text{Hom}_k(V, W)$  is a  $kG$ -module.

we define:  $(gf)v := gf(g^{-1}v)$

i.e.  $((\sum_g a_g g)f)v = \sum_g a_g (gf(g^{-1}v))$

The  $g^{-1}$  inside is needed for associativity:  $(g'g)f = g'(gf)$

Officially this is a functor.

$\text{Hom}_k(-, -) : (kG\text{-mod})^{op} \times kG\text{-mod} \rightarrow kG\text{-mod}$

Special case:

Dual Representation:  $W = k$ . Then,

$V^* = \text{Hom}_k(V, k)$ .

So,  $(gf)(v) = gf(g^{-1}v) = f(g^{-1}v)$

Exercise:  $\chi_{V^*} = ?$

## Wednesday, 9/18/2024

### Tensor Products

Motivation:

Product Structure:  $- \otimes -: kG\text{-mod} \times kG\text{-mod} \rightarrow kG\text{-mod}$  given by  $V \otimes_k W$ .

Group action works diagonally,  $g(x \otimes y) = (gx) \otimes (gy)$ , extended linearly.

Extension of scalars:

$$\mathbb{R}G \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}G$$

Product of Groups:  $k[G \times H] = kG \otimes_k kH$

When for  $k$  a field then modules are vector spaces  $k^m$  and  $k^n$  which are easy:

$$k^n \otimes_k k^m = k^{nm}$$

$$\dim(k^n \otimes_k k^m) = mn$$

$\{e_i\}$  a basis for  $k^n$

$\{f_j\}$  a basis for  $k^m$

Then  $\{e_i \otimes f_j\}$  is a basis for  $k^n \otimes k^m$ .

However, tensor product consists of more than 'pure' tensors.

**Definition** (Tensor Product). Let  $R$  be a commutative ring. Tensor product is a functor:

$$- \otimes_R - : R\text{-mod} \times R\text{-mod} \rightarrow R\text{-mod}$$

$$(A, B) \mapsto A \otimes_R B$$

[Functor meaning if we have homomorphism on the left we will have homomorphisms on the right]

Construction:

Let  $F(A \times B)$  be the free  $R$ -module with basis  $A \times B$ . Then a typical element of the basis is  $(a, b) \in A \times B$ .

Let  $S$  be the sub-module generated by the following:

- 1)  $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$
- 2)  $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$
- 3)  $r(a, b) - (ra, b)$
- 4)  $r(a, b) - (a, rb)$

Then, we define:

$$A \otimes_R B := \frac{F(A \times B)}{S}$$

and write  $a \otimes b$  for the image of  $(a, b)$ .

This means, a typical element of  $A \otimes_R B$  is:

$$\sum_{i=1}^n a_i \otimes b_i \in A \otimes_R B$$

We also have the following relations:

$$(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$$

$$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$$

$$r(a \otimes b) = (a \otimes rb) = (ra \otimes b)$$

**Exercise.**  $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/3 = 0$

**Proposition 30.** Suppose  $A, B, M$  are  $R$ -modules, and

$$\phi : A \times B \rightarrow M \text{ is } R\text{-bilinear}$$

Meaning,

- 1)  $\phi(a_1 + a_2, b) = \phi(a_1, b) + \phi(a_2, b)$
- 2)  $\phi(a, b_1 + b_2) = \phi(a, b_1) + \phi(a, b_2)$
- 3)  $r\phi(a, b) = \phi(ra, b) = \phi(a, rb)$

Then, by definition,

$$\pi : A \times B \rightarrow A \otimes_R B$$

is  $R$ -bilinear.

**Proposition 31** (Universal Property of Tensor Product).  $\pi$  is initial in the category of bilinear maps with domain  $A \times B$ . Meaning, every bilinear map from  $A \times B$  factors through  $\pi$ .

$$\begin{array}{ccc} A \times B & \xrightarrow{\forall \phi \text{ bilinear}} & M \\ \downarrow \pi & \searrow \exists! \bar{\phi} & \\ A \otimes_R B & & \end{array}$$

This diagram commutes

*Proof.* For uniqueness, note that,  $\bar{\phi}(a \otimes b) = \bar{\phi}(\pi(a, b)) = \phi(a, b)$

For existence, define  $\hat{\phi}(a, b) = \phi(a, b)$  where  $\hat{\phi} : F(A \times B) \rightarrow M$ . Then  $\bar{\hat{\phi}}(S) = 0$  so  $\bar{\phi} : A \otimes_R B \rightarrow M$  exists.  $\square$

**Proposition 32** (Rephrasing Universal Property in Terms of Adjoint Functors).

$$\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, \text{Hom}(B, C))$$

*Proof.*

$$f \mapsto (a \mapsto (b \mapsto f(a \otimes b)))$$

$$\begin{array}{ccc} & \text{Hom}(A \otimes -, C) & \\ & \curvearrowleft & \\ R\text{-mod} & & R\text{-mod} \\ & \curvearrowright & \\ & \text{Hom}(A, \text{Hom}(-, C)) & \end{array}$$

$\square$

**Proposition 33.** 1) Commutative  $A \otimes_R B \cong B \otimes_R A$

2) Identity  $R \otimes_R B \cong B$

3) Associative  $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$

4) Distributive  $(\bigoplus_{\alpha} A_{\alpha}) \otimes B \cong \bigoplus_{\alpha} (A_{\alpha} \otimes B)$

5) Functorial  $\begin{pmatrix} f : A \rightarrow A' \\ g : B \rightarrow B' \end{pmatrix} \implies f \otimes g : A \otimes B \rightarrow A' \otimes B'$

6) Exactness Short Exact Sequence  $0 \rightarrow A \xrightarrow{f} B \rightarrow C \rightarrow 0 \implies$  Short Exact Sequence  $0 \rightarrow A \otimes M \xrightarrow{f \otimes 1_M} B \otimes M \rightarrow C \otimes M \rightarrow 0$

7) Right Exactness  $M \text{ } R\text{-mod}, 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \implies$  Exact Sequence  $A \otimes M \rightarrow B \otimes M \rightarrow C \otimes M \rightarrow 0$

Friday, 9/20/2024

## Lang Section 2

### Tensor Product of Representation

Suppose  $V, W$  are  $k$ -vector spaces, then we have  $V \otimes_k W$  is also a  $k$ -vector space. But they all are  $kG$ -modules as well:

$$g(v \otimes w) = gv \otimes gw$$

**Proposition 34.** The character is multiplicative:

$$\chi_{v \otimes w} = \chi_v \chi_w$$

*Proof.* Let  $\{e_i\}$  be a basis for  $V$  and  $\{f_j\}$  a basis for  $w$ .

Suppose  $ge_i = \sum_k a_{ki} e_k$

And  $gf_j = \sum_l b_{lj} f_l$

Then,  $g(e_i \times f_j) = ge_i \times gf_j = \sum_{k,l} a_{ki} b_{lj} e_k \times f_l$

Take  $(k, l) = (i, j)$ .

Then,  $\chi_{v \otimes w}(g) = \sum_{i,j} a_{ii} b_{jj} = \chi_v(g) \chi_w(g)$  □

Consider  $f : G \rightarrow k$ . We have:

$\{1\text{d chars}\} \subset \{\text{simple chars}\} \subset \{\text{chars}\} \subset \{\text{virtual chars}\} \subset \{\text{class functions}\}$

We explain these later.

**Definition.**  $f$  is a character if  $\exists \rho : G \rightarrow GL_k(V)$  such that  $f = \chi_\rho = \text{Tr} \circ \rho$

**Definition.**  $f$  is a class function if  $\forall g, h \in G$  we have  $f(hgh^{-1}) = f(g)$

**Definition.**  $f$  is a virtual character if  $\exists \rho, \rho'$  such that  $f = \chi_\rho - \chi_{\rho'}$

**Definition.**  $f$  is simple (=irreducible) character if  $f = \chi_V$  where  $V$  is a simple  $kG$ -module.

**Definition.**  $f$  is 1-dimensional character if  $f : G \rightarrow k^\times$  is a homomorphism. eg trivial character  $\chi_1(g) \equiv 1$ .

**Proposition 35.** Class Functions are  $k$ -algebras. Virtual characters are a commutative ring.

Now, suppose  $\text{char } k \nmid |G|$ . Then,

$$kG \cong M_{n_1}(D_1) \times \cdots \times M_{n_s}(D_s)$$

Assume  $M_{n_1}(D_{n_1}) = k$ . Then we have the trivial representation:  $ga = a$ .

If  $L_i = D_i^{n_i}$  is a simple  $kG$ -module, then

$\chi_i = \chi_{L_i}$  is a simple characteristics.

We have  $1 = e_1 + \cdots + e_s$  [central non-trivial idempotents].

$\chi_i(e) = \text{Tr}(\text{Id}_{L_i}) = \dim_k L_i = n_i \dim_k D_i$ .

**Example.** Consider  $Q_8 \hookrightarrow \mathbb{H}^\times$ . Then,

$$\chi_{\mathbb{H}}(e) = 4$$

Now, consider  ${}_k G kG \cong \bigoplus_i n_i L_i$ , the ‘regular representation’.  $e_j L_i = 0$  for  $i \neq j$ . Then,

$$\chi_i(e_i) = \chi_i(1) = \chi_i(e) = \dim_k L_i$$

So,  $\text{char } \chi : G \rightarrow k$  extends to  $\chi : kG \rightarrow k$  by  $\sum a_g g \mapsto \sum a_g \chi(g)$ .

If  $V$  is a finitely generated  $kG$ -module, we have

$$V \cong m_1 L_1 \oplus \cdots \oplus m_s L_s$$

where  $m_i \geq 0$ .

**Theorem 36** (2.2, 2.3).  $\chi_v = \sum_i m_i \chi_i : G \rightarrow k$  with  $m_i$  uniquely determined if  $\text{char } k = 0$ .

**Theorem 37** (2.3). Characters Determine Representations: suppose  $\text{char } k = 0$ . Then,

$$V \cong V' \iff \chi_V = \chi_{V'}$$

*Proof.*  $\implies$  : Trace is independent of basis, so this is easy.

$\impliedby$  : We already gave a proof using projection operators. Second Proof: Assume  $\chi_V = \chi_{V'}$ . We decompose:

$$V \cong \oplus m_i L_i, V' \cong \oplus m'_i L_i$$

Note that we have  $\chi_V(e_i) = m_i \dim_k L_i = m'_i \dim_k L_i = \chi_{V'}(e_i)$ . Thus we must have  $m_i = m'_i$ . □

## Representation Ring

$R_k(G) = (\text{virtual char}, +, \times) \cong (\text{virtual rep}, \oplus, \otimes)$ .

Example:  $R_{\mathbb{Q}}[C_2] \cong \frac{\mathbb{Z}[X]}{(X^2-1)}$

**Monday, 9/23/2024**

## Dual Characters

Consider  $\rho : G \rightarrow GL_k(V)$

Dual  $V^* = \text{Hom}_k(V, k)$  is also a representation.

$$(g\phi)(v) = \phi(g^{-1}v)$$

Inverse because we want it to be a left module.

Claim:  $\rho : G \rightarrow GL(V) \rightarrow \rho^* : G \rightarrow GL(V^*)$

$$\rho^*(g) = (\rho(g)^{-1})^T$$

*Proof.*  $\rho^*(g) = (\rho(g^{-1}))^* = \rho(g^{-1})^T$  □

**Corollary 38.** a)  $\chi_{V^*}(g) = \chi_V(g^{-1})$

b)  $\chi_{\text{Hom}(V, W)}(g) = \chi_V(g^{-1})\chi_W(g)$

*Proof.* a follows from the claim.

b: Consider the slant homomorphism:

$$V^* \otimes W \rightarrow \text{Hom}(V, W)$$

$$\sum_i \phi_i \otimes w_i \mapsto \left( v \mapsto \sum_i \phi_i(v) w_i \right)$$

It is an isomorphism since  $V, W$  are both finite dimensional.

$$\chi_{\text{Hom}(V, W)}(g) = \chi_{V^* \otimes W}(g) = \chi_{V^*}(g)\chi_W(g) = \chi_V(g^{-1})\chi_W(g)$$

□

## 1 Dimensional Characters

**Definition.** 1 D representation is a homomorphism  $\rho : G \rightarrow k^\times$

$$\begin{array}{ccc} G & \xrightarrow{\quad} & k^\times \\ & \searrow & \nearrow \\ & G^{ab} & \end{array}$$

Question: What are the 1d representations for  $D_6$ ?

$$D_6 \cong \mathbb{Z}/3 \rtimes \mathbb{Z}/2$$

$$\text{So, } D_6^{ab} \cong \mathbb{Z}/2$$

So, we have  $k_T, k_-$

$$r \mapsto 1$$

$$s \mapsto -1$$

Exercise: Trivial Representation / Idempotent

$$e_T = \frac{\sum_{g \in G} g}{|G|} \in kG$$

$$e_T^2 = e_T$$

$$ge_T = e_T = e_T g$$

$$e_T \in Z(kG)$$

$$kG = (kG)e_T \oplus (kG)(1 - e_T)$$

$$kG \cong k \times \frac{kG}{\langle e_T \rangle}$$

**Lemma 39** (2). Any finite subgroup of  $k^\times$  is cyclic.

*Proof.* Key Fact:  $x^e - 1 \in k[x]$  has at most  $e$  roots [proof: long division].

Note:  $x^2 - 1 \in \mathbb{Z}/8[x]$  has 4 roots. This implies  $\mathbb{Z}/8$  is not a field.

Consider finite abelian  $A < k^\times$

Consider  $e = \text{exponent } A = \inf\{m \geq 1 \mid \forall a \in A, a^m = e\}$

Then,  $\forall a \in A, a^e - 1 = 0$ . From the key fact,  $|A| \leq e \leq |A|$

Thus,  $e = |A|$

□

**Corollary 40.**  $\forall \text{ hom } \rho : G \rightarrow k^\times, \exists \text{ Cyclic } C \text{ such that:}$

$$\begin{array}{ccc} G & \xrightarrow{\quad \rho \quad} & k^\times \\ & \searrow & \nearrow \\ & C & \end{array}$$

Recall only finite subgroup of  $\mathbb{Q}$  is  $\pm 1$ .

$1 - d$   $\mathbb{Q}$  reps of  $G \leftrightarrow$  trivial representation + index 2 subgroups

Now we suppose  $k$  is algebraically closed, eg  $k = \mathbb{C}$ . Then,

$$kG \cong \prod_i M_{n_i}(k)$$

If  $G$  is abelian, then,

$$kG \cong k \times \cdots \times k$$

**Corollary 41** (3).  $k$  is algebraically closed and  $G$  is abelian  $\iff$  all irreducible representations are 1-dimensional.

**Corollary 42.** Let  $|G| = n, k = \mathbb{C}$ .

- a)  $\forall V, \chi_V(G) \subset \mathbb{Q}(\zeta_n)$
- b)  $\forall V, \chi_{V^*}(g) = \overline{\chi_V(g)}$
- c)  $\forall V, W, \chi_{\text{Hom}(V, W)}(g) = \overline{\chi_V(g)} \chi_W(g)$

*Proof.* a) True for 1d representation from the lemma.

$\implies$  True for  $G$  abelian (corollary 3)

$\implies$  True for cyclic  $G$

$\implies$  always true:  $g \in G \implies \langle g \rangle$  cyclic.

$$\chi_\rho(g) = \chi_{\rho|_{\langle g \rangle}}(g)$$

Then,  $\rho : G \rightarrow GL(V)$ , consider  $g \in G$ .

Then  $\rho(g)^n = I \implies \text{Tr}(\rho_V(g)) \in \mathbb{Q}(\zeta_n)$ .

b) Same as (a).

$$\rho^*(g) = (\rho(g)^{-1})^t$$

For 1-dim,  $\rho^* = \bar{\rho}$ .

c)  $\chi_{\text{Hom}(V, W)}(g) = \chi_V(g^{-1}) \chi_W(g) = \overline{\chi_V(g)} \chi_W(g)$

□

## Two Bases for center $kG$

**Definition.**  $g \in G$  is conjugate to  $\sigma \in G$  if  $\exists \tau$  such that,

$$\tau g \tau^{-1} = \sigma$$

Write  $g \sim \sigma$

$$G = \coprod_{G/\sim} [g]$$

$[g] = \{\sigma \in G \mid g \sim \sigma\}$  conjugacy classes

**Proposition 43.**  $\{\sum_{\sigma \in [g]} \sigma\}_{[g] \in G/\sim}$  is a  $k$ -basis for center of  $kG$ .

*Proof.* Clearly these are linearly independent.

$$\alpha = \sum_{\sigma \in G} a_\sigma \sigma \in \text{center}$$

$$\iff \tau \alpha = \alpha \tau \iff \tau \alpha \tau^{-1} = \alpha$$

$$\sigma a_\sigma \tau \sigma \tau^{-1} = \sum a_\sigma \sigma \implies (g \sim \sigma \implies a_g = a_\sigma)$$

□

## Wednesday, 9/25/2024

Lang XVIII, 4

Two bases for  $Z(kG)$

conjugacy classes

primitive central idempotents [ $k$  algebraically closed]

**Exercise.**  $G \twoheadrightarrow Q$ , prove that  $kG \cong kQ \times R$

**Proposition 44** (4.1). Suppose  $\{\sum_{\sigma \in [g]} \sigma\}_{[g] \in G/\sim}$  form a  $\{\frac{k}{\mathbb{Z}}\}$ -basis for  $\{Z(kG)\}$

Consider a ring  $R$ .

**Definition.**  $e \in R$  is a primitive central idempotent if:

$e$  is a central idempotent [ $e^2 = e, e \in Z(R)$ ]

$e = e' + e''$  with  $e', e''$  central idempotent  $\implies \{e', e''\} = \{0, e\}$



Then,  $kG \ni 1 = e_1 + \cdots + e_s, kG \cong \prod M_{d_i}(D_i)$

$e_i \rightarrow (0, \dots, 0, 1, 0, \dots, 0)$

Now suppose  $n = |G|$

We have irreducible representations  $L_1, \dots, L_s$  and degrees  $d_1, \dots, d_s$  then  $L_i \cong D_i^{d_i}$ . We have irreducible characteristics  $\chi_1, \dots, \chi_s$  and primitive central idempotents (p.c.i.)  $e_1, \dots, e_s$

Facts: (\*):  $kGkG = \bigoplus_i d_i L_i$

(\*\*):  $\alpha \in kG, i \neq j$  then  $\chi_j(e_i \alpha) = 0$  since  $e_i L_j = 0, \chi_i(e_i \alpha) = \chi_i(1\alpha) = \chi_i(\alpha)$

We have:  $\chi_{\text{reg}} = \sum_i d_i \chi_i$

**Proposition 45** (4.3).  $\chi_{\text{reg}}(g) = \begin{cases} n, & \text{if } g = e; \\ 0, & \text{if } g \neq e \end{cases}$

*Proof.*  $\chi_{\text{reg}}(g) = \text{Tr}(\cdot g : kG \rightarrow kG)$

Thus,  $\chi_{\text{reg}}(e) = \text{Tr}(I) = n$

If  $g \neq e$  note that  $G$  has  $\{\sigma_1, \dots, \sigma_n\}$  and  $\rho_{\text{reg}}(g)(\sigma_j) = g\sigma_j \neq \sigma_j$  for all  $j$ . So, there is nothing in the diagonal matrix and trace is 0.  $\square$

Motivation for  $k$  algebraically closed:

Consider  $\mathbb{Q}C_3 \cong \mathbb{Q} \times \mathbb{Q}(\zeta_3)$ . We only have primitive central idempotents,  $1 = e_1 + e_2$ .

But the center has dimension 3:  $\dim_{\mathbb{Q}}(Z(\mathbb{Q}C_3)) = 3$ .

Assume  $k$  is algebraically closed.

Claim:  $k$  algebraically closed,  $D$  skew field,  $k < Z(D)$ ,  $\dim_k D < \infty$  implies  $k = D$

Now,  $kG \neq \prod M_{d_i}(k)$

Consider primitive central idempotents  $e_1, \dots, e_s$  for a basis.

$n = \sum_{i=1}^s d_i^2$

e.g.  $S_3 = D_6$ .  $s = ?$   $d_1, d_2, d_3 = ?$

We have representatives of conjugacy classes:  $(1), (12), (123)$ .

$s = 3, 6 = 1^2 + 1^2 + 2^2$

Char. Table:

	(1)	(12)	(123)
$\chi_1$	1	1	1
$\chi_2$	1	-1	1
$\chi_3$	2	0	-1

Table 3: characteristic table

We have  $\mathbb{C}S_3 = \mathbb{C}_+ \times \mathbb{C}_- \times M_2\mathbb{C}$

Our representatives are  $(1), (12), (123), (1234), (12)(34)$

$d_i = 1, 1, 2, 3, 3$

Goal: Express the p.c.i basis in terms of conjugacy class basis.

**Corollary 46** (4.2). If  $k$  is algebraically closed,

the number of conjugacy classes =  $\dim_k Z(G)$  = number of irreducible representation =  $s$

**Proposition 47** (4.4).  $k$  algebraically closed, then

$$e_i = \frac{d_i}{n} \sum_{\tau \in G} \chi_i(\tau^{-1}) \tau$$

*Proof.* Let  $e_i = \sum_{\tau \in G} a_{\tau} \tau$ .

We compute  $\chi_{\text{reg}}(e_i \tau^{-1})$  in two ways.

1:  $\chi_{\text{reg}}(e_i \tau^{-1}) = \chi_{\text{reg}}(\sum a_{\sigma} \sigma \tau^{-1}) = \sum a_{\sigma} \chi_{\text{reg}}(\sigma \tau^{-1}) = a_{\tau} n$

2:  $\chi_{\text{reg}}(e_i \tau^{-1}) \stackrel{(*)}{=} \sum_j d_j \chi_j(e_i \tau^{-1}) \stackrel{(**)}{=} d_i \chi_i(e_i \tau^{-1}) = d_i \chi_i(\tau^{-1})$

Thus,  $a_{\tau} n = d_i \chi_i(\tau^{-1}) \implies a_{\tau} = \frac{d_i}{n} \chi_i(\tau^{-1})$   $\square$

**Corollary 48** (4.5). Let  $m = \exp G$ . Then,

$$e_i \in \frac{1}{n} [\mathbb{Z}[\zeta_m]G] \subset \frac{1}{n} [\mathbb{Z}[\zeta_n]G]$$

**Corollary 49** (4.6).  $\text{char } k \nmid d_i$

*Proof.* If not,  $\text{char } k \mid d_i$  then  $e_i = 0$  which is a contradiction.  $\square$

**Corollary 50** (4.7).  $\chi_1, \dots, \chi_s$  are linearly independent over  $k$ . In fact they form a basis for the class functions  $f : G \rightarrow k$ .

*Proof.* Suppose  $0 = \sum a_i \chi_i$ .

$$\text{Then } 0 = \sum a_i \chi_i(e_j) = a_j \chi_j(e_j) = a_j d_j \implies a_j = 0 \quad \square$$

Then  $\dim_k(\text{class functions}) = \text{number of conjugacy classes} = s$ .

## Friday, 9/27/2024

Review:

$$e_i = \frac{d_i}{n} \sum_{\sigma \in G} \chi_i(\sigma) \sigma^{-1} \in kG \quad (*)$$

Is a primitive central idempotent.

$$\chi_{\text{reg}} = \chi_{kG} = \sum_i d_i \chi_i$$

$$\sigma = 1, n = \sum_i d_i^2 \\ d_i \mid n$$

$$\sum_{\sigma \in G} \chi_i(\sigma) \chi_j(\sigma^{-1}) = n \delta_{ij}$$

$$\sum_{i=1}^s \chi_i(\sigma) \chi_i(\tau^{-1}) = \begin{cases} \frac{n}{|\sigma|}, & \text{if } \tau = \sigma; \\ 0, & \text{otherwise.} \end{cases}$$

If  $G = S_3$  then:

	(1)	(12)	(123)	
$\chi_1$	1	1	1	6
$\chi_2$	1	-1	1	6
$\chi_3$	2	0	-1	6
	6	2	3	

Table 4: Characteristic Table of  $S_3$

$$0 = \chi_{\text{reg}}(123) = 1\chi_1(123) + 1\chi_2(123) + 2\chi_3(123)$$

$$k = \mathbb{C}, \chi(\sigma^{-1}) = \overline{\chi(\sigma)}$$

End of review

$X(G) = \{\text{class functions } f : G \rightarrow k\}$  so that  $f(\tau\sigma\tau^{-1}) = f(\sigma)$ .

**Definition** (Perfect Pairing). A perfect pairing of  $k$  vector space is a  $k$ -bilinear map  $\beta : V \times W \rightarrow k$  such that  $\exists$  basis  $\{v_i\}, \{w_j\}$  such that

$$\beta(v_i, w_j) = \delta_{ij}$$

$$\iff \text{Ad}_b : V \rightarrow W^*$$

$$v \mapsto (w \mapsto \beta(v, w))$$

**Theorem 51** (4.9).

$$X(G) \times Z(kG) \rightarrow k$$

$$(f, \alpha) \mapsto f(\alpha)$$

is a perfect pairing.

*Proof.* Dual basis:  $\left\{ \frac{1}{d_i} \chi_i \right\}, \{e_j\}$

$$\frac{1}{d_i} \chi_i(e_j) = \delta_{ij}$$

□

**Corollary 52** (4.8). Suppose  $k$  is algebraically closed,  $\text{char } k = 0$ . Then  $d_i = \dim_K L_i \mid n$

We need integrality theory (M502)

See Lang p 334.

$A$  subring of  $B$ ,  $\alpha \in B$ .

$\alpha$  is integral over  $A$  if  $\exists$  monic  $f(x) \in A[x]$  such that  $f(\alpha) = 0$ .

$\alpha \in \mathbb{Q} \implies \alpha \text{ int}/\mathbb{Z} \iff \alpha \in \mathbb{Z}$

Condition (\*\*):  $\alpha$  being integral is equivalent to the existence of a faithful  $A[\alpha]$ -module  $M$  which is finitely generated as  $A$ -module.

Faithful means:  $\forall \beta \in A[\alpha], \beta M = 0 \iff \beta = 0$ .

In other words,  $A[\alpha] \hookrightarrow \text{End}_{A[\alpha]}(M)$ .

Condition (\*\*)  $\iff \alpha \text{ int}/A$ . This is proved by a determinant trick.

Applying (\*\*) on  $A = \mathbb{Z}, \frac{n}{d_i} \in \mathbb{Q}$ ,

Multiplying  $e_i = \frac{d_i}{n} \sum_{\sigma \in G} \chi_i(\sigma) \sigma^{-1} \in kG$  with  $e_i$ ,

$$e_i = e_i^2 = \frac{d_i}{n} \sum_{\sigma} \chi_i(\sigma) \sigma^{-1} e_i$$

$$\frac{n}{d_i} e_i = \sum_{\sigma} \chi_i(\sigma) \sigma^{-1} e_i$$

$$M = \mathbb{Z} \langle \zeta_n^j \sigma e_i \rangle_{j, \sigma \in G} \text{ is a } \mathbb{Z} \left[ \frac{n}{d_i} \right] \text{-module}$$

We are done by (\*\*).  $d_i \mid n$ .

## Orthogonality, Lang XVIII, 5, Serre 2.3

**Theorem 53.** Suppose we have  $\langle, \rangle : X(G) \times X(G) \rightarrow k$  by:

$$\langle f, g \rangle = \frac{1}{n} \sum_{\sigma \in G} f(\sigma) g(\sigma^{-1})$$

is a nonsingular symmetric form and  $\{\chi_1, \dots, \chi_s\}$  forms an orthonormal basis.

*Proof.* Symmetric form,  $k$ -bilinear  $\langle f, g \rangle = \langle g, f \rangle$

Apply  $\chi_j$  to (\*)

$$d_i \delta_{ij} = \chi_j(e_i) = \frac{d_i}{n} \sum_{\sigma} \chi_i(\sigma) \chi_j(\sigma^{-1})$$

□

Remark: Irreducibility criterion:  $\langle \chi, \chi \rangle = 1 \iff \chi$  irreducible.

$$(\sum_i a_i \chi_i, \sum_i a_i \chi_i) = \sum_i a_i^2$$

**Proposition 54** (I.7, Serre p20). a)  $\sum_{i=1}^s \chi_i(\sigma) \chi_i(\sigma^{-1}) = \frac{n}{|\sigma|}$

b)  $[\sigma] \neq [\tau] \implies \sum_{i=1}^s \chi_i(\sigma) \chi_i(\tau^{-1}) = 0$

*Proof.* Consider the characteristic function for  $[\sigma]$ :

$f_\sigma = 1$  on  $[\sigma]$  and 0 everywhere else.

$f_\sigma = \sum_i \lambda_i \chi_i$ .

$\lambda_j = \langle f_\sigma, \chi_j \rangle = \frac{1}{n} \sum_{\tau \in G} f_\sigma(\tau) \chi_j(\tau^{-1}) = \frac{||[\sigma]||}{n} \chi_j(\sigma^{-1})$

$f_\sigma(-) = \sum_i \frac{||[\sigma]||}{n} \chi_i(\sigma^{-1}) \chi_i(-)$

□

This finishes the proof.

## Monday, 9/30/2024

### Serre Ch 4

What about representations of infinite groups?



**Definition** (Topological Group). Topological Group is a group  $(G, \cdot)$  such that  $G$  has a topology so that:

$$G \times G \rightarrow G$$

$$(g, h) \mapsto gh^{-1}$$

is continuous.

**Definition** (Lie Group). Lie Group is a topological lie group  $G$  where  $G$  is a smooth manifold and  $(g, h) \mapsto gh^{-1}$  is smooth.

Compact Lie Groups:

Torus  $T^r = S^1 \times \dots \times S^1$

$O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = I\}$

$U(n) = \{A \in M_n(\mathbb{C}) \mid AA^* = I\}$

Exceptional:  $G_2, F_4, E_6, E_7, E_8$

We also have compact groups are not lie groups;

$(\mathbb{Z}/p)^\infty = \prod \mathbb{Z}/p\mathbb{Z}$

$p$ -adic  $\mathbb{Z}_p = \lim \mathbb{Z}/p^n\mathbb{Z}$

Serre Ch 4 says that:

**Representation of compact groups is almost the same as finite group!**

We need Haar Measure.

**Proposition 55.** For locally compact Hausdorff topological group  $G$  there exists a unique Haar Measure:

$$\begin{aligned} dt : \{\text{Borel Subsets of } G\} &\rightarrow [0, 1] \\ B &\mapsto \int_B dt = \int_G \chi_B(t) dt \end{aligned}$$

So that  $\int_G dt = 1$  and  $dt$  is translation invariant:

$$\int_G f(t) dt = \int_G f(gt) dt = \int_G f(tg) dt$$

**Example.** If  $G$  is finite:

$$\int_G f \, dt = \frac{1}{|G|} \sum_{g \in G} f(g)$$

$$G = S^1$$

$$\int_{S^1} dt = 1 \quad \int_{\text{quarter circle}} dt = \frac{1}{4}$$

**Theorem 56** (Maschke's Theorem, Peter-Weyl Theorem). Let  $G$  be a compact group,  $k = \mathbb{C}$ . Let  $W \subset V$  be a subrepresentation of  $\rho : G \rightarrow GL(V)$ . Then  $\exists$  subrepresentation  $W'$  such that  $V = W \oplus W'$ .

*Proof.* Let  $\langle \cdot, \cdot \rangle' : V \times V \rightarrow \mathbb{C}$  be any inner product.

We define a new inner product by averaging this inner product.

$$\langle v, w \rangle = \int_G \langle \rho(t)v, \rho(t)w \rangle' dt$$

This gives us a  $G$ -invariant inner product.

We take  $W'$  to be orthogonal to  $W$  w.r.t. this inner product. □

**Corollary 57.** Any representation is the direct sum of irreducible representation (unique upto multiplicity).

Consider the regular representation  $L^2(G) \cong \bigoplus_i d_i L_i$ .

We don't have characteristic of regular representation

We don't have a group ring

Suppose  $G = S^1, n \in \mathbb{Z}$

$\chi_n : S^1 \rightarrow \mathbb{C}^\times$

$\chi_n(z) = z^n$  gives us  $\mathbb{C}_n$

$L^2(S^1) = \bigoplus \mathbb{C}_n$

Representation Ring:  $R(S^1) \ni \rho - \rho'$

$R(S^1) = \mathbb{Z}[\chi_1, \chi_1^{-1}]$ ,  $\chi_n = \chi_1 \otimes_G \cdots \otimes_G \chi_1$

Then,  $R(S^1 \times \cdots \times S^1) = \mathbb{Z}[\alpha_1, \alpha_1^{-1}, \dots, \alpha_r, \alpha_r^{-1}]$  where:

$$S^1 \times \cdots \times S^1 \xrightarrow{\text{proj}} S^1 \hookrightarrow \mathbb{C}^\times$$

Consider  $T^n \subset U(n)$

$\Sigma_n = U(n)/T^n$

$R(U(n)) \hookrightarrow R(T^n)$ .

image  $\mathbb{Z}[\sigma_1, \dots, \sigma_{n-1}, \sigma_n, \sigma_n^{-1}]$  where

$\sigma_i$  is the  $i$ -th elementary symmetric function in  $\alpha_1, \dots, \alpha_n$ .

## Infinite Discrete Groups

$C_\infty = \langle x \rangle$

$\mathbb{Z}C_\infty = \mathbb{Z}[x, x^{-1}]$  the Laurent Polynomial Ring.

We can think of it like the localization of  $\mathbb{Z}[x]$  at  $x$  [aka  $x^{-1}\mathbb{Z}[x]$ ] or  $\mathbb{Z}[x, x^{-1}] \subset \mathbb{Q}(x)$  the rational function field.

This is not a super well behaved domain since it has dimension 2.

$\mathbb{Q}[x, x^{-1}]$  is a Euclidean domain and hence a PID. But not  $\mathbb{Z}[x, x^{-1}]$ .

## Some Conjectures about Torsion-Free Groups

Torsion free: If  $g \in G - \{e\}$ ,  $n > 0$  then  $g^n \neq e$ .

**Proposition 58** (Farrell-Jones Conjecture). for  $R = \mathbb{Z}$  or a field, all finitely generated projective  $\mathbb{R}G$ -modules are stably-free.

Projective means it's a summand of a free module.

$P$  is stably free if  $P \oplus \text{free}$  is free.

It has been proved for the torsion-free groups we care about, but not generally.

**Proposition 59** (Kaplansky Idempotent Conjecture). Suppose  $R$  is an integral domain. Then the only idempotents in  $RG$  are 0 and 1.

**Proposition 60** (Zero Divisor Conjecture). Suppose  $R$  is an integral domain. Then  $RG$  has no zero divisor.

**Proposition 61** (Embedding Conjecture). Suppose  $R$  is an integral domain. Then  $RG$  is a subring of a skew field.

We have Embedding Conjecture  $\implies$  Zero Divisor Conjecture  $\implies$  Kaplansky Idempotent Conjecture

**Proposition 62** (Unit Conjecture). Suppose  $k$  is a field. Then,

$$(kG)^\times = \langle k^\times, G \rangle$$

## Wednesday, 10/2/2024

Serre Chapter 5

Examples

$k = \mathbb{C}$ : Use characters.

5.1:  $C_n = \langle r \rangle, \zeta_n = e^{2\pi i/n}$ .

$n = \# \text{conjugacy classes} \implies n = s$  irreducible representations.

$C_n$  is abelian  $\implies$  all irreducible representation (=char) is one dimensional.

$$\chi : C_n \rightarrow \mathbb{C}^\times$$

$$\chi(r)^n = \chi(r^n) = \chi(e) = 1$$

Irreducible representation  $\chi_h(r) = \zeta_n^h$ . We have characters  $\chi_0, \chi_1, \dots, \chi_{n-1}$ .

$$\chi_h \chi_{h'} = \chi_{h+h' \pmod n}$$

Representation Ring  $\mathbb{Z}[\text{characters}] = \mathbb{Z}[\chi_1] \cong \mathbb{Z}[x]/(x^n - 1)$ .

Trivial character is 1 in  $R(G)$ .

$$\begin{aligned} \phi : \mathbb{C}[C_n] &\rightarrow \mathbb{C} \times \dots \times \mathbb{C} \\ r &\mapsto (\rho^0, \rho^1, \dots, \rho^{n-1}) \end{aligned}$$

$$\Phi : \mathbb{Q}[C_n] \rightarrow \prod_{d|n} \mathbb{Q}(\zeta_d)$$

a

Question: How to justify that  $\phi$  and  $\Phi$  are isomorphisms?

Answer: CRT

For a non-abelian group  $G$ , recall that:

# of 1d rep =  $|G^{ab}| = |G/[G, G]|$

# of irreducible rep = # of conjugacy classes.

Suppose  $d_i = \dim_{\mathbb{C}} L_i$  then  $n = d_1^2 + \dots + d_s^2$  and  $d_i \mid |G|$ .

5.1 Dihedral Group  $D_{2n}$  (order  $2n$ )

Recal,

$$D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, srs = r^{-1} \rangle$$

isometries of a regular  $n$ -gon.

Here,  $(sr s^{-1})^k = s r^k s^{-1}$  so  $s r^k s^{-1} = r^{-k}$ . Also,  $r^k s r^{-k} = r^{2k} s$ .

Conjugacy classes are given by the following:

$$\begin{array}{cc} \{e\} & \{s\} \\ \{r, r^{-1}\} & \{r^2 s\} \\ \{r^2, r^{-2}\} & \{r^4 s\} \\ & \{r^6 s\} \end{array}$$

We have split based on whether  $n$  is even or odd.

$$\begin{array}{cc} n \text{ odd} & n \text{ even} \\ \{e\} & \{e\} \\ \{r, r^{-1}\} & \{r, r^{-1}\} \\ \vdots & \vdots \\ \{r^{\frac{n-1}{2}}, r^{-\frac{n-1}{2}}\} & \{r^{\frac{n-2}{2}}, r^{-\frac{n-2}{2}}\} \\ \{s, rs, r^2 s, \dots, r^{n-1} s\} & \{r^{\frac{n}{2}}\} \\ & \{r, r^2 s, \dots, r^{n-1} s\} \\ & \{rs, r^3 s, \dots, r^{n-2} s\} \end{array}$$

So, for  $n$  odd:

# of conjugacy class is  $\frac{n+3}{2}$

$$D_{2n}^{ab} = \{1, \bar{s}\} \cong C_2$$

$$Z(D_{2n}) = \{e\}$$

For  $n$  even,

# of conjugacy classes is  $\frac{n+6}{2}$

$$D_{2n}^{ab} = \{1, \bar{s}, \bar{r}, \bar{r}\bar{s}\} \cong C_2 \times C_2$$

1-dim representations:

$n$  odd implies we have representations  $\mathbb{C}_+, \mathbb{C}_-$

$$\chi_{\pm}(r) = 1, \chi_{\pm}(s) = \pm 1$$

$n$  even implies we have representations  $\mathbb{C}_{++}, \mathbb{C}_{+-}, \mathbb{C}_{-+}, \mathbb{C}_{--}$

$$\varepsilon_r = \pm 1, \varepsilon_s = \pm 1$$

$$\chi_{\varepsilon_r \varepsilon_s}(r) = \varepsilon_r \text{ and } \chi_{\varepsilon_r \varepsilon_s} = \varepsilon_s$$

2-dim representations:

$$\rho^h : D_{2n} \rightarrow GL_2(\mathbb{C})$$

$$\rho^h(r) = \begin{bmatrix} \zeta_n^h & 0 \\ 0 & \zeta_n^{-h} \end{bmatrix}$$

$$\rho^h(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

[Induced from  $C_n$ -representation  $\mathbb{C}_h$  later]

For  $0 < h < \frac{n}{2}$  it is irreducible [homework].

$$\chi_h(r^k) = e^{2\pi i h k / n} + e^{-2\pi i h k / n} = 2 \cos \frac{2\pi h k}{n}$$

$$\chi_h(r^k s) = 0$$

Since characters determine representation, we have  $\rho_h \cong \rho_{-h} = \rho_{n-h}$ .

Also, for  $0 < h < \frac{n}{2}$  the representations are distinct.

We have all irreducible 2-dim representations.

Remark:  $\exists$  real representations  $D_{2n} \rightarrow GL_2(\mathbb{R})$  [isometries in  $\mathbb{R}^2$ ]. Then,

$$\hat{\rho}^h(r) = \begin{bmatrix} \cos \frac{2\pi h}{n} & -\sin \frac{2\pi h}{n} \\ \sin \frac{2\pi h}{n} & \cos \frac{2\pi h}{n} \end{bmatrix}$$

$$\hat{\rho}(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We have  $\chi_h = \hat{\chi}_h$  and thus  $\rho_h \cong \hat{\rho}_h$

## Friday, 10/4/2024

Serre 5.4

Suppose  $G = D_{2n} \times C_2$ .

Then,  $\mathbb{C}G = \mathbb{C}D_{2n} \otimes_{\mathbb{C}} \mathbb{C}C_2 = (\mathbb{C}D_{2n})_+ \times (\mathbb{C}D_{2n})_-$ .

Twice as many irreducible representation as  $D_{2n}$ . 5.7 and 5.8

We have the following exact sequence:

$$1 \rightarrow A_4 \rightarrow S_4 \xrightarrow{\text{sign}} \{\pm 1\} \rightarrow 1$$

We have  $|S_4| = 24 = 4!$ ,  $|A_4| = 12$ .

$\left\{ \begin{smallmatrix} S_4 \\ A_4 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} S_4 \\ \text{o.p} \end{smallmatrix} \right\}$  isometries of a tetrahedron.

Conjugacy classes (c.c.) in  $\left\{ \begin{smallmatrix} S_4 \\ A_4 \end{smallmatrix} \right\}$  are  $\begin{matrix} (1), (12), (12)(34), (123), (1234) & s = 5 \\ (1), (12)(34), (123), (213) & s = 4 \end{matrix}$

Interestingly, not all 3-cycles are conjugates in  $A_4$ . For example,  $(123) \not\sim (124)$ .  
Intuition: we need to swap 3 and 4, but in  $A_4$  we need something else because swapping 3 and 4 is odd.

Also:  $A_4$  is not simple [even though  $A_5, A_6$  etc are].

$S_4 = C_2 \times C_2 \rtimes S_3$

$A_4 = C_2 \times C_2 \rtimes C_3$ .

Also:  $S_4^{ab} = C_2$

$A_4^{ab} = C_3$

Then,  $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$

$12 = 1^2 + 1^2 + 1^2 + 3^2$

$\mathbb{C}[A_4] = \underbrace{\mathbb{C} \times \mathbb{C} \times \mathbb{C}}_{C_3\text{-quotient}} \times \underbrace{M_3(\mathbb{C})}_{\text{geometry}}$

$\mathbb{C}[S_4] = \underbrace{\mathbb{C} \times \mathbb{C}}_{C_2\text{-quotient}} \times \underbrace{M_2\mathbb{C}}_{\text{geometry}} \times \underbrace{M_3\mathbb{C}}_{\text{geometry}} \times \underbrace{M_3\mathbb{C}}_{\text{geom} \otimes_{\mathbb{C}} \mathbb{C}_{\text{sign}}}$

### Chapter 6

Suppose we have a finite group  $G$  and  $(\text{char } k, |G|) = 1$ . Then  $kG$  is semisimple.

**Proposition 63** (10). Let  $A$  be semisimple ring. Suppose  $L_1, \dots, L_s$  are simple, non-isomorphic  $kG$ -modules such that  $\forall$  simple  $L$  we have  $L \cong L_i$  for some  $i$ . Then,

$$A \xrightarrow{\text{mul}} \prod \text{End}_A L_i$$

Corollary:  $t < s$  implies:

$$A \rightarrow \prod_{i=1}^t \text{End}_A L_i$$

is onto.

6.5:

Review: Corollary 2: if  $k$  is algebraically closed and  $\text{char } k = 0$  and  $d = \dim_k L$  where  $L$  is a simple  $kG$  module, then

$$d \mid |G|$$

We strengthen this.

**Proposition 64** (17). Let  $Z = Z(G)$  be the center of  $G$ . Then,

$$d \mid \frac{|G|}{|Z|}$$

*Proof.* Let  $\rho : G \rightarrow GL(L)$  be an irreducible representation and  $d = \dim$ . Define homomorphism  $\lambda : Z \rightarrow k^\times$  such that:

$$\rho(s) = \lambda(s) \text{id}$$



$\forall m \geq 1$  let  $\rho^m : G \times \cdots \times G \rightarrow GL(L \otimes \cdots \otimes L)$  which is irreducible.  
Then we have  $\lambda^m : Z \times \cdots \times Z \rightarrow k^\times$  with:

$$(s_1, \dots, s_m) \mapsto \lambda(s_1 \cdots s_m)$$

Let  $H = \{(s_i) \in Z^m \mid s_1 \cdots s_m = 1\} < Z^m < G^m$ .

$H \cong Z^{m-1}$  and  $H \subset \ker \rho^m$ .

Then  $\overline{\rho^m} : G^m/H \rightarrow GL(L \otimes \cdots \otimes L)$  irreducible.

Therefore,  $\forall m, d^m \mid \left| \frac{G^m}{H} \right| = \frac{|G|^m}{|Z|^{m-1}}$  which implies by taking  $m$  big enough that  $d \mid \frac{|G|}{|Z|}$ .  $\square$

## Tensor Product for Non-Commutative Rings

Suppose  $R$  is a non-commutative ring. Then, tensor product is a functor

$$- \otimes_R - : \begin{matrix} \text{mod } R \\ \text{right mod} \end{matrix} \times \begin{matrix} R \text{ mod} \\ \text{left mod} \end{matrix} \rightarrow \text{Ab}$$

$$\begin{aligned} A_R \otimes_R {}_R B &\ni a_1 \otimes b_1 + \cdots + a_k \otimes b_k \\ (a + a') \otimes b &= a \otimes b + a' \otimes b \\ a \otimes (b + b') &= a \otimes b + a \otimes b' \\ ar \otimes b &= a \otimes rb \end{aligned}$$

**Exercise.** Formulate adjoint proposition:

$$\text{Hom}_?(A \otimes B, C) \cong \text{Hom}_?(A, \text{Hom}_?(B, C))$$

**Definition** (Induced module). : Suppose  $k$  is a field and  $H < G$ . Then,

$$\text{Ind}_H^G : kH\text{-mod} \rightarrow kG\text{-mod}$$

$$\text{Ind}_H^G W = kG \otimes_{kH} W$$

eg. Suppose  $H = C_n = \langle r \mid r^n = 1 \rangle$  and  $G = D_{2n} = \langle r, s \mid r^n = 1 = s^2; srs = r^{-1} \rangle$ .  
If  $W = \mathbb{C}$  we have  $H \rightarrow \mathbb{C}^\times$  by  $r \mapsto \zeta_n$ .

$$V = \mathbb{C}D_{2n} \otimes_{\mathbb{C}[C_n]} \mathbb{C}_1 = (\mathbb{C}[C_n] \oplus s\mathbb{C}[C_n]) \otimes_{\mathbb{C}[C_n]} \mathbb{C}_1$$

$\mathbb{C}$ -basis of  $V$  is  $1 \otimes 1, s \otimes 1$ .

$$\text{Recall } r \mapsto \begin{bmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{bmatrix}, s \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$s(1 \otimes 1) = s \otimes 1$$

$$s(s \otimes 1) = s^2 \otimes 1 = 1 \otimes 1$$

$$r(1 \otimes 1) = r1 \otimes 1 = \zeta_n \otimes 1 = \zeta_n(1 \otimes 1)$$

$$r(s \otimes 1) = rs \otimes 1 = sr^{-1} \otimes 1 = s \otimes \zeta_n^{-1}1 = \zeta_n^{-1}(s \otimes 1)$$

## Monday, 10/7/2024

**Exercise.** Work out the representation theory of  $G = C_7 \rtimes_2 C_3 = \langle r, s \mid r^7 = 1, s^3 = 1, srs^{-1} = r^2 \rangle$ .

Meaning: find an isomorphism  $\mathbb{C}G \xrightarrow{\cong} M_{d_i} \mathbb{C}$

Suppose we have a (most likely non-commutative) ring  $R$  and

A tensor product functor  $- \otimes_R - : \text{mod-}R \times R\text{-mod} \rightarrow \text{Ab}$

**Proposition 65** (Universal Property). Suppose  $A$  is a right  $R$ -module and  $B$  is a left  $R$ -module and  $G$  is an abelian group.

$\pi : A \times B \rightarrow G$  is  $R$ -balanced. Meaning:  $\pi$  is  $\mathbb{Z}$ -bilinear and  $\pi(ar, b) = \pi(a, rb)$ .

There exists an  $R$ -balanced  $\pi : A \times B \rightarrow A \otimes_R B$  which is initial.

$$\begin{array}{ccc}
A \times B & & \\
\downarrow \pi & \searrow \forall R\text{-balanced} & \\
A \otimes_R B & \xrightarrow{\exists! \mathbb{Z}\text{-hom}} & G
\end{array}$$

Construction:

$$A \otimes_R B := \frac{F(A \times B)}{T}$$

Where  $F(A \times B)$  is the free abelian group with basis of  $A \times B$ . We write  $F(A \times B) = \mathbb{Z}[A \times B]$ .

$T$  is the subgroup generated by  $(a + a', b) - (a, b) - (a', b), (a, b + b') - (a, b) - (a, b'), (ar, b) - (a, rb)$ .

Main thing to remember:

$$\boxed{ar \otimes b = a \otimes rb}$$

**Proposition 66.** Suppose we have a ring homomorphism  $f : R \rightarrow S$  of possibly non-commutative rings. We preserve addition, multiplication and identity.

We then have the restriction functor

$$f^* : S\text{-mod} \rightarrow R\text{-mod}$$

$$f^* M = M \text{ (as abelian group)}$$

$$\begin{array}{ccc}
R \times f^* M & \rightarrow & f^* M \\
(r, m) & \mapsto & f(r)m
\end{array}$$

If we have inclusion  $\text{inc} : kH \rightarrow kG$  then we have:

$$\text{inc}^* = \text{Res}_H^G : kG\text{-mod} \rightarrow kH\text{-mod}$$

We also have the left adjoint of  $f^*$ .

$$f_* : R\text{-mod} \rightarrow S\text{-mod} \text{ "base change"}$$

$$f_* M = S \otimes_R M$$

$S$  is a right  $R$ -module. We have  $S \times R \rightarrow S$  given by  $(s, r) \mapsto sf(r)$  which trns  $S$  to a  $(S, R)$ -bimodule:  ${}_S S_R$ . So we can take the tensor product.

**Proposition 67.**

$$\boxed{\text{Hom}_S(f_* M, N) \cong \text{Hom}_R(M, f^* N)}$$

is an isomorphism of abelian groups.

So we can go back and forth between  $S$ -modules and  $R$ -modules.

$$\begin{array}{ccc}
& f^* & \\
S\text{-mod} & \xrightarrow{\quad} & R\text{-mod} \\
& f_* & \\
& \perp &
\end{array}$$

$f_*$  is left adjoint.

$f^*$  is right adjoint.

Adjoint of  $\text{Id}_{f^* N} : \boxed{f_* f^* N \rightarrow N}$  is the counit.

Adjoint of  $\text{Id}_{f_* M} : \boxed{M \rightarrow f^* f_* M}$  is the unit.

We also have:

$$\text{inc}_* = \text{Ind}_H^G : kH\text{-mod} \rightarrow kG\text{-mod}$$

Which gives us:

$$\text{Hom}_{kG}(\text{Ind}_H^G W, V) \cong \text{Hom}_{kH}(W, \text{Res}_H^G V)$$

Remark: If we have a module, how do we know it is induced?

**Proposition 68.** If  $V = \bigoplus_{i \in I} W_i$  and  $G$  permutes summands transitively and  $\exists W = W_{i_0}$  and  $H = \{g \in G \mid gW = W\}$  then  $V$  is induced.

Example:  $\mathbb{C}D_{2n} \otimes_{\mathbb{C}C_n} \mathbb{C}_1 = 1\mathbb{C}C_n \otimes \mathbb{C}_1 + s\mathbb{C}G_n \otimes \mathbb{C}_1$ .

**Proposition 69.**  $V$  is induced if  $\exists W < V$  invariant under  $H$ :

$$V = \bigoplus_{r \in R} rW$$

$R$  is a set of left coset representation for  $H$  in  $G$ .

## Character of Induced representation

**Theorem 70** (12, p30).  $V = \text{Ind}_H^G W$ .

$$\chi_V(u) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_W(r^{-1}ur) = \frac{1}{|H|} \sum_{\substack{g \in G \\ g^{-1}ug \in H}} \chi_W(g^{-1}ug)$$

*Proof.* Write  $V = \bigoplus_{r \in R} rW$ . We care about when  $urW = rW$ , since otherwise we have non-diagonal terms so they don't contribute to the trace.

$$urW = rW \iff r^{-1}urW = W \iff r^{-1}ur \in H$$

$$\chi_V(u) = \text{Tr}(u : V \rightarrow V) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \text{Tr}(u : rW \rightarrow rW)$$

$$= \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \text{Tr}(r^{-1}ur : rW \rightarrow rW) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_W(r^{-1}ur) = \frac{1}{|H|} \sum_{\substack{g \in G \\ g^{-1}ug \in H}} \chi_W(g^{-1}ug)$$

□

## Frobenius Reciprocity

$$\langle \text{Ind } \psi, \phi \rangle_G = \langle \psi, \text{Res } \phi \rangle_H$$

## Wednesday, 10/9/2024

Recall: If

$$V = \text{Ind}_H^G W$$

Then  $V$  as a  $k$ -vector space can be written as direct sum of  $k$ -vector spaces:

$$V = \bigoplus_{g \in G/H} gW$$

And action of  $H$  permutes the summands.

$$\text{Stab}(W) := \{g \in W \mid gW = W\} = H$$

Also recall Class Functions:

$$\text{Cl}(G) = \{f : G \rightarrow k \mid f(g\sigma g^{-1}) = f(\sigma)\}$$

The characters  $\chi_V$  are a basis of the vector space of class functions.  
For  $H < G$  we have restriction:

$$\begin{array}{ccc} \text{Res} : & \text{Cl}(G) & \rightarrow \text{Cl}(H) \\ & f & \mapsto f|_H \end{array}$$

We also have induction:

$$\text{Ind} : \text{Cl}(H) \rightarrow \text{Cl}(G)$$

$$(\text{Ind } f)(\sigma) := \frac{1}{|H|} \sum_{\substack{g \in G \\ g^{-1}\sigma g \in H}} f(g^{-1}\sigma g)$$

Last time we did:

$$\chi_{\text{Ind } W} = \text{Ind } \chi_W$$

Also we had the following:

$$\text{Hom}_{kG}(\text{Ind } W, V) \cong \text{Hom}_{kH}(W, \text{Res } V)$$

Today we give a character version of this.

## Frobenius Reciprocity

**Theorem 71** (Frobenius Reciprocity). Suppose  $k$  is algebraically closed. Then:

$$\langle \text{Ind } \psi, \phi \rangle_G = \langle \psi, \text{Res } \phi \rangle_H$$

where  $\psi \in \text{Cl}(H)$  and  $\phi \in \text{Cl}(G)$  with  $H < G$ .

Also, for review: if  $\alpha, \beta \in \text{Cl}(G)$  then,

$$\langle \alpha, \beta \rangle_G = \sum_{g \in G} \alpha(g) \beta(g^{-1}) \in k$$

And irreducible characters are an orthonormal basis w.r.t. this inner product.

$$\langle \chi_i, \chi_j \rangle_G = \delta_{ij}$$

*Proof.* Suppose

$$V \cong \bigoplus_i m_i V_i$$

where  $V_1, \dots, V_s$  are irreducible. We define multiplicity:  $m_{V_i}^V := m_i$ . Then,

$$\langle \chi_V, \chi_{V'} \rangle = \sum_{i=1}^s m_{V_i}^V m_{V_i}^{V'} \underset{\text{Schur}}{=} \dim_k \text{Hom}_{kG}(V, V')$$

We finally start the proof.

$$\text{Cl}(G) = \text{span}\{\chi_i\}$$

WLOG assume  $\psi, \phi$  are characters of  $W$  and  $V$ .

$$\dim_k \text{Hom}_{kG}(\text{Ind } W, V) = \dim_k(\text{Hom}_{kH}(W, \text{Res } V))$$

$$\implies \langle \text{Ind}(\chi_W), \chi_V \rangle_G = \langle \chi_W \cdot \text{Res } \chi_V \rangle_H$$

Since this is true for basis, it is true for general character.

□

## Mackey's Double Coset Formula

Suppose  $G$  is a group with subgroups  $H, K$ . aka  $H, K < G$ . Let  $W$  be a  $kH$ -module.

Question: What is  $\text{Res}_K^G \text{Ind}_H^G W$  as a  $kK$ -module?

Let  $s \in [K \backslash G / H]$  be the double coset representation. Meaning:

$$G = \coprod_{s \in S} KsH$$

i.e.

$$G \xrightarrow[\pi]{\quad \cdot \quad} K \backslash G / H$$

The above dotted map is  $[\cdot]$ . Then,

$$\pi \circ [\cdot] = \text{Id}$$

We have:

$$H_s := sHs^{-1} \cap K < K$$

$$\rho : H \rightarrow \text{GL}(W)$$

We thus have the twisted representation:

$$\rho^s : H_s \rightarrow \text{GL}(W)$$

$$\rho^s(x) = \rho_W(s^{-1}xs)$$

$W_s = W_{\rho^s}$  is a  $kH_s$ -module.

**Proposition 72** (Mackey's Double Coset Formula, MDCF).

$$\text{Res}_K^G \text{Ind}_H^G W \cong \bigoplus_{s \in [K \backslash G / H]} \text{Ind}_{H_s}^K W_s$$

*Proof.* Suppose  $V := \text{Ind}_H^G W$ . Then, from the definition of  $\text{Ind } W$ ,

$$V = \bigoplus_{x \in G/H} xW$$

Where  $\text{Stab}(W) = H$ .

$$V = \bigoplus_{x \in G/H} xW$$

Then, as  $kK$ -module,

$$V = \bigoplus_{s \in [K \backslash G / H]} KsW$$

Note that, since  $\text{Stab}^K(sW) = H_s$ ,

$$KsW = \bigoplus_{x \in K/H_s} xsW$$

$$= \text{Ind}_{H_s}^K sW$$

$$= \text{Ind}_{H_s}^K W_s$$

Since

$$\begin{aligned} W_s &\cong sW \\ w &\mapsto sw \end{aligned}$$

So we're done. □

## Mackey's Irreducibility Criterion, MIC

Suppose  $W = W_\rho$  is  $kH$ -module. TFAE:

- 1)  $V = \text{Ind}_H^G W$  is irreducible
- 2) a)  $W$  irreducible  
b)  $\forall s \in G \setminus H, \rho^s$  and  $\text{Res}_{H_s} \rho$  are disjoint.

Recall:  $V, V'$  are disjoint if  $\text{Hom}_{kG}(V, V') = 0$ .

*Proof.* We assume  $k$  is algebraically closed.

$$V \text{ irreducible} \iff \langle \chi_V, \chi_V \rangle_G = 1$$

$$\langle \chi_V, \chi_V \rangle_G = \langle \text{Ind } \chi_W, \text{Ind } \chi_W \rangle_G$$

$$= \langle \chi_W, \text{Res Ind } \chi_W \rangle_H [FR]$$

$$= \langle W, \bigoplus_{s \in [K \setminus G/H]} \text{Ind}_{H_s}^H(\rho_s) \rangle_H [MDCF]$$

$$= \sum_s \langle \text{Res}_{H_s} \rho, \rho^s \rangle_{H_s} [FR]$$

$$= \sum_s d_s$$

$$d_s = \langle \text{Res } \rho, \rho^s \rangle_{H_s}$$

$$d_1 = \langle \rho_W, \rho_W \rangle \geq 1$$

Thus,

$$1 = \langle V, V \rangle_G \iff \begin{matrix} d_1 = 1 \\ d_s = 0 \end{matrix}$$

So we're done. □

Example: Suppose  $G = H \times K$  where  $H = C_3, G = D_6 = S_3, K = C_2$ .  
Then,

$$\mathbb{C}[C_3] = \mathbb{C}_0 \times \mathbb{C}_1 \times \mathbb{C}_2$$

$$\mathbb{C}[D_6] = \mathbb{C}_+ \times \mathbb{C}_- \times M_2\mathbb{C}$$

$$\text{Res } \mathbb{C}_+ = \mathbb{C}_0$$

$$\text{Res } \mathbb{C}_- = \mathbb{C}_0$$

$$\text{Res } \mathbb{C}^2 \stackrel{?}{=} \mathbb{C}_1 \times \mathbb{C}_2$$

# Monday, 10/14/2024

Exercises 8-13 due Friday

Wed, Chapter 9

Suppose  $K, H < G$  and  $\rho : H \rightarrow GL(W)$ .

For  $s \in G$  consider  $H_s = sHs^{-1} \cap K < K$

Then  $\rho^s : H_s \rightarrow GL(W)$

$\rho^s(x) := \rho(s^{-1}xs)$

MDCT:

$$\text{Res}_K^G \text{Ind}_H^G \rho \cong \sum_{s \in [K \backslash G/H]} \text{Ind}_{H_s}^K \rho^s$$

Take  $K = H$ .

MIC:

$\text{Ind}_H^G \rho$  is irreducible

$\iff$

a)  $\rho$  irreducible

b)  $\forall s \in G - H, \rho^s$  and  $\rho|_{H_s}$  are disjoint.

Now take  $H = K \triangleleft G$  normal.

Corollary:  $\text{Ind } \rho$  is irreducible  $\iff \rho$  irreducible and  $\forall s \notin H$   $\rho$  is not isomorphic to conjugate  $\rho^s$ .

e.g.  $H = C_3 = \langle r \rangle$

$G = D_6 = S_3 = \langle r, s \rangle$

$\mathbb{C}H \cong \mathbb{C}_0 \times \mathbb{C}_1 \times \mathbb{C}_2$

$r \mapsto (1, \zeta_3, \zeta_3^2)$

$\mathbb{C}G \cong \mathbb{C}_+ \times \mathbb{C}_- \times M_2\mathbb{C}$

Only two dimensional irreducible reps are  $\mathbb{C}_+ \times \mathbb{C}_-$  and  $\mathbb{C}^2$

$\text{Ind}_H^G \mathbb{C}_0 \cong \mathbb{C}_+ \times \mathbb{C}_-$

$\text{Ind}_H^G \mathbb{C}_1 \cong \mathbb{C}^2$

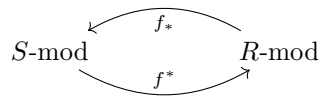
Corollary?:  $\text{Ind } \mathbb{C}_0$  is real since  $\rho \cong \rho^s, \rho^s = \rho(s^{-1}xs)$

$\text{Ind } \mathbb{C}_1$  is  $\square$ ,  $(\rho : H \rightarrow \mathbb{C})$ ,  $\rho \not\cong \rho^s$ .

More on MDCF "Mackey Functors"

Review

Ring  $f : R \rightarrow S$



"Res"  $f^*N = N$

"Ind"  $f_*M = S \otimes_R M$

MDCF:  $H, K < G$

$K^s = s^{-1}Ks$

${}^sH = sHs^{-1}$

$c_s : K^s \rightarrow K$   
 $g \mapsto sgs^{-1}$

$(\text{Ind } c_s)M = {}_kK \otimes_{{}_kK^s} M$

$$\text{Res}_K^G \text{Ind}_H^G = \sum_{s \in [K \backslash G/H]} \text{Ind}_{K \cap {}^sH}^K \text{Ind } c_s \text{Res}_{K^s \cap H}^H$$

**Definition.** A Mackey Functor  $M$  is:

$$M : \{\text{subgroups of } G\} \rightarrow \text{Ab}$$

$\forall H \leq K \leq G$ , we have:

Induction map  $I_H^K : M(H) \rightarrow M(K)$

Restriction map  $R_K^H M(K) \rightarrow M(H)$

Conjugation map  $\forall g \in K, c_g : M(K^s) \rightarrow M(K)$

Satisfies 6 axioms. Key one is MDCF.

$$H, K \leq J \leq G$$

$$R_K^J I_H^J = \sum_{K \setminus J/H} \cdots$$

Examples of Mackey Functors

$M(H) = R_K(H)$  representations.

Homology groups  $M(H)H_n(H; -)$

Cohomology groups  $M(H) = H^n(H; -)$

Stable Homotopy theory:  $M(H)$  equals  $X$  based  $G$ -space  $\Pi_n^H X$

Number theory: if we have  $K/\text{finite galois } L/\text{finite } \mathbb{Q}$ ,

$$M(H) = \text{Cl}(\mathcal{O}(K^H))$$