Partial Differential Equation 2 MATH 541

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This is dedicated to Sobolev Spaces, which we apply to Elliptic (linear) PDE.
Last part of the course is in a different direction. We talk about applying this to Parabolic/Hyperbolic PDE.
Also Schauder Theory

Monday, 8/26/2024

There are very explicit formula for certain PDE. For example D'Alembert, Poisson etc.
Weak Solutions to PDE: There is some notion of solution that doesn't have the requisite number of derivatives in the classical calculus sense. So we lower our notion of what a solution is.
ex. Conservation law: Burger's Equation: $u_t + uu_x = 0, u(x,0) = u_0(x)$. If we try to solve this for $-\infty < x < \infty, t > 0$, we have method of characteristics that attempts to give us explicit formula (solution will be constant along lines of slope $\frac{1}{u_0}$), but there is trouble for general u_0. These lines an bump into each other, so in that point our solution has to equal two different numbers.

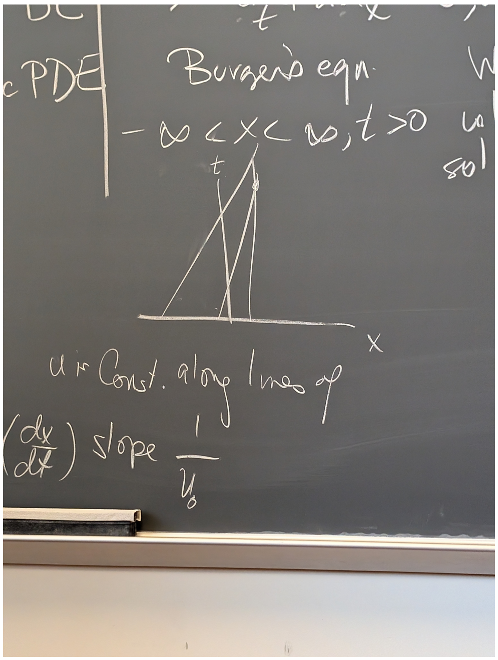


Figure 1: Burger's Equation

Can also happen that a classical solution does exists, BUT it is easiest to find a weak solution.
- Find a weak solution
- Show it's classical (regularity theory)

For linear elliptic PDE's ex. Laplace's Equation,

$$\Delta u = u_{x_1 x_1} + \cdots + u_{x_n x_n} = 0$$

in $\Omega \subset \mathbb{R}^n$

A weak solution satisfies the equation obtained through multiplication by a test function C_c^∞ and integrate by parts
ie take any $\phi \in C_c^\infty(\Omega)$
$\int_\Omega \phi \Delta u = 0$
$= -\int_\Omega \nabla\phi \cdot \nabla u + \int_{\partial\Omega} \phi \nabla u \cdot \nu$
Note that the second term is integrated over the boundary. So it goes to 0
So, if:

$$\int_\Omega \nabla\phi \cdot \nabla u = 0 \forall \phi \in C_c^\infty$$

We say u weakly solves Laplace's equation because it requires only one derivative (not two).

Definition 1 (Weak Derivative)**.** A locally integrable function u [notationally $u \in L^1_{loc}(\Omega)$] has a weak x_i derivative v if v is locally integrable and
$\int_\Omega u\phi_{x_i} \, \mathrm{d}x = -\int_\Omega v\phi \, \mathrm{d}x$ for all $\phi \in C_c^\infty(\Omega)$

u and v can be terrible near the boundary but ϕ vanishes so we don't care!
Recall: Multi-inde notation
If $u : \mathbb{R}^n \to \mathbb{R}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ where α_j is non-negative integer then,

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}}$$

eg in $\mathbb{R}^3$ for $\alpha = (1, 2, 1)$ then $D^\alpha u = u_{x_1 x_2 x_2 x_3}$

Definition 2. Given $u, v \in L^1_{loc}(\Omega), \Omega \subset \mathbb{R}^n$ and α a multi-index, we say v is the weak αth derivative of u if integration by parts works:
if $\int_\Omega u D^\alpha \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v\phi \, \mathrm{d}x, \forall \phi \in C_c^\infty(\Omega)$

If we are define weak anything, if our weak thing actually happened to be good, we want it to satisfy the strong definition as well! For example if differentiation is allowed, then integration by parts would actually work. So, if u were smooth, then our derivative would actually satisfy the solution, since

$$\int_\Omega (D^\alpha u - v)\phi \, \mathrm{d}x = 0$$

So $D^\alpha u = v$ almost everywhere.
Recall: $u \in L^p(\Omega), p > 0$ if $\int |u|^p \, \mathrm{d}x < \infty$
$u \in L^p_{loc}(\Omega)$ if $\forall \Omega_1 \subset\subset \Omega, \int_{\Omega_1} |u|^p \, \mathrm{d}x < \infty$

Sobolev Spaces

Definition 3 (Sobolev Spaces)**.** Fix $1 \leq p \leq \infty$ and a non-negative integer k. Let $\Omega \subset \mathbb{R}^n$ be open.
Then the Sobolev space $W^{k,p}(\Omega)$ consists of all functions $u \in L^1_{loc}(\Omega)$ such that for every multi-index α with $|\alpha| < k$, $D^\alpha u$ exists weakly and lies in $L^p(\Omega)$.

Example 1: Consider $\tilde{u}(x) = |x|$. Doesn't have a derivative. What about weak derivative?
Claim: u has weak 1st derivative $v(x) = \begin{cases} -1, & \text{if } x < 0; \\ 1, & \text{if } x > 0; \end{cases}$
We verify that using test function.
Let $\phi \in C_c^\infty(\mathbb{R})$

$$LHS = \int_{-\infty}^\infty v(x)\phi(x) \, \mathrm{d}x = -\int_{-\infty}^0 \phi(x) \, \mathrm{d}x + \int_0^\infty \phi(x) \, \mathrm{d}x$$

$$RHS = -\int_{-\infty}^\infty \tilde{u}(x)\phi'(x) \, \mathrm{d}x = \int_{-\infty}^0 x\phi'(x) \, \mathrm{d}x - \int_0^\infty x\phi'(x) \, \mathrm{d}x$$

By applying IBP

$$RHS = -\int_{-\infty}^{0} \phi(x)\,\mathrm{d}x + \int_{0}^{\infty} \phi(x)\,\mathrm{d}x$$

[boundary terms don't matter because either x or ϕ vanishes]
Since $|x|$ is locally integrable for any p, $\tilde{u} \in W^{1,p}_{loc}(\mathbb{R})$ for all p
Example 2: Consider the Heaviside function

$$u(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x < 0; \end{cases}$$

Is NOT going to be weakly differentiable!

Wednesday, 8/28/2024

Sobolev Space $W^{k,p}(\Omega)$

$u \in W^{k,p}(\Omega)$ if
$D^\alpha u \in L^p$ weakly for all α such that $|\alpha| \le k$
This is a normed space.

$$\|u\|_{W^{k,p}(\Omega)} = \left[\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p\right]^{\frac{1}{p}}$$

If $p = \infty$ we just take the sup norm.
We have convergence: $\{u_j\} \subset W^{k,p}(\Omega)$ converges in $W^{k,p}$
to $u \in W^{k,p}(\Omega)$ if $|u_j - u|_{W^{k,p}(\Omega)} \to 0$ as $j \to \infty$

Definition 4. $W^{k,p}_0(\Omega) :=$ closure in $W^{k,p}(\Omega)$-norm of $C^\infty_o(\Omega)$

Remark: $W^{k,p}(\Omega)$ and $W^{k,p}_0(\Omega)$ are Banach spaces [normed, complete [since Lp is complete]]
For $p = 2$ we don't usually use $W^{k,2}(\Omega)$. We use $H^k(\Omega)$. H is for Hilbert. This is a Hilbert space [there exists an inner product].

$$(u,v)_{H^k(\Omega)} := \sum_{|\alpha|\le k} \int_\Omega D^\alpha u D^\alpha v$$

Hölder's inequality implies:

$$\left|\int_\Omega D^\alpha u D^\alpha v\right| \le \|D^\alpha u\|_{L^2} \|D^\alpha v\|_{L^2}$$

Let's go back to the example.
Let $u(x)$ be the heaviside function, 0 for negative, 1 for positive. Is it weakly differentiable? Is it in some sobolev space?
Is $u \in W^{k,p}(\mathbb{R})$ for some k and p? Does u have a weak solution?
Answer: No! We use contradiction.
Suppose there is such a $v \in L^1_{loc}(\mathbb{R})$ such that IBP holds:

$$\int_{\mathbb{R}} u\phi'\,\mathrm{d}x = -\int_{\mathbb{R}} v\phi\,\mathrm{d}x$$

for all $\phi \in C^\infty_c(\mathbb{R})$
Then, $\int_0^\infty \phi'(x)\,\mathrm{d}x = -\int_{-\infty}^{\infty} v\phi\,\mathrm{d}x$
Therefore, $\phi(0) = \int_{-\infty}^{\infty} v(x)\phi(x)\,\mathrm{d}x$
We use this for contradiction. Consider a sequence $\{\phi_j\}$ of test functions so that $\phi_j(0) = 1$ for all j and $0 \le \phi_j(x) \le 1$ for all x.
Further suppose that the support for ϕ_j shrinks to the origin.

$$\phi_j(0) = 1 = -\int_{\mathbb{R}} v\phi_j \,\mathrm{d}x$$

Now, $v\phi_j \to 0$ pointwise a.e.
$|v\phi_j| \leq |v| \in L^1_{loc}$
This gives us the desired contradiction.
Notice: $\phi(0) = \int v(x)\phi(x)$ is true for the 'function' dirac delta. This is not really a function, this is a distribution.
Moral of the story: We can't have too big of a discontinuity [here we have a jump discontinuity] and still be in Sobolev spaces.
Example: $f(x) = \frac{1}{|x|^\alpha}$ for $x \in B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$ and $\alpha > 0$
For which k, p, n, α is $f \in W^{k,p}(B(0,1))$?
Question 1: Is this in any Lp space? If no then game over.
Is $f \in L^p(B(0,1))$

$$\int_{B(0,1)} \frac{1}{|x|^{\alpha p}} \,\mathrm{d}x = \int_0^1 \int_{\partial B(0,r)} \frac{1}{|x|^{\alpha p}} \,\mathrm{d}S \,\mathrm{d}r$$
$$= \int_0^1 \frac{1}{r^{\alpha p}} \mu(\partial B(0,r)) \,\mathrm{d}r$$
$$= \omega_n \int_0^1 \frac{1}{r^{\alpha p}} r^{n-1} \,\mathrm{d}r = \omega_n \int_0^1 r^{-\alpha p + n - 1} \,\mathrm{d}r$$

So, $f \in L^p$ if $\alpha p \leq n$
Note that $|D^\alpha f| \leq L^p$ for $|\alpha| = 1$ provided $(\alpha + 1)p \leq n$
Is f weakly differentiable for $|\alpha| = 1$?
Consider $\epsilon < 1$

$$\int_{B(0,1)\setminus B(0,\epsilon)} f\phi_{x_i} \,\mathrm{d}x = \int_{B(0,1)\setminus B(0,\epsilon)} \nabla \cdot (0, 0, f\phi, 0, 0) \,\mathrm{d}x - \int_{B(0,1)\setminus B(0,\epsilon)} \phi f_{x_i} \,\mathrm{d}x$$
$$\int_{B(0,1)\setminus B(0,\epsilon)} f\phi_{x_i} \,\mathrm{d}x = -\int_{B(0,1)\setminus B(0,\epsilon)} \phi f_{x_i} \,\mathrm{d}x + \int_{\partial(B(0,1)\setminus B(0,\epsilon))} f\phi\nu_i \,\mathrm{d}S$$

Note that the outer normal dissapears, we only have the inner normal.

$$\int_{\partial(B(0,1)\setminus B(0,\epsilon))} f\phi\nu_i \,\mathrm{d}S = \int_{\partial B(0,\epsilon)} f\phi\nu_i \,\mathrm{d}S$$

Setting $\epsilon \to 0$ we see that the integral converges. Also,

$$\left| \int_{\partial(B(0,1)\setminus B(0,\epsilon))} f\phi\nu_i \,\mathrm{d}S \right| \leq \int_{\partial B(0,\epsilon)} |f\phi\nu_i| \leq c \int_{\partial B(0,\epsilon)} |f| \,\mathrm{d}S \leq \frac{c}{\epsilon^\alpha} \omega_n \epsilon^{n-1} \to 0$$

provided $n - 1 \geq \alpha$
So,

$$\int_{B(0,1)} f\phi_{x_i} \,\mathrm{d}x = -\int_{B(0,1)} \phi f_{x_i} \,\mathrm{d}x$$

So f is weakly differentiable for $|\alpha| = 1$
(See Appendix 5 in Evans)
Mollification:
There are lot of situation in PDE where you have a function and you don't know how nice is it in terms of derivative. So you convolve it so that it is nice and take some limit.
Let η satisfy $\eta \in C_c^\infty(\mathbb{R}^n)$
Suppose $\eta \equiv 0$ for $|x| \geq 1$
$\int_{\mathbb{R}^n} \eta(x) \,\mathrm{d}x = 1$

Suppose η is radial, $\eta = \eta(|x|)$ just a function of radial distance
Note that there is no such analytic function. But there are infinitely differentiable ones.
Define $\eta_\epsilon \coloneqq \epsilon^{-n}\epsilon(\frac{x}{\epsilon})$
So we rescale the function.
Then given $u \in L^1_{loc}(?)$ we define the mollification

$$u_\epsilon(x) = u * \eta_\epsilon = \int u(x-y)\eta_\epsilon(y)\,\mathrm{d}y = \int u(y)\eta_\epsilon(x-y)\,\mathrm{d}y$$

Friday, 8/30/2024

Given $\Omega \subset \mathbb{R}^n$, open, bounded
$\forall \epsilon$ define $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\} = \Omega_\epsilon$
Let $\eta \in C_c^\infty(\mathbb{R}^n)$ such that
$0 \le \epsilon \le 1$ and $\int_{B(0,1)} \eta(x)\,\mathrm{d}x = 1$ and $\operatorname{supp}(\eta) \subset \overline{B(0,1)}$
Define $\eta_\epsilon(x) \coloneqq \epsilon^{-n}\eta(\frac{x}{\epsilon})$
Then, $\operatorname{supp}(\eta_\epsilon) \subset \overline{B(0,\epsilon)}$

$$\int_{B(0,\epsilon)} \eta_\epsilon(x)\,\mathrm{d}x = \epsilon^{-n}\int_{B(0,\epsilon)} \eta(\frac{x}{\epsilon})\,\mathrm{d}x = \int_{B(0,1)} \eta(x)\,\mathrm{d}x = 1$$

Theorem 1. Given $f \in L^1_{loc}(\Omega)$ define $f_\epsilon \coloneqq \eta_\epsilon * f$ for $x \in \Omega_\epsilon$

$$f_\epsilon(x) = \int_\Omega \eta_\epsilon(x-y)f(y)\,\mathrm{d}y$$

Then, i: $f_\epsilon(x)$ is infinitely differentiable.
ii: $f_\epsilon \to f$ pointwise a.e.
iii: If f is continuous then $f_\epsilon \to f$ uniformly on compact subsets of Ω
iv: If for $1 \le p < \infty, f \in L^p(\Omega)$ then $f_\epsilon \to f$ in $L^p(\Omega)$. Also true for $f \in L^p_{loc}$

Proof. (i): Convolution is a child, and it inherits the nicest properties of the parent. η_ϵ is nicer.
Idea: it is legal to bring derivatives inside the integral.

$$\frac{f_\epsilon(x+he_i) - f_\epsilon(x)}{h} = \int_\Omega \left[\frac{\eta_\epsilon(x+he_i-y) - \eta_\epsilon(x-y)}{h}\right] f(y)\,\mathrm{d}y$$

Stuff in brackets converges uniformly in y to $\eta_{\epsilon_{x_i}}(x-y)$ using Taylor remainder theorem.
Then Lebesgue Dominated Convergence finishes the job.
(iii) Fix $K \subset \Omega$ where K compact.
$\forall x \in K$ we have $|f_\epsilon(x) - f(x)| = \left|\int_\Omega (\eta_\epsilon(y)f(x-y))\,\mathrm{d}y - f(x)\right|$

$$= \left|\int_\Omega \eta_\epsilon(y)\left[f(x-y) - f(x)\right]\,\mathrm{d}y\right|$$

$$\le \int_\Omega \eta_\epsilon(y)\left|f(x-y) - f(x)\right|\,\mathrm{d}y$$

Since K is compact, f is uniformly continuous on K.
So, $\forall \beta > 0$ we have ϵ_0 such that $\forall \epsilon < \epsilon_0$ such that $\forall x, y$ such that $|y| < \epsilon$ we have $|f(x-y) - f(x)| < \beta$
So,

$$\le \int_\Omega \eta_\epsilon(y)\beta\,\mathrm{d}x = \beta$$

$\square$

What is the relevance to Sobolev spaces?
Often we are going to try to prove some estimates. We want to prove some inequalities. Then, we mollify and prove it for the mollification. If we mollify a Sobolev function, we get an approximation.

Theorem 2 (Local Approximation Away from $\partial\Omega$)**.** Assume $u \in W^{k,p}(\Omega)$ [aka function has k'th weak derivatives which are locally integrable in order p]. Define $u_\epsilon = \eta_\epsilon * u$ in $\Omega_\epsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \epsilon\}$
THen, $u_\epsilon \in C^\infty(\Omega_\epsilon)$
And $u_\epsilon \to u$ in $W^{k,p}_{loc}(\Omega)$

Proof. Note infinite derivative we already have by property of mollification.
Fix the multi-index α suh that $|\alpha| \leq k$.
<u>Claim 1</u>: $D^\alpha u_\epsilon = \eta_\epsilon * D^\alpha u$
In other words, mollification and derivatives commute.
To see this, consider $x \in \Omega_\epsilon$

$$D^\alpha u_\epsilon = D^\alpha \int_\Omega \eta_\epsilon(x-y)u(y)\,\mathrm{d}y$$

$$= \int_\Omega D^\alpha_x \eta_\epsilon(x-y)u(y)\,\mathrm{d}y$$

$$= (-1)^{|\alpha|} \int_\Omega D^\alpha_y \eta_\epsilon(x-y)u(y)\,\mathrm{d}y$$

$$= (-1)^{|\alpha|}(-1)^{|\alpha|} \int_\Omega \eta_\epsilon(x-y)D^\alpha u(y)\,\mathrm{d}y$$

$$= \eta_\epsilon * D^\alpha u$$

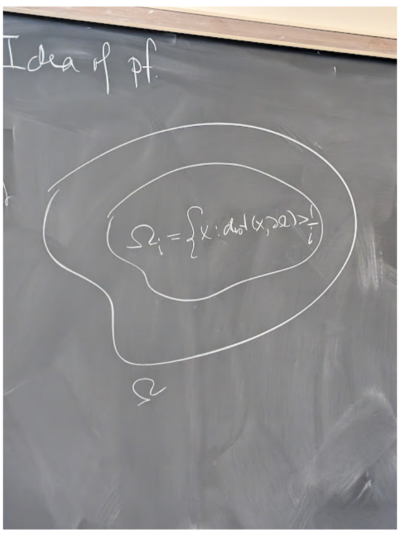
proving the claim.
Now, fix $V \subset \Omega$ with open $\overline{V} \subset \Omega$ ($V \subset\subset \Omega$)
Apply previous theorem, item iv and the claim
$D^\alpha u_\epsilon \to D^\alpha u$ in $L^p(V)$ for all α such that $|\alpha| \leq k$

□

Theorem 3 (Global Approximation)**.** Theme:
Sobolev can be approximated by smooth sobolev
Let $\Omega \subset \mathbb{R}^n$ be open, bounded. Assume $u \in W^{k,p}(\Omega)$ for $1 \leq p < \infty$
Then, $\exists \{u_m\} \subset C^\infty(\Omega) \cap W^{k,p}(\Omega)$ [infinitely smooth AND sobolev] such that $u_m \to u$ in $W^{k,p}(\Omega)$. Meaning:

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(\Omega)} \to 0$$

Idea of Proof:



We have a bunch of Ω_i with $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$
Use a partition of unity $\{\phi_i\}$ so that $\sum \phi_i(x) = 1$, mollify.
Proof is on Evans.

Wednesday, 9/4/2024

Traces

We are going to solve PDEs using Sobolev spaces. We typically specify boundary conditions in PDEs. But in first glance, since Sobolev functions are defined upto a set of measure zero, it seems ill suited to dealing with boundary conditions [since boundary $\partial\Omega$ has measure 0].
How to define boundary values for a Sobolev functions?
We are going to establish an inequality for smooth function.

Theorem 4. Assume $\partial\Omega$ is C^1 surface[1]. Then $\exists$ a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ for $1 \leq p < \infty$ such that:

i) $Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$

ii) $\|Tu\|_{L^p(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}$ for some $C = C(p, \Omega)$

Proof. Outline:

1) Fix $x_0 \in \partial\Omega$. Suppose $\partial\Omega$ is 'flat' near x_0. In fact, define ball B centered at x_0 such that inside B, $\partial\Omega$ is flat.

   [insert picture]

   Then we can define $x' = (x_1, \cdots, x_{n-1})$. If the last coordinate is positive, we are inside Ω.

   Inside B, $\partial\Omega$ is flat. we can define $B' \subset B$ centered at x_0.

   Let $\zeta \in C_0^\infty(B)$ so that $\zeta \equiv 1$ on B' and $\zeta \geq 0$.

   Define $\Gamma \coloneqq \partial\Omega \cap B'$.

   Assume $u \in C^1(\overline{\Omega})$.

$$\int_\Gamma |u|^p \,\mathrm{d}x' \leq \int_{\{x_n=0\}\cap B} \zeta |u|^p \,\mathrm{d}x'$$

$$= -\int_{\{x_n=0\}\cap B} (0, \cdots, 0, \zeta|u|^p) \cdot (0, \cdots, 0, -1) \,\mathrm{d}x'$$

   We are almost set up for divergence theorem. Since $\zeta = 0$ on the boundary of B, we actually have the whole boundary! Applying the divergence theorem, we get:

$$= -\int_{B\cap\Omega} \frac{\partial}{\partial x_n} \left[\zeta(x)|u(x)|^p\right] \mathrm{d}x$$

$$= -\int_{B\cap\Omega} \left[\zeta_{x_n}|u|^p + \zeta p |u|^{p-1} \frac{u}{|u|} u_{x_n}\right] \mathrm{d}x$$

$$= -\int_{B\cap\Omega} \left[\zeta_{x_n}|u|^p + \zeta p |u|^{p-1} \operatorname{sgn}(u) u_{x_n}\right] \mathrm{d}x$$

   Now we use Young's Inequality.

[1] meaning $\partial\Omega$ can be described as the graph of a C^1 function.

Theorem 5 (Young's Inequality)**.** For any $a, b > 0$ and p, q such that $\frac{1}{p} + \frac{1}{q} = 1$, we have:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

We are estimating abolute value, so we can take absolute value. Using Young's inequality on $a = |u_{x_n}|$ and $b = |u|^{p-1}$,

$$\int_\Gamma |u|^p \, \mathrm{d}x'$$

$$\leq \int_B^\infty \left[|\zeta_{x_n}||u|^p + p|\zeta| \left[\frac{(|u|^{p-1})^{p/(p-1)}}{p/(p-1)} + \frac{|u_{x_n}|^p}{p} \right] \right] \mathrm{d}x$$

$$\leq C \int_{B\cap\Omega} [|u|^p + |\nabla u|^p] \, \mathrm{d}x \leq C \int_\Omega [|u|^p + |\nabla u|^p] \, \mathrm{d}x$$

Now, $\partial\Omega \in C^1$ means, centered at any point $x_0 \in \partial\Omega$ $\partial\Omega$ can be written as a graph $x_n = f(x')$ where $|f|_{C_1}$ is bounded.[2]

Therefore, Ball inside of which $\partial\Omega$ is a graph has radius R depending on C^1 norm.

2) For the next step, we flatten the boundary of Ω by a change of variables.

[insert picture]

[insert picture 2]

Change variables: $y = (y', y_n)$.

Set $y' = x'$

$y_n := x_n - f(x')$

$y = G(x)$.

So, $y_n = 0$ means $x_n = f(x')$, which means <u>we're on the graph</u>.

We can think of this in terms of the <u>Inverse Function Theorem</u>. What is the Jacobian of G?

$$\mathrm{Jac}(G) = \det \nabla G = \det \begin{pmatrix} & I & & 0 \\ & & & \vdots \\ -f_{x_1} - f_{x_2} - \cdots - f_{x_{n-1}} & & & 1 \end{pmatrix} = 1$$

So, G^{-1} is C^1.

Then, $|\nabla G| \leq C(\partial\Omega)$ and $|\nabla(G^{-1}) \leq C(\partial\Omega)|$.

u is a given function. We can define a new function that has a flat boundary which we can use instead of u.

Define $\tilde{u}(y) := u(G^{-1}(y))$.

Then, $u(x) = \tilde{u}(G(x))$

Suppose $G(\Gamma) = \tilde{G}$.

Then, Step 1 implies,

$$\int_{\tilde{\Gamma}} |\tilde{u}^p| \, \mathrm{d}y' \leq C \int_{\text{shaded region}} [|\tilde{u}|^p + |\nabla\tilde{u}|^p] \, \mathrm{d}y$$

Where the shaded region is $\{x \mid x_n > f(x'), x \in B_R\}$. Now,

[2] $|f|_{C_1} = \sup |f| + \sup |\nabla f|$

$$|\nabla \tilde{u}(y)| \leq \left|Du \cdot \nabla(G^{-1})\right| \leq C\,|\nabla u|$$

Continuing,

$$\int_{\tilde{\Gamma}} |\tilde{u}^p| \,\mathrm{d}y' \leq C \int_{\text{shaded region}} [|\tilde{u}|^p + |\nabla \tilde{u}|^p] \,\mathrm{d}y$$

$$\implies \int_{\Gamma} |u|^p \,\mathrm{d}x' \leq C \int_{\Omega \cap B_R} [|u|^p + |\nabla u|^p] \,\mathrm{d}x \leq C \int_{\Omega} [|u|^p + |\nabla u|^p] \,\mathrm{d}x$$

Which finishes Step 2.

3) Decompose $\partial\Omega$ into N_R pieces, and add them up using Step 2.
Since $\partial\Omega$ is compact, $N_R < N(\partial\Omega)$.
[insert picture]
We have: $\forall u \in C^1(\overline{\Omega})$,

$$\int_{\partial\Omega} |u|^p \,\mathrm{d}S \leq C(p, \Omega) \int_{\Omega} [|u|^p + |\nabla u|^p] \,\mathrm{d}x$$

Now, suppose $u \in W^{1,p}(\Omega)$. Approximate u in $W^{1,p}$ norm by $\{u_m\} \subset W^{1,p}(\Omega) \cap C^1(\overline{\Omega})$. Then,

$$\underbrace{\|u_m - u_l\|_{L^p(\partial\Omega)}}_{\text{cauchy sequence}} \leq C\,\|u_m - u_L\|^p_{W^{1,p}}$$

$RHS \to 0$, so $LHS \to 0$ for $m, l \gg 1$.
Therefoe, $\exists Tu \in L^p(\partial\Omega)$ which is the limit of this cauchy sequence.

$\square$

Friday, 9/6/2024

A few comments about traces.
Last time: If $u \in W^{1,p}(\Omega)$ for $1 \leq p < \infty$ and $\partial\Omega \in C^1$, we can define the trace of u on $\partial\Omega$, $Tu \in L^p(\partial\Omega)$ and $\exists C = C(p, \Omega)$ such that,

$$\|Tu\|_{L^p(\partial\Omega)} \leq C\,\|u\|_{W^{1,p}(\Omega)}$$

Recall: $W_0^{k,p}(\Omega) =$ closure of $C_c^\infty(\Omega)$ in the $W^{k,p}$ norm.
Suppose $k = 1, u \in W_0^{1,p}(\Omega)$. Then, $Tu = 0$ on $\partial\Omega$
Also, if $k > 1$ and $u \in w_0^{k,p}(\Omega)$, we have $Tu = 0$.
In fact, $T(D^\alpha u) = 0 \forall \alpha$ such that $|\alpha| \leq k - 1$.

Calculus of Variations

Our objects of interest are functionals.
Consider an integral functional of the form:

$$E(u) = \int_{\Omega} L(x, u, \nabla u) \,\mathrm{d}x$$

We use E because this often denotes Energy. It comes from physics often.
In that case, we call L an energy density. It is also often called the energy density.
A fundamental problem is:
Determine the infimum of E among all functions u in an admissible set $\mathscr{A}$.
We define:

$$m := \inf_{u \in \mathscr{A}} E(u)$$

"First Variation of E"

This is the 'calculus of variations' version of a derivative.
If $\Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$,

$$L = L(x, z, p)$$

where: x is a point in $\Omega \subset \mathbb{R}^n$
$z \in \mathbb{R}$ [we put u here]
$p \in \mathbb{R}^n$ [we put ∇u here]
We put u in E, but we purtrube it a little bit.
Suppose $u \in \mathscr{A}$, and v such that $u + tv \in \mathscr{A}$ for $|t|$ small.
$E(u + tv)$
Then, $t \mapsto E(u + tv)$ is a function $\mathbb{R} \to \mathbb{R}$. We take derivative and set $t = 0$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(u + tv) = \delta E(u; v)$$

Note: if u is a minimum of E, i.e. $E(u) = m$, then the first variation $\delta E(u; v) = 0$ for all v such that $u + tv \in \mathscr{A}$.

Definition 5. u is called a critical point of E in $\mathscr{A}$ if $\delta E(u; v) = 0 \forall v$ such that $u + tv \in \mathscr{A}$.

Question: What is true about a critical point u of an integral functional of form $(*)$?

$$\begin{aligned} 0 &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(u + tv) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_\Omega L(x, u + tv, \nabla u + t\nabla v)\,\mathrm{d}x \\ &= \int_\Omega \left[\frac{\partial L}{\partial z}(x, u, \nabla u)v + \sum_{j=1}^n \frac{\partial L}{\partial p_j}(x, u, \nabla u)v_{x_j}\right] \mathrm{d}x \\ &= \int_\Omega \left[\frac{\partial L}{\partial z}(x, u, \nabla u)v + \nabla_p L \cdot \nabla_x v\right] \mathrm{d}x \end{aligned}$$

Applying IBP,

$$\begin{aligned} = \int_\Omega \left[\frac{\partial L}{\partial z}(x, u, Du) - \sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial p_j} L(x, u, \nabla u)\right)\right] v\,\mathrm{d}x \\ + \int_{\partial\Omega} v \nabla_p L(x, u, \nabla u) \cdot \nu\,\mathrm{d}S \end{aligned}$$

Now, suppose all allowable v's are included in $C_c^\infty(\Omega)$ functions. That would make the boundary term 0. Then, since we can choose v however we want, the big integral is 0.

$$\sum_{j=1}^n \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial p_j} L(x, u, \nabla u)\right] = \frac{\partial L}{\partial z}(x, u, \nabla u)$$

This is a 2nd order PDE. This is called the Euler-Lagrange equation for E.
Example: Take $E(u) = \displaystyle\int_\Omega \frac{1}{2}|\nabla u|^2 + W(x, u)\,\mathrm{d}x$
Here $L(x, z, p) = \frac{1}{2}|p|^2 + W(x, z)$
Then, we find the Euler-Lagrangian.

$$\frac{\partial L}{\partial p_j}(x, z, p) = \frac{\partial}{\partial p_j}\left[\frac{1}{2}|p|^2 + W(x, u)\right] = p_j$$

$$\frac{\partial L}{\partial p_j}(x, u, \nabla u) = u_{x_j}$$

Also,

$$\frac{\partial L}{\partial z}(x, z, p) = \frac{\partial W}{\partial z}(x, z)$$

So, the Euler-Lagrangian equation gives us:

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(u_{x_j}) = \frac{\partial W}{\partial z}(x, u)$$

$$\Delta u = \frac{\partial W}{\partial z}(x, u)$$

This is a Nonlinear Poisson Equation.
Question: What should we choose $\mathscr{A}$ to be to make our life easiest?
We want integration by parts to be justified, but we want our topology to be weak enough so that finding minimizers is easy.
In the previous example, we have integral of $|\nabla u|^2$. So, ideally we want it to be L^2. So, in the example, best choice is $H^1(\Omega)$.
How does one find a minimzer?
Direct Method: Our problem is, we want to find $u \in \mathscr{A}$ such that:

$$m := \inf_{u \in \mathscr{A}} E(u)$$

Idea: We find a sequence $\{u_j\}$ [called a minimizing sequence]: so that $\{u_j\} \subset \mathscr{A}$ and $E(u_j) \to m$.
Step 1: Try to get a convergent subsequence of $u_{j_k} \to u_* \in \mathscr{A}$. [Compactness]
In this step, if we choose $\mathscr{A}$ to be 'too strong', we lose. Ideally we want $\mathscr{A}$ to be sequentally compact.
Step 2: E might not be continuous in this topology, then we don't have $E(u_*) = m$. So, we want lower semi-continuity.

$$\liminf_{k \to \infty} E(u_{j_k}) \geq E(u_*)$$

Then we have $E(u_*) \leq m$.
This tells us that u_* is a minimum.

Monday, 9/9/2024

Theorem 6 (Extension of Sobolev Functions)**.** Assume $\Omega \subset \mathbb{R}^n$, open , bounded with $\partial\Omega \in C^1$. Then, $\exists C = C(p, \Omega)$ and an extension $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ such that:

i) $Eu = u$ in Ω

ii) Eu has compact support

iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)}$

Idea is, we can let a function down to zero 'gently' and extend that way.

Theorem 7 (Gagliardo - Nirenberg - Sobolev Inequality)**.** Assume $1 \leq p < n$. Then, there exists $C = C(p, n)$ such that for every $u \in C_0^1(\mathbb{R}^n)$ [compactly supported continuously differentiable function] one has the following inequality:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

Where $p^* = \frac{np}{n-p}$ is the critical sobolev exponent.
Note that $\frac{np}{n-p} = \frac{np-p^2+p^2}{n-p} = p + \frac{p^2}{n-p} > p$.

Question: Where does p^* come from?
Given $u \in C_0^1(\mathbb{R}^n)$, define:

$$u_\lambda(x) := u(\lambda x)$$

for $\lambda > 0$.
Suppose $|u|_{L^q(\mathbb{R}^n)} \leq C\, |\nabla u|_{L^p(\mathbb{R}^n)}$
For which q could this be true? Consider the norm of the scaled function:

$$\|u_\lambda\|_{L^q} = \left[\int_{\mathbb{R}^n} |u(\lambda x)|^q \,\mathrm{d}x\right]^{\frac{1}{q}} = \frac{1}{\lambda^{n/q}} \left[\int_{\mathbb{R}^n} |u(y)|^q \,\mathrm{d}y\right]^{\frac{1}{q}}$$

$$\|\nabla u_\lambda\|_{L^p} = \left[\int_{\mathbb{R}^n} |\nabla_x u(\lambda x)|^p \,\mathrm{d}x\right]^{\frac{1}{p}} = \underbrace{\frac{1}{\lambda^{\frac{n}{p}}} (\lambda^p)^{\frac{1}{p}}}_{=\lambda^{1-\frac{n}{p}}} \left[\int_{\mathbb{R}^n} |\nabla_y u(y)|^p \,\mathrm{d}y\right]^{\frac{1}{p}}$$

Applying the same inequality,

$$\frac{1}{\lambda^{\frac{n}{q}}} \left[\int_{\mathbb{R}^n} |u(y)|^q \,\mathrm{d}y\right]^{\frac{1}{q}} \leq C\lambda^{1-\frac{n}{p}} \left[\int_{\mathbb{R}^n} |\nabla u(y)|^p \,\mathrm{d}y\right]^{\frac{1}{p}}$$

$$\|u\|_{L^q} \leq C\lambda^{1-\frac{n}{p}+\frac{n}{q}} \|\nabla u\|_{L^p} \quad \forall \lambda > 0$$

This means $1 - \frac{n}{p} + \frac{n}{q}$ must be 0.
Therefore, $q = p^*$
So, if the theorem is true we must have the q.
Now we prove the theorem.

Proof. First take $n = 3$.
Take $p = 1$.

$$u(x) = \int_{-\infty}^{x_1} u_{x_1}(y_1, x_2, x_3) \,\mathrm{d}y_1$$

$$u(x) = \int_{-\infty}^{x_2} u_{x_2}(x_1, y_2, x_3) \,\mathrm{d}y_2$$

$$u(x) = \int_{-\infty}^{x_3} u_{x_3}(x_1, x_2, y_3) \,\mathrm{d}y_3$$

Therefore,

$$|u(x)| \leq \int_{-\infty}^{x_1} |\nabla u(y_1, x_2, x_3)| \,\mathrm{d}y_1$$

$$|u(x)| \leq \int_{-\infty}^{x_2} |\nabla u(x_1, y_2, x_3)| \,\mathrm{d}y_2$$

$$|u(x)| \leq \int_{-\infty}^{x_3} |\nabla u(x_1, x_2, y_3)| \,\mathrm{d}y_3$$

Multiplying, $|u(x)|^3$

$$\leq \left[\int_{-\infty}^{x_1} |\nabla u(y_1, x_2, x_3)| \,\mathrm{d}y_1\right] \left[\int_{-\infty}^{x_2} |\nabla u(x_1, y_2, x_3)| \,\mathrm{d}y_2\right] \left[\int_{-\infty}^{x_3} |\nabla u(x_1, x_2, y_3)| \,\mathrm{d}y_3\right]$$

Therefore, $|u(x)|^{\frac{3}{2}} \leq$

$$\underbrace{\left[\int_{-\infty}^{\infty} |\nabla u(y_1, x_2, x_3)| \,\mathrm{d}y_1\right]^{\frac{1}{2}}}_{=I_1(x_2,x_3)} \underbrace{\left[\int_{-\infty}^{\infty} |\nabla u(x_1, y_2, x_3)| \,\mathrm{d}y_2\right]^{\frac{1}{2}}}_{=I_2(x_1,x_3)} \underbrace{\left[\int_{-\infty}^{\infty} |\nabla u(x_1, x_2, y_3)| \,\mathrm{d}y_3\right]^{\frac{1}{2}}}_{=I_3(x_1,x_2)}$$

Integrating with respect to x_1 and applying Hölder,

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{3}{2}}\,\mathrm{d}x_1 \le (I_1(x_2,x_3))^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} I_2(x_1,x_3)\,\mathrm{d}x_1\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} I_3(x_1,x_2)\,\mathrm{d}x_1\right]^{\frac{1}{2}}$$

repeat for x_2 and x_3 and use the fact:

$$\int |f||g| \le \left[\int f^2\right]^{\frac{1}{2}} \left[\int g^2\right]^{\frac{1}{2}}$$

We see that,

$$\iiint_{\mathbb{R}^3} |u(x)|^{\frac{3}{2}}\,\mathrm{d}x \le \left[\iiint_{\mathbb{R}^3} |\nabla u(x)|\,\mathrm{d}x\right]^{\frac{3}{2}}$$

Therefore, $\|u\|_{L^{\frac{3}{2}}} \le \|\nabla u\|_{L^1}$
For $p = 1, n \ge 3$, same proof, but use 'generalized Hölder inequality' given by:

$$\int |u_1 \cdots u_m| \le \|u_1\|_{L^{p_1}} \cdots \|u_m\|_{L^{p_m}}$$

provided $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$

$\square$

Now, for any $1 < p < n$, let $v(x) = |u(x)|^\gamma$ for some $\gamma > 1$
$v \in C_0^1$ since $u \in C_0^1$. Use previous case $[p = 1]$. Note that, in that case, $p^* = \frac{n}{n-1}$.
Also note that, $\nabla v = \gamma |u|^{\gamma-1}\operatorname{sgn}(u)\nabla u$
Therefore $|\nabla v| \le \gamma |u|^{\gamma-1}|\nabla u|$
Then,

$$\left[\int |v(x)|^{\frac{n}{n-1}}\right] \le \int |\nabla v(x)|$$

$$\left[\int |u|^{\frac{\gamma n}{n-1}}\right]^{\frac{n-1}{n}} \le \gamma \int |u|^{\gamma-1}|\nabla u| \underset{\text{Hölder}}{\le} \gamma \left[\int |u|^{(\gamma-1)\frac{p}{p-1}}\right]^{\frac{p-1}{p}} \left[\int |\nabla u|^p\right]^{\frac{1}{p}}$$

We pick $\gamma = \frac{p(n-1)}{n-p} > 1$. This gives us,

$$\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1} \implies \frac{\gamma n}{n-1} = p^*$$

So we get:

$$\left[\int |u|^{p^*}\right]^{\frac{n-1}{n} - \frac{p-1}{p}} \le \gamma \|\nabla u\|_{L^p}$$

$$\left[\int |u|^{p^*}\right]^{\frac{n-p}{np}} \le \gamma \|\nabla u\|_{L^p}$$

$$\|u\|_{L^{p^*}} \le \gamma \|\nabla u\|_{L^p}$$

Wednesday, 9/11/2024

Theorem 8. Assume $u \in W_0^{1,p}(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open, bounded, $\partial\Omega \in C^1$. Then, for $1 \le p < n$ there exists $C = C(p, n, \Omega)$ such that,

$$\|u\|_{L^{p^*}} \le C \|\nabla u\|_{L^p(\Omega)}$$

for $p^* = \frac{np}{n-p}$
[Note that before we had condition $u \in C_0^1(\mathbb{R}^n)$, so this is a stronger theorem]

Proof. Extend u to be $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ so that it is compactly supported
[insert picture here]
Then we approximate $\tilde{u}$ by smooth functions $\{u_m\} \subset C_0^\infty(\mathbb{R}^n)$.
Apply previous result to get:

$$\|u_m - u_j\|_{L^{p^*}} \leq C\|\nabla u_m - \nabla u_j\|_{L^p} \to 0$$

Therefore, $\{u_m\}$ is cauchy in $L^{p^*}(\Omega)$.
So we have $u_m \overset{L^{p^*}(\Omega)}{\to} u$
To prove the inequality, we let $m \to \infty$ in the previous result.

□

<u>Corollary</u>: For $1 \leq q \leq p^*$ and $1 \leq p < n$, $\exists C = C(p, n, \Omega)$ such that,

$$\|u\|_{L^q(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$$

Proof. By Hölder,

$$\|v\|_{L^q} \leq C'\|v\|_{L^{p^*}} \quad \forall q < p^*$$

□

This is only half the story since we have $p < n$. What if $p > n$? What about $p = n$?

$p = n$ case

Here $u \in W_0^{1,n}(\Omega)$
Note that $p^* = \frac{np}{n-p}$ which is undefined. So we have problem. For that, we do:
Suppose $\epsilon > 0$. Apply GNS inequality for $p_\epsilon = n - \epsilon$.
Then $p_\epsilon^* = \frac{np_\epsilon}{n-p_\epsilon} = \frac{n(n-\epsilon)}{\epsilon}$. We have:

$$\|u\|_{L^{p_\epsilon^*}(\Omega)} < C_\epsilon\|\nabla u\|_{L^n(\Omega)}$$

Such an inequality exists for every ϵ. So, u is in every $L_q(\Omega)$ space for $1 \leq q < \infty$.
Note that $q = \infty$ not necessary, there are counterexamples.
<u>Corollary</u>[Poincaré Inequality]: For $\Omega \subset \mathbb{R}^n$ open, bounded and for $1 \leq p < \infty$ there exists $C = C(p, n, \Omega)$ such that

$$\|u\|_{L^p(\Omega)} \leq C(p, n, \Omega)\|\nabla u\|_{L^p(\Omega)}$$

This is true for all $u \in W_0^{1,p}(\Omega)$.
This is weaker than previous inequalities, since $p < p^*$.

What if $u \in W^{1,p}$ or $u \in W_0^{1,p}$ and $\Omega \subset \mathbb{R}^n$ and $p > n$?

These functions are even better!
But first we need to talk about Hölder Continuous Functions.

Definition 6 (Hölder Quotient)**.** Let $0 < \alpha < 1$. Define <u>Hölder quotient</u>:

$$[u]_{C^{0,\alpha}(\overline{\Omega})} = \sup_{x,y \in \overline{\Omega}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

Definition 7 (Hölder Space)**.** $u \in C^{0,\alpha}(\overline{\Omega})$ if

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} := \sup_{x \in \overline{\Omega}} \left[|u(x)| + [u]_{C^{0,\alpha}(\overline{\Omega})}\right] < \infty$$

This is a <u>Banach Space</u>.

Note: if $u \in C^{0,\alpha}(\overline{\Omega})$ then u is uniformly continuous.
Example: Suppose $\Omega = [0,1]$ and $u(x) = x^\beta$ for any $\beta \in (0, \alpha]$.
[graph]
It's derivative goes to ∞ but it is still Hölder continuous.

Theorem 9 (Morrey's Inequality)**.** Assume $n < p \leq \infty$. Then $\exists C = C(p,n)$ such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C_0^1(\mathbb{R}^n)$ where $\gamma = 1 - \frac{n}{p}$.

We can approximate so this is also true for $u \in W_0^{1,p}(\Omega)$.
Thus, when $u \in W_0^{1,p}(\Omega)$ we can say that u is in fact Hölder continuous.

Proof. We have to prove that both sup and the Hölder quotient are controlled by the Sobolev norm.
Step 1: Claim: There exists $C = C(n)$ such that $\forall x \in \mathbb{R}^n$ such that $r > 0$,

$$\fint_{B(x,r)} |u(y) - u(x)| \, \mathrm{d}y \leq C \int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} \, \mathrm{d}y \quad \forall u \in C^1(\mathbb{R}^n)$$

Proof of Claim: Fix x. Fix r. Fix $w \in \partial B(0,1)$. Then,

$$|u(x+sw) - u(x)| = \left| \int_0^s \frac{\mathrm{d}}{\mathrm{d}t} u(x+tw) \, \mathrm{d}t \right| \leq \int_0^s |\nabla_x u(x+tw)| \cdot \underbrace{|w|}_{=1} \, \mathrm{d}t$$

Integrating over all $w \in \partial B(0,1)$ we get

$$\int_{w \in \partial B(0,1)} |u(x+sw) - u(x)| \, \mathrm{d}S_w \leq \int_{w \in \partial B(0,1)} \int_0^s |\nabla_x u(x+tw)| \, \mathrm{d}t \, \mathrm{d}S_w$$

$$= \int_0^s \int_{w \in \partial B(0,1)} |\nabla_x u(x+tw)| \underbrace{t^{n-1} \, \mathrm{d}S_w}_{= \mathrm{d}S \text{ on } \partial B(0,t)} \, \mathrm{d}t$$

Let $y = x + tw \implies t = |y - x|$

$$= \int_{B(x,s)} \frac{|\nabla_y u(y)|}{|y-x|^{n-1}} \, \mathrm{d}y$$

$$\leq \int_{B(x,r)} \frac{|\nabla_y u(y)|}{|y-x|^{n-1}} \, \mathrm{d}y$$

TBC

Friday, 9/13/2024

Continuing Morrey's Inequality.
We had:

$$\int_{\partial B(0,1)} |u(x+sw) - u(x)| \, \mathrm{d}S_w \leq \int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} \, \mathrm{d}y$$

Multiplying by s^{n-1},

$$\int_{\partial B(0,1)} |u(x+sw) - u(x)| s^{n-1} \, \mathrm{d}S_w \leq \int_{B(x,r)} \frac{|\nabla u(y)| s^{n-1}}{|y-x|^{n-1}} \, \mathrm{d}y$$

Integrating over s from 0 to r,

$$\int_0^r \int_{\partial B(0,1)} |u(x+sw) - u(x)| s^{n-1} \, \mathrm{d}S_w \, \mathrm{d}s \leq \int_0^r \int_{B(x,r)} \frac{|\nabla u(y)| s^{n-1}}{|y-x|^{n-1}} \, \mathrm{d}y \, \mathrm{d}s$$

$$\int_0^r \int_{\partial B(x,s)} |u(y)-u(x)|\,\mathrm{d}S_y\,\mathrm{d}s \leq \int_0^r \int_{B(x,r)} \frac{|\nabla u(y)| s^{n-1}}{|y-x|^{n-1}}\,\mathrm{d}y\,\mathrm{d}s$$

$$\int_{B(x,r)} |u(y)-y(x)|\,\mathrm{d}y \leq \frac{r^n}{n} \int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\,\mathrm{d}y$$

$$\frac{1}{\alpha(n) r^n} \int_{B(x,r)} |u(y)-y(x)|\,\mathrm{d}y \leq \frac{1}{n\alpha(n)} \int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\,\mathrm{d}y$$

which is the claim.
Step 2: (Bounding $\sup |u|$)
Fix $x \in \mathbb{R}^n$.

$$|u(x)| = \fint_{B(x,1)} |u(x)|\,\mathrm{d}y$$

$$\leq \fint_{B(x,1)} |u(x)-u(y)|\,\mathrm{d}y + \fint_{B(x,1)} |u(y)|\,\mathrm{d}y$$

First term: apply step 1. Second term: apply Young's inequality

$$|u(x)| \leq C \int_{B(x,1)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\,\mathrm{d}y + C\left(\int 1^{\frac{p}{p-1}}\,\mathrm{d}y\right)^{\frac{p-1}{p}} \left(\int_{B(x,1)} |u(y)|^p\right)^{\frac{1}{p}} \quad (*)$$

Now apply Hölder's inequality on the first part:

$$\int_{B(x,1)} \frac{1}{|x-y|^{n-1}} |\nabla u(y)|\,\mathrm{d}y \leq \left[\int_{B(x,1)} \left[\frac{1}{|x-y|^{n-1}}\right]^{\frac{p}{p-1}}\,\mathrm{d}y\right]^{\frac{p-1}{p}}$$

We simplify:

$$\left[\int_{B(x,R)} \left[\frac{1}{|y-x|^{n-1}}\right]^{\frac{p}{p-1}}\,\mathrm{d}y\right]^{\frac{p-1}{p}} = \left[\int_0^R \int_{\partial B(0,s)} s^{(n-1)\frac{p}{p-1}}\,\mathrm{d}S\,\mathrm{d}s\right]^{\frac{p-1}{p}}$$

$$= \left[\int_0^R s^{(n-1)\frac{p}{p-1}} \omega(n) s^{n-1}\,\mathrm{d}s\right]^{\frac{p-1}{p}} = C\left[\int_0^R s^{n-1-(n-1)\frac{p}{p-1}}\,\mathrm{d}s\right]^{\frac{p-1}{p}}$$

$$= C\left[R^{n-(n-1)\frac{p}{p-1}}\right]^{\frac{p-1}{p}} = CR^{n\frac{p-1}{p}-(n-1)} = CR^{1-\frac{n}{p}} \quad (**)$$

Applying $R=1$ on $(**)$ and substituting to $()*)$,
$|u(x)| \leq C \left\|\nabla u\right\|_{L^p(\mathbb{R}^n)}$
Step 3: Bounding the Hölder quotient $[u]_{C^{0,\gamma}}$:
Fix x, y in $\mathbb{R}^n$ and let $r = |x-y|$.
Define $W = B(x,r) \cap B(y,r)$.
[insert picture]

$$|u(x)-u(y)| = \fint_{z\in W} |u(x)-u(y)|\,\mathrm{d}z$$

$$\leq \fint_W |u(x)-u(z)|\,\mathrm{d}z + \fint_W |u(y)-u(z)|\,\mathrm{d}z$$

Now,

$$\fint |u(x)-u(z)|\,\mathrm{d}z \leq \frac{1}{|W|} \int_{B(x,r)} |u(x)-u(z)|\,\mathrm{d}z$$

$$\leq C(n) \fint_{B(x,r)} |u(x)-u(z)|\,\mathrm{d}z$$

Applying the claim,

$$\leq C \int_{B(x,r)} \frac{|\nabla u(z)|}{|x-z|^{n-1}} \, \mathrm{d}z$$

Applying Hölder and $(*)$,

$$\leq C \, \|\nabla u\|_{L^p(\mathbb{R}^n)} \, r^{1-\frac{n}{p}}$$

So we're done.

□

Theorem 10. Let $\Omega \subset \mathbb{R}^n$ be bounded, open with $\partial\Omega \in C^1$. Assume $n < p < \infty$. Then, $\exists C(n, p, \Omega)$ such that: for every $u \in W^{1,p}(\Omega)$ one has $u \in C^{0,\gamma}(\Omega)$ where $\gamma = 1 - \frac{n}{p}$ and:

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} \leq C \, \|u\|_{W^{1,p}(\Omega)}$$

So, if our Sobolev L^p space is better than our dimension, our function must be continuous, or even better, Hölder Continuous!
Actually, Sobolev Functions are defined upto a set of measure 0. So, we actually have a continuous representative.

Proof. Extension Theorem □

If we have $W^{k,p}$ we can have better statements.
Morrey and G-N-S inequalities can be concatenated.
[See Evans]
Example: Suppose $u \in W^{2,2}(\Omega)$. Suppose $\Omega \subset \mathbb{R}^3$ and bounded.
$u \in H^2(\Omega)$ therefore $u_{x_j} \in W^{1,2}(\Omega)$.
$p = 2, n = 3$. Use GNS. $2^* = \frac{3 \times 2}{3-2} = 6$.
Thus, $u_{x_j} \in L^{2^*} = L^6$.
Since also $u \in W^{1,2}$, $u \in L^6$.
Therefore, $u \in W^{1,6}$. We have jumped above the dimension, since $p = 6, n = 3$.
$n < p$ so we can use Morrey.
$\implies u \in C^{0,\gamma}(\Omega)$ where $\gamma = 1 - \frac{n}{p} = 1 - \frac{3}{6} = \frac{1}{2}$.
So, $u \in W^{2,2} \implies u \in C^{0,\frac{1}{2}}(\Omega)$.

Monday, 9/16/2024

Definition 8 (continuously / compactly embedded)**.** Given two Banach spaces X, Y with $X \subseteq Y$ we say X is continuously embedded in Y if $\exists C > 0$ such that:

$$\|x\|_Y \leq C\|x\|_X \forall x \in X$$

X is compactly embedded in Y if it is continuously embedded and every bounded sequence in X is pre-compact in Y.
In other words, if every bounded sequence in X has a Y-convergent subsequence.
Typically people use $X \subset\subset Y$ to denote this.

Theorem 11 (Rellich-Kondrachov Compactness)**.** Let $\Omega \subset \mathbb{R}^n$ open, bounded with $\partial\Omega \subset C^1$. If $1 \leq p < n$ then $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$ for all $q \in [1, p^*)$
[here $p^* = \frac{np}{n-p}$]

Remark: We already have continuous embedding for $q \in [1, p^*]$ by GNS inequality.

Theorem 12 (Hölder Generalization)**.** Assume $1 \leq s < r < t \leq \infty$ and $1 = \frac{r\theta}{s} + \frac{r(1-\theta)}{t}$ for some $\theta \in (0, 1)$.
Then, if $u \in L^s(\Omega) \cap L^t(\Omega)$ then $u \in L^r(\Omega)$ and:

$$\|u\|_{L^r} \leq \|u\|_{L^s}^{\theta} \|u\|_{L^t}^{1-\theta}$$

Proof.

$$\int_\Omega |u|^r = \int_\Omega |u|^{r\theta}|u|^{(1-\theta)r} \leq \left[\int_\Omega |u|^{r\theta\frac{s}{r\theta}}\right]^{\frac{r\theta}{s}} \left[\int_\Omega |u|^{(1-\theta)r\frac{t}{r(1-\theta)}}\right]^{\frac{r(1-\theta)}{t}}$$

$$= \left[\int_\Omega |u|^s\right]^{\frac{r\theta}{s}} \left[\int_\Omega |u|^t\right]^{\frac{r(1-\theta)}{t}}$$

□

Proof of Rellich-Kondrachov. Assume $\|u_m\|_{W^{1,p}(\Omega)} < C_0$

Step 1: Mollify $u_m \rightsquigarrow u_m^\epsilon$ and argue that u_m^ϵ approximates u_m [uniformly in m] in L^q.

Step 2: Apply Arzela-Ascoli to $\{u_m^\epsilon\}$ for each ϵ fixed.

Step 3: Use diagonalization argument. We control things when m is fixed, we control things when ϵ is fixed so we can vary both.

First extend u_m to be compactly supported in $V' \supset \Omega$. Still have $\|u_m\|_{W^{1,p}(V')} < \tilde{C}_0$.

Now Mollify: $u_m^\epsilon := \eta_\epsilon * u_m$. $\eta_\epsilon(x) = \epsilon^{-n}\eta\left(\frac{x}{\epsilon}\right)$ with $\eta \in C_0^\infty$ and $\text{supp}(\eta) \subset B(0,1)$ with $\int \eta = 1$

Note u_m^ϵ is compactly supported. Say $\text{supp}(u_m^\epsilon) \subset V$. $V' \subset V$.

Claim: $\{u_m^\epsilon\}$ converges as $\epsilon \to 0$ to u_m in $L^q(\Omega)$ uniformly in m. That is, we claim: $\forall \delta > 0$ there exists $\epsilon_0 > 0$ such that $\forall \epsilon < \epsilon_0$

$$\|u_m^\epsilon - u_m\|_{L^q(V)} < \delta$$

$\forall m$.

To see this, first assume u_m is smooth.

$$u_m^\epsilon(x) - u_m(x) = \int_{B(0,\epsilon)} \eta_\epsilon(y)\left(u_m(x-y) - u_m(x)\right) \mathrm{d}y$$

Let $z = \frac{y}{\epsilon}$

$$= \int_{B(0,1)} \eta(z)\left[u_m(x-\epsilon z) - u_m(x)\right] \mathrm{d}z$$

$$= \int_{B(0,1)} \eta(z) \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} u_m(x-\epsilon t z)\,\mathrm{d}t\,\mathrm{d}z$$

$$= -\epsilon \int_{B(0,1)} \eta(z) \int_0^1 \nabla u_m(x-\epsilon t z)\cdot z\,\mathrm{d}t\,\mathrm{d}z$$

$$\leq \epsilon \int_V \int_{B(0,1)} \int_0^1 \eta(z)|\nabla u_m(x-\epsilon t z)|\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}x$$

Let $y' = x - \epsilon t z$

$$= \epsilon \int_{\tilde{V}} \int_{B(0,1)} \int_0^1 \eta(z)|u_m(y')|\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}y'$$

$$= \epsilon \int_{\tilde{V}} |\nabla u_m(y')|\,\mathrm{d}y'$$

Apply Hölder:

$\leq \epsilon C\|\nabla u_m\|_{L^p} \leq \epsilon\tilde{\tilde{C}}_0$

If u_m is not smooth then approximate u_m by $\tilde{u}_m$ smooth such that $\|\tilde{u}_m - u_m\|_{W^{1,p}} < \delta$ for all m.Then,

$$\int_V |u_m^\epsilon(x) - u_m(x)|\,\mathrm{d}x \leq$$

$$\int_V |u_m^\epsilon(x) - \tilde{u}_m(x)|\,\mathrm{d}x + \int_V |\tilde{u}_m(x) - u_m(x)|\,\mathrm{d}x < \delta + \delta$$

Now apply interpolation:

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \|u_m^\epsilon - \tilde{u}_m\|_{L^1}^\theta \|\tilde{u}_m - u_m\|_{L^{p^*}}^{1-\theta}$$

Here $s = 1, t = p^*$
$\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$
Recall GNS gave us $\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \tilde{C}_0$

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$$|u_m^\epsilon| \leq \int_V \epsilon^{-n} \eta\left(\frac{x-y}{\epsilon}\right) |u_m(y)| \,\mathrm{d}y$$

$$\leq \epsilon^{-n} \int_V |u_m(y)| \,\mathrm{d}y$$

$$\leq \epsilon^{-n} \|u_m\|_{W^{1,p}(V)} < \text{constant}$$

Note that

$$\nabla u_m^\epsilon(x) = \int_V \epsilon^{-n} \nabla\eta\left(\frac{x-y}{\epsilon}\right) \frac{1}{\epsilon} u_m(y) \,\mathrm{d}y$$

$$|\nabla u_m^\epsilon(x)| \leq \int_V \epsilon^{-n} \left|\nabla\eta\left(\frac{x-y}{\epsilon}\right)\right| \frac{1}{\epsilon} |u_m(y)| \,\mathrm{d}y$$

$$\leq \epsilon^{-(n+1)} \sup|\nabla\eta| \int |u_m(y)| \,\mathrm{d}y$$

$$\leq \epsilon^{n+1} \sup|\nabla\eta| \|u_m\|_{W^{1,p}(V)} \leq \text{constant}$$

Therefore, u_m^ϵ is equi-Lipschitz and thus equicontinuous.
<u>Step 3</u>:
<u>Claim</u>: We can find a subsequence $\{u_{m_j}\}$ of the u_m so that:

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} < \delta$$

Use step 1 to find ϵ such that $\|u_m^\epsilon - u_m\|_{L^q(V)} < \frac{\delta}{2}$ for all m.
Apply Arzela-Ascoli for that ϵ to get subsequence u_{m_j} such that:

$$\limsup_{j,k\to\infty} \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\| \to 0$$

$$\implies \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\|_{L^q(V)} = 0$$

Therefore,

$$\left\|u_{m_j} - u_{m_k}\right\|_{L^q(V)} \leq \left\|u_{m_j} - u_{m_j}^\epsilon\right\|_{L^q} + \left\|u_{m_j}^\epsilon - u_{m_k}^\epsilon\right\|_{L^q} + \left\|u_{m_k}^\epsilon - u_{m_k}\right\|_{L^q}$$

Take $\limsup_{j,k\to\infty}$ to see $< \frac{\delta}{2} + 0 + \frac{\delta}{2} = \delta$
Then take $\delta = 1, \frac{1}{2}, \frac{1}{4} \cdots$ [subsequences of subsequences] to get $u_{m_{l_k}} \xrightarrow{L^q} \overline{u}$.

$\square$

<u>Remark</u>: What if $\|u_m\|_{W^{1,p}(\Omega)} < C_0$ with $n < p$?
Morrey's inequality gives us:

$$\|u_m\|_{C^{0,\alpha}} \leq \text{constant}$$

$$|u_m(x) - u_m(y)| \leq C|x-y|^\alpha$$

Thus we also have equicontiunity.

Elliptic PDE Definition

Suppose $u =$ density, amount/volume, $u = u(x,t), x \in \mathbb{R}^3$.
Let $\Omega \subset \mathbb{R}^3$ be any domain where the whole process is taking place.
Suppose we want to know the amount of stuff in Ω in time t. It is:

$$\int_\Omega u(x,t)\,\mathrm{d}x$$

Now we ask: how does it change w.r.t. time?

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x,t)\,\mathrm{d}x$$

This depends on the stuff enterring or exiting [= flux accross $\partial\Omega$] + sources/sinks.
What is flux? It must be a vector $\vec{Q}(x,t)$. So, total contribution of flux is:

$$-\int_{\partial\Omega} \vec{Q}(x,t)\cdot\nu\,\mathrm{d}S$$

Plus sources/sinks density.

$$+\int_\Omega F(x,t)\,\mathrm{d}x$$

Thus we have,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_\Omega u(x,t)\,\mathrm{d}x = -\int_{\partial\Omega} \vec{Q}(x,t)\cdot\nu\,\mathrm{d}S + \int_\Omega F(x,t)\,\mathrm{d}t$$

The choice of flux $\vec{Q}$ distinguishes different physical settings.
Let's rewite our equation as 1 volume integral.

$$\int_\Omega \left[u_t(x,t) + \operatorname{div}\vec{Q}(x,t) - F(x,t)\right]\,\mathrm{d}x = 0$$

This is true for arbitrary Ω so we have:

$$u_t = -\operatorname{div}(\vec{Q}) + F$$

Naturally, we pick Q to model a diffusion process.
If there's a lot of stuff 'inside' then stuff will go outside and vice versa. So, we can take,

$$\vec{Q} = -k\nabla u$$

Or more generally,

$$\vec{Q} = -A(x)\nabla u$$

where A is positive definite 3×3 matrix.
For $Q = -k\nabla u$ we have:

$$u_t = \operatorname{div}(k\nabla u) + F = k\Delta u + F$$

which is the heat equation.
For $Q = -A(x)\nabla u$,

$$u_t = \operatorname{div}(A(x)\nabla u) + F$$

This is a model where stuff 'smooths out / settles down'.
As $t\to\infty$ we expect $u(x,t)\to\tilde{u}(x)$.
In that case, $-\operatorname{div}(A(x)\nabla u) = F$.
If A has entries $a_{ij}(x)$, then,

$$\operatorname{div}(A(x)\nabla u) = \sum_{i,j} \frac{\partial}{\partial x_i}(a_{ij}(x)u_{x_j})$$

This is an elliptic operator [if $a_{ij} = a_{ji}, A$ positive definite].
This is the divergence form.

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Assume A is symmetric. We define the linear elliptic operator L by:

i) $Lu = -\frac{\partial}{\partial x_j}(a_{ij}u_{x_i}) + b_i(x)u_{x_i} + c(x)u$ [divergence form elliptic operator]

ii) $Lu = -a_{ij}(x)u_{x_ix_j} + b_i(x)u_{x_i} + c(x)u$ [non divergence form] where $b_i : \Omega \to \mathbb{R}, i = 1, \cdots, n, c : \Omega \to \mathbb{R}, \Omega \subset \mathbb{R}^n$. If a_{ij} are C^1 then $a_{ij}(x)u_{x_ix_j} = \frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) - \left(\frac{\partial}{\partial x_j}a_{ij}(x)\right)u_{x_i}$

Why elliptic? $A = (a_{ij})$ and $a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \quad \forall \xi \in \mathbb{R}^n$ for some $\theta > 0$.

Definition 9. We say $u \in H_0^1(\Omega)$ [same as $W_0^{1,2}$] is a weak solution to the equation $Lu = f$ for some $f \in L^2(\Omega)$ and $u = 0$ on $\partial\Omega$ [homogeneous Dirichlet condition] if:

$$\int_\Omega A\nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega b_i(x)u_{x_i}v \, \mathrm{d}x + \int_\Omega c(x)uv \, \mathrm{d}x = \int_\Omega vf \, \mathrm{d}x \quad (*)$$

for all $v \in H_0^1(\Omega)$.

Definition 10. We say $u \in H^1(\Omega)$ is a weak solution to

$$\begin{cases} Lu = f, & \text{in } \Omega; \\ A\nabla u \cdot \nu_{\partial\Omega} = 0, & \text{on } \partial\Omega \end{cases}$$

[Neumann boundary condition]
if $(*)$ holds for all $v \in H^1(\Omega)$.

Suppose

$$\int_\Omega a_{ij}(x)u_{x_i}v_j + b_i(x)u_{x_i}v + cuv \, \mathrm{d}x + \int_{\partial\Omega} vA\nabla u \cdot \nu \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x \quad (**)$$

why? Suppose u was a classical solution. Then integrating by part yields the second equation.

Functional Analysis

Background on Hilbert Spaces
Recall: a Hilbert Space H is a Banach space [normed and complete] that posses an inner product $(\cdot,\cdot)_H$ such that $\|\cdot\|$ is inherited from the inner product.
Basically: complete space with inner product.
Example: $\mathbb{R}^n$ with dot product, $L^2(\Omega)$ with $(u,v)_{L^2} = \int_\Omega uv \, \mathrm{d}x$
Or, $H^1(\Omega)$ with $(u,v)_H = \int_\Omega uv \, \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$
Or, most importantly, in $H_0^1(\Omega)$ we have $(u,v)_{H_0^1} = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$
By Poincare, then $C_1\|u\|_{H^1}^2 \leq \|u\|_{H_0^1}^2 = \int_\Omega |\nabla u|^2 \, \mathrm{d}x \leq C_2\|u\|^2$
Given $u, v \in H$ we'll say u is orthogonal to v if $(u,v) = 0$.
Given a subspace $M \subset H$, we'll write $M^\perp := \{v \in H : (u,v) = 0 \forall u \in M\}$

Proposition 1. Let $M \subset H$ be a closed subspace of H. Then $\forall x \in H, \exists y \in M, z \in M^\perp$ such that $x = y + z$.

Proof. Idea: y is the point closest to x in M.
Consider $x \notin M$. Define

$$d = \inf_{x' \in M} \|x - x'\|$$

Then there must be a sequence $\{y_n\} \subset M$ such that $\|x - y_n\| \to d$.
Recall:

$$\|X - Y\|^2 + \|X + Y\|^2 = 2\|X\|^2 + 2\|Y\|^2$$

Pick $X = x - y_n, Y = x - y_m$. Then,

$$\|y_m - y_n\|^2 + 4\left\|x - \frac{1}{2}(y_n + y_m)\right\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2$$

Therefore,

$$4d^2 + \|y_n - y_m\|^2 \leq 2\|x - y_n\|^2 + 2\|x - y_m\|^2$$

$$\|y_n - y_m\|^2 \leq 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4d^2$$

Taking limsup we see that $\{y_n\}$ is cauchy.
Define $z = x - y$. It is easy to see that $z \in M^\perp$.

□

Monday, 9/23/2024

Divergence from Linear Elliptic Operator

$$L(u) = -\frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) + b_i(x)u_{x_i} + c(x)u$$

Uniform ellipticity:
$a_{ij} = a_{ji}$ symmetric
$a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \forall \xi \in \mathbb{R}^n \forall x \in \Omega$ for some $\theta > 0$.
Meaning minimum eigenvalue is uniformly bigger than some θ
Review: $u \in H_0^1(\Omega)$ is a weak solution to $Lu = f$ in Ω for $f \in L^2(\Omega)$ and $u = 0$ on $\partial\Omega$ if:

$$B(u, v) := \int_\Omega \left[a_{ij}(x)u_{x_i}v_{x_j} + b_i(x)u_{x_i}v + c(x)uv\right] \mathrm{d}x = \int_\Omega fv \,\mathrm{d}x \,\forall v \in H_0^1(\Omega)$$

u is a weak solution to $Lu = f$ in Ω and $A\nabla u \cdot \nu_{\partial\Omega} = 0$ on $\partial\Omega$ if

$$B(u, v) = \int_\Omega fv \,\mathrm{d}x \,\forall v \in H^1(\Omega)$$

Note that B is a bilinear form.

Definition 11. A bounded linear operator $L : X \to Y$ that is <u>linear</u> that satisfies:

$$\|L\| := \sup_{\|x\|_X \leq 1} \|L(x)\|_Y < \infty$$

For linear operators, boundedness is the same as continuity.

Definition 12. A bounded linear functional on X is a bounded linear operator $L : X \to \mathbb{R}$

<u>Notation</u>: If u^* is a bounded linear functional we'll often write $\langle u^*, x\rangle$ for the evaluation of u^* at x.

$$\|u^*\| = \sup_{\|x\| \leq 1} \langle u^*, x\rangle$$

Definition 13. The dual of a Banach Space X is the set of bounded linear functionals on X. Notation: X^*

Theorem 13 (Riesz Representation Theorem)**.** Assume H is a Hilbert space with inner product $(\cdot,\cdot)$.
For every bounded linear functional $u^* : H \to \mathbb{R}$ there exists a unique $u \in H$ such that $(u^*, v) = (u, v) \forall v \in H$ and $\|u^*\| = \|u\|$

Proof. Suppose $u^* \in H^*$.
Let N be the nullspace of u^*:

$$N = \{v \in H : \langle u^*, v\rangle = 0\}$$

If $N = H$ then $\langle u^*, -\rangle = 0$ so u must be 0. Assume N is a proper subspace. N must be closed since u^* is continuous.
Fix $z \in N^\perp$, $\langle u^*, z\rangle \neq 0$.
Then $\forall x \in H$ we have:

$$\left\langle u^*, x - \frac{\langle u^*, x\rangle}{\langle u^*, z\rangle} z \right\rangle = \langle u^*, x\rangle - \langle u^*, x\rangle = 0$$

Thus, $x - \frac{\langle u^*, x\rangle}{\langle u^*, z\rangle} \in N$.

$$\left(x - \frac{\langle u^*, x\rangle}{\langle u^*, z\rangle} z, z \right) = 0$$

$$\implies (x, z) = \frac{\langle u^*, x\rangle}{\langle u^*, z\rangle} \|z\|^2$$

$$\implies \langle u^*, x\rangle = \left(\frac{\langle u^*, z\rangle}{\|z\|^2} z, x \right)$$

We just take $u := \frac{\langle u^*, z\rangle}{\|z\|^2} z$
Uniqueness: if $\langle u^*, x\rangle = (u_1, x) = (u_2, x)$ then $(u_1 - u_2, x) = 0$, choose $x = u_1 - u_2$ to deduce that $u_1 = u_2$.

$$\|u^*\| = \sup_{\|x\| \leq 1} \langle u^*, x\rangle = \sup_{\|x\| \leq 1} (u, x) \leq \sup_{\|x\| \leq 1} \|u\|_H \|x\|_H \leq \|u\|_H$$

On the other hand,

$$\|u\| = \frac{(u, u)}{\|u\|} = \left\langle u^*, \frac{u}{\|u\|} \right\rangle \leq \|u^*\|$$

Thus $\|u\| = \|u^*\|$

□

Solving a PDE [finally]

Poisson: Find a weak solution to $-\Delta u = f$ where $f \in L^2(\Omega)$ in Ω and $u = 0$ on $\partial\Omega$
We seek $u \in H_0^1(\Omega)$ satisfying

$$B(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x \, \forall v \in H_0^1(\Omega)$$

Recall: By Poincaré's inequality, an inner product for H_0^1 can be taken as:

$$(u, v)_{H_0^1} = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$$

We seek u such that:

$$(u, v)_{H_0^1} = \int_\Omega f v \, \mathrm{d}x \, \forall v \in H_0^1(\Omega)$$

Existence of unique u follows from Riesz Representation. We just need to show that $v \mapsto \int_\Omega fv \,\mathrm{d}x$ is a bounded linear functional on $H_0^1(\Omega)$.
It is obviously linear.

$$\sup_{\|v\|_{H_0^1}\le 1} \int_\Omega fv \,\mathrm{d}x \le \sup_{\|v\|_{H_0^1}\le 1} \|f\|_{L^2}\|v\|_{L^2} \le c_p \cdot 1\|f\|_{L^2}$$

Where c_p is the Poincaré constant: $\int u^2 \le c_p \int |\nabla u|^2 \,\forall u \in H_0^1$

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Last time, we used Riesz Representation Theorem to get existene of a weak solution to $Lu = f$ in Ω and $u = 0$ on $\partial\Omega$
for $Lu = -\frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) = f$
a_{ij} elliptic.
Weak formulation: seek $u \in H_0^1(\Omega)$ such that:

$$B[u,v] := \int_\Omega a_{ij}u_{x_i}v_{x_j}\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x \quad \forall v \in H_0^1(\Omega)$$

For $-\Delta u = f$ we seek u such that $B[u,v] = \int_\Omega \nabla u \cdot \nabla v$ where $B[u,v] = (u,v)_{H_0^1(\Omega)}$
How do we obtain a weak solution for

$$Lu := -\frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) + b_i(x)u_{x_i} + c(x)u = f$$

Consider:

$$B[u,v] = \int_\Omega a_{ij}u_{x_i}v_{x_j} + b_i(x)u_{x_i}v + c(x)uv\,\mathrm{d}x$$

Not symmetric so not an inner product but that's not the only problem. Consider 1-dim.

$$-u'' - u = 1 \quad \Omega = (0,\pi)$$

$u(0) = u(\pi) = 0$.
Suppose u solves this.
Multiply the ODE by $\sin x$ and integrate.

$$\int_0^\pi -\sin x \cdot u'' - \sin x \cdot u\,\mathrm{d}x = \int_0^\pi \sin x\,\mathrm{d}x = 2$$

Note

$$\int_0^\pi -\sin x \cdot u''\,\mathrm{d}x = [-\sin x \cdot u']_0^\pi - \int_0^\pi \cos x \cdot u'\,\mathrm{d}x = [-\cos x \cdot u]_0^\pi + \int_0^\pi \sin x \cdot u\,\mathrm{d}x$$

Therefore, by plugging it into the original argument,

$$0 = 2$$

Thus we don't have solutions!
Today: Lax-Milgram Lemma

Theorem 14 (Lax-Milgram Lemma)**.** Assume $B : H \times H \to \mathbb{R}$ is a bilinear form on a hilbert space H. Suppose,

i) $\exists \alpha > 0$ such that $B(u,v) \le \alpha\|u\| \cdot \|v\| \forall u,v \in H$

ii) $\exists \beta > 0$ such that $B[u,u] \ge \beta\|u\|^2$

Then we have the same conclusion as Riesz: $\forall f \in H^* \exists! u \in H$ such that $B[u,v] = \langle f, v\rangle \forall v \in H$

Proof. For any fixed $u \in H$ consider the map $v \mapsto B[u, v]$.
It is a <u>bounded linear functional</u> in H.
Apply Riesz Representation: for each fixed u there exists unique $w \in H$ such that $B[u, v] = (w, v) \forall v$.
We write $Au = w$
Then $B[u, v] = (Au, v)$
<u>Claim</u>: $A : H \to H$ is linear, bounded, 1-1
Linearity: $B[c_1u_1 + c_2u_2, v] = (A(c_1u_1 + c_2u_2), v)$
$= c_1B[u_1, v] + c_2B[u_2, v] = c_1A(u_1, v) + c_2A(u_2, v)$.
A is bounded since $\|Au\|^2 = (Au, Au) = B[u, Au] \le \alpha\|u\|\|Au\|$
A is 1-1 since $\beta\|u\|^2 \le B[u, u] = (Au, u) \le \|Au\|\|u\|$
So $\beta\|u\| \le \|Au\|$
So, if $u \ne 0$ we have $Au \ne 0$ so 0 is the only element in the kernel, so it is 1-1.
Claim: range of $A, R(A)$ is closed.
Consider $\{w_j\} \subset R(A)$. Suppose $w_j \to w$. $\exists\{u_j\} \subset H$ so that $Au_j = w_j$
Since w_j are cauchy, u_j are cauchy:
$\|w_j - w_k\| = \|Au_j - Au_k\| \ge \beta\|u_j - u_k\|$
$\implies$ u_j are cauchy. So $u_j \to u \in H$ and by continuity $Au_j \to Au = w$ and thus $R(A)$ is closed.
Claim: $R(A) = H$.
If not, apply projection lemma. $\exists w \in R(A) \perp$ so that $w \ne 0$.

$$\beta\|u\|^2 \le B[w, w] = (Aw, w)$$

Since $Aw \in R(A)$ we have the inner product is 0 and thus $w = 0$.
Now, let $f \in H^*$ be any bounded linear functional. Apply Riesz to show that there exists unique $w \in H$ such that $\langle f, v\rangle = (w, v) = (Au, v) = B[u, v]$ for unique u.

□

Theorem 15. For $Lu = -\frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) + b_i(x)u_{x_i} + c(x)u$ with a_{ij} elliptic and with $a_{ij}, b_i, c \in L^\infty(\Omega)$.
Then there exists a number $\gamma > 0$ such that $\forall \mu \ge \gamma$:
a weak $H_0^1(\Omega)$ solution exists to:

$$Lu + \mu u = f \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

$\forall f \in L^2(\Omega)$

Proof. We just apply Lax-Milgrim
We will prove:

i) $|B[u, v]| \le \alpha\|u\|_{H_0^1}\|v\|_{H_0^1}$

ii) $B[u, u] + \gamma\|u\|_{L^2}^2 \ge \beta\|u\|_{H_0^1}^2$

First Condition: Need to check $B[u, v] \le \alpha\|u\|_{H_0^1}\|v\|_{H_0^1}$ for some α

$$B[u, v] = \int_\Omega a_{ij}(x)u_{x_i}v_{x_j}b_i(x)u_{x_i}v + c(x)uv\,\mathrm{d}x$$

$$\le B[u, v] \le \int_\Omega n^2M|\nabla u||\nabla v| + nM|\nabla u||v| + M|u||v|\,\mathrm{d}x$$

$$\le n^2M\|u\|_{H_0^1}\|v\|_{H_0^1} + nM\|u\|_{H_0^1}\|v\|_{L^2} + M\|u\|_{L^2}\|v\|_{L^2}$$

Apply Poincaré

$$\le \alpha\|u\|_{H_0^1}\|v\|_{H_0^1}$$

So we have the first condition.

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Now we check the second condition.

$$B[u,u] = \int_\Omega a_{ij}(x)u_{x_i}u_{x_j} + b_i(x)u_{x_i}u + c(x)u^2 \, \mathrm{d}x$$

Use the fact that a_{ij} is elliptic, meaning $a_{ij}(x)\zeta_i\zeta_j \geq \theta|\zeta|^2$

$$\geq \theta \int_\Omega |\nabla u|^2 - nM|\nabla u||u| - M|u|^2 \, \mathrm{d}x$$

We don't have to worry about the $-M|u|^2$ because we can choose γ. We need to deal with $-nM|\nabla u||u|$.
Recall that AM-GM implies:

$$\varepsilon^2 a^2 + \frac{1}{4\varepsilon^2}b^2 \geq ab$$

$$\geq \theta \int_\Omega |\nabla u|^2 - nM\left(\varepsilon|\nabla u|^2 + \frac{1}{4\varepsilon^2}|u|^2\right) - M|u^2| \, \mathrm{d}x$$

$$\geq \theta \int_\Omega (1 - nM\varepsilon)\,|\nabla u|^2 - \left(M + \frac{nM}{4\varepsilon^2}\right)|u|^2 \, \mathrm{d}x$$

We can choose ε so that $nM\varepsilon = \frac{\theta}{2}$ and then choose appropriate γ. So we're done. □

Theorem 16. For L as defined, $\forall \mu \geq \gamma \forall f \in L^2(\Omega)$ there exists a unique weak solution to $L_\mu u = f, u = 0$ on $\partial\Omega$ where $L_\mu \coloneqq L + \mu$

Proof. Lax-Milgrim: $B_\mu[u,v] \coloneqq B[u,v] + \mu \int uv$ so L_μ has inverse. □

Functional Analysis

Definition 14 (Adjoint)**.** Given a bounded linear operator on Hilbert spaces

$$A : H \to H \quad H \text{ is a Hilbert space}$$

The adjoint of A is the bounded linear operator $A^* : H \to H$ defined by:

$$(x, A^*y) = (Ax, y) \quad \forall x, y \in H$$

If $A = A^*$ we say A is self-adjoint.
Example: If $H = \mathbb{R}^n$ then A is a matrix and A^* is the transpose.

Definition 15 (Compact bounded linear operator)**.** Let X, Y be Banach spaces. A bounded linear operator $K : X \to Y$ is compact if for every sequence $\{x_j\} \subset X$ such that $\|x_j\|_X$ is uniformly bounded then there exists a subsequence and $y \in Y$ such that:

$$Kx_{j_l} \to y$$

Example: Suppose $X = C([0,1])$ and $Y = C^1([0,1])$. $\forall f \in X$ define $Kf = u$ provided u solves $u'' = f$ on $(0,1)$ and $u(0) = 0, u'(0) = 0$. We have a formula:

$$u(x) = \int_0^x \int_0^y f(s) \, \mathrm{d}s \, \mathrm{d}y$$

Suppose $\|f_j\|_{C([0,1])} \leq M$. Then,

$$u_j(x) = \int_0^x \int_0^y f_j(s) \, \mathrm{d}s \, \mathrm{d}y$$

$$|u_j| \leq M$$

$$|u_j'| \le \left| \int_0^x f_j(s)\,\mathrm{d}s \right| \le M$$

$$\|u_j''\| = \|f_j\| \le M$$

Apply Arzela Ascoli to find:

$$u_{j_k} \xrightarrow{C^1} u_0$$

Thus K must be compact.

Theorem 17 (Fredhome Alternative)**.** Assume $K : H \to H$ is a compact operator on a Hilbert space. Then either:

i) The homogeneous equation $x - Kx = 0$ has a non-trivial solution. OR

ii) $\forall y \in H, \exists! x \in H$ such that $x - Kx = y$

Proof. There are 4 steps.
Step 1: Instead of $x - Kx$ we write $S := I - K$. $N(S)$ denotes the nullspace of S.
Claim: $\exists C$ such that:

$$\operatorname{dist}(x, N(S)) \le C\|Sx\| \,\forall x \in H$$

Proof: Suppose we cannot find such a C. Then we can find a sequence $\{\tilde{x}_k\} \subset H$ such that:

$$\operatorname{dist}(\tilde{x}_k, N(S)) \ge k\|S\tilde{x}_k\|$$

Replace with $x_k = \frac{\tilde{x}_k}{\|S\tilde{x}_k\|}$. Then $\|Sx_k\| = 1$ and

$$d_k := \operatorname{dist}(x_k, N(S)) \to \infty$$

Thus, we can find $\{y_k\} \subset N(S)$ such that:

$$d_k \le \|x_k - y_k\| \le 2d_k$$

define $z_k := \frac{x_k - y_k}{\|x_k - y_k\|}$ and so $\|z_k\| = 1$.
But $\|Sz_k\| = \frac{1}{d_k}\|Sx_k - Sy_k\| = \frac{1}{d_k}\|Sx_k\| = \frac{1}{d_k} \to 0$
Therefore $Sz_k \to 0$
K is compact, so we have subsequence $Kz_{k_j} \to y_0 \in H$.
$S_{z_{k_j}} \to 0 \implies z_{k_j} - Kz_{k_j} \to 0$ and since $K_{z_{k_j}} \to y_0$ we have $z_{k_j} \to y_0$.
Since $Sz_{k_j} = 0$ we have $Sy_0 = 0$ by continuity. Thus $y_0 \in N(S)$
However, $\operatorname{dist}(z_{k_j}, N(S)) = \inf_{y \in N(S)} \|z_{k_j} - y\| = \inf_{y \in N(S)} \left\| \frac{x_{k_j} - y_{k_j}}{\|x_{k_j} - y_{k_j}\|} - y \right\|$
$= \frac{1}{\|x_{k_j} - y_{k_j}\|} \inf_{y \in N(S)} \|x_{k_j} - \underbrace{(y\|x_{k_j} - y_{k_j}\| + y_k)}_{\in N(S)}\|$
$\ge \frac{1}{\|x_{k_j} - y_{k_j}\|} d(x_{k_j}, N(S)) \ge \frac{d_{k_j}}{2d_{k_j}} = \frac{1}{2}$
z_{k_j} converges to something in $N(S)$ but is a set distance away from $N(S)$, which is impossible. Thus we have proved the claim.

Monday, 9/30/2024

Step 2: Claim: Let $R(S)$ = range of S. Then, $R(S)$ is a closed subspae of H.
Proof: Consider a sequence $\{x_k\} \subset H$ so that $Sx_k \to y$ for some y. We must show that $y \in R(S)$.
From step 1, define $d_k := \operatorname{dist}(x_k, N(S)) \le C\|Sx_k\| \to \|y\|$.
d_k is uniformly bounded.
By projection theorem, we can find:

$$d_k = \|\underbrace{x_k - y_k}_{w_k}\| \quad y_k = \text{closest point to } x_k \text{ in } N(S).$$

Then $\|w_k\| \leq$ const.
Since K is compact, $Kw_{k_j} \to w_0 \in H$
Thus, $Sx_{k_j} = x_{k_j} - Kx_{k_j} \to y$
Thus, $x_{k_j} \to y + w_0$
Since S is continuous, $S(y + w_0) = y$
So, $y \in R(S)$.
Step 3: If $N(S) = \{0\}$ then $R(S) = H$.
Let R_j = range of $S^j [= S(S(\cdots S(H)))]$.
By Step 2, $\{R_j\}$ is a sequence of closed subspaces. Furthermore, it is non-increasing.
We claim that it eventually stabilizes.
Suppose, for contradiction, the sequence keeps decreasing.
By projection theorem, $\forall j \exists y_j \in R_j$ such that $\|y_j\| = 1$ and $\text{dist}(y_j, R_{j+1}) \geq \frac{1}{2}$.
Let $n > m$. We look at $Ky_m - Ky_n$.

$$Ky_m - Ky_n = (I - S)y_m - (I - S)y_n = y_m - (\underbrace{Sy_m}_{\in R_{m+1}} - \underbrace{Sy_n}_{\in R_{n+1} \subset R_{m+1}} + \underbrace{y_n}_{\in R_n \subset R_{m+1}})$$

Thus, $\|Ky_m - Ky_n\| \geq \frac{1}{2}$.
This contradicts the compactness of K.
Therefore, $\exists k$ such that $R_j = R_k \forall j > k$.
So, assume $N(S) = \{0\}$. Let $y \in H$.
We have $S^k y \in R_k = R_{k+1}$.
Therefore, $S^k y = S^{k+1} x$ for some x.
Therefore, $S^k(y - Sx) = 0$.
Since $N(S) = \{0\}$ we have $y = Sx$.
Step 4. If $R(S) = H$ then $N(S) = \{0\}$.
Let $N_j :=$ nullspace of S^j. Now, N_j is a non-decreasing sequence of closed subspaces.
Claim: $\exists k$ such that $N_j - N_k \forall j > k$. Argue by contradiction as before.
Assume $R(S) = H$.
$\forall y \in N_k, S^k y = 0$ and furthermore $\exists x_1$ such that $y = Sx_1$. Repeating, $y = S^k x$.
So, $S^{2k} x = 0$. Since null space stabilzes after k we have $S^k x = 0$. Therefore $y = 0$.
Thus $N_k = \{0\} \implies N(S) = \{0\}$.
In case ii why is $(I - K)^{-1}$ bounded?
Use Step 1: $\text{dist}(x, N(S)) \leq C\|Sx\| = C\|(I - K)x\|$
For ii we have $N(S) = \{0\}$ so:

$$\|x\| \leq C\|Sx\|$$

Writing $Sx = y$,

$$\|(I - K)^{-1} y\| \leq C\|y\|$$

□

Given a bounded linear operator $T : H \to H$ [could be a normed linear space as well],

Definition 16. The resolvent set of T is:

$$\rho(T) = \{\lambda \in \mathbb{R} : T - \lambda I \text{ is bijective}\}$$

The spectrum of T is:

$$\sigma(T) = \mathbb{R} \setminus \rho(T)$$

We can substitute real for complex.

Definition 17. $\lambda \in \sigma(T)$ is called an eigenvalue of T if $\exists x \in H$ such that $Tx = \lambda x$.

Note: $(\sigma(T) \setminus \{\text{eigenvalues}\})$ is called the continuous spectrum.

Wednesday, 10/2/2024

$$Lu := -(a^{ij}u_{x_i})_{x_j} + b^i(x)u_{x_i} + c(x)u$$

$$a^{ij} = a^{ji}, a^{ij}(x)\zeta_i\zeta_j \geq \theta|\zeta|^2, \theta > 0$$

$$a^{ij}, b^i, c \in L^\infty$$

Recall that adjoint is defined by

$$(Ax, y) = (x, A^*, y)$$

Can we find the adjoint of L?
Formal Adjoint of $L : L^*$

$$(Lu, v)_{L^2} = (u, L^*v)_{L^2}$$

$$(Lu, v)_L^2 = \int_\Omega -(a^{ij}u_{u_{x_i}})_{x_j}v + b^i u_{x_i}v + cuv \, \mathrm{d}x$$

$$= \int_\Omega a^{ij}u_{x_i}u_{x_j} - \mathrm{div}(v\vec{b})u + cuv \, \mathrm{d}x$$

$$= \int_\Omega -(a^{ij}v_{x_i})_{v_{x_j}}u - b^i v_{x_i}cx \, \mathrm{d}x = \int_\Omega -b^i v_{x_i} - b^i_{x_i} - b^i_{x_i}v + cv \, \mathrm{d}x$$

Define that to be L^*v
Lemma: $R(I + K) = N(I + K^*)$

Proof. $w \in N(I + K^*)o_p$
$\iff (x, (I + K^*)w = 0) \ \forall x \in H$
$\iff ((I + K)x, w) = 0 \ \forall x \in H$
$\iff w \in R(I + K) \perp$. ☐

Theorem 18. For L as defined, for $\Omega \subset \mathbb{R}^n$ bounded, $\partial\Omega \in C^1$, then either:

i) $\exists$ weak $H^1_0(\Omega)$ solution to $Lu = 0, u = 0$ on $\partial\Omega$.

ii) $\forall f \in L^2(\Omega)\exists!$ weak $H^1_0(\Omega)$ solution to $Lu = f, u = 0$ on $\partial\Omega$

Furthermore if i holds thereen $Lu = f, u = 0$ on $\partial\Omega$ has a weak solution $\iff$ $(f, v)_{L^2} = 0 \forall v$ such that $L^*v = 0, v = 0$ in $\partial\Omega$.

Example: $-u'' - u = \sin x$ and $u(0) = u(\pi) = 0$.
Here $Lu = -u'' - u$. We found $\nexists$ solution.
Note that L has non-trivial nullspace. $\sin x$ is in the nullspace.

$$\int \sin x \cdot \sin x \neq 0$$

Theorem not applicable.

Proof. Recall $\exists\gamma > 0$ such that

$$B_\gamma[u, v] = B[u, v] + \gamma \int_\Omega uv$$

where:

$$B(u, v) := \int_\Omega (a^{ij}u_{x_i}v_{x_j} + b^i u_{x_i}v + cuv) \, \mathrm{d}x$$

We can apply Lax-Milgrim to obtain a unique weak solution to $L_\gamma u = f$ where $L_\gamma u = Lu + \gamma u$. ch that: So $B_\gamma[u, v] = (f, v) \forall v \in H^1_0(\Omega)$
We say $L^{-1}_\gamma = u$ if this holds.

We seek a function $u \in H_0^1(\Omega)$ such that:

$$B_\gamma[u,v] = (f,v)_{L^2} + \gamma(u,v)_{L^2}$$

We want u such that:

$$u = L_\gamma^{-1}(f+\gamma u) = \underbrace{L_\gamma^{-1}(f)}_{=h} + \gamma L_\gamma^{-1}(u)$$

Note: $L_\gamma^{-1} : L^2 \to H_0^1$ or $L_\gamma^{-1} : L^2 \to L^2$
Let $h \coloneqq L_\gamma^{-1}(f)$.
So we're trying to solve $u - \gamma L_\gamma^{-1}(u) = h$.
Define $K \coloneqq \gamma L_\gamma^{-1}$
So we want to solve $(I-K)u = h$
$K : L^2 \to L^2$.
We want to use Lax Milgrim. Why must K be bounded?
Let $g \in L^2(\Omega)$, $\frac{1}{\gamma}K(g) = L_\gamma^{-1}(g) =: u$
$\|K(g)\|_{H_0^1} = \|\gamma u\| H_0^1$. $B_\gamma[u,v] = (g,v)\, \forall v$.
Pick $v = u$ then,

$$B_\gamma[u,u] \geq \beta \|u\|_{H_0^1}^2$$

$$\beta\|u\|^2 \leq B_\gamma[u,u] = (g,u) \leq \|g\|_{L^2}\|u\|_{L^2} \leq C_p\|g\|_{L^2}\|u\|_{H_0^1}$$

$$\gamma\beta\|u\|_{H_0^1} \leq \gamma C_p \|g\|_{L^2}$$

$$\|K(g)\|_{L^2} \leq C\|K(g)\|_{H_0^1} \leq \frac{\gamma C_p}{p}\|g\|_{L^2}$$

<u>Claim</u>: $K : L^2 \to L^2$ is compact.
Let $\{g_k\} \subset L^2$ $\|g_k\| \leq C$.

$$\|K(g_k)\|_{H_0^1} \leq \tilde{C}\|g_k\|_{L^2} \leq \text{const}$$

By Rellich-Kondrachov,

$$K(g_{k_j}) \xrightarrow{L^2} W$$

Now, apply Fredholme alternative. Then, $\forall h \in L^2 \exists!$ solution u to $(I-K)u = h$ or else the nullspace of $(I-K^*)$ is non-trivial.

$$(I-K)u = h \iff u \text{ weakly solution to } Lu = f$$

Then apply the Lemma.

$\square$

Friday, 10/4/2024

Recall: Given a bounded linear operator $T : H \to H$, the <u>resolvent set</u>:

$$\rho(T) \coloneqq \{\lambda \in \mathbb{R} : T - \lambda I \text{ is bijective}\}$$

Then the <u>spectrum</u> of T is:

$$\sigma(T) = \mathbb{R} \setminus \rho(T)$$

$\lambda \in \sigma(T)$ is an <u>eigenvalue</u> if $\exists x \in H$ such that $(T - \lambda I)x = 0$.
$\{\lambda \in \sigma(T) : \lambda \text{ is not an eigenvalue}\}$ is called the <u>continuous spectrum</u>.
Now, what if $T = K$ compact, linear operator?

We have seen that $I-K$ has a nontrivial nullspace or else it is invertible, and $(I-K)^{-1}$ is bounded.
Further, we have seen that for $\lambda \neq 0$ either $I - \frac{1}{\lambda}K$ has a nontrivial nullspace or it is invertible.
Then, $(I - \frac{1}{\lambda}K)^{-1}$ is bounded.
Nontrivial nullspace implies λ is an eigenvalue of K.
Thus, K compat $\implies$ no continuous spectrum [except perhaps 0].

Theorem 19. A compact operator $K : H \to H$ possesses at most a countable set of eigenvalues having no limit pointsexcept possibly 0.
Furthermore, Each eigenvalue has a finite multiplicity [dim $N(K - \lambda I)$ is finite].

Proof. Suppose not. Then we have an accumulation point $\exists\{\lambda_n\}$ of eigenvalues such that $\lambda_n \to \lambda \in (\mathbb{R} \cup \{\pm\infty\}) \setminus \{0\}$ and a sequence of linearly independent eigenvectors x_n.
Let $M_n = \operatorname{span}\{x_1, \cdots, x_n\}$.
M_n is a closed subspace.
Projection Lemma $\implies \exists\{y_n\} \subset M_n \setminus M_{n-1}$ so that $\|y_n\| = 1, \operatorname{dist}(y_n, M_{n-1}) \geq \frac{1}{2}$.
Let $S_\lambda \coloneqq \lambda I - K$.
For $n > m$: we have:

$$\lambda_n^{-1} K y_n - \lambda_m^{-1} K y_m = y_n - \lambda_n^{-1} S_{\lambda_n} y_n - y_m + \lambda_m^{-1} S_{\lambda_m} y_m = y_n + z$$

Claim: $z \in M_{n-1}$. To prove this, note that z is sum of elements of M_{n-1}.
$y_m \in M_m \subseteq M_{n-1}$.
Write $y_n = \sum_{j=1}^n c_j x_j$
$S_{\lambda_n} y_n = (\lambda_n I - K)\left(\sum_{j=1}^n c_j x_j\right) = \sum_{j=1}^n (\lambda_n c_j x_j - c_j \lambda_j x_j) = \sum_{j=1}^{n-1} (\lambda_n c_j x_j - c_j \lambda_j x_j)$
Thus $S_{\lambda_n} y_n \in M_{n-1}$
Also $S_{\lambda_m} y_m \in M_{m-1} \subseteq M_{n-1}$.
Now, $y_n = \lambda_n^{-1} K y_n$.
Therefore, $\|\lambda_n^{-1} K y_n - \lambda_m^{-1} K y_m\| \geq \frac{1}{2}$.
$\lambda_n^{-1} \|K y_n - \frac{\lambda_n}{\lambda_m} K y_m\| \geq \frac{1}{2}$.
If λ_m approaches finite value, by taking m, n large enough we get $\|Ky_n - Ky_m\| \geq \frac{\lambda}{4}$.
Contradiction.
If we have an infinite limit then LHS approaches 0 which is also not possible.

$\square$

Theorem 20. i) $\exists$ an at most countable set $\Sigma \subset \mathbb{R}$ such that $Lu = \lambda u + f$ in Ω and $u = 0$ on $\partial\Omega$ has a solutio $\forall f \in L^2$ provided $\lambda \neq \Sigma$.

ii) If Σ is infinite then writing $\Sigma = \{\lambda_n\}, \lambda_1 \leq \lambda_2 \leq \cdots$ then $\lambda_n \to \infty$.

iii) If $\lambda \in \Sigma$ then $Lu = \lambda u + f$ for $f \in L^2$ with $u = 0$ on $\partial\Omega$ is <u>solvable</u> $\iff$ $(f, v)_{L^2} = 0 \forall v \in N(L^* - \lambda I)$.

Recall: u is a weak solution to $Lu = \lambda u$ in $\Omega, u = 0$ on $\partial\Omega \iff B[u, v] = \lambda(u, v) \forall v \in H_0^1(\Omega)$.
Where $B[u, v] = \int_\Omega a_{ij}(x) u_{x_i} v_{x_j} + b_i(x) u_{x_i} v + c(x) uv \, \mathrm{d}x$.
Now, $B[u, v] = \lambda(u, v) \iff B_\gamma[u, v] \coloneqq B[u, v] + \gamma \int uv = (\gamma + \lambda)(u, v)$.
Where γ is sufficiently large to make $B_\gamma[u, v] \geq \beta \|u\|_{H_0^1}^2$
So u solves $(P) \iff$

$$u = L_\gamma^{-1}\left((\lambda + \gamma) u\right)$$

$$\iff u = \gamma L_\gamma^{-1}\left(\frac{\lambda + \gamma}{\gamma} u\right) = \frac{\lambda + \gamma}{\gamma} K u \text{ where } K = \gamma L_\gamma^{-1}$$

Thus, u solves $P \iff \left(K - \frac{\gamma}{\gamma + \lambda} I\right) u = 0$.
Then the only posssible limit point of eigenvalues of K is $0 \iff$ only possible limit point of eigenvalues of $L = \infty$.

Monday, 10/7/2024

Weak Convergence

Given a banach space X let X^* be the space of bounded linear functionals.
Given $x^* \in X^*$:

$$\|x^*\|_{X^*} := \sup_{\|x\|_X \le 1} \langle x^*, x \rangle$$

Definition 18. We say $\{x_n\} \subset X$ converges weakly to $x \in X$ if:

$$\forall x^* \in X^*,\ \langle x^*, x_n \rangle \to \langle x^*, x \rangle$$

Notation: $x_n \overset{X}{\rightharpoonup} x$

Example: for $1 < p < \infty$ If $X = L^p(\Omega)$ then when $\frac{1}{p} + \frac{1}{q} = 1$ we have $X^* \cong L^q(x)$.
$u_n \overset{L^p}{\rightharpoonup} u$ means:

$$\forall v \in L^q,\ \int_\Omega u_n v \,\mathrm{d}x \to \int_\Omega uv \,\mathrm{d}x$$

Facts:

- $x_n \to x \implies x_n \rightharpoonup x$
- $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\| \implies x_n \to x$
- $x_n \rightharpoonup x \implies \|x_n\| \le M$ for some M.

Definition 19. A reflexive Banach space X is one such that:

$$(X^*)^* = X$$

Example: $L^p(\Omega)$ for $1 < p < \infty$

If X is reflexive and $\|x_n\| \le M$ then $\exists \{x_{n_j}\}, x \in X$ such that $x_{n_j} \rightharpoonup x$.
If $x_n \rightharpoonup x$ then $\liminf_{n\to\infty} \|x_n\| \ge \|x\|$.

Proof. If $x^* \in X^*, \|x^*\| \le 1$ then,

$$\liminf_{n\to\infty} \|x_n\| \ge \liminf_{n\to\infty} \langle x^*, x_n \rangle = \langle x^*, x \rangle$$

We see our aswer by taking sup over all $\|x^*\| \le 1$. □

Basic Question:
Suppose $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ in Ω where L is an elliptic operator and $f \in L^2$ [eg $-\Delta u = f$]. Can one argue that u is better than $H^1(\Omega)$?
A (formal) calculation suggesting that this isn't a ridiculous question:
Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is smooth and compactly supported and solving $-\Delta u = f$ for $f \in L^2$.

$$\begin{aligned}
\infty > \int_{\mathbb{R}^\ltimes} f^2 \,\mathrm{d}x &= \int_{\mathbb{R}^n} (\Delta u)^2 \,\mathrm{d}x = \int_{\mathbb{R}^n} \nabla \cdot (\nabla u) \Delta u \,\mathrm{d}x \\
&= -\int_{\mathbb{R}^n} \nabla u \cdot \nabla(\Delta u) \,\mathrm{d}x = -\int_{\mathbb{R}^n} u_{x_j} \frac{\partial}{\partial x_j}(u_{x_i x_i}) \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} u_{x_i x_j} u_{x_i x_j} \,\mathrm{d}x = \int_{\mathbb{R}^n} |D^2 u|^2 \,\mathrm{d}x
\end{aligned}$$

So we have control over all second derivatives of u.

Difference Quotients

Definition 20. Given $u : \underset{\subset \mathbb{R}^n}{\Omega} \to \mathbb{R}$ given $h \in \mathbb{R}$ given $k \in 1, \cdots, n$, define:

$$D_k^h := \frac{u(x+he_k) - u(x)}{h}$$

We want to bound the difference quotient of the derivative to control the second derivaive.

Lemma 1: For $1 < p < \infty$, let $u \in W^{1,p}(\Omega)$ for $\Omega \subset \mathbb{R}^n$. Then $\forall \Omega' \subset\subset \Omega$ and for h such that $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$ we have:

$$\|D_k^h u\|_{L^p(\Omega')} \leq \|u_{x_k}\|_{L^p(\Omega)}$$

Proof. First assume $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$.

$$D_k^h u(x) = \frac{1}{h} \int_0^h u_{x_k} u(x_1, \cdots, x_k + s, \cdots, x_n) \, \mathrm{d}s$$

$$|D_k^h(x)|^p \underset{\text{Hölder}}{\leq} \frac{1}{h^p} \left(\int_0^h 1^{\frac{p}{p-1}} \, \mathrm{d}x \right)^{p-1} \left(\int_0^h |u_{x_k}(\cdots, x_k + s, \cdots)|^p \, \mathrm{d}x \right)$$

$$\int_{\Omega'} |D_k^h u(x)|^p \, \mathrm{d}x \leq \frac{1}{h} \int_{\{\operatorname{dist}(x, \partial\Omega) > h\}} \int_0^h |u_{x_k}(\cdots x_k + s \cdots)|^p \, \mathrm{d}s \, \mathrm{d}x$$

$$\underset{\text{Fubini}}{\leq} \frac{1}{h} \int_0^h \int_\Omega |u_{x_k}(x)|^p \, \mathrm{d}x \, \mathrm{d}s = \|u_{x_k}\|_{L^p(\Omega)}^p$$

Now we approximabe by C^1 functions.

□

Lemma 2: Let $u \in L^p(\Omega)$ for $1 < p < \infty$ and suppose for some $k \in \{1, \cdots, n\}$, $\exists M$ such that

$$\|D_k^h u\|_{L^p(\Omega')} \leq M$$

$\forall \Omega' \subset\subset \Omega, \forall h$ such that $\operatorname{dist}(\Omega', \partial\Omega) > h$. Then,

$$\|u_{x_k}\|_{L^p(\Omega)} \leq M$$

Proof. $\exists \{h_j\} \to 0, \exists v \in L^p(\Omega)$ such that:

$$D_k^{h_j} u \overset{L^p(\Omega)}{\rightharpoonup} v$$

[We are using diagonalization. We cannot directly go near $\partial\Omega$ but by making $h_j \to 0$ and taking subsequences, we get the convergence for whole Ω].

Furthermore, by lower semicontinuity of $\|\cdot\|_{L^p}$ under weak convergence: $\|v\|_{L^p(\Omega)} \leq M$.

Must still sho v is the weak k'th derivative of u.

Fix $\phi \in C_0^1(\Omega)$.

We have:

$$\lim_{h_j \to 0} \int_\Omega \phi D_k^{h_j} u \, \mathrm{d}x = \int_\Omega \phi v \, \mathrm{d}x$$

Now we do integration by parts on difference quotient. We also have:

$$\int_\Omega \phi D_k^{h_j} u \, \mathrm{d}x = \int_\Omega \phi(x) \left(\frac{u(x + h_j e_k) - u(x)}{h_j} \right) \mathrm{d}x$$

For first term, let $y = x + h_j e_k$.

$$= \int_{\operatorname{supp}(\phi + h_j e_k)} \frac{\phi(y - h_j e_k) u(y)}{h_j} \, \mathrm{d}y - \int_{\operatorname{supp}(\phi)} \frac{\phi(x) u(x)}{h_j} \, \mathrm{d}x$$

$$= -\int_\Omega \frac{\phi(y - h_j e_k) - \phi(y)}{-h_j} u(y)\,\mathrm{d}y = -\int_\Omega D_k^{-h_j}\phi(y)u(y)\,\mathrm{d}y$$

Now let $h_j \to 0$

$$\implies \lim_{h_j \to 0} \int_\Omega \phi D_k^{h_j} u\,\mathrm{d}x = -\int_\Omega \phi_{x_k} u\,\mathrm{d}x$$

Thus v is the weak x_k derivative of u.

□

Wednesday, 10/9/2024

Recall:

$$D_k^h u(x) := \frac{u(x + he_k) - u(x)}{h}$$

$$\int D_k^h u_1(x) u_2(x)\,\mathrm{d}x = -\int D_k^{-h} u_2(x) u_1(x)\,\mathrm{d}x$$

Lemma 1: Suppose $1 < p < \infty$. Let $u \in W^{1,p}(\Omega)$. Then, $\forall \Omega' \subset\subset \Omega$ such that $\operatorname{dist}(\Omega', \partial\Omega) > |h|$ we have:

$$\|D_k^h u\|_{L^p(\Omega)} \leq \|u_{x_k}\|_{L^p(\Omega)}$$

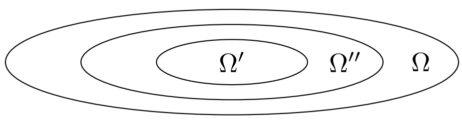
Lemma 2: Let $u \in L^p(\Omega), 1 < p < \infty$. Assume $\exists M > 0$ such that $D_k^h u \in L^p(\Omega')$ and $\|D_k^h u\|_{L^p(\Omega')} \leq M$ $\forall k, \forall \Omega' \subset\subset \Omega$ such that $\operatorname{dist}(\Omega', \partial\Omega) \geq |h|$.
Then u_{x_k} exists $\forall k$ and $\|u_{x_k}\|_{L^p(\Omega')} \leq M$.
Suppose $Lu := -\frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) + b_i(x)u_{x_i} + c(x)u$
Assume $a_{ij}(x)\zeta_i\zeta_j \geq \theta|\zeta|^2, a_{ij} = a_{ji}$
Assume $a_{ij} \in C^1(\Omega), b_i, c \in L^\infty(\Omega)$

Theorem 21 (Interior Regularity)**.** Assume $u \in H^1(\Omega)$ weak solution to $Lu = f$ in Ω for $f \in L^2(\Omega)$. Then $u \in H^2_{loc}(\Omega)$ with:

$$\|u\|_{H^2(\Omega')} \leq C(\Omega', \Omega, \theta, \cdots)\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

$\forall \Omega' \subset\subset \Omega$.

Proof. Consider $\Omega' \subset\subset \Omega'' \subset\subset \Omega$



Let ζ be a smooth cutoff function:
$\zeta \equiv 1$ in Ω'
$\zeta \equiv 0$ on $\Omega \setminus \Omega''$
$0 \leq \zeta \leq 1$
We have:

$$\int_\Omega a_{ij}(x)u_{x_i}v_{x_j} + b_i(x)u_{x_i}v + c(x)uv\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x$$

$\forall v \in H_0^1(\Omega)$
Note that,

$$\int_\Omega a_{ij}(x)u_{x_i}v_{x_j} + \boxed{b_i u_{x_i}}v + \boxed{cu}v\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x$$

Choose $v = -D_k^{-h}(\zeta^2 D_k^h u)$

in Ω' where $\zeta \equiv 1$ we have:

$$v = -\left(D_k^{-h}\left(\frac{u(x+he_k)-u(x)}{h}\right)\right)$$

$$= -\left(\frac{u(x-he_k+he_k)-u(x-he_k)-u(x+he_k)+u(x)}{-h^2}\right)$$

$$= -\left(\frac{u(x+he_k)+u(x-he_k)-2u(x)}{h^2}\right)$$

Using this v, we get:

$$I = \int_\Omega -a_{ij}u_{x_i}D_k^{-h}(\zeta^2 D_k^h u)_{x_i} - b_i u_{x_i} D_k^{-h}(\zeta^2) - cuD_k^{-h}(\zeta^2 D_k^h u)\,\mathrm{d}x$$

$$= -\int_\Omega f D_k^{-h}(\zeta^2 D_k^h u)\,\mathrm{d}x$$

$$I = \int_\Omega D_k^h(a_{ij}u_{x_i})\left(\zeta^2 D_k^h u_{x_i} + 2\zeta\varphi_{x_j}D_k^h u\right)\,\mathrm{d}x$$

$$= \int_\Omega \left(a_{ij}(x+he_k)D_k^h(u_{x_i}) + D_k^h(a_{ij})u_{x_i}\right)\left(\zeta^2 D_k^h u_{x_j} + 2\zeta\varphi_{x_j}D_k^h u\right)\,\mathrm{d}x$$

$$= \int_\Omega \zeta^2 a_{ij}(x+he_k)D_k^h(u_{x_i})D_k^h(u_{x_j}) + \text{others }\,\mathrm{d}x$$

$$\geq \theta\int_\Omega \zeta^2|D_k^h(\nabla u)|^2 + \text{ others }\,\mathrm{d}x$$

Write $\tilde{f} \coloneqq f - b_i u_{x_i} - cu \in L^2$.
Weak form becomes:

$$\int_\Omega a_{ij}(x)u_{x_i}v_{x_j}\,\mathrm{d}x = \int_\Omega \tilde{f}v\,\mathrm{d}x$$

We have:

$$\theta\int_\Omega \zeta^2|D_k^h(\nabla u)|^2\,\mathrm{d}x \leq \int_\Omega \tilde{f}D_k^{-h}(\zeta^2 D_k^h u) - a_{ij}(x+he_k)2\zeta\zeta_{x_j}D_k^h(u_{x_i})D_k^h u$$

$$-D_k^h(a_{ij})u_{x_i}\zeta^2 D_k^h u_{x_j} - D_k^h(a_{ij})u_{x_i}2\zeta\zeta_{x_j}D_k^h u\,\mathrm{d}x$$

Use $ab \leq \varepsilon^2 a^2 + \frac{1}{4\varepsilon^2}b^2$ to estimate terms 2,3,4.

$$\leq C\int_{\Omega''}\left(|D_k^h(\nabla u)||D_k^h u| + |D_k^h(\nabla u)||\nabla u| + |\nabla u||D_k^h u|\right)\zeta\,\mathrm{d}x$$

Note: $C \to \infty$ as $\Omega'' \to \Omega$ since it involves the derivative of the cutoff function.

Monday, 10/14/2024

Recall: weak formulation:

$$\int_\Omega a_{ij}u_{x_i}v_{x_i}\,\mathrm{d}x = \int_\Omega fv - b_i u_{x_i} - cuv\,\mathrm{d}x$$

$$=: \int_\Omega \tilde{f}v\,\mathrm{d}x, \tilde{f} \in L^2(\Omega)$$

Choose: $v = -D_k^{-h}(\zeta^2 D_k^h u)$
Start with LHS.
We arrived at:

$$LHS = \int_\Omega a_{ij}(x+he_k) D_k^h D_k^h (u_{x_j}) \zeta^2 \,\mathrm{d}x$$

$$+ \int_\Omega \Big[a_{ij}(x+he_k) 2\zeta\zeta_{x_i} D_k^h u D_k^h(u_{x_i}) + D_k^h(a_{ij}) 2\zeta\zeta_{x_j} D_k^h(u) u_{x_i}$$

$$+ D_k^h(a_{ij}) u_{x_i} D_k^h(u_{x_j}) \zeta^2 \Big] \,\mathrm{d}x$$

$$\implies LHS \geq \theta \int_\Omega |D^h(\nabla u)|^2 \zeta^2 \,\mathrm{d}x$$

$$-C \left\{ \int_\Omega |D_k^u||D_k^h(\nabla u)| + |D_k^h u||\nabla u| + |D_k^h(\nabla u)||\nabla u| \,\mathrm{d}x \right\}$$

Now use $\left(\varepsilon a - \frac{1}{2\varepsilon} b\right)^2 \geq 0 \implies ab \leq \varepsilon^2 a^2 + \frac{1}{4\varepsilon^2} b^2$
Now use our lemmas.

$$LHS \geq \theta \int |D^h(\nabla u)|^2 - \varepsilon^2 |D_k^h(\nabla u)|^2 - C_\varepsilon |\nabla u|^2$$

Pick $\varepsilon^2 = \frac{\theta}{2}$

$$LHS \geq \frac{\theta}{2} \int_\Omega |D_k^h(\nabla u)|^2 \zeta^2 \,\mathrm{d}x - C_\varepsilon \int_\Omega |\nabla u|^2 \,\mathrm{d}x$$

Now we estimate RHS with a little of L^2 norm v + a lot of L^2 norm $\tilde{f}$
By lemma 1:

$$\int v^2 \leq \int |\nabla(\zeta^2 D_k^h u)|^2$$

$$\leq C \int |D_k^h|^2 + |D_k^h(\nabla u)|^2 \,\mathrm{d}x$$

$$v \tilde{f} \,\mathrm{d}x \leq \varepsilon^2 \int v^2 + C_\varepsilon \int \tilde{f}^2$$

Pick $\varepsilon^2 = \frac{\theta}{4}$

$$\implies \frac{\theta}{4} \int_{\Omega'} \left| D_k^(\nabla u) \right|^2 \,\mathrm{d}x \leq C \int_\Omega (f^2 + |\nabla u|^2 + u^2) \,\mathrm{d}x$$

$$\leq C \int_\Omega f^2 + \|u\|_{H^1(\Omega)}^2 \,\mathrm{d}x$$

Apply Lemma 2
$u \in H^2_{loc}(\Omega)$ and $\forall \Omega' \subset\subset \Omega$

$$\int_{\Omega'} |D^2 u|^2 \,\mathrm{d}x \leq C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 \right)$$

Finally, we need to replace $\|u\|_{H^1}^2$ with $\|u\|_{L^2}^2$ on RHS.
Using a new cut-off function ζ such that $\zeta \equiv 0$ on $\Omega \setminus \Omega''$, we have:

$$\|u\|_{H^2(\Omega')} \leq C \left(\|f\|_{L^2(\Omega'')}^2 + \|u\|_{H^1(\Omega'')}^2 \right)$$

Now go back to $(*)$ weak formulation:
Choose $v = \zeta^2 u$

$$\int_\Omega a_{ij} u_{x_i} u_{x_j} \zeta^2 + a_{ij} u_{x_i} 2\zeta\zeta_{x_i} \,\mathrm{d}x = \int \tilde{f} \zeta^2 u$$

Again, by ellipticity and $ab \leq \varepsilon a^2 + C_\varepsilon b^2$,
Choosing ε small in terms of θ,

$$\int_{\Omega'} |\nabla u|^2 \,\mathrm{d}x \leq C \int_{\Omega''} f^2 + u^2 \,\mathrm{d}x$$

so we're done!

□

Wednesday, 10/16/2024

Higher Interior Regularity

Suppose $Lu \coloneqq -(a_{ij}u_{x_i})_{x_j} + b_i(x)u_{x_i} + c(x)u$
So far: Assume a_{ij} elliptic, $a_{ij} \in C^1$, $b_i, c \in C^\infty$ $f \in L^2(\Omega)$.
Then if $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ then $u \in H^2_{\text{loc}}(\Omega)$ and $\forall \Omega' \subset\subset \Omega \exists C(\Omega')$ such that:

$$\|u\|_{H^2(\Omega')} \leq C(\Omega')(\|f\|_{L^2} + \|u\|_{L^2})$$

What if a_{ij}, b_i, c are nicer, as is f? Then u should be nicer.
Idea: Consider the PDE satisfied (weakly) by u_{x_k} for some $k \in \{1, \cdots, n\}$.

$$\frac{\partial}{\partial x_k} L(u) = \frac{\partial}{\partial x_k} f$$

$$-(a_{ij}(x)(u_{x_k})_{x_i})_{x_j} - (a_{ij}(x))_{x_k}(u_{x_k})_{x_i} + (b_i(x))_{x_k}u_{x_i} + b_i(x)(u_{x_k})_{x_i}$$
$$+(c(x))_{x_k}u + c(x)u_{x_k} = f_{x_k}$$

This is not exactly allowed. So we express it weakly.
Then u_{x_k} weakly solves a new elliptic PDE.

Theorem 22. Let m = non-neg integer. Assume $a_{ij}, b_i, c \in C^{m+1}(\Omega)$. Assume $f \in H^m(\Omega)$
Ten if $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ we have:

$$u \in H^{m+2}_{\text{loc}}(\Omega)$$

and $\forall \Omega' \subset\subset \Omega$

$$\|u\|_{H^{m+2}(\Omega')} \leq C(\|f\|_{H^m} + \|u\|_{L^2})$$

Proof. By induction. □

Boundary Regularity

Theorem 23. Take $\Omega \subset \mathbb{R}^n$, open, bounded, $\partial\Omega \in C^2$. Take a_{ij} elliptic, $a_{ij} \in C^1, b_i, c \in L^\infty$.
Take $f \in L^2$.
Then if $u \in H^1_0(\Omega)$ is a weak solution to $Lu = f$ in Ω and $u = 0$ on $\partial\Omega$ then,

$$u \in H^2(\Omega)$$

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2} + \|u\|_{L^2})$$

Proof. First we assume the boundary is flat. If not we flatten the boundary
Suppose first that $B(0,1) \cap \Omega = B(0,1) \cap \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$.
Let ζ be a cutoff function.
$\zeta \in C^\infty, \zeta \equiv 1$ in $B(0, \frac{1}{2})$
$\zeta \equiv 0$ in $\mathbb{R}^n \setminus B(0,1)$

$$B[u,v] = \int fv \,\forall v \in H^1_0(\Omega)$$

Rewrite:

$$\int_\Omega a_{ij} u_{x_i} v_{x_j} \,\mathrm{d}x = \int_\Omega \widetilde{f} v \,\mathrm{d}x$$

For $|h|$ small take:

$$v = -D_k^{-h}(\zeta^2 D_k^h u)$$

for any $k \in \{1, \cdots, n-1\}$
To be a legal v for this, we need $v \in H_0^1(\Omega)$ in light of the cut-off function and $\operatorname{Tr}\big|_{\partial B(0,1)^+} u = 0$
So we can use this v in $*$.
The rest of the proof is the same for interior regularity. We obtain:

$$\|D_k^h(\nabla u)\|_{L^2(B(0,1)^+)} \leq C\left(\|f\|_{L^2} + \|u\|_{L^2}\right)$$

Since difference quotients are uniformly bounded we must have weak derivatives.
$\implies$ by lemma 2,

$$\|u_{x_i x_j}\| \leq C\left(\|f\|_{L^2} + \|u\|_{L^2}\right)$$

for all i, j except $i = j = n$.
How to control $u_{x_n x_n}$?
By interior regularity, $Lu = f$ at a.e. $x \in \Omega$.
We rewrite the PDE:

$$-a_{nn} u_{x_n x_n} = \underbrace{\sum_{(i,j) \neq (n,n)} (a_{ij}(x) u_{x_i})_{x_j} - b_i(x) u_{x_i} - c(x) u + f}_{:= \tilde{\tilde{f}}}$$

We have $\|\tilde{\tilde{f}}\| < \text{const}(\|f\|_{L^2} + \|u\|_{L^2})$

$$\implies \|u_{x_n x_n}\| \leq \frac{\text{const}}{\theta}\left(\|f\|_{L^2} + \|u\|_{L^2}\right)$$

Friday, 10/18/2024

We have proved the theorem for flat portion of $\partial\Omega$. For proving the general case, we flatten the boundary.
Locally write $\partial\Omega$ as a graph:

$$x_n = f(x_1, \cdots, x_{n-1})\, f \in C^2$$

We change variables.

$$y_j = x_j\, j = 1, \cdots, n-1$$

$$y_n = x_n - f(x_1, \cdots, x_{n-1})$$

$$y = \Phi(x)$$

[insert picture figure]
Then,

$$D\Phi(x) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -f_{x_1} & -f_{x_2} & \cdots & 1 \end{bmatrix}$$

$\det D\Phi = J\Phi = 1$.

By inverse function theorem, Φ is invertible. Let $\Psi = \Phi^{-1}$. Define:

$$\widetilde{u}(y) \coloneqq u(\Psi(y))$$

$$u(x) = \widetilde{u}(\Phi(x))$$

By chain rule,

$$u_{x_i} = \widetilde{u}_{y_k} \Phi^{(k)}_{x_i}$$

Weak form:

$$\int_{\Phi(B(x_0,R)\cap\Omega)} \left(\underbrace{a_{ij}(\Psi(y))\widetilde{u}_{u_k}\Phi^{(k)}_{x_k}\widetilde{v}_{y_l}\Phi^{(l)}_{x_j}}_{\widetilde{a}_{kl}\widetilde{u}_{y_k}\widetilde{v}_{y_l}} + b_i(\Psi(y))\widetilde{u}_{x_k}\Phi^{(k)}_{x_i} + c(\Psi(y))\widetilde{u} \right) \cdot 1 \,\mathrm{d}y$$

$$= \int_{\Phi(B(x_0,R)\cap\Omega)} f(\psi(y))\widetilde{v}(y)\,\mathrm{d}y$$

<u>Claim</u>: $\widetilde{u}$ solves an elliptic PDE weakly on a <u>flat</u> domain.
Define $\widetilde{a}_{kl}(y) \coloneqq a_{ij}\Phi^{(k)}_{x_i}\Phi^{(l)}_{x_j}$.
We're done one we check that:

$$\widetilde{a}_{kl} = \widetilde{a}_{lk}, \widetilde{a_{kl}}\eta_k\eta_l \geq \widetilde{\theta}|\eta|^2 \,\forall \eta \in \mathbb{R}^n$$

Firstly,

$$\widetilde{a}_{lk} = a_{ij}\Phi^{(l)}_{x_i}\Phi^{(k)}_{x_j} = a_{ji}\Phi^{(l)}_{x_i}\Phi^{(k)}_{x_j} = \widetilde{a}_{kl}$$

Then let $\eta \in \mathbb{R}^n$.

$$\widetilde{a}_{kl}\eta_k\eta_l = a_{ij}\Phi^{(k)}_{x_i}\Phi^{(l)}_{x_j}\eta_k\eta_l = a_{ij}\underbrace{\Phi^{(k)}_{x_i}\eta_k}_{=\xi_i}\underbrace{\Phi^{(l)}_{x_j}\eta_l}_{=\xi_j} \geq \theta|\xi|^2$$

Let $\xi_i = \Phi^{(k)}_{x_i}\eta_k$ and $\xi_j = \Phi^{(l)}_{x_j}\eta_l$.
Then, $\xi = D\Phi\eta \implies (D\Phi)^{-1}\xi = \eta \implies D\Psi\xi = \eta$.

$$|\eta| \leq |D\Psi||\xi| \leq C_{d\Omega}|\xi|$$

Therefore,

$$\widetilde{a}_{kl}\eta_k\eta_l \geq \theta|\xi|^2 \geq \frac{\theta}{C_{\partial\Omega}}|\eta|^2$$

So, this reduces to the case of flat boundary, which we have already proven.

□

Maximum Principles

Consider L in non-divergence form:

$$Lu \coloneqq -a_{ij}(x)u_{x_ix_j} + b_i(x)u_{x_i} + c(x)u$$

Theorem 24 (Weak Maximum Principle)**.** Assume u is a 'nice classical solution', meaning $u \in C^2(\Omega) \cap C(\overline{\Omega})$ for some bounded open $\Omega \subset \mathbb{R}^n$.
Further assume $c(x) \equiv 0$ in the definition of L.

i) If $Lu \leq 0$ in Ω then,

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

ii) If $Lu \geq 0$ in Ω then,
$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$$

Remarks:

1) If $Lu \leq 0$ we call u a subsolution. If $Lu \geq 0$ we call u a supersolution.

2) If $Lu = 0$ then both maximum and minimum are achieved on the boundary.

3) A weak maximum principle does not preclude the max also being achieved inside Ω.

4) If $c \not\equiv 0$, the maximum principle may fail.

   Example: If $\Omega = (0, \pi)$ and $Lu = -u'' - u = 0$ then $a_{11} = 1, b_1 = 0, c(x) = -1$. For Dirichlet boundary condition $u(0) = u(\pi) = 0$. Our answer can be $\sin x$. Then the maximum happens at $\frac{\pi}{2}$ not in boundary for $A > 0$.

Monday, 10/21/2024

Recap:

$$Lu \coloneqq -a_{ij}(x)u_{x_ix_j} + b_i()u_{x_i} + c(x)u_i$$

a_{ij} uniformly elliptic $a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2, \theta > 0, a_{ij} = a_{ji}.a_{ij}, b_i, c$ are continuous on $\overline{\Omega}, \Omega \subset \mathbb{R}^n$ is open bounded.
Then we have the weak maximum principle as shown above.

Proof. Case 1: Suppose $Lu < 0$ [strictly less than 0]. We proceed by contradiction to show a strong maximum principle.
Suppose $\exists x_0 \in \Omega$ such that $u(x_0) = \max_{\overline{\Omega}} u(x)$.
Consider $Lu(x_0)$. Note that $u_{x_i}\big|_{x_0} = 0$ since $\nabla u(x_0) = 0$. Thus,

$$Lu(x_0) = -a_{ij}(x_0)u_{x_ix_j}(x_0)$$

Linear Algebra Fact: If a matrix A is symmetric positive definite then A can be diagonalizable by an orthogonal matrix $\mathcal{O}$ so that $\mathcal{O}\mathcal{O}^T = I$. Then,

$$\mathcal{O}A\mathcal{O}^T = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\lambda_1, \cdots, \lambda_n > 0$$

Assume $u \in C^2$ and has a min at x_0.
Change variables: $y = x_0 + \mathcal{O}(x - x_0)$. Then,

$$u_{x_i} = u_{y_k}\mathcal{O}_{ki}$$

$$u_{x_ix_j} = u_{y_ky_l}\mathcal{O}_{ki}\mathcal{O}_{lj}$$

$$a_{ij}(x_0)u_{x_ix_j}(x_0) = \sum_{i,j}\left[\sum_{k,l} a_{ij}(x_0)u_{y_ky_l}\mathcal{O}_{ki}\mathcal{O}_{lj}\right]$$

$$= \sum_{k,l} u_{y_ky_l}\left[\sum_{i,j} a_{ij}(x_0)\mathcal{O}_{ki}\mathcal{O}_{lj}\right]$$

$$= \sum_{k,l} u_{y_ky_l}(\mathcal{O}A)\mathcal{O}^T_{jl}u_{y_ky_l}(x_0) = \lambda_1 u_{y_1y_1} + \cdots + \lambda_n u_{y_ny_n}$$

At a max, $u_{y_jy_j} \leq 0$ for all j. Then $Lu(x_0) \geq 0$ which gives us the contradiction.
Case 2: Suppose $Lu \leq 0$. We pertrub L to get back to the first case. For example:

$$u^\epsilon(x) := u(x) + \epsilon e^{\lambda x_1}$$

$$Lu^\epsilon = \underbrace{Lu}_{\leq 0} + L(\epsilon e^{\lambda x_1}) \leq -\epsilon\lambda^2 a_{11}(x)e^{\lambda x_1} + \epsilon b_1(x)\lambda e^{\lambda x_1}$$

$$= \epsilon\lambda e^{\lambda x_1}(-\lambda a_{11}(x) + b_1(x)) \leq \epsilon e^{\lambda x_1}(-\theta\lambda + \|b\|_{L^\infty})$$

Pick λ big enough so that $Lu^\epsilon < 0$. By case 1,

$$\max_{\Omega} u \leq \max_{\Omega} u^\epsilon(x) = \max_{\partial\Omega} u^\epsilon(x) \leq \max_{\partial\Omega} u + \epsilon \max_{\partial\Omega} e^{\lambda x_1} \leq \max_{\partial\Omega} u + \epsilon e^{\lambda R}$$

where $\Omega \subset B(0,R)$. Let $\epsilon \to 0$ to finish the proof.

□

Now we try to make sense of the case $c(x) \geq 0$.

Theorem 25 (Weak max princ. for $c(x) \geq 0$). Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$.
Assume $c(x) \geq 0 \forall x \in \Omega$.

i) If $Lu \leq 0$ in Ω then $\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+$

ii) If $Lu \geq 0$ in Ω then $\min_{\overline{\Omega}} u \geq -\max_{\partial\Omega} u^-$

Where $u^+(x) := \max(u(x), 0), u^-(x) = -\min(u(x), 0)$.

Note: If $Lu = 0$ had a solution, then $\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|$
Example: Let $Lu = -u'' + (4x^2 + 1)u$.
$b_i \equiv 0, c(x) = 4x^2 + 1 > 0$
$\Omega = (-1, 1)$.
Consider $u(x) = e^{-x^2} - 4$
$u' = -2xe^{-x^2}$
$u'' = -2e^{-x^2} + 4x^2e^{-x^2}$
$Lu = (2 - 4x^2)e^{-x^2} + (4x^2 + 1)(e^{-x^2} - 4) = 3e^{-x^2} - 16x^2 - 4 < 0$ on $(-1, 1)$
Max of u comes in the origin, which is -3. But it is not at the boundary! We need to be careful about the sign and positive part and negative parts.

Proof. Let $\Omega' = \{x \in \Omega : u(x) > 0\}$.
$\Omega' = \varnothing \implies u(x) \leq 0$ in $\overline{\Omega}$ so we're done.
If $\Omega' \neq \varnothing$ then we have:
[picture figure]
Let $Ku := Lu - c(x)u$, and K doesn't have the c term. So we can apply theorem 1 to Ω'.

$$Ku \leq -c(x)u \leq 0 \,\text{in}\, \Omega'$$

$$\max_{\Omega} u \leq \max_{\Omega'} u \leq \max_{\partial\Omega'} u = \max_{\partial\Omega} u^+$$

In $\partial\Omega$ for the negative part $u^+ \equiv 0$ so we can ignore that boundary, we're done!

□

Wednesday, 10/23/2024

Lemma 1 (Hopf Lemma)**.** Assume $u \in C^2(\Omega) \cap C^1(\Omega)$ and assume $c(x) \equiv 0$ Suppose $Lu \le 0$ in Ω and $\exists x_0 \in \partial\Omega$ such that:
$u(x_0) > u(x) \forall x \in \Omega$and
Ω satisfies an interior ball condition at x_0, namely $\exists y_0 \in \Omega, \exists r > 0$ such that $B(y_0, r) \subset \Omega$ with $x_0 \in \partial B(y_0, r)$.
Then $\frac{\partial u}{\partial \nu}(x_0) > 0$ where $\nu = \frac{x_0 - y_0}{|x_0 - y_0|}$ [outer normal].
If $c(x) \ge 0$ then the same conclusion holds provided $u(x_0) \ge 0$.
[pictures / fig bad example]
A <u>sufficient</u> condition for this to hold: if the boundary is given by a C^2 function [so the curvature never gets too extreme] it is enough for the boundary to be C_2.

<u>Note</u>: $\frac{\partial u}{\partial \nu}(x_0) \ge 0$ is immediate since $u(x_0) > u(x) \forall x \in \Omega$.
So the significance is the <u>strict inequality</u>.

Proof. Define $v(x) = e^{-\lambda |x|^2} - e^{-\lambda r^2}$, $\lambda > 0$ to be specified later.

$$v_{x_i} = -2\lambda x_i e^{-\lambda |x|^2}$$

$$v_{x_i x_j} = (4\lambda^2 x_i x_j - 2\lambda \delta_{ij}) e^{-\lambda |x|^2}$$

$$Lv = -a_{ij}(x)(4\lambda^2 x_i x_j - 2\lambda\delta_{ij})e^{-\lambda|x|^2} - 2\lambda b_i(x) x_i e^{-\lambda|x|^2} + c(x)e^{-\lambda|x|^2} - c(x)e^{-\lambda r^2}$$

$$\implies Lv \le \left(-4\lambda^2\theta|x|^2 + 2\lambda \operatorname{Tr} A + 2\lambda |b|_{L^\infty}|x| + |c|_{L^\infty}\right) e^{-\lambda|x^2|} - \cancel{c(x)e^{-\lambda r^2}}$$

This is < 0 for some choice of $\lambda = \lambda(\theta, \Omega, \operatorname{Tr} A, |b|_{L^\infty}, |c|_{L^\infty})$
WLOG $y_0 = 0$. Consider the annulus:

$$\left\{x : \frac{r}{2} < |x| < r\right\} = \mathcal{A}$$

[insert picture figure]
$\exists \varepsilon > 0$ such that:

$$u(x_0) \ge u(x) + \varepsilon v(x) \, \forall x \in \partial B\left(0, \frac{r}{2}\right)$$

since $u(x_0) > \max_{\partial B(0, \frac{r}{2})} u$
On $\partial B(0, r)$:

$$u(x_0) \ge u(x) + \varepsilon v(x) = u(x)$$

Note $L(u + \varepsilon v - u(x_0))$:

$$= \underset{\le 0}{Lu} + \varepsilon \underset{<0}{Lv} - \underset{\le 0}{c(x)u(x_0)} < 0$$

And we've shown that $u(x) + \varepsilon v(x) - u(x_0) \le 0$ on $\partial\mathcal{A}$.
By weak maximum principle,

$$u(x) + \varepsilon v(x) - u(x_0) \le 0 \,\text{in}\, \mathcal{A}$$

Also note:

$$u(x_0) + \varepsilon v(x_0) - u(x_0) = 0$$

Therefore,

$$\frac{\partial}{\partial \nu}(u(x) + \varepsilon v(x) - u(x_0)) \ge 0$$

$$\implies \frac{\partial}{\partial\nu}u(x_0) \geq -\varepsilon\frac{\partial}{\partial\nu}v(x_0)$$

Note that v is a radial function so $(-\varepsilon)\frac{\partial}{\partial\nu}v(x_0) = 2\varepsilon\lambda(x_0)e^{-\lambda r^2} > 0$.

$\square$

Theorem 26 (Strong Maximum Principle)**.** Assume $\Omega \subset \mathbb{R}^n$ is open, bounded and connected. Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.
Suppose $c \equiv 0$.

i) If $Lu \leq 0$ in Ω and if u attains its maximum over $\overline{\Omega}$ at an interior point, then $u \equiv \text{const}$.

ii) If $Lu \geq 0$ in Ω and if u attains its minimum over $\overline{\Omega}$ at an interior point then $u \equiv \text{const}$.

Proof. Let $M := \max_{\overline{\Omega}} u$. Let $S := \{x \in \Omega : u(x) = M\}$.
If $S = \Omega$ we're done.
If $S = \varnothing$ then we're done.
So suppose, by contradiction, $S \neq \Omega, \varnothing$.
[insert picture]
Choose $y \in \Omega \setminus S$ so that:

$$\operatorname{dist}(y, S) < \operatorname{dist}(y, \partial\Omega)$$

Draw the largest open ball B centered at y that doesn't intersect S.
Necessarily, $\exists x_0 \in \partial B \cap S$.
Thus $\Omega \setminus S$ satisfies an interior ball condition at x_0.
$u(x_0) > u(x) \forall x \in \Omega \setminus S$.
We apply Hopf lemma.
The outer normal derivative $\frac{\partial u}{\partial \nu}(x_0) > 0$.
This cannot be true, since $\frac{\partial u}{\partial\nu}(x_0) = \nabla u(x_0) \cdot \nu = 0$ since $\nabla u(x_0) = 0$ at an internal max. $\square$

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Theorem 27 (Strong Maximum Principle with $c(x) \geq 0$)**.** Assume $\Omega \subset \mathbb{R}^n$, bounded open with $\partial\Omega \subset C^2$. Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

i) If $Lu \leq 0$ in Ω, u achieves a non-negative max inside Ω then $u \equiv \text{const}$.

ii) If $Lu \geq 0$ in Ω, u achieves a non-positive min inside Ω then $u \equiv \text{const}$.

Proof. Identical to $c \equiv 0$ case. $\square$

Uniqueness

For $Lu = f, c(x) \geq 0, u = 0$ on $\partial\Omega$ we have seen a uniqueness result.

Theorem 28. $\Omega \subset \mathbb{R}^n$ open bounded an $\partial\Omega \in C^2$.Suppose u_1 and u_2 both solve $Lu = f$ in Ω with $c(x) \equiv 0$ and $\nabla u \cdot \nu = g$ on $\partial\Omega$ where $\nu =$ outer unit normal.
Then $u_1 - u_2 \equiv \text{const}$.

Proof. Let $v := u_1 - u_2$.
Then $Lv = Lu_1 - Lu_2 = f - f = 0$ in Ω.

$$\nabla v \cdot \nu = \nabla u_1 \cdot \nu - \nabla u_2 \cdot \nu = g - g = 0 \,\text{on}\, \partial\Omega$$

By the max principle either $v \equiv \text{const}$ or v attains its max at a point $x_0 \in \partial\Omega$.
Then, $v(x_0) > v(x) \forall x \in \Omega$.
We can use Hopf Lemma:

$$\nabla v(x_0) \cdot \nu > 0$$

this is a contradiction, so we're done. $\square$

Theorem 29. Assume $Lu \leq f$ in a open connected bounded domain $\Omega \subset \mathbb{R}^n$ and assume $c(x) \geq 0$.
Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then,

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C_1 \max_{\overline{\Omega}} f^+$$

where C_1 depends on the coefficients of L and Ω.
If $Lu = f$ then we can say

$$\max_{\overline{\Omega}} |u| \leq \max_{\partial\Omega} |u| + C_1 \max_{\overline{\Omega}} |f|$$

Proof. We use a Barrier construction.
Without loss of generality let's assume

$$\Omega \subset \{x \in \mathbb{R}^n : 0 < x_1 < d\}$$

for some d.
Let $Ku := Lu - cu = -a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i}$.
For $\lambda > 0$ to be chosen let's compute

$$K(e^{\lambda x_1}) = (-a_{11}(x)\lambda^2 + b_1\lambda)e^{\lambda x_1}$$

$$\leq (-\theta\lambda^2 + |b|_{L^\infty}\lambda)e^{\lambda x_1}$$

$$= -\theta\left(\lambda^2 - \frac{\lambda |b|_{L^\infty}}{\theta}\right)e^{\lambda x_1}$$

$$\leq -\theta \text{ for } \lambda \text{ large enough}$$

Goal: Pick a v such that $L(u-v) \leq 0, u - v \leq 0$ on $\partial\Omega$.
Pick $v(x) = \max_{\partial\Omega} u^+ + \left(\frac{e^{\lambda d} - e^{\lambda x_1}}{\theta}\right) \max_{\overline{\Omega}} f^+$
$Lv = Kv + cv \geq Kv$

$$= \frac{\max_{\overline{\Omega}} f^+}{\theta} K(e^{\lambda x_1}) \geq \max f^+$$

Note: $cv \geq 0$ since $c \geq 0, v \geq 0$.
Then, $L(u-v) \leq f - \max f^+ \leq 0$.
On $\partial\Omega$,

$$u - v = u - \max_{\partial\Omega} u^+ - \text{positive} \leq 0$$

By maximum principle,

$$\max_{\overline{\Omega}}(u-v) \leq \max_{\partial\Omega}(u-v)^+ \leq 0$$

Thus, $u - v \leq 0 \implies u \leq v \leq \max_{\partial\Omega} u^+ + \frac{e^{\lambda d}}{\theta} \max_{\overline{\Omega}} f^+$.
Choosing $C_1 = \frac{e^{\lambda d}}{\theta}$ solves our problem. $\square$

Recall that we had

$$\|u\|_{H^2(\Omega')} \leq C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

$\forall \Omega' \subset\subset' \Omega, C = C(\Omega', \Omega)$.
If, for example $Lu = f$ in Ω and $u = h$ on $\partial\Omega$ then,

$$|u| \leq \text{const}(f, h)$$