M702 ANT

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Abstract

Chapter 1: Local Class Field Theory (LCFT).

Chapter 2: p-divisible groups (eg LT formal groups) and associated Galois representations V and the Hodge-Tate Decomposition of $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ and also the diagonal action of $\mathscr{G}_K$.

Tate: p-divisible groups.

Chapter 3: Sen theory, Fontaine's period rings (φ, Γ)-modules.

1 Local Class Field Theory (LCFT)

1.1 Lubin Tate Theory

[N] Neukirch, Alg. NT
[S] Serre, Local Class Field Theory (Cassels-Frohlich)
[LT] Lubin, Tate Formal complex multiplication
K = non-archimedean local field (locally compact) $\supset \mathcal{O} = \mathcal{O}_K$ = valuation ring $\supset P_K$ = valuation ideal.
Residue Field $k = \mathcal{O}/P_K$, $\mathrm{char}(k) = p > 0, q := |k| = p^f$.
Normalized Valuation $v = v_K : K \twoheadrightarrow \mathbb{Z} \cup \{\infty\}, |a| = q^{-v(a)}$.
$U_K = \mathcal{O}_K^\times$.

Definition. $e(x) \in \mathcal{O}[[x]]$ (a formal power series) is called a Lubin-Tate (LT) series for the uniformizer π (fixed) if the following conditions are fulfilled:

- $e(x) \equiv \pi x \mod \deg 2$.
- $e(x) \equiv x^q \mod \pi$.

Set $\mathscr{E}_\pi$ = set of LT series for the uniformizer π.
Recall: Let R be any $\mathcal{O}$-algebra ($i : \mathcal{O} \to R$ ring homomorphism).
A formal $\mathcal{O}$-module over R is a 1-dimensional commutative formal group $F(x, y) \in R[[x, y]]$ over R (some people call it a formal group law) together with a unital (sending 1 to 1) ring homomorphism:

$$[\cdot]_F : \mathcal{O} \to \mathrm{End}_R(F) = \{f(x) \in R[[x]] \mid f(0) = 0, f(F(x, y)) = F(f(x), f(y))\}$$

such that $\forall a \in \mathcal{O} : [a]_F(x) = i(a)x \mod \deg 2$.
We have the following properties:
$F(x, y) = x + y +$ higher order terms
Associativity: $F(x, F(y, z)) = F(F(x, y), z)$
Commutativity: $F(x, y) = F(y, x)$.
$\implies \exists! \iota(x) \in R[[x]] : F(x, \iota(x)) = 0$. Also, $\iota(x) = -x +$ higher order terms.
If R is a local $\mathcal{O}$-algebra with maximal ideal M ($i^{-1}(M) = P_K, k = \mathcal{O}/P_K \to R/M$) then a formal $\mathcal{O}$-module F over R is callled a LT $\mathcal{O}$-module over R if in addition it is a formal $\mathcal{O}$-module and for any uniformizer π of K: $[\pi]_F(x) \equiv x^q \mod M$.

Remark. If F is a LT $\mathcal{O}$-module over $\mathcal{O}$ [$i : \mathcal{O} \xrightarrow{\mathrm{id}} \mathcal{O}$] then $[\pi]_F(x) \in \mathscr{E}_\pi$ [meaning it is a Lubin Tate series] for any uniformizer π.

Example. 1) $K = \mathbb{Q}_p, F = \widehat{\mathbb{G}}_m, \widehat{\mathbb{G}}_m(x,y) = x + y + xy = (1+x)(1+y) - 1$.

Then, $[\cdot] : \mathbb{Z}_p \to \mathrm{End}_{\mathbb{Z}_p}(\widehat{\mathbb{G}}_m), [a](x) = (1+x)^a - 1 := \sum_{n=1}^{\infty} \binom{a}{n} x^n, \binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!} \in \mathbb{Z}_p$ for any $a \in \mathbb{Z}_p, n \geq 1$.

Exercise. 1) $\forall a \in \mathbb{Z}_p \forall n \geq 0, \binom{a}{n}$ as defined above is in $\mathbb{Z}_p$.

2) If K is a proper extension of $\mathbb{Q}_p$ then $\binom{a}{n} \notin \mathcal{O}_K$ for infinitely many $a \in \mathcal{O}_K$.

2) $K = \mathbb{F}_q((t)), F = \widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a(x,y) \equiv x + y$. Set $[t](x) = tx + x^q$. Then,

$$\Bigg[\underbrace{\sum_{\nu=0}^{\infty} \alpha_\nu t^\nu}_{a}\Bigg](x) := \sum_{\nu=0}^{\infty} \alpha_\nu [t]^{\circ \nu}(x) = \sum_{n=1}^{\infty} a_n x^n \text{ where } a_1 = a$$

gives $F = \widehat{\mathbb{G}}_a$ the structure of a LT $\mathcal{O}$-module over $\mathcal{O}$.

Theorem 1.1.1. i) For all uniformizer π of K and any $e \in \mathscr{E}_\pi$ there exists unique LT $\mathcal{O}$-module F_e over $\mathcal{O}$ such that:

$$[\pi]_{F_e}(x) = e(x)$$

ii) $\forall e, e' \in \mathscr{E}_\pi$ there is an isomorphism of formal $\mathcal{O}$-modules $f : F_e \to F_{e'}$ ($f \in x\mathcal{O}[[x]], f(F_e(x,y)) = F_{e'}(f(x), f(y))$.

$\forall a \in \mathcal{O} : f([a]_{F_e}(x)) = [a]_{F_{e'}}(f(x))$.

$f'(0) \in \mathcal{O}^\times$.

iii) Let K^{nr} be the maximal unramified extension of K (inside some fixed algebraic closure $\overline{K}$) and let $K_{nr} := \widehat{K^{nr}}$ be the completion of K^{nr} and let $\mathcal{O}_{K_{nr}}$ be its valuation ring. Then for any two uniformizers π, π' of K and LT series $e \in \mathscr{E}_\pi$ and $e' \in \mathscr{E}_{\pi'}$, $\exists$ an isomorphism of formal $\mathcal{O}$-modules $F_e \to F_{e'}$ over $\mathcal{O}_{K_{nr}}$.

Formal Complex Multiplication

Let $\overline{K}$ be the fixed algebraic closure of $K \supset \mathcal{O}_{\overline{K}} \supset P_{\overline{K}}$. Let $\pi =$ the fixed uniformizer, $e \in \mathscr{E}_\pi, F_e =$ LT $\mathcal{O}$-module over $\mathcal{O}$.

Set $F[\pi^m] = \left\{ \alpha \in P_{\overline{K}} \mid \underbrace{[\pi^m]_{F_e}}_{e^{\circ m}(x)}(\alpha) = 0 \right\}$. This can be shown to be finite from Theorem 1.1.1.ii by setting $e'(x) = \pi x + x^q$. Then the isomorphism will provide a bijection to $F_{e'}[\pi^m, e' \in \mathscr{E}_\pi]$. Then the zeros of the power series are the zeros of the iteration of the polynomial. Hence the set is finite.

$L_{\pi,m} := K(F_e(\pi^m))$ called the field of π^m-torsion points of F_e. It doesn't depend on e, though it does depend on π.

Example. if $K = \mathbb{Q}_p$ and $e(x) = (1+x)^p - 1$ then, $L_{p,m} = \mathbb{Q}_p(\zeta - 1 \mid \zeta^{p^m} = 1) = \mathbb{Q}_p(\mu_{p^m})$.

If we take $e'(x) = px + x^p$, the power series and the torsion points $F_e[p^m]$ and $F_{e'}[p^m]$ are different but the fields $\mathbb{Q}_p(F_{e[p^m]})$ and $\mathbb{Q}_p(F_{e'}[p^m])$ has to be the same!

Theorem 1.1.2. i) $F_e[\pi^m]$ is a free $\mathcal{O}/(\pi^m)$ module of rank 1 [note that $[\pi^m]$ annihilates $[a](x)$ since $[\pi^m]_{F_e}(\alpha) = 0$].

ii) $\forall m \geq 1$ the maps $\mathcal{O}/(\pi^m) \to \mathrm{End}_{\mathcal{O}}(F_e[\pi^m]), a \mod \pi^m \mapsto [\alpha \mapsto [a](\alpha)]$.

Also, $\mathcal{O}^\times/(1 + (\pi^m)) \to \mathrm{Aut}_{\mathcal{O}}(F_e[\pi^m])$, same formula are isomorphism (of finite groups).

iii) $L_{\pi,m}$ does not depend on $e \in \mathscr{E}_\pi$ but depends on π. In particular, if $e'(x) = \pi x + x^q$ then $L_{\pi,m} = K(F_{e'}[\pi^m])$.

iv) $L_{\pi,m}$ is a finite purely ramified Galois extension (so it does not contain a proper unramified extension) of K of degree $(q-1)q^{m-1}$.

The map $G(L_{\pi,m}/K) \to \mathrm{Aut}_{\mathcal{O}}(F_e[\pi^m]) \overset{\text{ii, canonical}}{\cong} \mathcal{O}^\times/(1+(\pi^m))$ given by $\sigma \mapsto a \mod (1+(\pi^m))$.

If $\forall \alpha \in F_e[\pi^m]\colon \sigma(\alpha) = [a]_{F_e}(\alpha)$, is an isomorphism.

v) If $L_\pi = \bigcup_{m\geq 1} L_{\pi,m}$, then the maps in iv induce an isomorphism:

$$G(L_\pi/K) = \varprojlim_m G(L_{\pi,m}/K) \xrightarrow{\cong} \varprojlim \mathcal{O}^\times/(1+(\pi^m)) \cong \mathcal{O}^\times$$

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Recall: we fixed an algebraic closure $\overline{K}$. Residue field of $\overline{K} = \overline{k} =$ algebraic closure of $k = \mathbb{F}_q$.

Theorem 1.1.3. If L/K is abelian, $L_\pi \subset L$, and L/L_π is purely rammified, then $L_\pi = L$.

Proof. Proof uses the Hasse-Arf theorem, which says that the jumps (or breaks) of the upper ramification filtration $(G(L/K)^t, t \geq -1)$ are integers. □

Remark. $G(L_\pi/K)^m = \mathrm{Gal}(L_\pi/L_{\pi,m}), m \geq 0$.
$L_{\pi,0} := K$.

Let $K^{ab} \subset \overline{K}$ be the maximal abelian subextension.

Theorem 1.1.4. For any uniformizer π one has $K^{ab} = K^{nr}$.
$K^{nr} =$ maximal unramified extension $= K(\mu_n \mid p \nmid n)$.

Proof. Set $L_\pi^{nr} := K^{nr}.L_\pi \subset K^{ab}$. This gives us an exact sequence:

$$1 \longrightarrow G(K^{ab}/L_\pi^{nr}) \longrightarrow G(K^{ab}/L_\pi) \twoheadrightarrow G(L_\pi^{nr}/L_\pi) \longrightarrow 1$$

$$G(L_\pi^{nr}/L_\pi) \xrightarrow{\cong} G(\overline{k} \mid k) = \langle \varphi \rangle^{\mathrm{top}}$$

Where $\varphi(\overline{\alpha}) = \overline{\alpha}^q, \overline{\alpha} \in \overline{k}$.

$$\langle \varphi \rangle^{\mathrm{top}} := \varprojlim \varphi^{\mathbb{Z}}/\varphi^{n\mathbb{Z}} \cong \varprojlim \mathbb{Z}/n\mathbb{Z} =: \widehat{\mathbb{Z}}$$

Choose $\widetilde{\varphi} \in G(K^{ab}/L_\pi)$ such that $\widetilde{\varphi}\big|_{K^{nr}} = \varphi$

□

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<u>Recall</u>: $K =$ local nonarch.. field, $\pi =$ uniformizer, $e \in \mathscr{E}_\pi$ a LT seriess for π, F_e a LT formal $\mathcal{O}$-module, $L_\pi = \bigcup K(F_e[\pi^m]) \subset K^{ab}$ with topological isomorphism $\mathrm{Gal}(L_\pi/K) \underset{\iota_\pi}{\xrightarrow{\cong}} U_K = \mathcal{O}_K^\times$.
1.1.6 $\Longrightarrow$

$$K^\times = U_K \times \pi^{\mathbb{Z}} \longrightarrow G(K^{ab}/K)$$

$$(a, \pi^n) \longmapsto \underbrace{\iota_\pi^{-1}(a)}_{\text{acts trivially on } K^{nr}} \quad \widetilde{\varphi}^n$$

$\widetilde{\varphi} =$ Frobenius element of $G(K^{ab}/K)$.
The map:

$$U_K \xrightarrow[\cong]{\text{can}} G(L_\pi/K) \xleftarrow[\cong]{\text{can}} G(L_\pi K^{nr}/K^{nr}) \hookrightarrow G(K^{ab}/K)$$

is canoncial. Here, $K^{ab} = L_\pi K^{nr}$.

Definition. The Weil group W_K is defined by:

$$W_K = \left\{ \sigma \in G(\overline{K}/K) \mid \sigma\Big|_{K^{nr}} \in \varphi_{K^{nr}}^{\mathbb{Z}} \right\}$$

Here $\varphi_{K^{nr}} =$ <u>the</u> arithmetic Frobenius of K^{nr}.
We equip W_K with the coarsest topology which makes the <u>inertia subgroup</u>:

$$I_K = \left\{ \sigma \in G(\overline{K}/K) \mid \sigma\Big|_{K^{nr}} = \mathrm{id}_{K^{nr}} \right\}$$

an open subgroup, and I_K is equipped witth its profinite topology. Then,

$$W_K = \sqcup_{n\in\mathbb{Z}} I_L \widetilde{\varphi}^{\mathbb{Z}}$$

(disjoint union of open cosets) with $\widetilde{\varphi}$ as in 1.1.6.

Proposition 1.1.5. The abelianization $W_K^{ab} = W_K/\overline{[W_K, W_K]}$ is isomorphic to:

$$\left\{ \sigma \in G(K^{ab} \mid K) \mid \sigma\Big|_{K^{nr}} \in \varphi^{\mathbb{Z}} \right\}$$

The image of the homomorphism:

$$K^\times \to G(K^{ab}/K)$$

of 1..1.6 is W_K^{ab}.
$U_K \supset 1 + (p^m)$ is open.

Definition. Let Γ be a topological group and $\rho : \Gamma \to \mathrm{Aut}(V)$ be a representation of Γ as an E-vector space ($E =$ any field). ρ is called <u>smooth</u> if $\forall v \in V$ we have:

$$\mathrm{Stab}_\rho(v) = \{\gamma \in \Gamma \mid \rho(\gamma)(v) = v\}$$

is open.

Proposition 1.1.6 (ℓ-adic local Langlands correspondence for GL_1)**.** Let $\ell \neq p$ be a prime. Then the isomorphism $K^\times \to W_K^{ab}$ from 1.1.7 induces a bijection:

$$\left\{ \begin{matrix} \text{continuous homomorphisms} \\ W_K \to GL_1(\overline{\mathbb{Q}_\ell}) = \overline{\mathbb{Q}_\ell}^\times \end{matrix} \right\} \to \left\{ \begin{matrix} \text{smooth irreducible} \\ \text{rep's of } GL_1(K) = K^\times \\ \text{on } \overline{\mathbb{Q}_\ell}\text{-vector space} \end{matrix} \right\} / \cong$$

$$\chi \mapsto \left[K^\times \xrightarrow{\cong} W_K^{ab} \overset{\chi}{\dashrightarrow} \overline{\mathbb{Q}_\ell}^\times \right]$$

$$\begin{tikzcd} \chi \arrow[r, mapsto] & {[K^\times} \arrow[r, "\cong"] & W_K^{ab} \arrow[r, dash, "\chi"] & \overline{\mathbb{Q}}_\ell^\times] \\ & & W_K \arrow[u] \arrow[ur, "\chi"] & \end{tikzcd}$$

Proof. Main point: a smooth irreducible representation of $K^\times$ on a $\overline{\mathbb{Q}}_\ell^\times$ vector space is 1-dimensional. □

Remark. Proposition 1.1.8 is also true when $\overline{\mathbb{Q}}_\ell$ is replaced by $\mathbb{C}$ and with the appropriate modifications, when $\overline{\mathbb{Q}}_\ell$ is replaced by $\overline{Q}_p$.

1.2 1-dim formal groups: the functional equation lemma

Cf. Hazewinkel, Formal groups and Applications = [H1 Formal]
Here we let:

- K = any commutative ring
- $A \subset K$ subring
- p prime
- q power of p
- $\sigma : K \to K$ ring homomorphism
- $I \subset A$ ideal
- $s_1, s_2, s_3, \dots \in K$

We assume:

- $\sigma(A) \subset A$
- $\forall a \in A : \sigma(a) \cong a^q \mod I$
- $p \in I$ so A/I is an $\mathbb{F}_p$-algebra
- $\forall i \geq 1 : s_i I \subset A$
- $\forall b \in K \forall r \geq 0 : bI^r \subset A \implies \sigma(b) I^r \subset A$.

Lemma 1.2.1. Let $g(x) = \sum_{i=1}^{\infty} b_i x^i \in xA[[x]]$.
By HW1, $\exists! f_g(x) = \sum_{i=1}^{\infty} d_i x^i \in xK[[x]]$ so that,

$$f(x) = g(x) + \sum_{i=1}^{\infty} s_i (\sigma_*^i f)(x^{q^i}) \tag{1.2.1}$$

where $\sigma_*^i f_g$ is power series obtained from f_g obtained by applying σ^i to all coefficients.

Indeed, $d_n = \begin{cases} b_n, & \text{if } q \nmid n; \\ b_n + s_1\sigma(d_{n/q}) + \cdots + s_r \sigma(d_{n/q^r}), & \text{if } n = q^r m, q \nmid m. \end{cases}$

Lemma 1.2.2 (The functional equation lemma (FEL))**.** Let ll data be as above. Let $g(x) = \sum_{i=1}^{\infty} b_i x^i$ and $\overline{g}(x) = \sum_{i=1}^{\infty} \overline{b_i} x^i$ be in $xA[[x]]$ and assume $b_1 \in A^\times$. Then, $f_g(x) = b_1 x +$ higher order terms $\implies$ f_g has inverse f_g^{-1} w.r.t. composition. Then,

i) $F_g(x, y) \coloneqq f_g^{-1}(f_g(x) + f_g(y))$ is a formal group over A.

ii) $f_g^{-1}(f_{\overline{g}}(x)) \in xA[[x]]$.

iii) Given $h(x) = \sum_{i=1}^{\infty} c_n x^n \in xA[[x]], \exists \widehat{h}(x) = \sum_{n=1}^{\infty} \widehat{c_n} x^n$ s.t. $f_g(h(x)) = f_{\widehat{h}}(x)$.

iv) If $\alpha(x) \in xA[[x]], \beta(x) \in K[[X]]$, then $\forall r \geq 0 : \alpha(x) \equiv \beta(x) \mod I^r A[[x]] \iff f_g(\alpha(x)) \equiv f_g(\beta(x)) \mod I^r A[[x]]$

Lemma 1.2.3 (HW1)**.** Write $f_g(x) = \sum_{i=1}^{\infty} d_i x^i$ and write $n = q^r m, q \nmid m$. Then $d_n I^r \subset A$.

Lemma 1.2.4. Let $G(x, y) \in A[[x, y]]$ and $n = q^r m$, and $\ell > 0$. Then,

$$G(x, y)^{q^\ell n} \cong \left((\sigma_*^\ell G)(x^{q^\ell}, y^{q^\ell}) \right)^n \mod I^{r+1}$$

$(\sigma(a) \equiv a^q \mod I)$

Proof of (i) of FEL. Note that $f_g^{-1}(x) = b_1^{-1}x + h.o.t.$ Then,

$$F_g(x,y) = b_1^{-1}(b_1x + b_1y + h.o.t) = x + y + h.o.t \tag{1}$$

and associativity follows from the definition.
Write $F(x,y) = F_1(x,y)+F_2(x,y)+F_3(x,y)+\cdots$ with $F_d(x,y) \in K[x,y]$ homogeneous of degree d.
We want to show, $\forall d \geq 1, F_d(x,y) \in A[x,y]$.
We prove this by induction. Case $d = 1$ already done.
Assume $d \geq 2$ and the statement is true for $F_1, \cdots, F_{d-1}$.
Note:

$$\forall r \geq 2 : (F_1(x,y) + \cdots + F_{d-1}(x,y))^r \equiv F(x,y)^r \mod \deg d + 1 \tag{2}$$

(2) and 1.2.4 together imply that $\forall i \geq 1, n = q^r m, q \nmid m$ ($n = 1, r = 0$ are ok).

$$F(x,y)^{q^i n} \cong \left((\sigma_*^i F)(x^{q^i}, y^{q^i})\right)^n \mod \deg d + 1, I^{n+1} \tag{3}$$

By definition,

$$f(F(x,y)) = f(x) + f(y) \tag{4}$$

(4) $\implies$ (5):

$$(\sigma_* f)((\sigma_*^i F)(x,y)) = (\sigma_*^i f)(x) + (\sigma_*^i f)(y) \tag{5}$$

(1.1.2) = (6)

$$f(x) = g(x) + \sum_{i=1}^{\infty} s_i (\sigma_*^i f)(x^{q^i}) \tag{6}$$

Substitute $F(x,y)$ for x in (6). We get (7):

$$f(F(x,y)) = g(F(x,y)) + \sum_{i=1}^{\infty} s_i \sum_{n=1}^{\infty} \sigma^i(d_n) F(x,y)^{q^i n}$$

Then we use the 12.4 congruence and our knowledge about the integrality of s_i. Eventually it turns out that $F_d(x,y) \equiv 0 \mod A[[x,y]]$. Thus F_d has coefficients in A.

□