

Group Representations MATH 607

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Texts: Lang, Algebra, Revised Third Edition, Chapter 17 (sections 1-5) and 18 (sections 1-8)

Serre, Linear Representations of Finite Groups, Parts II and III

Monday, 8/26/2024

Today:

History

Modular

Quotients

Matrices

Lang XVII, Section 1

(Fake) History

History of Groups

Most notions (let's say what is a vector space, what is a group) were vague.

Originally, groups were seen as:

- Symmetry Groups S_n
- $GL_n(\mathbb{R})$ aka $n \times n$ invertible matrices
- Subgroups of the above
- Representations of the above

For representation, consider G and a homomorphism $G \rightarrow S_n$ [which is a group action $G \curvearrowright \{1, 2, \dots, n\}$] or a homomorphism $G \rightarrow GL_n$ [which is a group action on vector space].

Part I of this course will be Ring Theory.

Part I: Ring Theory

Module

Convention: R = Ring with unity

Definition (Left Module). Left Module is an abelian group M with a function $R \times M \rightarrow M$ so that $(r, m) \mapsto rm$ such that $R \times M \rightarrow M$ is \mathbb{Z} -bilinear.

Meaning, we have:

$$(r + r')m = rm + r'm$$

$$r(m + m') = rm + rm'$$

$$\text{Also } (rr')m = r(r'm)$$

$$\text{And finally } 1m = m$$

By default, module = left module (since Jim doesn't want Trump to get reelected, he prefers left module)

module / field [module over field] = vector space

We can have submodules $M' \triangleleft M$

We have quotients M/M'

We have the short exact sequence:

$$0 \rightarrow M' \rightarrow M \rightarrow M/M' \rightarrow 0$$

which means in each homomorphism, $\text{im} = \ker$

So, $M' \rightarrow M$ is injective and $M \rightarrow M/M'$ is surjective.

Also, kernel of $M \rightarrow M/M'$ is M'

Remark. Note that R is itself an R -module.

Convention: Submodule M of R = left ideal of R .

Left ideals are not enough to take quotients (like how we need normal subgroup for group quotients).

So we need two sided ideals.

Definition (Two Sided Ideals). $I \subset R$ is 2-sided ideal if I is abelian subgroup and $ri \in I, ir \in I$ aka “closed”.

Example. Consider a homomorphism $f : R \rightarrow R'$. Then $\ker f$ is a 2-sided ideal of R .

For ring homomorphism we need:

$$f(r + r') = f(r) + f(r')$$

$$f(rr') = f(r)f(r')$$

$$f(1) = 1$$

If $I \subset R$ is 2-sided then R/I is a quotient ring.

For example, $M_2(\mathbb{R})$ has no proper 2-sided ideal. But there exists left ideals!

$$\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \text{ is a left ideal}$$

Matrices are a good ‘source’ of non-commutative rings.

Given any ring R we can consider ring $M_n(R)$ of $n \times n$ matrices.

Given R -module M we can get $\text{End}_R(M) = \{f : M \rightarrow M, f \text{ is } R\text{-module map}\}$

We have $(f + g)m = f(m) + g(m), (fg)m = f(g(m))$.

This is a ‘coordinate free approach’ to matrices.

Remark. $M_n(R)$ and $\text{End}_R(R^n)$ often looks the same, but in general $M_n(R) \not\cong \text{End}_R(R^n)$.

Let’s first take $n = 1$. Let $r_0 \in R$.

Consider $R \rightarrow R$ map $r \mapsto r_0 r$

We don’t like this because this is not a left module map!!!

So this is not even in $\text{End}_R(R)$

What if we consider $r \mapsto r r_0$?

This is a left module map, aka $\in \text{End}_R(R)$

But $R \rightarrow \text{End}_R(R)$ is not a ring homomorphism.

So we are going to take the opposite ring.

Fix 1:

Given ring R , we can look into the mirror and find opposite ring R^{op}

Elements of R^{op} = elements of R .

$0, 1, +$ remain the same

But multiplication is reversed: define $r \cdot_{op} r' = r' r$

Alternate notation, we write op on elements.

$$\text{Then } r^{op}(r')^{op} = (r' r)^{op}$$

Then we have isomorphism $R^{op} \cong \text{End}_R(R)$ which is a ring homomorphism!

Exercise. 1) $R \cong R^{op} \iff \exists$ antiautomorphism $\alpha : R \rightarrow R$

Antiautomorphism means α preserves $0, 1, +$ but reverses multiplication

2) R commutative, then $(M_n R) \cong (M_n R)^{op}$

3) Real quaternions $\mathbb{H} \cong \mathbb{H}^{op}$

Remark. If you take right modules, you don’t need op .

There is a contravariant endofunctor in the category of rings which takes objects of rings to their opposite.

$\text{Ring}^{op} \rightarrow \text{Ring}$ [opposite category, not the same thing]

$R \mapsto R^{op}$

Fix 2: [From Lang]

Suppose we have module homomorphism $\phi : E = E_1 \oplus \cdots \oplus E_n \rightarrow F_1 \oplus \cdots \oplus F_m = F$

Then we have $E_j \rightarrow E \xrightarrow{\phi} F \rightarrow F_i$ which we define to be $E_j \xrightarrow{\phi_{ij}} F_i$

Then we have a matrix $M(\phi)$ so that $M(\phi) = (\phi)_{ij}$

Then for $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in E_1 \oplus \cdots \oplus E_n$

Then $\phi(x) = (\phi_{ij}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

So, if we have $E^n = E \oplus \cdots \oplus E$ [n times]

Lang says, there is a ring isomorphism

$$\text{End}_R(E^n) \xrightarrow{\cong} M_n(\text{End}_R E)$$

$$\phi \mapsto (\phi_{ij})$$

If $E = R$ as left module, then $\text{End}_R R \cong R^{op}$

By combining these, $\text{End}_R(R^n) \cong M_n(R^{op})$

Wednesday, 8/28/2024

Today:

Group ring

Category

Simple modules

Question: The course is about ‘group representations’. So why study rings?

Answer: A group representation [homomorphism $G \rightarrow GL_n(\mathbb{R})$] is exactly the same as a module over the ring $\mathbb{R}G$.

So knowing everything about modules would tell us everything about representation.

Abelian Category!

Suppose we have a ring R and a group G . We can get a ring out of G

Definition (Group Ring RG). As an abelian group, this is the free R -module with basis the elements of G .

Elements are symbols of the form $r_1 g_1 + \cdots + r_n g_n$ [finite linear combination].

0 is the trivial linear combination. So $0 = 0$

$1 = 1e = 1_R e_G$

Multiplication is defined in the obvious way.

$$(\sum_i r_i g_i)(\sum_j r'_j g'_j) = \sum_{i,j} r_i r'_j g_i g'_j$$

Suppose V is a R -module.

Then a homomorphism $\rho : G \rightarrow \text{Aut}_R(V) \leftrightarrow V$ is RG -module.

$$\rho \mapsto (\sum_i r_i g_i)v := \sum_i r_i \rho(g_i)v$$

$g \mapsto (v \mapsto gv) \leftarrow V$ RG module.

Example. $C_2 = \{1, t\}$

Then we have $\mathbb{Z}C_2 = \{a + bt \mid a, b \in \mathbb{Z}, t^2 = 0\} = \mathbb{Z}[t]/(t^2)$

Note that $(1+t)(1-t) = 1 - t^2 = 0$ so we have zero divisors.

Take $C_\infty = \langle t \rangle$

Then $\mathbb{Z}C_\infty = \mathbb{Z}[t, t^{-1}]$ the laurent polynomial ring.

$\mathbb{Q}C_\infty = \mathbb{Q}[t, t^{-1}]$ is a PID [since it is a euclidean ring]

Now we see categories.

If we fix R then we have a functor $\text{Group} \rightarrow \text{Ring}$ given by $G \mapsto RG$

Or we could say we have a functor $\text{Ring} \times \text{Group} \rightarrow \text{Ring}$ given by $(R, G) \mapsto RG$

Definition. A category \mathcal{C} consists of:

- objects $\text{Ob } \mathcal{C}$
- morphism $C(X, Y)$ for $X, Y \in \text{Ob } \mathcal{C}$
- compositions $C(X, Y) \times C(Y, Z) \rightarrow C(X, Z)$ given by $(g, f) \mapsto f \circ g$
- identity $\text{Id}_X \in C(X, X) \forall X \in \text{Ob } \mathcal{C}$

Such that we have:

- associativity: $(f \circ g) \circ h = f \circ (g \circ h)$
- composition with identity: $\text{Id}_Y \circ f = f = f \circ \text{Id}_X$ for $f \in C(X, Y)$

For example in the category of groups, we have objects groups and morphisms homomorphism.

Morphism notations: $f : X \rightarrow Y$ or $X \xrightarrow{f} Y$ for $f \in C(X, Y)$

Definition. $f : X \rightarrow Y$ is isomorphism if $\exists g : Y \rightarrow X$ such that $f \circ g = \text{Id}$, $g \circ f = \text{Id}$. Then we say X and Y are isomorphic and write $X \cong Y$.

Example. Example of Categories:

- Set
- Ring
- Group
- Ab (Abelian Groups)
- R -modules (objects are modules, morphisms are homomorphisms $h(rm) = rh(m)$)
- Given a group G we can get a category BG such that:
 $\text{Ob } BG = \{*\}$ and $BG(*, *) = G$

In this category, there is only one object $*$. The elements of the group are morphisms.

Definition. Functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is $F : \text{Ob } \mathcal{C} \rightarrow \text{Ob } \mathcal{D}$ given by $X \mapsto F(X)$

And $F : C(X, Y) \rightarrow \mathcal{D}(F(X), F(Y))$ such that

$X \xrightarrow{f} Y$ gives us $F(X) \xrightarrow{F(f)} F(Y)$

such that $F(f \circ g) = F(f) \circ F(g)$ and $F(\text{Id}_X) = \text{Id}_{F(X)}$

Example. Unit Functor $\text{Ring} \rightarrow \text{Group}$ given by $R \mapsto R^\times = \{r \in R \mid \exists s \in R, rs = 1 = sr\}$

For example, $\mathbb{Q}^\times \cong C_2 \oplus \mathbb{Z}^\infty [= \pm p_1^{e_1} p_2^{e_2} \dots]$

$\mathbb{Z}^\times \cong \{\pm 1\} = C_2$

$(\mathbb{Z}C_2)^\times \cong \{\pm 1, \pm t\} \cong C_2 \times C_2$

Definition. R is a division ring (= skew field) if $1 \neq 0$ and $R^\times = R - 0$.

Definition. Quaternions

$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d, \in \mathbb{R}\}$

Where $i^2 = j^2 = k^2 = -1$

$ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$

This is a division ring since we can write down inverses.

$\alpha = a + bi + cj + dk$ gives us $\bar{\alpha} = a - bi - cj - dk$

So, $\text{norm}(\alpha) = \alpha \bar{\alpha} = a^2 + b^2 + c^2 + d^2$

So, $\alpha^{-1} = \frac{\bar{\alpha}}{\text{norm}(\alpha)}$

Remark. Note that the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is a subgroup of $\mathbb{H}^\times = GL_1(\mathbb{H})$.
So, \mathbb{H} is a $\mathbb{R}Q_8$ module.

Theorem 1 (Weddenburn's Little Theorem). a. A finite commutative domain is a field [easy]
b. A finite skew field is a field [aka commutative]

a is easy: suppose F is finite commutative domain. For $0 \neq f \in F$, consider multiplication by f as a map $F \rightarrow F$. It is injective, and finiteness implies surjective. So, it is bijective, and there exists inverse.
eg \mathbb{Z}/p is a field.

Simple Modules

These are like primes. We also have some analogue of prime factorization.

Definition. R -module E is simple if:
 $E \neq 0$

No proper submodules, aka $M \triangleleft E \implies M = 0$ or E

In other words, E is a simple module if it only has two submodules: 0 and E .

eg simple \mathbb{R} -modules are 1 dim vector spaces, aka \mathbb{R}

Exercise. a) \mathbb{R}^2 is a simple $M_2(\mathbb{R})$ -module

b) Express $M_2(\mathbb{R})$ as direct sum of simple modules.

Friday, 8/30/2024

Exercise. Suppose finite $G \neq 1$ and $R \neq 0$ Prove that RG has zero divisors.

Definition. Direct product of rings $R \times S$, addition and multiplication is done componentwise.

It is a product in the category of rings. aka:

$$\begin{array}{ccccc} & & T & & \\ & f_1 \swarrow & \vdots f \downarrow & \searrow f_2 & \\ R & \xleftarrow{\pi_1} & R \times S & \xrightarrow{\pi_2} & S \end{array}$$

for any pair of ring homomorphisms $T \xrightarrow{f_1} R$ and $T \xrightarrow{f_2} S$ we have a unique ring homomorphism $f : T \xrightarrow{f} R \times S$ so that the diagram commutes.

Definition. $e \in R$ is an idempotent if $e^2 = e$.

0, 1 are trivial idempotents.

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is an idempotent in $M_2(\mathbb{R})$

$(0, 1)$ is an idempotent in $\mathbb{R} \times \mathbb{R}$

If e is an idempotent so is $1 - e$

Definition. Idempotent $e \in R$ is central if $\forall r$ we have $er = re$

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is not central, but $(0, 1)$ is.

Exercise. A ring can be written as a product ring, aka $R \cong R_1 \times R_2$ with $R_i \neq 0$ if and only if there exists a nontrivial central idempotent.

Semisimple Modules

Definition. E is a simple R -module if it doesn't have any nontrivial submodules.
If $E \neq 0$ and $M \triangleleft E$ then $M \neq 0$ or $M = E$

Example. R^2 is a simple $M_2\mathbb{R}$ -module.

$\mathbb{R} \times 0$ is a simple $\mathbb{R} \times \mathbb{R}$ module.

$\mathbb{Z}/p\mathbb{Z}$ is a simple \mathbb{Z} -module

Lemma 2. [Schur's Lemma]: Let E, F be simple R -modules. Then any nonzero homomorphism $f : E \rightarrow F$ is an isomorphism.

Proof. $f \neq 0$ means $\ker f \neq E$ and $\text{im } f \neq 0$.

Since they are submodules, $\ker f = 0$ and $\text{im } f = F$

So f is bijective. □

Corollary 3. If E is simple, then $\text{End}_R E$ is a skew field [any non-zero element is invertible]

Example. Commutative example: $\text{End}_{M_2\mathbb{R}}(\mathbb{R}^2)$ is a skew field.

In fact, $\text{End}_{M_2\mathbb{R}}(\mathbb{R}^2) \cong \mathbb{R}$

Definition (Direct Sum). Suppose $M_i \triangleleft M$ for $i \in I$

Then, $M = \bigoplus_{i \in I} M_i$ means, $\forall m \in M$ we have $m = \sum_{i \in I} m_i$ with $m_i \in M_i$ uniquely.

There are notions of internal and external direct sums. The above is an internal direct sum.

External direct sum: given $\{M_i\}_{i \in I}$ we can construct $\bigoplus_{i \in I} M_i$

Proposition 4 (Universal Property). Given a collection of homomorphisms $\{t_i : M_i \rightarrow N\}_{i \in I}$, it extends directly to a homomorphism $\bigoplus M_i \rightarrow N$. We denote this by $\bigoplus f_i$

Remark. Note: Maps to product are easy, maps from direct sum are easy.

Proposition 5 (1.2, Lang XVII). Suppose we have isomorphism $E_1^{n_1} \oplus \dots \oplus E_r^{n_r} \xrightarrow{\cong} F_1^{m_1} \oplus \dots \oplus F_s^{m_s}$ with E_i and F_j simple and non-isomorphic [ie for all $k \neq i, E_k \not\cong E_i$ and $k \neq j, F_k \not\cong F_j$]

Then $r = s$ and there exists a permutation $\sigma \in S_r$ so that $E_j \cong F_{\sigma(j)}$ and $n_j = m_{\sigma(j)}$

Corollary: If E is a finite direct sum of simple modules, then the isomorphism class of simple components of E and multiplicities are well-defined.

Proof. We use Schur's Lemma.

We write ϕ as a matrix $(\phi_{ji} : E_i^{n_i} \rightarrow F_j^{m_j})$

Since ϕ is injective, for all i there exists a j such that $\phi_{ji} \neq 0$

Then, $E_i \cong F_j$ by Schur's Lemma

Note that F_j are isomorphic. So, for all i , the j such that $\phi_{ji} \neq 0$ is unique!

We also get $\sigma : \{1, \dots, r\} \rightarrow \{1, \dots, s\}$ so that $\sigma(i) = j$

Since σ^{-1} exists σ^{-1} exists, and thus $r = s$

Since ϕ is an isomorphism, individual $\phi_{ji} : E_i^{n_i} \rightarrow F_{\sigma(i)}^{m_{\sigma(i)}}$ are isomorphisms.

To complete the proof, we need a lemma

Lemma: Let E be simple. If $E^n \cong E^m$ then $n = m$

Proof of lemma; Let $D = \text{End}_R E$. By Schur's Lemma, D is a division ring.

Since $E^n \cong E^m$, we have $\text{End}_R(E^n) \cong \text{End}_R(E^m)$

So, $M_n(D) \cong M_m(D)$

Also, isomorphism not just as rings, but also as D -modules.

Every module over a skew field is free, and the number of dimensions is the same.

So, $n^2 = m^2 \implies n = m$

This finishes the proof. □

Lang XVII section 2

Theorem 6. Let E be an R -module. Then TFAE:

SS1: E is a sum of simple modules [so, we can write $m \in E$ as sum of m_i but it is not unique]

SS2: E is a direct sum of simple modules [we can write as a sum, and it's unique]

SS3: Every submodule of E is a summand.

$F \triangleleft E \iff$ we can find F' so that $E = F \oplus F'$

SS3' : any monomorphism $F \rightarrow E$ 'splits'

SS3'' Short exact sequence

$$0 \rightarrow F \rightarrow E \rightarrow H \rightarrow 0$$

splits.

This leads us to:

Definition. E is semisimple if it satisfies one of the above.

Davies: SS2 is best

eg: $R = \mathbb{R} \times \mathbb{R}$

$E = \mathbb{R} \times \mathbb{R}$ is semisimple but not simple.

Because: $E = \mathbb{R} \times 0 \oplus 0 \times \mathbb{R}$

Wednesday, 9/4/2024

Recap: Semisimple modules.

Lemma 7. If $E = \sum_{i \in I} E_i$ with E_i simple. Then, $\exists J \subset I$ such that $E = \bigoplus_{j \in J} E_j$

Corollary 8. SS1 \implies SS2

Proof. Let $J \subset I$ be maximal such that $\sum_{j \in J} E_j = \bigoplus_{j \in J} E_j$

This exists by Zorn's lemma.

$\forall i \in I - J$, we have $E_i \cap \bigoplus_{j \in J} E_j \neq \emptyset$ by maximality.

Since E_i is simple, $E_i \subset \bigoplus_{j \in J} E_j$. Therefore, $E = \bigoplus_{j \in J} E_j$. □

True or False? Every module has a maximal proper submodule.

False!!! Exercise.

Exercise. a) If $M \triangleleft F$ proper and M maximal, then F/M is simple.

b) Find a ring R , module M which does not have proper maximal submodules.

c) If F is a finitely generated R -module, then it is contained in a proper maximal submodule.

Proof of SS2 \implies SS3. Suppose $F \triangleleft E = \bigoplus_{i \in I} E_i$ with E_i simple. Let $J \subset I$ be maximal such that:

$$F + \bigoplus_{j \in J} E_j = F \oplus \bigoplus_{j \in J} E_j$$

Take any $i \in I - J$. Then, $E_i \cap \left[F \oplus \bigoplus_{j \in J} E_j \right] \neq 0$ by maximality of J .

Since E_i is simple, $E_i \subset F \oplus \bigoplus_{j \in J} E_j$.

Therefore, $E = F \oplus \underbrace{\bigoplus_{j \in J} E_j}_{F'}$.

We have found F' , which proves SS3. □

Proof of SS3 \implies SS1.

Lemma 9. $0 \neq F \triangleleft E$ and E satisfies SS3. Then, there exists simple finitely generated $S \triangleleft F$.

Plan: $M \triangleleft F_0 \triangleleft F \triangleleft E$.
 \neq f.g.

Then, choose $0 \neq v \in F$. Let $F_0 = Rv$.

Exercise. M exists. [Zorn's Lemma]

Let $E = \sum_{\text{simple } S \triangleleft E} S$.

Then, by SS3, $E = E_0 \oplus E'_0$.

Lemma and definition of E_0 implies: $E'_0 = 0$. So, E is indeed a sum of simple R -modules. We're done! □

Proposition 10 (2.2). Every quotient module and submodule of a semisimple module is semisimple.

Proof. Quotients: Suppose $M = E/N$. We have surjective $f : E \rightarrow M$ with E semisimple.

SS1 implies $E = \sum_{i \in I} S_i$ with S_i simple.

Then, $M = \sum_{i \in I} f(S_i)$

Schur's lemma implies $f(S_i)$ is either 0 or simple, so M satisfies SS1.

Submodules: Suppose $F \triangleleft E$ with E semisimple. SS3 implies $E = F \oplus F'$. Thus $E \cong E/F'$, so it is semisimple by the quotient result. □

Preview:

Definition. A ring R is semisimple if and only if all R -modules are semisimple.

Lang defines semisimple differently: A ring R is semisimple if it is semisimple as an R -module.

Theorem 11 (Artin-Weddenburn Theorem). A ring is semisimple if and only if it is isomorphic to a finite product of matrix rings over division algebras:

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$$

$\mathbb{C}G, \mathbb{R}G$ are semisimple. We also have the result:

Theorem 12 (Maschke's Theorem). The group ring kG is semisimple if G is finite and k is a field of characteristic prime to G .

This also works with $\text{char } k = 0$. It is in fact an if and only if.

So $\mathbb{F}_p G$ is also semisimple given $p \nmid |G|$

Proof. Outline: let $|G| = n$. We will verify SS3.

Let $F \triangleleft E$ be kG modules.

k is a field, so there exists a k -linear projection $\pi : E \rightarrow F$ such that $\pi(f) = f$ for $f \in F$ [take a basis of F as a k -vector space, complete it to a basis of E].

Now, define an 'average'.

$$\pi'(e) = \frac{\sum_{g \in G} g\pi(g^{-1}e)}{n}$$

Then, $\pi' : E \rightarrow F$ is a kG -linear projection, meaning $\pi'(ge) = g\pi'(e)$.

Then $E = \text{im } \pi' \oplus \ker \pi'$

□

Friday, 9/6/2024

Lang XVII, Section 3

“Density Theorem”

Suppose R is a ring and E is a R -module. Then we have maps $R \times E \rightarrow E$ by multiplication on the left.

Definition (Commutant). $R' = R'(E) = \text{End}_R(E)$ is a ring.

$\phi \in R' \iff \phi : E \rightarrow E$ such that $\phi(re) = r\phi(e)$. It ‘commutes with E ’.

Note that E is also an R' -module, with $R' \times E \rightarrow E$ given by $(\phi, e) \mapsto \phi(e)$.

Definition (Double Commutant). We can iterate on the previous definition.

$$R'' = R'(R'E) = \text{End}_{R'}(E)$$

Therefore,

$$R'' = \text{End}_{R'}(E) = \text{End}_{\text{End}_R(E)}(E)$$

This means, $f \in R'' \iff f : E \rightarrow E, \forall \phi \in R', f \circ \phi = \phi \circ f$. So, things in R'' :

commute with things which commute with $r \in R$.

Example. Suppose $R = \mathbb{R}$ and $E = \mathbb{R}^n$. Then,

$$R' = \text{End}_{\mathbb{R}}(\mathbb{R}^n) = M_n(\mathbb{R})$$

$$R'' = \text{End}_{M_n(\mathbb{R})}(\mathbb{R}^n) \xleftarrow[r]{rI} \mathbb{R}$$

Suppose $V =$ vector space.

$V^* = \text{Hom}(V, \mathbb{R})$

Then we have evaluation map $ev : V \rightarrow V^*$ given by $v \mapsto (\phi \mapsto \phi(v))$.

ev is 1-1.

ev is onto iff $\dim V < \infty$.

With inspiration from this, we define,

Definition (Evaluation map). $ev : R \rightarrow R''$ given by $r \mapsto (e \mapsto re)$

We define $f_r : E \rightarrow E$ given by $f_r = ev(r)$

Proposition 13. a) $f_r \in R''$

b) ev is a ring homomorphism.

Proof. a) $f_r(\phi(e)) = r\phi(e) = \phi(re)\phi(f_r(e))$

b) $ev(r + r') = ev(r) + ev(r'), ev(1) = 1$.

$$(ev(r))(ev(r'))e = ev(r)(r'e) = rr'e = ev(rr')e$$

□

Lemma 14 (3.1). Suppose E is semisimple over R , $e \in E$ and $f \in R''$

Then $\exists r \in R$ such that $re = f(e)$ [i.e. $f(e) = ev(r)(e)$]

Proof. E is semisimple, and Re is a submodule. Therefore, we can write $E = Re \oplus F$.

Define $\pi : E \rightarrow E$ be projection to Re .

Then $\pi \in E' \implies f \circ \pi = \pi \circ f \implies f(e) = f(\pi(e)) = \pi(f(e)) = re$ for some $r \in R$. □

We will prove a stronger version of this lemma called the Jacobson Density Theorem.

Theorem 15 (3.2, Jacobson Density Theorem). Suppose E is semisimple over R

$e_1, \dots, e_n \in E$

$f \in R''$

Then, $\exists r \in R$ such that $re_i = f(e_i) \forall i$.

Therefore, if E is finitely generated over R , then $R \rightarrow R''$ is onto.

Proof. We use a diagonal trick.

Special Case: E is simple.

Idea: Apply the lemma on E with $\underline{e} = (e_1, \dots, e_n)$ and $f^n : E^n \rightarrow E^n$ such that $f(y_1, \dots, y_n) = (f(y_1), \dots, f(y_n))$.

We need to check that $f \in R'(R'(E))$ to apply it.

This would imply that $f^n \in R'(M_n R) \underset{E \text{ simple}}{=} R'(R'(E^n))$

Therefore, $\exists r$ such that $r\underline{e} = f^n(\underline{e})$. This finishes the proof.

For E semisimple, key idea is $f^n \in R'(R'(E))$ as above. [Complicated for infinite sums. We avoid.]

□

Application:

Theorem 16 (Burnside's Theorem). Suppose k is an algebraically closed field.

Take subring R such that $k \subset R \subset M_n(k)$

If $k^n (= E)$ is a simple R -module, then prove that:

$$R = M_n(k)$$

Exercise. Suppose D_{2n} is the dihedral group of order $2n$, aka

$$D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, sr s^{-1} = r^{-1} \rangle$$

Let $\zeta_n = e^{2\pi i/n} \in \mathbb{C}$

Then we can define a homomorphism $D_{2n} \rightarrow GL_2(\mathbb{C})$ given by:

$$\begin{aligned} r &\mapsto \begin{bmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{bmatrix} \\ s &\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

This gives us a ring map $\pi : \mathbb{C}D_{2n} \rightarrow M_2\mathbb{C}$

Prove the following:

- Prove that \mathbb{C}^2 is a simple $\mathbb{C}D_{2n}$ module [can be done without technology]
- Use Burnside's theorem to show that π is onto.

Note that Burnside's theorem doesn't work if k is not algebraically closed.

We have:

$$\mathbb{R} \subset \mathbb{C} \subset M_2\mathbb{R}$$

since we can embed \mathbb{C} into $M_2\mathbb{R}$.

\mathbb{C} is a simple R module, but $\mathbb{C} \neq M_2\mathbb{R}$

Proof of Burnside's Theorem. Step 1: We show that $\text{End}_R(E) = k$

Note that, $k \subset \underset{\text{central}}{\text{End}_R(E)} \subset \underset{\text{finite dim}/k}{\overline{\text{End}_k(E)}}$

$\forall \alpha \in \text{End}_R(E), k(\alpha)$ is a field and finite dimensional $/k$.

Therefore, $k(\alpha) = k$ since k is algebraically closed.

Thus, $\alpha \in k$. This finishes Step 1.

Step 2: We show that $R = \text{End}_k(E)$.

$\overline{R} \subset \overline{\text{End}_k(E)}$ by hypothesis.

Suppose $A \in \text{End}_k(E)$. Let e_1, \dots, e_n be a k -basis for $E = k^n$.

Density theorem implies: $\exists r \in R$ such that $Ae_i = re_i$ for all i .

Therefore, $A = r \in R$.

□

Monday, 9/9/2024

Today:

Density Theorem

Characters determine representation

Artin-Wedderburn Theorem

Homework due Monday 9/16, Exercises 1-7

Recall Jacobson Density Theorem:

If E is semisimple over R , $e_1, \dots, e_n \in E$ and $f \in R''$ then,

$$\exists r \in R \text{ s.t. } f(e_i) = re_i \forall i$$

Recall that R'' is defined as follows:

$$f \in R'' \iff f : E \rightarrow E \text{ s.t. } \forall \phi \in R' = \text{End}_R E, f \circ \phi = \phi \circ f$$

Also recall Burnside's Theorem:

Suppose k is an algebraically closed field, and $k \subset R \subset M_n(k)$ are subrings

If k^n is a simple R -module, then

$$R = M_n(k)$$

3.7 Existence of Projection Operators

Theorem 17. Suppose $E = V_1 \oplus \dots \oplus V_m$, simple non-isomorphic R -modules. Then, for any i , there exists $r_i \in R$ such that,

$$r_i v = \begin{cases} v, & \text{if } v \in V_i; \\ 0, & \text{if } v \in V_j, i \neq j \end{cases}$$

So, each projection map is just multiplication.

Proof. This is a consequence of the density theorem.

Choose nonzero $e_k \in V_k$.

Let $f = \pi_i : E \rightarrow E$ which is a projection on V_i .

Note that $f \in R''$ since for all $\phi \in R', \phi(V_k) \subset V_k$ [Schur's Lemma, non-isomorphic].

Density theorem $\implies \exists r_i \in R$ such that $r_i e_k = \pi_i(e_k)$.

Note that $V_k = Re_k$ so $\forall v \in V_k, v = re_k$.

So, $r_i v = r_i re_k = r \pi_i(e_k) = \pi_i(re_k) = \pi_i(v)$

Which is what we wanted. □

Correction to the Existence of Projection Operators

Suppose k is a field, R is a k -algebra so that R is semisimple. Suppose R -module $E = V \oplus V', \dim_k E < \infty$.

For all simple $L \triangleleft V, \forall L' \triangleleft V'$ then $L \cong L'$

Then, $\exists r \in R$ such that for all $e \in E$,

$$re = \begin{cases} e, & \text{if } e \in V; \\ 0, & \text{if } e \in V'; \end{cases}$$

Proof. We apply density theorem. Since we have finite dimension, we have:

$$\{e_1, \dots, e_n\} = (k\text{-basis of } V) \cup (k\text{-basis of } V')$$

Let $\pi_V : E \rightarrow E$ be the projection on V .

Then, $\pi_V \in R''$ [the second commutant] since $\forall \phi \in R', \phi(v) \subset V, \phi(v') \subset V'$.

Density theorem implies $\exists r$ such that $re_i = \pi_V(e_i)$.

Then $\forall a \in k \subset \text{center } R$,

$$r(ae_k) = a(re_k) = a\pi_V(e_k) = \pi_V(ae_k)$$

Therefore, $re = \pi_V(re)$. □

Question: What is a k -algebra?

Following Atiyah-McDonald, let k be a commutative ring [often but not always a field]. Then,

$$R \text{ is a } k\text{-algebra} \stackrel{\text{def}}{\iff} \text{homomorphism } h : k \rightarrow R, h(k) \subset \text{center}(R)$$

Example. Any ring is a \mathbb{Z} -algebra, homomorphism sends n to $1 + 1 + \cdots + 1$

k field, $R \neq 0 \implies k \hookrightarrow R$

k -algebra $\iff k \subset \text{center}(R)$

Corollary 18 (3.8). Suppose $\text{char } k = 0$, R is a k -algebra, E, F semisimple over R , finite dimensional over k .

For $r \in R$, let:

$f_r^E : E \rightarrow E$ be $f_r^E(e) = re$

$f_r^F : F \rightarrow F$ be $f_r^F(f) = rf$

If $\text{Tr}(f_r^E) = \text{Tr}(f_r^F)$ for all $r \in R$,

Then $E \cong F$ as R -modules.

Proof. Let V be a simple R -module.

Suppose $E = V^n \oplus$ direct sum of simple R -modules not isomorphic to V

$F = V^m \oplus$ direct sum of simple R -modules not isomorphic to V

We want to show $n = m$

Let $r_v \in R$ be the projection operation from 3.7.

Then, $\text{Tr}(f_{r_v}^E) = \text{Tr}(r_v \cdot : E \rightarrow E) = \dim_k V^n = n \dim_k V$

Similarly, $\text{Tr}(f_{r_v}^F) = m \dim_k V \implies n = m$

□

Corollary 19 (Characters determine representations). Suppose k is a field and $\text{char } k = 0$. Let G be a finite group. Suppose:

$\rho : G \rightarrow GL_n(k)$

$\rho' : G \rightarrow GL_m(k)$

with kG -modules $E = k^n$ over ρ and $F = k^m$ over ρ'

If $\text{Tr}(\rho(g)) = \text{Tr}(\rho'(g))$ for all g ,

Then $E \cong F$ as kG -modules.

Note that, substituting $g = 1$ gives us:

$\text{Tr}(\rho(1)) = \text{Tr}(\rho'(1)) \implies \text{Tr}(I) = \text{Tr}(I) \implies n = m$.

Definition ((semi)simple rings). Note that if R is a ring, then R is a left module as well. We write ${}_R R$ when we're considering it as a left module, and ${}_R R_R$ when we are considering a two sided ideal.

R is called a semisimple ring if ${}_R R$ is a semisimple R -module.

R is called a simple ring if R is a semisimple ring, and for all simple $L, L' \triangleleft_R R \implies L \cong L'$

This means, ${}_R R = \oplus_{i \in I} L_i$ where L_i are simple (left) ideals such that $L_i \cong L_j$ for all i, j .

Recall that an ideal is simple if it has no proper sub-ideals.

Example. $M_2(\mathbb{H})$ is a simple ring. We can write it as direct sum of two ideals

$$\begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$$

Example. $M_2(\mathbb{H}) \times \mathbb{R}$ is semisimple.

$$\begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix} \times 0 \oplus \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix} \times 0 \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \times \mathbb{R}$$

Artin-Wedderburn generalizes this.

Theorem 20 (Artin-Wedderburn Theorem). i) R simple $\iff R \cong M_n(D)$
where D is a skew-field.

ii) R semisimple $\iff R \cong R_1 \times \cdots \times R_s$ simple rings.

Wednesday, 9/11/2024

Today we discuss the Artin-Wedderburn Theorem.

Exercise: $C_2 = \{1, g\}$, prove that $\mathbb{Q}C_2$ is a semisimple ring.

$\mathbb{Q}C_2 = B_1 \oplus B_2$ 2-sided ideals

$\mathbb{Q}C_2 \cong \mathbb{Q} \times \mathbb{Q}$.

Lemma 21. Suppose we have a ring R which is decomposed as a sum of (left) ideals:

$${}_R R = \bigoplus_{i \in I} L_i \quad \text{with } L_i \neq 0$$

Then $|I| < \infty$.

Proof. Suppose ${}_R R = \bigoplus_{j \in J} L_j$ where L_j are ideals. We want to prove that only finitely many are non-zero.

Note that, $1 = \sum_{j \in J} e_j$. We use only finitely many elements here, so $1 = \sum_{i \in I} e_i$ where $e_i \neq 0, I \subset J, |I| < \infty$.

For all $r \in R$ we have $r = r \cdot 1 = r \sum_{i \in I} e_i = \sum_{i \in I} r e_i \in \sum_{i \in I} L_i$.

Therefore, ${}_R R = \bigoplus_{i \in I} L_i$ a finite sum! \square

Now we go to the theorem.

Proof of Artin-Wedderburn Theorem Part I. We want to prove: R simple ring $\iff R \cong M_n D$ where D is a skew field.

First, note that ${}_R R \cong L^n$ where L is a simple ideal [so no proper sub-ideals]. Therefore,

$$R^{op} \cong \text{End}_R({}_R R) \cong \text{End}_R(L^n) \cong M_n(\underbrace{\text{End}_R L}_{\text{division ring}})$$

Taking transpose,

$$R \cong M_n(\text{End}_R L)^{op} \cong M_n((\text{End}_R L)^{op}) = M_n(D)$$

So we are done with one direction!

The other direction is an exercise. Here are the steps:

Step 1: $M_n D = \begin{bmatrix} * & 0 & \cdots & 0 \\ * & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 0 & \cdots & * \\ 0 & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{bmatrix}$

Step 2: Each summand is isomorphic to $D^n = \begin{bmatrix} * \\ * \\ \vdots \\ * \end{bmatrix}$

Step 3: D^n is a simple module. \square

Remark. R simple $\iff R$ artinian, R has no proper 2-sided ideals. Some definitions forgo the artinian condition, in this case these are called artinian simple rings.

Lemma 22 (4.2). Suppose L is a simple ideal and M is a simple module so that $L \not\cong M$. Then $LM = 0$.

Proof. This is a direct consequence of Schur's lemma. Consider the map $\phi_m : L \rightarrow M$ given by $l \mapsto lm$ for $m \in M$. Since this can't be an isomorphism, it must be the zero map. Thus, $lm = 0$. \square

Proof of Artin-Wedderburn Theorem Part II. Idea: Decompose R as direct sum of simple ideals. Partition the set of simple ideals so that members of a partition are isomorphic to each other, members of a partition are not isomorphic to members of another partition. Direct sum of each partition gives us one R_j .

Suppose R is semisimple.

Let L_1, \dots, L_s be a set of pairwise non-isomorphic simple ideals [meaning $L_i \not\cong L_j$]

So that, for all simple $L <_R R$, $L \cong L_i$ for some i .

Let $B_i = \sum_{L \cong L_i} L$.

Claim: B_i is a 2-sided ideal.

Proof of Claim:

$$B_i R \underset{4.2}{=} B_i B_i \subset R B_i \underset{B_i \text{ is a left ideal}}{=} B_i$$

Thus the claim is proven.

Claim: We have a ‘block decomposition of R ’, meaning,

Proof of Claim:

$${}_R R_R = B_1 \oplus \dots \oplus B_s$$

Subclaim: $B_i \cap \sum_{j \neq i} B_j = 0$

Proof of Subclaim: Every $r \in R$, we have that $r \in L$ where L is simple. $L \subset B_i \implies L \cong L_i$. $L \subset \sum_{j \neq i} B_j \implies L \cong B_j$ for some $j \neq i$ which is not possible.

Now, we go back to the main proof.

We can write $1 = e_1 + \dots + e_s$.

Then, $R_i := (B_i, e_i)$ is a ring!

We have $R \cong (R_1, e_1) \times \dots \times (R_s, e_s)$, so we’re done.

The other direction is an exercise. □

Friday, 9/13/2024

Key idea:

$${}_R R = L^n \implies \text{End}_R R \cong M_n(\text{End}_R L)$$

Note that $R^{op} \cong \text{End}_R R$ [function composition is written in the opposite direction].

Suppose L_1, \dots, L_s are non-isomorphic simple R -ideals.

L simple $\implies L \cong L_i$.

Define $B = \sum_{\text{simple } L \cong L_i} L \triangleleft_R R_R$.

We can prove that it is a two sided ideals.

Then we can write $R \cong R_1 \times \dots \times R_s$ simple, where

$R_i = (B_i, e_i)$ [e_i is the identity in B_i].

Theorem 23 (4.4). Suppose E is a R -module.

$$E_i := \sum_{\substack{\text{simple } M \triangleleft E \\ M \cong L_i}} M$$

Then, $E = \bigoplus_{i=1}^s E_i$

$E_i = e_i E = B_i M$.

Corollary 24 (4.5). If R is semisimple, M a simple R -module, then $M \cong L_i$ for some i .

Corollary 25 (4.6). All simple modules of a simple ring are isomorphic.

$$M \cong \oplus L$$

External Product vs. Internal Product

Definition (External Product). If we have [finite] rings R_1, \dots, R_s we can construct the ring:

$$R_1 \times R_2 \times \dots \times R_s$$

Definition (Internal Product). ‘Block Decomposition’: If we have a ring R and we can write it as sum of 2 sided ideals:

$${}_R R_R = B_1 \oplus \dots \oplus B_s$$

Then we have $e_j \in B_j$ so that:

$$1 = e_1 + \dots + e_s$$

Then, each B_j has a ring structure with e_j as identity. Then,

$$R \cong (B_1, e_1) \times \dots \times (B_s, e_s)$$

Just for clarity:

Definition (Direct Sum of Ideals).

$${}_R R_R = B_1 \oplus \dots \oplus B_s$$

If and only if for every $r \in R$,

$$r = b_1 + \dots + b_s$$

where $b_j \in B_j$ and the expression is unique.

Jim’s Rant: A subring has to have the same identity. So, (B_j, e_j) is not a subring.

Block Decomposition is not a direct sum of rings!

This is because in category theory, sum refers to the co-product.

Lemma 26. Let k be a field, and let D be a skew-field which is a k -algebra such that $\dim_k D < \infty$. Then,

- a) $\forall \alpha \in D$ we have $k[\alpha]$ is a field.
- b) k algebraically closed $\implies D = k$.

Example. If $k \in \mathbb{R}, D = \mathbb{H}, \alpha \in \mathbb{H} - \mathbb{R}$ then $k[\alpha] \cong \mathbb{C}$.

It is not completely obvious since $k[i + j] \cong \mathbb{C}$ as well.

Proof. a) D is a k -algebra. Therefore, $k[\alpha]$ is commutative. We just need to find inverse.

Let $0 \neq \beta \in k[\alpha]$. It is enough to prove that for $\beta \in k[\alpha]$, multiplication map $\cdot\beta : k[\alpha] \rightarrow k[\alpha]$ is bijective.

$\cdot\beta$ is a finite dimensional linear transformation so those are true.

- b) For all $\alpha \in D$ we have: $k[\alpha] = k$ since k is closed. So, $\alpha \in K$. Thus $D = k$. □

Corollary 27. Suppose G is finite. Then,

$$\mathbb{C}G \cong \prod_{i=1}^s M_{n_i}(\mathbb{C})$$

Proof. Artin-Wedderburn Theorem plus the previous lemma. □

Example. Suppose $C_n = \langle g \rangle$ cyclic and $\zeta_n = e^{2\pi i/n}$. Then,
 $\mathbb{Q}C_2 \cong \mathbb{Q}_+ \times \mathbb{Q}_-$ where $g \mapsto (1, -1)$.
If p is prime we can write:
 $\mathbb{Q}(C_p) \cong \mathbb{Q} \times \mathbb{Q}(\zeta_p)$ where $g \mapsto (1, \zeta_p)$.
 $\mathbb{C}[C_n] \cong \mathbb{C}^n$ where:
 $g \mapsto (1, \zeta_n, \dots, \zeta_n^{n-1})$
 $\mathbb{Q}[C_2 \times C_2] \cong \mathbb{Q}^4$ where:

$$(1, g) \mapsto (1, 1, -1, -1)$$

$$(g, 1) \mapsto (1, -1, 1, -1)$$

$\mathbb{R}[Q_8] \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{H}$ where $\mathbb{R}[Q_8] \twoheadrightarrow \mathbb{R}[C_2 \times C_2]$
Some other examples: $\mathbb{Q}[C_n], \mathbb{C}[Q_8], \mathbb{Q}[D_{2n}], \mathbb{R}[D_{2n}], \mathbb{C}[D_{2n}]$

Representation Theory

Here, G is a finite group and k is a field.

Representations	Modules over kG	Characters
$\rho : G \rightarrow GL(V)$ where V is a finite dimensional vector space	V is a kG module	$\chi : G \rightarrow k, \chi_\rho(g) = \text{Tr } \rho(g)$

Table 1: Representations, Modules and Characters

Monday, 9/16/2024

We have:

$$\text{representation} \iff \text{modules over } kG \implies [\iff \text{only if char } k = 0] \text{ characters.}$$

rep \rightarrow kG -module

$$\rho \mapsto V_\rho \text{ by } (\sum_g a_g g)v := \sum_g a_g \rho(g)v$$

$$\rho_v \leftarrow V$$

$$\rho_V(g)v := gv$$

Recall the definition of character:

We have the trace map:

$$\text{Tr} : M_n k \rightarrow k$$

Where $\text{Tr}(a_{ij}) = \sum_j a_{jj}$ [or the sum of eigenvalues]

We have $\text{Tr}(AB) = \text{Tr}(BA)$ which implies $\text{Tr}(PAP^{-1}) = \text{Tr}(A)$.

So, Tr is basis independent. Thus,

$$\text{Tr} : \text{End}_k V \rightarrow k$$

Definition (character). Trace is an endomorphism map. This gives us:

$$G \xrightarrow{\rho} GL(V) \xrightarrow{\text{Tr}} k$$

χ_ρ

This is called the character of ρ

There's a correspondence between kG modules and Representations concepts:

Representations	Modules over kG
irreducible	simple
	isomorphism
	direct sum
	Hom
	dual
	tensor product

Table 2: Rep and kG -mod

Irreducible vs Simple

We say irreducible representation, when we on the other hand say simple modules.

Same concept!

Isomorphism

Suppose we have two representations:

$$\rho : G \rightarrow GL(V)$$

$$\rho' : G \rightarrow GL(V')$$

We say two representations are isomorphic when:

$$\rho \cong \rho' \stackrel{\text{def}}{\iff} V_\rho \stackrel{\phi}{\cong} V_\rho \stackrel{\phi}{\cong} V_{\rho'} \iff \exists k \text{ isomorphism s.t.} \\ \phi(gv) = g\phi(v)$$

$\phi : V \rightarrow V'$ s.t. $\forall g \in G$ we have the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ \downarrow \phi & & \downarrow \phi \\ V' & \xrightarrow{\rho'(g)} & V' \end{array}$$

ϕ is called the intertwining map.

Corollary 28. $\rho \cong \rho' \implies \chi_\rho = \chi_{\rho'}$

Direct Sum

Suppose $V \oplus W$ is a kG -module.

$$\rho_{V \oplus W} : G \rightarrow GL(V \oplus W)$$

is given by:

$$\rho_{V \oplus W} = \begin{bmatrix} \rho_V & 0 \\ 0 & \rho_W \end{bmatrix}$$

We also have $\chi_{V \oplus W} = \chi_V + \chi_W$.

Two Representations

Definition (Trivial Representations).

$$\rho : G \rightarrow GL(k)$$

$$g \mapsto 1$$

Is the trivial representation. Also, $\chi_\rho \equiv 1$.

Definition (Regular Representation). Consider the kG -module ${}_kGkG$. We have:

$$\rho_{kG} : G \rightarrow GL(kG)$$

This is injective.

Note that $G \curvearrowright G$ by multiplication, this is a free action. For finite group G with $|G| = n$,
 $G \hookrightarrow \text{Bijection}(G, G)$ so G is a subgroup of S_n . So we have:

$$\begin{array}{c} \text{regular rep.} \\ \curvearrowright \\ G \longrightarrow S_n \longrightarrow GL(k^n) \end{array}$$

With the action of ‘permuting the standard basis’.

Exercise: Compute character of Regular Representation.

We have, in line of the previous theorem:

Theorem 29 (Maschke’s Theorem). If $V \subset W$ as kG -modules and $\text{char } k \nmid |G|$ then $\exists V'$ such that $W = V \oplus V'$

Proof. First, find a k -linear map $\pi : W \rightarrow V$ such that $\pi(v) = v$ for all $v \in V$.

We average it to make it kG -linear:

$\pi' : W \rightarrow V$ given by:

$$\pi'(w) := \frac{\sum_g g\pi(g^{-1}w)}{|G|}$$

We have: π' is kG -linear and $\pi'(v) = v$

We can take $V' := \ker \pi$

□

Thus, for $w \in W$ we can write $w = \pi'(w) + (w - \pi'(w))$.

Note that Maschke’s theorem implies kG is semisimple. Artin Wedderburn implies semisimple kG module is a direct sum of irreducible modules.

$$\begin{aligned} V &\cong \bigoplus_i n_i V_i \\ \chi_V &= \sum_i n_i \chi_i \end{aligned}$$

Homomorphisms:

Suppose V, W are kG -modules, “representations”. Then,

$\text{Hom}_{kG}(V, W)$ is a k -vector space.

$\text{Hom}_k(V, W)$ is a kG -module.

we define: $(gf)v := gf(g^{-1}v)$

i.e. $((\sum_g a_g g)f)v = \sum_g a_g (gf(g^{-1}v))$

The g^{-1} inside is needed for associativity: $(g'g)f = g'(gf)$

Officially this is a functor.

$\text{Hom}_k(-, -) : (kG\text{-mod})^{op} \times kG\text{-mod} \rightarrow kG\text{-mod}$

Special case:

Dual Representation: $W = k$. Then,

$V^* = \text{Hom}_k(V, k)$.

So, $(gf)(v) = gf(g^{-1}v) = f(g^{-1}v)$

Exercise: $\chi_{V^*} = ?$

Wednesday, 9/18/2024

Tensor Products

Motivation:

Product Structure: $- \otimes -: kG\text{-mod} \times kG\text{-mod} \rightarrow kG\text{-mod}$ given by $V \otimes_k W$.

Group action works diagonally, $g(x \otimes y) = (gx) \otimes (gy)$, extended linearly.

Extension of scalars:

$$\mathbb{R}G \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}G$$

Product of Groups: $k[G \times H] = kG \otimes_k kH$

When for k a field then modules are vector spaces k^m and k^n which are easy:

$$k^n \otimes_k k^m = k^{nm}$$

$$\dim(k^n \otimes_k k^m) = mn$$

$\{e_i\}$ a basis for k^n

$\{f_j\}$ a basis for k^m

Then $\{e_i \otimes f_j\}$ is a basis for $k^n \otimes k^m$.

However, tensor product consists of more than 'pure' tensors.

Definition (Tensor Product). Let R be a commutative ring. Tensor product is a functor:

$$- \otimes_R - : R\text{-mod} \times R\text{-mod} \rightarrow R\text{-mod}$$

$$(A, B) \mapsto A \otimes_R B$$

[Functor meaning if we have homomorphism on the left we will have homomorphisms on the right]

Construction:

Let $F(A \times B)$ be the free R -module with basis $A \times B$. Then a typical element of the basis is $(a, b) \in A \times B$.

Let S be the sub-module generated by the following:

- 1) $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$
- 2) $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$
- 3) $r(a, b) - (ra, b)$
- 4) $r(a, b) - (a, rb)$

Then, we define:

$$A \otimes_R B := \frac{F(A \times B)}{S}$$

and write $a \otimes b$ for the image of (a, b) .

This means, a typical element of $A \otimes_R B$ is:

$$\sum_{i=1}^n a_i \otimes b_i \in A \otimes_R B$$

We also have the following relations:

$$(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$$

$$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$$

$$r(a \otimes b) = (a \otimes rb) = (ra \otimes b)$$

Exercise. $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/3 = 0$

Proposition 30. Suppose A, B, M are R -modules, and

$$\phi : A \times B \rightarrow M \text{ is } R\text{-bilinear}$$

Meaning,

- 1) $\phi(a_1 + a_2, b) = \phi(a_1, b) + \phi(a_2, b)$
- 2) $\phi(a, b_1 + b_2) = \phi(a, b_1) + \phi(a, b_2)$
- 3) $r\phi(a, b) = \phi(ra, b) = \phi(a, rb)$

Then, by definition,

$$\pi : A \times B \rightarrow A \otimes_R B$$

is R -bilinear.

Proposition 31 (Universal Property of Tensor Product). π is initial in the category of bilinear maps with domain $A \times B$. Meaning, every bilinear map from $A \times B$ factors through π .

$$\begin{array}{ccc} A \times B & \xrightarrow{\forall \phi \text{ bilinear}} & M \\ \downarrow \pi & \searrow \exists! \bar{\phi} & \\ A \otimes_R B & & \end{array}$$

This diagram commutes

Proof. For uniqueness, note that, $\bar{\phi}(a \otimes b) = \bar{\phi}(\pi(a, b)) = \phi(a, b)$

For existence, define $\hat{\phi}(a, b) = \phi(a, b)$ where $\hat{\phi} : F(A \times B) \rightarrow M$. Then $\bar{\hat{\phi}}(S) = 0$ so $\bar{\phi} : A \otimes_R B \rightarrow M$ exists. \square

Proposition 32 (Rephrasing Universal Property in Terms of Adjoint Functors).

$$\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, \text{Hom}(B, C))$$

Proof.

$$f \mapsto (a \mapsto (b \mapsto f(a \otimes b)))$$

$$\begin{array}{ccc} & \text{Hom}(A \otimes -, C) & \\ & \curvearrowleft & \\ R\text{-mod} & & R\text{-mod} \\ & \curvearrowright & \\ & \text{Hom}(A, \text{Hom}(-, C)) & \end{array}$$

\square

Proposition 33. 1) Commutative $A \otimes_R B \cong B \otimes_R A$

2) Identity $R \otimes_R B \cong B$

3) Associative $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$

4) Distributive $(\bigoplus_{\alpha} A_{\alpha}) \otimes B \cong \bigoplus_{\alpha} (A_{\alpha} \otimes B)$

5) Functorial $\begin{pmatrix} f : A \rightarrow A' \\ g : B \rightarrow B' \end{pmatrix} \implies f \otimes g : A \otimes B \rightarrow A' \otimes B'$

6) Exactness Short Exact Sequence $0 \rightarrow A \xrightarrow{f} B \rightarrow C \rightarrow 0 \implies$ Short Exact Sequence $0 \rightarrow A \otimes M \xrightarrow{f \otimes 1_M} B \otimes M \rightarrow C \otimes M \rightarrow 0$

7) Right Exactness $M \text{ } R\text{-mod}, 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \implies$ Exact Sequence $A \otimes M \rightarrow B \otimes M \rightarrow C \otimes M \rightarrow 0$

Friday, 9/20/2024

Lang Section 2

Tensor Product of Representation

Suppose V, W are k -vector spaces, then we have $V \otimes_k W$ is also a k -vector space. But they all are kG -modules as well:

$$g(v \otimes w) = gv \otimes gw$$

Proposition 34. The character is multiplicative:

$$\chi_{v \otimes w} = \chi_v \chi_w$$

Proof. Let $\{e_i\}$ be a basis for V and $\{f_j\}$ a basis for w .

Suppose $ge_i = \sum_k a_{ki} e_k$

And $gf_j = \sum_l b_{lj} f_l$

Then, $g(e_i \times f_j) = ge_i \times gf_j = \sum_{k,l} a_{ki} b_{lj} e_k \times f_l$

Take $(k, l) = (i, j)$.

Then, $\chi_{v \otimes w}(g) = \sum_{i,j} a_{ii} b_{jj} = \chi_v(g) \chi_w(g)$ □

Consider $f : G \rightarrow k$. We have:

$\{1\text{d chars}\} \subset \{\text{simple chars}\} \subset \{\text{chars}\} \subset \{\text{virtual chars}\} \subset \{\text{class functions}\}$

We explain these later.

Definition. f is a character if $\exists \rho : G \rightarrow GL_k(V)$ such that $f = \chi_\rho = \text{Tr} \circ \rho$

Definition. f is a class function if $\forall g, h \in G$ we have $f(hgh^{-1}) = f(g)$

Definition. f is a virtual character if $\exists \rho, \rho'$ such that $f = \chi_\rho - \chi_{\rho'}$

Definition. f is simple (=irreducible) character if $f = \chi_V$ where V is a simple kG -module.

Definition. f is 1-dimensional character if $f : G \rightarrow k^\times$ is a homomorphism. eg trivial character $\chi_1(g) \equiv 1$.

Proposition 35. Class Functions are k -algebras. Virtual characters are a commutative ring.

Now, suppose $\text{char } k \nmid |G|$. Then,

$$kG \cong M_{n_1}(D_1) \times \cdots \times M_{n_s}(D_s)$$

Assume $M_{n_1}(D_{n_1}) = k$. Then we have the trivial representation: $ga = a$.

If $L_i = D_i^{n_i}$ is a simple kG -module, then

$\chi_i = \chi_{L_i}$ is a simple characteristics.

We have $1 = e_1 + \cdots + e_s$ [central non-trivial idempotents].

$\chi_i(e) = \text{Tr}(\text{Id}_{L_i}) = \dim_k L_i = n_i \dim_k D_i$.

Example. Consider $Q_8 \hookrightarrow \mathbb{H}^\times$. Then,

$$\chi_{\mathbb{H}}(e) = 4$$

Now, consider ${}_k G kG \cong \bigoplus_i n_i L_i$, the ‘regular representation’. $e_j L_i = 0$ for $i \neq j$. Then,

$$\chi_i(e_i) = \chi_i(1) = \chi_i(e) = \dim_k L_i$$

So, $\text{char } \chi : G \rightarrow k$ extends to $\chi : kG \rightarrow k$ by $\sum a_g g \mapsto \sum a_g \chi(g)$.

If V is a finitely generated kG -module, we have

$$V \cong m_1 L_1 \oplus \cdots \oplus m_s L_s$$

where $m_i \geq 0$.

Theorem 36 (2.2, 2.3). $\chi_v = \sum_i m_i \chi_i : G \rightarrow k$ with m_i uniquely determined if $\text{char } k = 0$.

Theorem 37 (2.3). Characters Determine Representations: suppose $\text{char } k = 0$. Then,

$$V \cong V' \iff \chi_V = \chi_{V'}$$

Proof. \implies : Trace is independent of basis, so this is easy.

\impliedby : We already gave a proof using projection operators. Second Proof: Assume $\chi_V = \chi_{V'}$. We decompose:

$$V \cong \oplus m_i L_i, V' \cong \oplus m'_i L_i$$

Note that we have $\chi_V(e_i) = m_i \dim_k L_i = m'_i \dim_k L_i = \chi_{V'}(e_i)$. Thus we must have $m_i = m'_i$. □

Representation Ring

$R_k(G) = (\text{virtual char}, +, \times) \cong (\text{virtual rep}, \oplus, \otimes)$.

Example: $R_{\mathbb{Q}}[C_2] \cong \frac{\mathbb{Z}[X]}{(X^2-1)}$

Monday, 9/23/2024

Dual Characters

Consider $\rho : G \rightarrow GL_k(V)$

Dual $V^* = \text{Hom}_k(V, k)$ is also a representation.

$$(g\phi)(v) = \phi(g^{-1}v)$$

Inverse because we want it to be a left module.

Claim: $\rho : G \rightarrow GL(V) \rightarrow \rho^* : G \rightarrow GL(V^*)$

$$\rho^*(g) = (\rho(g)^{-1})^T$$

Proof. $\rho^*(g) = (\rho(g^{-1}))^* = \rho(g^{-1})^T$ □

Corollary 38. a) $\chi_{V^*}(g) = \chi_V(g^{-1})$

b) $\chi_{\text{Hom}(V, W)}(g) = \chi_V(g^{-1})\chi_W(g)$

Proof. a follows from the claim.

b: Consider the slant homomorphism:

$$V^* \otimes W \rightarrow \text{Hom}(V, W)$$

$$\sum_i \phi_i \otimes w_i \mapsto \left(v \mapsto \sum_i \phi_i(v) w_i \right)$$

It is an isomorphism since V, W are both finite dimensional.

$$\chi_{\text{Hom}(V, W)}(g) = \chi_{V^* \otimes W}(g) = \chi_{V^*}(g)\chi_W(g) = \chi_V(g^{-1})\chi_W(g)$$

□

1 Dimensional Characters

Definition. 1 D representation is a homomorphism $\rho : G \rightarrow k^\times$

$$\begin{array}{ccc} G & \xrightarrow{\quad} & k^\times \\ & \searrow & \nearrow \\ & G^{ab} & \end{array}$$

Question: What are the 1d representations for D_6 ?

$$D_6 \cong \mathbb{Z}/3 \rtimes \mathbb{Z}/2$$

$$\text{So, } D_6^{ab} \cong \mathbb{Z}/2$$

So, we have k_T, k_-

$$r \mapsto 1$$

$$s \mapsto -1$$

Exercise: Trivial Representation / Idempotent

$$e_T = \frac{\sum_{g \in G} g}{|G|} \in kG$$

$$e_T^2 = e_T$$

$$ge_T = e_T = e_T g$$

$$e_T \in Z(kG)$$

$$kG = (kG)e_T \oplus (kG)(1 - e_T)$$

$$kG \cong k \times \frac{kG}{\langle e_T \rangle}$$

Lemma 39 (2). Any finite subgroup of k^\times is cyclic.

Proof. Key Fact: $x^e - 1 \in k[x]$ has at most e roots [proof: long division].

Note: $x^2 - 1 \in \mathbb{Z}/8[x]$ has 4 roots. This implies $\mathbb{Z}/8$ is not a field.

Consider finite abelian $A < k^\times$

Consider $e = \text{exponent } A = \inf\{m \geq 1 \mid \forall a \in A, a^m = e\}$

Then, $\forall a \in A, a^e - 1 = 0$. From the key fact, $|A| \leq e \leq |A|$

Thus, $e = |A|$

□

Corollary 40. $\forall \text{ hom } \rho : G \rightarrow k^\times, \exists \text{ Cyclic } C \text{ such that:}$

$$\begin{array}{ccc} G & \xrightarrow{\quad \rho \quad} & k^\times \\ & \searrow & \nearrow \\ & C & \end{array}$$

Recall only finite subgroup of \mathbb{Q} is ± 1 .

$1 - d$ \mathbb{Q} reps of $G \leftrightarrow$ trivial representation + index 2 subgroups

Now we suppose k is algebraically closed, eg $k = \mathbb{C}$. Then,

$$kG \cong \prod_i M_{n_i}(k)$$

If G is abelian, then,

$$kG \cong k \times \cdots \times k$$

Corollary 41 (3). k is algebraically closed and G is abelian \iff all irreducible representations are 1-dimensional.

Corollary 42. Let $|G| = n, k = \mathbb{C}$.

- a) $\forall V, \chi_V(G) \subset \mathbb{Q}(\zeta_n)$
- b) $\forall V, \chi_{V^*}(g) = \overline{\chi_V(g)}$
- c) $\forall V, W, \chi_{\text{Hom}(V, W)}(g) = \overline{\chi_V(g)} \chi_W(g)$

Proof. a) True for 1d representation from the lemma.

\implies True for G abelian (corollary 3)

\implies True for cyclic G

\implies always true: $g \in G \implies \langle g \rangle$ cyclic.

$$\chi_\rho(g) = \chi_{\rho|_{\langle g \rangle}}(g)$$

Then, $\rho : G \rightarrow GL(V)$, consider $g \in G$.

Then $\rho(g)^n = I \implies \text{Tr}(\rho_V(g)) \in \mathbb{Q}(\zeta_n)$.

b) Same as (a).

$$\rho^*(g) = (\rho(g)^{-1})^t$$

For 1-dim, $\rho^* = \bar{\rho}$.

c) $\chi_{\text{Hom}(V, W)}(g) = \chi_V(g^{-1}) \chi_W(g) = \overline{\chi_V(g)} \chi_W(g)$

□

Two Bases for center kG

Definition. $g \in G$ is conjugate to $\sigma \in G$ if $\exists \tau$ such that,

$$\tau g \tau^{-1} = \sigma$$

Write $g \sim \sigma$

$$G = \coprod_{G/\sim} [g]$$

$[g] = \{\sigma \in G \mid g \sim \sigma\}$ conjugacy classes

Proposition 43. $\{\sum_{\sigma \in [g]} \sigma\}_{[g] \in G/\sim}$ is a k -basis for center of kG .

Proof. Clearly these are linearly independent.

$$\alpha = \sum_{\sigma \in G} a_\sigma \sigma \in \text{center}$$

$$\iff \tau \alpha = \alpha \tau \iff \tau \alpha \tau^{-1} = \alpha$$

$$\sigma a_\sigma \tau \sigma \tau^{-1} = \sum a_\sigma \sigma \implies (g \sim \sigma \implies a_g = a_\sigma)$$

□

Wednesday, 9/25/2024

Lang XVIII, 4

Two bases for $Z(kG)$

conjugacy classes

primitive central idempotents [k algebraically closed]

Exercise. $G \twoheadrightarrow Q$, prove that $kG \cong kQ \times R$

Proposition 44 (4.1). Suppose $\{\sum_{\sigma \in [g]} \sigma\}_{[g] \in G/\sim}$ form a $\{\frac{k}{\mathbb{Z}}\}$ -basis for $\{Z(kG)\}$

Consider a ring R .

Definition. $e \in R$ is a primitive central idempotent if:

e is a central idempotent [$e^2 = e, e \in Z(R)$]

$e = e' + e''$ with e', e'' central idempotent $\implies \{e', e''\} = \{0, e\}$

Then, $kG \ni 1 = e_1 + \cdots + e_s, kG \cong \prod M_{d_i}(D_i)$

$e_i \rightarrow (0, \dots, 0, 1, 0, \dots, 0)$

Now suppose $n = |G|$

We have irreducible representations L_1, \dots, L_s and degrees d_1, \dots, d_s then $L_i \cong D_i^{d_i}$. We have irreducible characteristics χ_1, \dots, χ_s and primitive central idempotents (p.c.i.) e_1, \dots, e_s

Facts: (*): $kGkG = \bigoplus_i d_i L_i$

(**): $\alpha \in kG, i \neq j$ then $\chi_j(e_i \alpha) = 0$ since $e_i L_j = 0, \chi_i(e_i \alpha) = \chi_i(1\alpha) = \chi_i(\alpha)$

We have: $\chi_{\text{reg}} = \sum_i d_i \chi_i$

Proposition 45 (4.3). $\chi_{\text{reg}}(g) = \begin{cases} n, & \text{if } g = e; \\ 0, & \text{if } g \neq e \end{cases}$

Proof. $\chi_{\text{reg}}(g) = \text{Tr}(\cdot g : kG \rightarrow kG)$

Thus, $\chi_{\text{reg}}(e) = \text{Tr}(I) = n$

If $g \neq e$ note that G has $\{\sigma_1, \dots, \sigma_n\}$ and $\rho_{\text{reg}}(g)(\sigma_j) = g\sigma_j \neq \sigma_j$ for all j . So, there is nothing in the diagonal matrix and trace is 0. \square

Motivation for k algebraically closed:

Consider $\mathbb{Q}C_3 \cong \mathbb{Q} \times \mathbb{Q}(\zeta_3)$. We only have primitive central idempotents, $1 = e_1 + e_2$.

But the center has dimension 3: $\dim_{\mathbb{Q}}(Z(\mathbb{Q}C_3)) = 3$.

Assume k is algebraically closed.

Claim: k algebraically closed, D skew field, $k < Z(D)$, $\dim_k D < \infty$ implies $k = D$

Now, $kG \neq \prod M_{d_i}(k)$

Consider primitive central idempotents e_1, \dots, e_s for a basis.

$$n = \sum_{i=1}^s d_i^2$$

e.g. $S_3 = D_6$. $s = ?$ $d_1, d_2, d_3 = ?$

We have representatives of conjugacy classes: $(1), (12), (123)$.

$$s = 3, 6 = 1^2 + 1^2 + 2^2$$

Char. Table:

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Table 3: characteristic table

We have $\mathbb{C}S_3 = \mathbb{C}_+ \times \mathbb{C}_- \times M_2\mathbb{C}$

Our representatives are $(1), (12), (123), (1234), (12)(34)$

$$d_i = 1, 1, 2, 3, 3$$

Goal: Express the p.c.i basis in terms of conjugacy class basis.

Corollary 46 (4.2). If k is algebraically closed,

the number of conjugacy classes = $\dim_k Z(G)$ = number of irreducible representation = s

Proposition 47 (4.4). k algebraically closed, then

$$e_i = \frac{d_i}{n} \sum_{\tau \in G} \chi_i(\tau^{-1}) \tau$$

Proof. Let $e_i = \sum_{\tau \in G} a_{\tau} \tau$.

We compute $\chi_{\text{reg}}(e_i \tau^{-1})$ in two ways.

$$1: \chi_{\text{reg}}(e_i \tau^{-1}) = \chi_{\text{reg}}(\sum a_{\sigma} \sigma \tau^{-1}) = \sum a_{\sigma} \chi_{\text{reg}}(\sigma \tau^{-1}) = a_{\tau} n$$

$$2: \chi_{\text{reg}}(e_i \tau^{-1}) \stackrel{(*)}{=} \sum_j d_j \chi_j(e_i \tau^{-1}) \stackrel{(**)}{=} d_i \chi_i(e_i \tau^{-1}) = d_i \chi_i(\tau^{-1})$$

$$\text{Thus, } a_{\tau} n = d_i \chi_i(\tau^{-1}) \implies a_{\tau} = \frac{d_i}{n} \chi_i(\tau^{-1})$$

\square

Corollary 48 (4.5). Let $m = \exp G$. Then,

$$e_i \in \frac{1}{n} [\mathbb{Z}[\zeta_m]G] \subset \frac{1}{n} [\mathbb{Z}[\zeta_n]G]$$

Corollary 49 (4.6). $\text{char } k \nmid d_i$

Proof. If not, $\text{char } k \mid d_i$ then $e_i = 0$ which is a contradiction. \square

Corollary 50 (4.7). χ_1, \dots, χ_s are linearly independent over k . In fact they form a basis for the class functions $f : G \rightarrow k$.

Proof. Suppose $0 = \sum a_i \chi_i$.

$$\text{Then } 0 = \sum a_i \chi_i(e_j) = a_j \chi_j(e_j) = a_j d_j \implies a_j = 0 \quad \square$$

Then $\dim_k(\text{class functions}) = \text{number of conjugacy classes} = s$.

Friday, 9/27/2024

Review:

$$e_i = \frac{d_i}{n} \sum_{\sigma \in G} \chi_i(\sigma) \sigma^{-1} \in kG \quad (*)$$

Is a primitive central idempotent.

$$\chi_{\text{reg}} = \chi_{kG} = \sum_i d_i \chi_i$$

$$\sigma = 1, n = \sum_i d_i^2 \\ d_i \mid n$$

$$\sum_{\sigma \in G} \chi_i(\sigma) \chi_j(\sigma^{-1}) = n \delta_{ij}$$

$$\sum_{i=1}^s \chi_i(\sigma) \chi_i(\tau^{-1}) = \begin{cases} \frac{n}{|\sigma|}, & \text{if } \tau = \sigma; \\ 0, & \text{otherwise.} \end{cases}$$

If $G = S_3$ then:

	(1)	(12)	(123)	
χ_1	1	1	1	6
χ_2	1	-1	1	6
χ_3	2	0	-1	6
	6	2	3	

Table 4: Characteristic Table of S_3

$$0 = \chi_{\text{reg}}(123) = 1\chi_1(123) + 1\chi_2(123) + 2\chi_3(123)$$

$$k = \mathbb{C}, \chi(\sigma^{-1}) = \overline{\chi(\sigma)}$$

End of review

$X(G) = \{\text{class functions } f : G \rightarrow k\}$ so that $f(\tau\sigma\tau^{-1}) = f(\sigma)$.

Definition (Perfect Pairing). A perfect pairing of k vector space is a k -bilinear map $\beta : V \times W \rightarrow k$ such that \exists basis $\{v_i\}, \{w_j\}$ such that

$$\beta(v_i, w_j) = \delta_{ij}$$

$$\iff \text{Ad}_b : V \rightarrow W^*$$

$$v \mapsto (w \mapsto \beta(v, w))$$

Theorem 51 (4.9).

$$X(G) \times Z(kG) \rightarrow k$$

$$(f, \alpha) \mapsto f(\alpha)$$

is a perfect pairing.

Proof. Dual basis: $\left\{ \frac{1}{d_i} \chi_i \right\}, \{e_j\}$

$$\frac{1}{d_i} \chi_i(e_j) = \delta_{ij}$$

□

Corollary 52 (4.8). Suppose k is algebraically closed, $\text{char } k = 0$. Then $d_i = \dim_K L_i \mid n$

We need integrality theory (M502)

See Lang p 334.

A subring of B , $\alpha \in B$.

α is integral over A if \exists monic $f(x) \in A[x]$ such that $f(\alpha) = 0$.

$\alpha \in \mathbb{Q} \implies \alpha \text{ int}/\mathbb{Z} \iff \alpha \in \mathbb{Z}$

Condition (**): α being integral is equivalent to the existence of a faithful $A[\alpha]$ -module M which is finitely generated as A -module.

Faithful means: $\forall \beta \in A[\alpha], \beta M = 0 \iff \beta = 0$.

In other words, $A[\alpha] \hookrightarrow \text{End}_{A[\alpha]}(M)$.

Condition (**) $\iff \alpha \text{ int}/A$. This is proved by a determinant trick.

Applying (**) on $A = \mathbb{Z}, \frac{n}{d_i} \in \mathbb{Q}$,

Multiplying $e_i = \frac{d_i}{n} \sum_{\sigma \in G} \chi_i(\sigma) \sigma^{-1} \in kG$ with e_i ,

$$e_i = e_i^2 = \frac{d_i}{n} \sum_{\sigma} \chi_i(\sigma) \sigma^{-1} e_i$$

$$\frac{n}{d_i} e_i = \sum_{\sigma} \chi_i(\sigma) \sigma^{-1} e_i$$

$$M = \mathbb{Z} \langle \zeta_n^j \sigma e_i \rangle_{j, \sigma \in G} \text{ is a } \mathbb{Z} \left[\frac{n}{d_i} \right] \text{-module}$$

We are done by (**). $d_i \mid n$.

Orthogonality, Lang XVIII, 5, Serre 2.3

Theorem 53. Suppose we have $\langle, \rangle : X(G) \times X(G) \rightarrow k$ by:

$$\langle f, g \rangle = \frac{1}{n} \sum_{\sigma \in G} f(\sigma) g(\sigma^{-1})$$

is a nonsingular symmetric form and $\{\chi_1, \dots, \chi_s\}$ forms an orthonormal basis.

Proof. Symmetric form, k -bilinear $\langle f, g \rangle = \langle g, f \rangle$

Apply χ_j to (*)

$$d_i \delta_{ij} = \chi_j(e_i) = \frac{d_i}{n} \sum_{\sigma} \chi_i(\sigma) \chi_j(\sigma^{-1})$$

□

Remark: Irreducibility criterion: $\langle \chi, \chi \rangle = 1 \iff \chi$ irreducible.

$$\left(\sum_i a_i \chi_i, \sum_i a_i \chi_i \right) = \sum_i a_i^2$$

Proposition 54 (I.7, Serre p20). a) $\sum_{i=1}^s \chi_i(\sigma) \chi_i(\sigma^{-1}) = \frac{n}{|\sigma|}$

b) $[\sigma] \neq [\tau] \implies \sum_{i=1}^s \chi_i(\sigma) \chi_i(\tau^{-1}) = 0$

Proof. Consider the characteristic function for $[\sigma]$:

$f_\sigma = 1$ on $[\sigma]$ and 0 everywhere else.

$f_\sigma = \sum_i \lambda_i \chi_i$.

$\lambda_j = \langle f_\sigma, \chi_j \rangle = \frac{1}{n} \sum_{\tau \in G} f_\sigma(\tau) \chi_j(\tau^{-1}) = \frac{||[\sigma]||}{n} \chi_j(\sigma^{-1})$

$f_\sigma(-) = \sum_i \frac{||[\sigma]||}{n} \chi_i(\sigma^{-1}) \chi_i(-)$

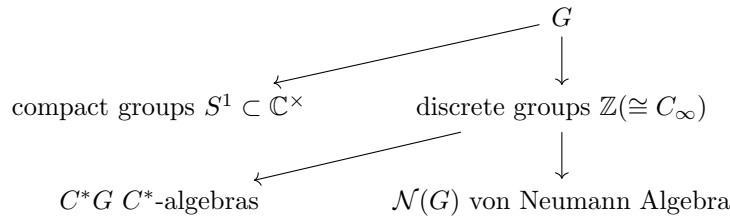
□

This finishes the proof.

Monday, 9/30/2024

Serre Ch 4

What about representations of infinite groups?



Definition (Topological Group). Topological Group is a group (G, \cdot) such that G has a topology so that:

$$G \times G \rightarrow G$$

$$(g, h) \mapsto gh^{-1}$$

is continuous.

Definition (Lie Group). Lie Group is a topological lie group G where G is a smooth manifold and $(g, h) \mapsto gh^{-1}$ is smooth.

Compact Lie Groups:

Torus $T^r = S^1 \times \dots \times S^1$

$O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = I\}$

$U(n) = \{A \in M_n(\mathbb{C}) \mid AA^* = I\}$

Exceptional: G_2, F_4, E_6, E_7, E_8

We also have compact groups are not lie groups;

$(\mathbb{Z}/p)^\infty = \prod \mathbb{Z}/p\mathbb{Z}$

p -adic $\mathbb{Z}_p = \lim \mathbb{Z}/p^n\mathbb{Z}$

Serre Ch 4 says that:

Representation of compact groups is almost the same as finite group!

We need Haar Measure.

Proposition 55. For locally compact Hausdorff topological group G there exists a unique Haar Measure:

$$\begin{aligned} dt : \{\text{Borel Subsets of } G\} &\rightarrow [0, 1] \\ B &\mapsto \int_B dt = \int_G \chi_B(t) dt \end{aligned}$$

So that $\int_G dt = 1$ and dt is translation invariant:

$$\int_G f(t) dt = \int_G f(gt) dt = \int_G f(tg) dt$$

Example. If G is finite:

$$\int_G f \, dt = \frac{1}{|G|} \sum_{g \in G} f(g)$$

$$G = S^1$$

$$\int_{S^1} dt = 1 \quad \int_{\text{quarter circle}} dt = \frac{1}{4}$$

Theorem 56 (Maschke's Theorem, Peter-Weyl Theorem). Let G be a compact group, $k = \mathbb{C}$. Let $W \subset V$ be a subrepresentation of $\rho : G \rightarrow GL(V)$. Then \exists subrepresentation W' such that $V = W \oplus W'$.

Proof. Let $\langle \cdot, \cdot \rangle' : V \times V \rightarrow \mathbb{C}$ be any inner product.

We define a new inner product by averaging this inner product.

$$\langle v, w \rangle = \int_G \langle \rho(t)v, \rho(t)w \rangle' dt$$

This gives us a G -invariant inner product.

We take W' to be orthogonal to W w.r.t. this inner product. □

Corollary 57. Any representation is the direct sum of irreducible representation (unique upto multiplicity).

Consider the regular representation $L^2(G) \cong \bigoplus_i d_i L_i$.

We don't have characteristic of regular representation

We don't have a group ring

Suppose $G = S^1, n \in \mathbb{Z}$

$\chi_n : S^1 \rightarrow \mathbb{C}^\times$

$\chi_n(z) = z^n$ gives us \mathbb{C}_n

$L^2(S^1) = \bigoplus \mathbb{C}_n$

Representation Ring: $R(S^1) \ni \rho - \rho'$

$R(S^1) = \mathbb{Z}[\chi_1, \chi_1^{-1}]$, $\chi_n = \chi_1 \otimes_G \cdots \otimes_G \chi_1$

Then, $R(S^1 \times \cdots \times S^1) = \mathbb{Z}[\alpha_1, \alpha_1^{-1}, \dots, \alpha_r, \alpha_r^{-1}]$ where:

$$S^1 \times \cdots \times S^1 \xrightarrow{\text{proj}} S^1 \hookrightarrow \mathbb{C}^\times$$

Consider $T^n \subset U(n)$

$\Sigma_n = U(n)/T^n$

$R(U(n)) \hookrightarrow R(T^n)$.

image $\mathbb{Z}[\sigma_1, \dots, \sigma_{n-1}, \sigma_n, \sigma_n^{-1}]$ where

σ_i is the i -th elementary symmetric function in $\alpha_1, \dots, \alpha_n$.

Infinite Discrete Groups

$C_\infty = \langle x \rangle$

$\mathbb{Z}C_\infty = \mathbb{Z}[x, x^{-1}]$ the Laurent Polynomial Ring.

We can think of it like the localization of $\mathbb{Z}[x]$ at x [aka $x^{-1}\mathbb{Z}[x]$] or $\mathbb{Z}[x, x^{-1}] \subset \mathbb{Q}(x)$ the rational function field.

This is not a super well behaved domain since it has dimension 2.

$\mathbb{Q}[x, x^{-1}]$ is a Euclidean domain and hence a PID. But not $\mathbb{Z}[x, x^{-1}]$.

Some Conjectures about Torsion-Free Groups

Torsion free: If $g \in G - \{e\}$, $n > 0$ then $g^n \neq e$.

Proposition 58 (Farrell-Jones Conjecture). for $R = \mathbb{Z}$ or a field, all finitely generated projective $\mathbb{R}G$ -modules are stably-free.

Projective means it's a summand of a free module.

P is stably free if $P \oplus \text{free}$ is free.

It has been proved for the torsion-free groups we care about, but not generally.

Proposition 59 (Kaplansky Idempotent Conjecture). Suppose R is an integral domain. Then the only idempotents in RG are 0 and 1.

Proposition 60 (Zero Divisor Conjecture). Suppose R is an integral domain. Then RG has no zero divisor.

Proposition 61 (Embedding Conjecture). Suppose R is an integral domain. Then RG is a subring of a skew field.

We have Embedding Conjecture \implies Zero Divisor Conjecture \implies Kaplansky Idempotent Conjecture

Proposition 62 (Unit Conjecture). Suppose k is a field. Then,

$$(kG)^\times = \langle k^\times, G \rangle$$

Wednesday, 10/2/2024

Serre Chapter 5

Examples

$k = \mathbb{C}$: Use characters.

5.1: $C_n = \langle r \rangle, \zeta_n = e^{2\pi i/n}$.

$n = \# \text{conjugacy classes} \implies n = s$ irreducible representations.

C_n is abelian \implies all irreducible representation (=char) is one dimensional.

$$\chi : C_n \rightarrow \mathbb{C}^\times$$

$$\chi(r)^n = \chi(r^n) = \chi(e) = 1$$

Irreducible representation $\chi_h(r) = \zeta_n^h$. We have characters $\chi_0, \chi_1, \dots, \chi_{n-1}$.

$$\chi_h \chi_{h'} = \chi_{h+h' \pmod n}$$

Representation Ring $\mathbb{Z}[\text{characters}] = \mathbb{Z}[\chi_1] \cong \mathbb{Z}[x]/(x^n - 1)$.

Trivial character is 1 in $R(G)$.

$$\begin{aligned} \phi : \mathbb{C}[C_n] &\rightarrow \mathbb{C} \times \dots \times \mathbb{C} \\ r &\mapsto (\rho^0, \rho^1, \dots, \rho^{n-1}) \end{aligned}$$

$$\Phi : \mathbb{Q}[C_n] \rightarrow \prod_{d|n} \mathbb{Q}(\zeta_d)$$

a

Question: How to justify that ϕ and Φ are isomorphisms?

Answer: CRT

For a non-abelian group G , recall that:

of 1d rep = $|G^{ab}| = |G/[G, G]|$

of irreducible rep = # of conjugacy classes.

Suppose $d_i = \dim_{\mathbb{C}} L_i$ then $n = d_1^2 + \dots + d_s^2$ and $d_i \mid |G|$.

5.1 Dihedral Group D_{2n} (order $2n$)

Recal,

$$D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, srs = r^{-1} \rangle$$

isometries of a regular n -gon.

Here, $(sr s^{-1})^k = s r^k s^{-1}$ so $s r^k s^{-1} = r^{-k}$. Also, $r^k s r^{-k} = r^{2k} s$.

Conjugacy classes are given by the following:

$$\begin{array}{cc} \{e\} & \{s\} \\ \{r, r^{-1}\} & \{r^2 s\} \\ \{r^2, r^{-2}\} & \{r^4 s\} \\ & \{r^6 s\} \end{array}$$

We have split based on whether n is even or odd.

$$\begin{array}{cc} n \text{ odd} & n \text{ even} \\ \{e\} & \{e\} \\ \{r, r^{-1}\} & \{r, r^{-1}\} \\ \vdots & \vdots \\ \{r^{\frac{n-1}{2}}, r^{-\frac{n-1}{2}}\} & \{r^{\frac{n-2}{2}}, r^{-\frac{n-2}{2}}\} \\ \{s, rs, r^2 s, \dots, r^{n-1} s\} & \{r^{\frac{n}{2}}\} \\ & \{r, r^2 s, \dots, r^{n-1} s\} \\ & \{rs, r^3 s, \dots, r^{n-2} s\} \end{array}$$

So, for n odd:

of conjugacy class is $\frac{n+3}{2}$

$$D_{2n}^{ab} = \{1, \bar{s}\} \cong C_2$$

$$Z(D_{2n}) = \{e\}$$

For n even,

of conjugacy classes is $\frac{n+6}{2}$

$$D_{2n}^{ab} = \{1, \bar{s}, \bar{r}, \bar{r}\bar{s}\} \cong C_2 \times C_2$$

1-dim representations:

n odd implies we have representations $\mathbb{C}_+, \mathbb{C}_-$

$$\chi_{\pm}(r) = 1, \chi_{\pm}(s) = \pm 1$$

n even implies we have representations $\mathbb{C}_{++}, \mathbb{C}_{+-}, \mathbb{C}_{-+}, \mathbb{C}_{--}$

$$\varepsilon_r = \pm 1, \varepsilon_s = \pm 1$$

$$\chi_{\varepsilon_r \varepsilon_s}(r) = \varepsilon_r \text{ and } \chi_{\varepsilon_r \varepsilon_s} = \varepsilon_s$$

2-dim representations:

$$\rho^h : D_{2n} \rightarrow GL_2(\mathbb{C})$$

$$\rho^h(r) = \begin{bmatrix} \zeta_n^h & 0 \\ 0 & \zeta_n^{-h} \end{bmatrix}$$

$$\rho^h(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

[Induced from C_n -representation \mathbb{C}_h later]

For $0 < h < \frac{n}{2}$ it is irreducible [homework].

$$\chi_h(r^k) = e^{2\pi i h k / n} + e^{-2\pi i h k / n} = 2 \cos \frac{2\pi h k}{n}$$

$$\chi_h(r^k s) = 0$$

Since characters determine representation, we have $\rho_h \cong \rho_{-h} = \rho_{n-h}$.

Also, for $0 < h < \frac{n}{2}$ the representations are distinct.

We have all irreducible 2-dim representations.

Remark: \exists real representations $D_{2n} \rightarrow GL_2(\mathbb{R})$ [isometries in \mathbb{R}^2]. Then,

$$\hat{\rho}^h(r) = \begin{bmatrix} \cos \frac{2\pi h}{n} & -\sin \frac{2\pi h}{n} \\ \sin \frac{2\pi h}{n} & \cos \frac{2\pi h}{n} \end{bmatrix}$$

$$\hat{\rho}(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We have $\chi_h = \hat{\chi}_h$ and thus $\rho_h \cong \hat{\rho}_h$

Friday, 10/4/2024

Serre 5.4

Suppose $G = D_{2n} \times C_2$.

Then, $\mathbb{C}G = \mathbb{C}D_{2n} \otimes_{\mathbb{C}} \mathbb{C}C_2 = (\mathbb{C}D_{2n})_+ \times (\mathbb{C}D_{2n})_-$.

Twice as many irreducible representation as D_{2n} . 5.7 and 5.8

We have the following exact sequence:

$$1 \rightarrow A_4 \rightarrow S_4 \xrightarrow{\text{sign}} \{\pm 1\} \rightarrow 1$$

We have $|S_4| = 24 = 4!$, $|A_4| = 12$.

$\left\{ \begin{smallmatrix} S_4 \\ A_4 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} S_4 \\ \text{o.p} \end{smallmatrix} \right\}$ isometries of a tetrahedron.

Conjugacy classes (c.c.) in $\left\{ \begin{smallmatrix} S_4 \\ A_4 \end{smallmatrix} \right\}$ are $\begin{matrix} (1), (12), (12)(34), (123), (1234) & s = 5 \\ (1), (12)(34), (123), (213) & s = 4 \end{matrix}$

Interestingly, not all 3-cycles are conjugates in A_4 . For example, $(123) \not\sim (124)$.
Intuition: we need to swap 3 and 4, but in A_4 we need something else because swapping 3 and 4 is odd.

Also: A_4 is not simple [even though A_5, A_6 etc are].

$S_4 = C_2 \times C_2 \rtimes S_3$

$A_4 = C_2 \times C_2 \rtimes C_3$.

Also: $S_4^{ab} = C_2$

$A_4^{ab} = C_3$

Then, $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$

$12 = 1^2 + 1^2 + 1^2 + 3^2$

$\mathbb{C}[A_4] = \underbrace{\mathbb{C} \times \mathbb{C} \times \mathbb{C}}_{C_3\text{-quotient}} \times \underbrace{M_3(\mathbb{C})}_{\text{geometry}}$

$\mathbb{C}[S_4] = \underbrace{\mathbb{C} \times \mathbb{C}}_{C_2\text{-quotient}} \times M_2\mathbb{C} \times \underbrace{M_3\mathbb{C}}_{\text{geometry}} \times \underbrace{M_3\mathbb{C}}_{\text{geom} \otimes_{\mathbb{C}} \mathbb{C}_{\text{sign}}}$

Chapter 6

Suppose we have a finite group G and $(\text{char } k, |G|) = 1$. Then kG is semisimple.

Proposition 63 (10). Let A be semisimple ring. Suppose L_1, \dots, L_s are simple, non-isomorphic kG -modules such that \forall simple L we have $L \cong L_i$ for some i . Then,

$$A \xrightarrow{\text{mul}} \prod \text{End}_A L_i$$

Corollary: $t < s$ implies:

$$A \rightarrow \prod_{i=1}^t \text{End}_A L_i$$

is onto.

6.5:

Review: Corollary 2: if k is algebraically closed and $\text{char } k = 0$ and $d = \dim_k L$ where L is a simple kG module, then

$$d \mid |G|$$

We strengthen this.

Proposition 64 (17). Let $Z = Z(G)$ be the center of G . Then,

$$d \mid \frac{|G|}{|Z|}$$

Proof. Let $\rho : G \rightarrow GL(L)$ be an irreducible representation and $d = \dim$. Define homomorphism $\lambda : Z \rightarrow k^\times$ such that:

$$\rho(s) = \lambda(s) \text{id}$$

$\forall m \geq 1$ let $\rho^m : G \times \cdots \times G \rightarrow GL(L \otimes \cdots \otimes L)$ which is irreducible.
Then we have $\lambda^m : Z \times \cdots \times Z \rightarrow k^\times$ with:

$$(s_1, \dots, s_m) \mapsto \lambda(s_1 \cdots s_m)$$

Let $H = \{(s_i) \in Z^m \mid s_1 \cdots s_m = 1\} < Z^m < G^m$.

$H \cong Z^{m-1}$ and $H \subset \ker \rho^m$.

Then $\overline{\rho^m} : G^m/H \rightarrow GL(L \otimes \cdots \otimes L)$ irreducible.

Therefore, $\forall m, d^m \mid \left| \frac{G^m}{H} \right| = \frac{|G|^m}{|Z|^{m-1}}$ which implies by taking m big enough that $d \mid \frac{|G|}{|Z|}$. \square

Tensor Product for Non-Commutative Rings

Suppose R is a non-commutative ring. Then, tensor product is a functor

$$- \otimes_R - : \begin{matrix} \text{mod } R \\ \text{right mod} \end{matrix} \times \begin{matrix} R \text{ mod} \\ \text{left mod} \end{matrix} \rightarrow \text{Ab}$$

$$\begin{aligned} A_R \otimes_R {}_R B &\ni a_1 \otimes b_1 + \cdots + a_k \otimes b_k \\ (a + a') \otimes b &= a \otimes b + a' \otimes b \\ a \otimes (b + b') &= a \otimes b + a \otimes b' \\ ar \otimes b &= a \otimes rb \end{aligned}$$

Exercise. Formulate adjoint proposition:

$$\text{Hom}_?(\overset{?}{A} \otimes \overset{?}{B}, \overset{?}{C}) \cong \text{Hom}_?(A, \text{Hom}_?(B, C))$$

Definition (Induced module). : Suppose k is a field and $H < G$. Then,

$$\text{Ind}_H^G : kH\text{-mod} \rightarrow kG\text{-mod}$$

$$\text{Ind}_H^G W = kG \otimes_{kH} W$$

eg. Suppose $H = C_n = \langle r \mid r^n = 1 \rangle$ and $G = D_{2n} = \langle r, s \mid r^n = 1 = s^2; srs = r^{-1} \rangle$.
If $W = \mathbb{C}$ we have $H \rightarrow \mathbb{C}^\times$ by $r \mapsto \zeta_n$.

$$V = \mathbb{C}D_{2n} \otimes_{\mathbb{C}[C_n]} \mathbb{C}_1 = (\mathbb{C}[C_n] \oplus s\mathbb{C}[C_n]) \otimes_{\mathbb{C}[C_n]} \mathbb{C}_1$$

\mathbb{C} -basis of V is $1 \otimes 1, s \otimes 1$.

$$\text{Recall } r \mapsto \begin{bmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{bmatrix}, s \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$s(1 \otimes 1) = s \otimes 1$$

$$s(s \otimes 1) = s^2 \otimes 1 = 1 \otimes 1$$

$$r(1 \otimes 1) = r1 \otimes 1 = \zeta_n \otimes 1 = \zeta_n(1 \otimes 1)$$

$$r(s \otimes 1) = rs \otimes 1 = sr^{-1} \otimes 1 = s \otimes \zeta_n^{-1}1 = \zeta_n^{-1}(s \otimes 1)$$

Monday, 10/7/2024

Exercise. Work out the representation theory of $G = C_7 \rtimes_2 C_3 = \langle r, s \mid r^7 = 1, s^3 = 1, srs^{-1} = r^2 \rangle$.

Meaning: find an isomorphism $\mathbb{C}G \xrightarrow{\cong} M_{d_i} \mathbb{C}$

Suppose we have a (most likely non-commutative) ring R and

A tensor product functor $- \otimes_R - : \text{mod-}R \times R\text{-mod} \rightarrow \text{Ab}$

Proposition 65 (Universal Property). Suppose A is a right R -module and B is a left R -module and G is an abelian group.

$\pi : A \times B \rightarrow G$ is R -balanced. Meaning: π is \mathbb{Z} -bilinear and $\pi(ar, b) = \pi(a, rb)$.

There exists an R -balanced $\pi : A \times B \rightarrow A \otimes_R B$ which is initial.

$$\begin{array}{ccc}
A \times B & & \\
\downarrow \pi & \searrow \forall R\text{-balanced} & \\
A \otimes_R B & \xrightarrow{\exists! \mathbb{Z}\text{-hom}} & G
\end{array}$$

Construction:

$$A \otimes_R B := \frac{F(A \times B)}{T}$$

Where $F(A \times B)$ is the free abelian group with basis of $A \times B$. We write $F(A \times B) = \mathbb{Z}[A \times B]$.

T is the subgroup generated by $(a + a', b) - (a, b) - (a', b), (a, b + b') - (a, b) - (a, b'), (ar, b) - (a, rb)$.

Main thing to remember:

$$\boxed{ar \otimes b = a \otimes rb}$$

Proposition 66. Suppose we have a ring homomorphism $f : R \rightarrow S$ of possibly non-commutative rings. We preserve addition, multiplication and identity.

We then have the restriction functor

$$f^* : S\text{-mod} \rightarrow R\text{-mod}$$

$$f^* M = M \text{ (as abelian group)}$$

$$\begin{array}{ccc}
R \times f^* M & \rightarrow & f^* M \\
(r, m) & \mapsto & f(r)m
\end{array}$$

If we have inclusion $\text{inc} : kH \rightarrow kG$ then we have:

$$\text{inc}^* = \text{Res}_H^G : kG\text{-mod} \rightarrow kH\text{-mod}$$

We also have the left adjoint of f^* .

$$f_* : R\text{-mod} \rightarrow S\text{-mod} \text{ "base change"}$$

$$f_* M = S \otimes_R M$$

S is a right R -module. We have $S \times R \rightarrow S$ given by $(s, r) \mapsto sf(r)$ which trns S to a (S, R) -bimodule: ${}_S S_R$. So we can take the tensor product.

Proposition 67.

$$\boxed{\text{Hom}_S(f_* M, N) \cong \text{Hom}_R(M, f^* N)}$$

is an isomorphism of abelian groups.

So we can go back and forth between S -modules and R -modules.

$$\begin{array}{ccc}
& f^* & \\
S\text{-mod} & \xrightarrow{\quad} & R\text{-mod} \\
& f_* & \\
& \perp &
\end{array}$$

f_* is left adjoint.

f^* is right adjoint.

Adjoint of $\text{Id}_{f^* N} : \boxed{f_* f^* N \rightarrow N}$ is the counit.

Adjoint of $\text{Id}_{f_* M} : \boxed{M \rightarrow f^* f_* M}$ is the unit.

We also have:

$$\text{inc}_* = \text{Ind}_H^G : kH\text{-mod} \rightarrow kG\text{-mod}$$

Which gives us:

$$\text{Hom}_{kG}(\text{Ind}_H^G W, V) \cong \text{Hom}_{kH}(W, \text{Res}_H^G V)$$

Remark: If we have a module, how do we know it is induced?

Proposition 68. If $V = \bigoplus_{i \in I} W_i$ and G permutes summands transitively and $\exists W = W_{i_0}$ and $H = \{g \in G \mid gW = W\}$ then V is induced.

Example: $\mathbb{C}D_{2n} \otimes_{\mathbb{C}C_n} \mathbb{C}_1 = 1\mathbb{C}C_n \otimes \mathbb{C}_1 + s\mathbb{C}G_n \otimes \mathbb{C}_1$.

Proposition 69. V is induced if $\exists W < V$ invariant under H :

$$V = \bigoplus_{r \in R} rW$$

R is a set of left coset representation for H in G .

Character of Induced representation

Theorem 70 (12, p30). $V = \text{Ind}_H^G W$.

$$\chi_V(u) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_W(r^{-1}ur) = \frac{1}{|H|} \sum_{\substack{g \in G \\ g^{-1}ug \in H}} \chi_W(g^{-1}ug)$$

Proof. Write $V = \bigoplus_{r \in R} rW$. We care about when $urW = rW$, since otherwise we have non-diagonal terms so they don't contribute to the trace.

$$urW = rW \iff r^{-1}urW = W \iff r^{-1}ur \in H$$

$$\chi_V(u) = \text{Tr}(u : V \rightarrow V) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \text{Tr}(u : rW \rightarrow rW)$$

$$= \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \text{Tr}(r^{-1}ur : rW \rightarrow rW) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_W(r^{-1}ur) = \frac{1}{|H|} \sum_{\substack{g \in G \\ g^{-1}ug \in H}} \chi_W(g^{-1}ug)$$

□

Frobenius Reciprocity

$$\langle \text{Ind } \psi, \phi \rangle_G = \langle \psi, \text{Res } \phi \rangle_H$$

Wednesday, 10/9/2024

Recall: If

$$V = \text{Ind}_H^G W$$

Then V as a k -vector space can be written as direct sum of k -vector spaces:

$$V = \bigoplus_{g \in G/H} gW$$

And action of H permutes the summands.

$$\text{Stab}(W) := \{g \in W \mid gW = W\} = H$$

Also recall Class Functions:

$$\text{Cl}(G) = \{f : G \rightarrow k \mid f(g\sigma g^{-1}) = f(\sigma)\}$$

The characters χ_V are a basis of the vector space of class functions.
For $H < G$ we have restriction:

$$\begin{array}{ccc} \text{Res} : & \text{Cl}(G) & \rightarrow \text{Cl}(H) \\ & f & \mapsto f|_H \end{array}$$

We also have induction:

$$\text{Ind} : \text{Cl}(H) \rightarrow \text{Cl}(G)$$

$$(\text{Ind } f)(\sigma) := \frac{1}{|H|} \sum_{\substack{g \in G \\ g^{-1}\sigma g \in H}} f(g^{-1}\sigma g)$$

Last time we did:

$$\chi_{\text{Ind } W} = \text{Ind } \chi_W$$

Also we had the following:

$$\text{Hom}_{kG}(\text{Ind } W, V) \cong \text{Hom}_{kH}(W, \text{Res } V)$$

Today we give a character version of this.

Frobenius Reciprocity

Theorem 71 (Frobenius Reciprocity). Suppose k is algebraically closed. Then:

$$\langle \text{Ind } \psi, \phi \rangle_G = \langle \psi, \text{Res } \phi \rangle_H$$

where $\psi \in \text{Cl}(H)$ and $\phi \in \text{Cl}(G)$ with $H < G$.

Also, for review: if $\alpha, \beta \in \text{Cl}(G)$ then,

$$\langle \alpha, \beta \rangle_G = \sum_{g \in G} \alpha(g) \beta(g^{-1}) \in k$$

And irreducible characters are an orthonormal basis w.r.t. this inner product.

$$\langle \chi_i, \chi_j \rangle_G = \delta_{ij}$$

Proof. Suppose

$$V \cong \bigoplus_i m_i V_i$$

where V_1, \dots, V_s are irreducible. We define multiplicity: $m_{V_i}^V := m_i$. Then,

$$\langle \chi_V, \chi_{V'} \rangle = \sum_{i=1}^s m_{V_i}^V m_{V_i}^{V'} \underset{\text{Schur}}{=} \dim_k \text{Hom}_{kG}(V, V')$$

We finally start the proof.

$$\text{Cl}(G) = \text{span}\{\chi_i\}$$

WLOG assume ψ, ϕ are characters of W and V .

$$\dim_k \text{Hom}_{kG}(\text{Ind } W, V) = \dim_k(\text{Hom}_{kH}(W, \text{Res } V))$$

$$\implies \langle \text{Ind}(\chi_W), \chi_V \rangle_G = \langle \chi_W \cdot \text{Res } \chi_V \rangle_H$$

Since this is true for basis, it is true for general character. □

Mackey's Double Coset Formula

Suppose G is a group with subgroups H, K . aka $H, K < G$. Let W be a kH -module.

Question: What is $\text{Res}_K^G \text{Ind}_H^G W$ as a kK -module?

Let $s \in [K \backslash G / H]$ be the double coset representation. Meaning:

$$G = \coprod_{s \in S} KsH$$

i.e.

$$G \xrightarrow[\pi]{\quad \cdot \quad} K \backslash G / H$$

The above dotted map is $[\cdot]$. Then,

$$\pi \circ [\cdot] = \text{Id}$$

We have:

$$H_s := sHs^{-1} \cap K < K$$

$$\rho : H \rightarrow \text{GL}(W)$$

We thus have the twisted representation:

$$\rho^s : H_s \rightarrow \text{GL}(W)$$

$$\rho^s(x) = \rho_W(s^{-1}xs)$$

$W_s = W_{\rho^s}$ is a kH_s -module.

Proposition 72 (Mackey's Double Coset Formula, MDCF).

$$\text{Res}_K^G \text{Ind}_H^G W \cong \bigoplus_{s \in [K \backslash G / H]} \text{Ind}_{H_s}^K W_s$$

Proof. Suppose $V := \text{Ind}_H^G W$. Then, from the definition of $\text{Ind } W$,

$$V = \bigoplus_{x \in G/H} xW$$

Where $\text{Stab}(W) = H$.

$$V = \bigoplus_{x \in G/H} xW$$

Then, as kK -module,

$$V = \bigoplus_{s \in [K \backslash G / H]} KsW$$

Note that, since $\text{Stab}^K(sW) = H_s$,

$$KsW = \bigoplus_{x \in K/H_s} xsW$$

$$= \text{Ind}_{H_s}^K sW$$

$$= \text{Ind}_{H_s}^K W_s$$

Since

$$\begin{aligned} W_s &\cong sW \\ w &\mapsto sw \end{aligned}$$

So we're done. □

Mackey's Irreducibility Criterion, MIC

Suppose $W = W_\rho$ is kH -module. TFAE:

- 1) $V = \text{Ind}_H^G W$ is irreducible
- 2) a) W irreducible
 b) $\forall s \in G \setminus H, \rho^s$ and $\text{Res}_{H_s} \rho$ are disjoint.

Recall: V, V' are disjoint if $\text{Hom}_{kG}(V, V') = 0$.

Proof. We assume k is algebraically closed.

$$V \text{ irreducible} \iff \langle \chi_V, \chi_V \rangle_G = 1$$

$$\langle \chi_V, \chi_V \rangle_G = \langle \text{Ind } \chi_W, \text{Ind } \chi_W \rangle_G$$

$$= \langle \chi_W, \text{Res Ind } \chi_W \rangle_H [FR]$$

$$= \langle W, \bigoplus_{s \in [K \setminus G/H]} \text{Ind}_{H_s}^H(\rho_s) \rangle_H [MDCF]$$

$$= \sum_s \langle \text{Res}_{H_s} \rho, \rho^s \rangle_{H_s} [FR]$$

$$= \sum_s d_s$$

$$d_s = \langle \text{Res } \rho, \rho^s \rangle_{H_s}$$

$$d_1 = \langle \rho_W, \rho_W \rangle \geq 1$$

Thus,

$$1 = \langle V, V \rangle_G \iff \begin{matrix} d_1 = 1 \\ d_s = 0 \end{matrix}$$

So we're done. □

Example: Suppose $G = H \times K$ where $H = C_3, G = D_6 = S_3, K = C_2$.
 Then,

$$\mathbb{C}[C_3] = \mathbb{C}_0 \times \mathbb{C}_1 \times \mathbb{C}_2$$

$$\mathbb{C}[D_6] = \mathbb{C}_+ \times \mathbb{C}_- \times M_2\mathbb{C}$$

$$\text{Res } \mathbb{C}_+ = \mathbb{C}_0$$

$$\text{Res } \mathbb{C}_- = \mathbb{C}_0$$

$$\text{Res } \mathbb{C}^2 \stackrel{?}{=} \mathbb{C}_1 \times \mathbb{C}_2$$