

# Partial Differential Equation 2 MATH 541

Thanic Nur Samin

This is dedicated to Sobolev Spaces, which we apply to Elliptic (linear) PDE.

Last part of the course is in a different direction. We talk about applying this to Parabolic/Hyperbolic PDE.

Also Schauder Theory

## Monday, 8/26/2024

There are very explicit formula for certain PDE. For example D'Alembert, Poisson etc.

Weak Solutions to PDE: There is some notion of solution that doesn't have the requisite number of derivatives in the classical calculus sense. So we lower our notion of what a solution is.

ex. Conservation law: Burger's Equation:  $u_t + uu_x = 0, u(x, 0) = u_0(x)$ . If we try to solve this for  $-\infty < x < \infty, t > 0$ , we have method of characteristics that attempts to give us explicit formula (solution will be constant along lines of slope  $\frac{1}{u_0}$ ), but there is trouble for general  $u_0$ . These lines can bump into each other, so in that point our solution has to equal two different numbers.



Figure 1: Burger's Equation

Can also happen that a classical solution does exist, BUT it is easiest to find a weak solution.

- Find a weak solution
- Show it's classical (regularity theory)

For linear elliptic PDE's ex. Laplace's Equation,

$$\Delta u = u_{x_1 x_1} + \cdots + u_{x_n x_n} = 0$$

in  $\Omega \subset \mathbb{R}^n$

A weak solution satisfies the equation obtained through multiplication by a test function  $C_c^\infty$  and integrate by parts  
ie take any  $\phi \in C_c^\infty(\Omega)$

$$\begin{aligned}\int_{\Omega} \phi \Delta u &= 0 \\ &= - \int_{\Omega} \nabla \phi \cdot \nabla u + \int_{\partial \Omega} \phi \nabla u \cdot \nu\end{aligned}$$

Note that the second term is integrated over the boundary. So it goes to 0

So, if:

$$\int_{\Omega} \nabla \phi \cdot \nabla u = 0 \forall \phi \in C_c^\infty$$

We say  $u$  weakly solves Laplace's equation because it requires only one derivative (not two).

**Definition 1** (Weak Derivative). A locally integrable function  $u$  [notationally  $u \in L_{loc}^1(\Omega)$ ] has a weak  $x_i$  derivative  $v$  if  $v$  is locally integrable and  $\int_{\Omega} u \phi_{x_i} dx = - \int_{\Omega} v \phi dx$  for all  $\phi \in C_c^\infty(\Omega)$

$u$  and  $v$  can be terrible near the boundary but  $\phi$  vanishes so we don't care!

Recall: Multi-index notation

If  $u: \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\alpha = (\alpha_1, \dots, \alpha_n)$  where  $\alpha_j$  is non-negative integer then,

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}$$

eg in  $\mathbb{R}^3$  for  $\alpha = (1, 2, 1)$  then  $D^\alpha u = u_{x_1 x_2 x_2 x_3}$

**Definition 2.** Given  $u, v \in L_{loc}^1(\Omega)$ ,  $\Omega \subset \mathbb{R}^n$  and  $\alpha$  a multi-index, we say  $v$  is the weak  $\alpha$ th derivative of  $u$  if integration by parts works:

$$\text{if } \int_{\Omega} u D^\alpha \phi dx = (-1)^{|\alpha|} \int_{\Omega} v \phi dx, \forall \phi \in C_c^\infty(\Omega)$$

If we define weak anything, if our weak thing actually happened to be good, we want it to satisfy the strong definition as well! For example if differentiation is allowed, then integration by parts would actually work. So, if  $u$  were smooth, then our derivative would actually satisfy the solution, since

$$\int_{\Omega} (D^\alpha u - v) \phi dx = 0$$

So  $D^\alpha u = v$  almost everywhere.

Recall:  $u \in L^p(\Omega)$ ,  $p > 0$  if  $\int |u|^p dx < \infty$

$u \in L_{loc}^p(\Omega)$  if  $\forall \Omega_1 \subset \subset \Omega$ ,  $\int_{\Omega_1} |u|^p dx < \infty$

## Sobolev Spaces

**Definition 3** (Sobolev Spaces). Fix  $1 \leq p \leq \infty$  and a non-negative integer  $k$ . Let  $\Omega \subset \mathbb{R}^n$  be open.

Then the Sobolev space  $W^{k,p}(\Omega)$  consists of all functions  $u \in L_{loc}^1(\Omega)$  such that for every multi-index  $\alpha$  with  $|\alpha| < k$ ,  $D^\alpha u$  exists weakly and lies in  $L^p(\Omega)$ .

Example 1: Consider  $\tilde{u}(x) = |x|$ . Doesn't have a derivative. What about weak derivative?

Claim:  $u$  has weak 1st derivative  $v(x) = \begin{cases} -1, & \text{if } x < 0; \\ 1, & \text{if } x > 0; \end{cases}$

We verify that using test function.

Let  $\phi \in C_c^\infty(\mathbb{R})$

$$LHS = \int_{-\infty}^{\infty} v(x) \phi(x) dx = - \int_{-\infty}^0 \phi(x) dx + \int_0^{\infty} \phi(x) dx$$

$$RHS = - \int_{-\infty}^{\infty} \tilde{u}(x) \phi'(x) dx = \int_{-\infty}^0 x \phi'(x) dx - \int_0^{\infty} x \phi'(x) dx$$

By applying IBP

$$RHS = - \int_{-\infty}^0 \phi(x) dx + \int_0^{\infty} \phi(x) dx$$

[boundary terms don't matter because either  $x$  or  $\phi$  vanishes]  
 Since  $|x|$  is locally integrable for any  $p$ ,  $\tilde{u} \in W_{loc}^{1,p}(\mathbb{R})$  for all  $p$

Example 2: Consider the Heaviside function

$$u(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x < 0; \end{cases}$$

Is NOT going to be weakly differentiable!

## Wednesday, 8/28/2024

### Sobolev Space $W^{k,p}(\Omega)$

$u \in W^{k,p}(\Omega)$  if

$D^\alpha u \in L^p$  weakly for all  $\alpha$  such that  $|\alpha| \leq k$

This is a normed space.

$$\|u\|_{W^{k,p}(\Omega)} = \left[ \sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha u|^p \right]^{\frac{1}{p}}$$

If  $p = \infty$  we just take the sup norm.

We have convergence:  $\{u_j\} \subset W^{k,p}(\Omega)$  converges in  $W^{k,p}$

to  $u \in W^{k,p}(\Omega)$  if  $\|u_j - u\|_{W^{k,p}(\Omega)} \rightarrow 0$  as  $j \rightarrow \infty$

**Definition 4.**  $W_0^{k,p}(\Omega) :=$  closure in  $W^{k,p}(\Omega)$ -norm of  $C_0^\infty(\Omega)$

Remark:  $W^{k,p}(\Omega)$  and  $W_0^{k,p}(\Omega)$  are Banach spaces [normed, complete [since  $L^p$  is complete]]

For  $p = 2$  we don't usually use  $W^{k,2}(\Omega)$ . We use  $H^k(\Omega)$ .  $H$  is for Hilbert. This is a Hilbert space [there exists an inner product].

$$(u, v)_{H^k(\Omega)} := \sum_{|\alpha| \leq k} \int_{\Omega} D^\alpha u D^\alpha v$$

Hölder's inequality implies:

$$\left| \int_{\Omega} D^\alpha u D^\alpha v \right| \leq \|D^\alpha u\|_{L^2} \|D^\alpha v\|_{L^2}$$

Let's go back to the example.

Let  $u(x)$  be the heaviside function, 0 for negative, 1 for positive. Is it weakly differentiable? Is it in some sobolev space?

Is  $u \in W^{k,p}(\mathbb{R})$  for some  $k$  and  $p$ ? Does  $u$  have a weak solution?

Answer: No! We use contradiction.

Suppose there is such a  $v \in L_{loc}^1(\mathbb{R})$  such that IBP holds:

$$\int_{\mathbb{R}} u \phi' dx = - \int_{\mathbb{R}} v \phi dx$$

for all  $\phi \in C_c^\infty(\mathbb{R})$

Then,  $\int_0^\infty \phi'(x) dx = - \int_{-\infty}^\infty v \phi dx$

Therefore,  $\phi(0) = \int_{-\infty}^\infty v(x) \phi(x) dx$

We use this for contradiction. Consider a sequence  $\{\phi_j\}$  of test functions so that  $\phi_j(0) = 1$  for all  $j$  and  $0 \leq \phi_j(x) \leq 1$  for all  $x$ .

Further suppose that the support for  $\phi_j$  shrinks to the origin.

$$\phi_j(0) = 1 = - \int_{\mathbb{R}} v \phi_j \, dx$$

Now,  $v \phi_j \rightarrow 0$  pointwise a.e.

$$|v \phi_j| \leq |v| \in L^1_{loc}$$

This gives us the desired contradiction.

Notice:  $\phi(0) = \int v(x) \phi(x)$  is true for the ‘function’ dirac delta. This is not really a function, this is a distribution.

Moral of the story: We can’t have too big of a discontinuity [here we have a jump discontinuity] and still be in Sobolev spaces.

Example:  $f(x) = \frac{1}{|x|^\alpha}$  for  $x \in B(0, 1) = \{x \in \mathbb{R}^n : |x| < 1\}$  and  $\alpha > 0$

For which  $k, p, n, \alpha$  is  $f \in W^{k,p}(B(0, 1))$ ?

Question 1: Is this in any  $L^p$  space? If no then game over.

Is  $f \in L^p(B(0, 1))$

$$\begin{aligned} \int_{B(0,1)} \frac{1}{|x|^{\alpha p}} \, dx &= \int_0^1 \int_{\partial B(0,r)} \frac{1}{|x|^{\alpha p}} \, dS \, dr \\ &= \int_0^1 \frac{1}{r^{\alpha p}} \mu(\partial B(0, r)) \, dr \\ &= \omega_n \int_0^1 \frac{1}{r^{\alpha p}} r^{n-1} \, dr = \omega_n \int_0^1 r^{-\alpha p + n-1} \, dr \end{aligned}$$

So,  $f \in L^p$  if  $\alpha p \leq n$

Note that  $|D^\alpha f| \leq L^p$  for  $|\alpha| = 1$  provided  $(\alpha + 1)p \leq n$

Is  $f$  weakly differentiable for  $|\alpha| = 1$ ?

Consider  $\epsilon < 1$

$$\begin{aligned} \int_{B(0,1) \setminus B(0,\epsilon)} f \phi_{x_i} \, dx &= \int_{B(0,1) \setminus B(0,\epsilon)} \nabla \cdot (0, 0, f \phi, 0, 0) \, dx - \int_{B(0,1) \setminus B(0,\epsilon)} \phi f_{x_i} \, dx \\ \int_{B(0,1) \setminus B(0,\epsilon)} f \phi_{x_i} \, dx &= - \int_{B(0,1) \setminus B(0,\epsilon)} \phi f_{x_i} \, dx + \int_{\partial(B(0,1) \setminus B(0,\epsilon))} f \phi \nu_i \, dS \end{aligned}$$

Note that the outer normal disappears, we only have the inner normal.

$$\int_{\partial(B(0,1) \setminus B(0,\epsilon))} f \phi \nu_i \, dS = \int_{\partial B(0,\epsilon)} f \phi \nu_i \, dS$$

Setting  $\epsilon \rightarrow 0$  we see that the integral converges. Also,

$$\left| \int_{\partial(B(0,1) \setminus B(0,\epsilon))} f \phi \nu_i \, dS \right| \leq \int_{\partial B(0,\epsilon)} |f \phi \nu_i| \leq c \int_{\partial B(0,\epsilon)} |f| \, dS \leq \frac{c}{\epsilon^\alpha} \omega_n \epsilon^{n-1} \rightarrow 0$$

provided  $n - 1 \geq \alpha$

So,

$$\int_{B(0,1)} f \phi_{x_i} \, dx = - \int_{B(0,1)} \phi f_{x_i} \, dx$$

So  $f$  is weakly differentiable for  $|\alpha| = 1$

(See Appendix 5 in Evans)

Mollification:

There are lot of situation in PDE where you have a function and you don’t know how nice is it in terms of derivative. So you convolve it so that it is nice and take some limit.

Let  $\eta$  satisfy  $\eta \in C_c^\infty(\mathbb{R}^n)$

Suppose  $\eta \equiv 0$  for  $|x| \geq 1$

$$\int_{\mathbb{R}^n} \eta(x) \, dx = 1$$

Suppose  $\eta$  is radial,  $\eta = \eta(|x|)$  just a function of radial distance

Note that there is no such analytic function. But there are infinitely differentiable ones.

Define  $\eta_\epsilon := \epsilon^{-n} \eta(\frac{x}{\epsilon})$

So we rescale the function.

Then given  $u \in L^1_{loc}(\mathbb{R}^n)$  we define the mollification

$$u_\epsilon(x) = u * \eta_\epsilon = \int u(x-y) \eta_\epsilon(y) dy = \int u(y) \eta_\epsilon(x-y) dy$$

## Friday, 8/30/2024

Given  $\Omega \subset \mathbb{R}^n$ , open, bounded

$\forall \epsilon$  define  $\{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\} = \Omega_\epsilon$

Let  $\eta \in C_c^\infty(\mathbb{R}^n)$  such that

$0 \leq \eta \leq 1$  and  $\int_{B(0,1)} \eta(x) dx = 1$  and  $\text{supp}(\eta) \subset \overline{B(0,1)}$

Define  $\eta_\epsilon(x) := \epsilon^{-n} \eta(\frac{x}{\epsilon})$

Then,  $\text{supp}(\eta_\epsilon) \subset \overline{B(0,\epsilon)}$

$$\int_{B(0,\epsilon)} \eta_\epsilon(x) dx = \epsilon^{-n} \int_{B(0,\epsilon)} \eta(\frac{x}{\epsilon}) dx = \int_{B(0,1)} \eta(x) dx = 1$$

**Theorem 1.** Given  $f \in L^1_{loc}(\Omega)$  define  $f_\epsilon := \eta_\epsilon * f$  for  $x \in \Omega_\epsilon$

$$f_\epsilon(x) = \int_{\Omega} \eta_\epsilon(x-y) f(y) dy$$

Then, i:  $f_\epsilon(x)$  is infinitely differentiable.

ii:  $f_\epsilon \rightarrow f$  pointwise a.e.

iii: If  $f$  is continuous then  $f_\epsilon \rightarrow f$  uniformly on compact subsets of  $\Omega$

iv: If for  $1 \leq p < \infty$ ,  $f \in L^p(\Omega)$  then  $f_\epsilon \rightarrow f$  in  $L^p(\Omega)$ . Also true for  $f \in L^p_{loc}$

*Proof.* (i): Convolution is a child, and it inherits the nicest properties of the parent.  $\eta_\epsilon$  is nicer.

Idea: it is legal to bring derivatives inside the integral.

$$\frac{f_\epsilon(x + he_i) - f_\epsilon(x)}{h} = \int_{\Omega} \left[ \frac{\eta_\epsilon(x + he_i - y) - \eta_\epsilon(x - y)}{h} \right] f(y) dy$$

Stuff in brackets converges uniformly in  $y$  to  $\eta_{\epsilon x_i}(x - y)$  using Taylor remainder theorem.

Then Lebesgue Dominated Convergence finishes the job.

(iii) Fix  $K \subset \Omega$  where  $K$  compact.

$\forall x \in K$  we have  $|f_\epsilon(x) - f(x)| = \left| \int_{\Omega} (\eta_\epsilon(y) f(x-y)) dy - f(x) \right|$

$$\begin{aligned} &= \left| \int_{\Omega} \eta_\epsilon(y) [f(x-y) - f(x)] dy \right| \\ &\leq \int_{\Omega} \eta_\epsilon(y) |f(x-y) - f(x)| dy \end{aligned}$$

Since  $K$  is compact,  $f$  is uniformly continuous on  $K$ .

So,  $\forall \beta > 0$  we have  $\epsilon_0$  such that  $\forall \epsilon < \epsilon_0$  such that  $\forall x, y$  such that  $|y| < \epsilon$  we have  $|f(x-y) - f(x)| < \beta$

So,

$$\leq \int_{\Omega} \eta_\epsilon(y) \beta dx = \beta$$

□

What is the relevance to Sobolev spaces?

Often we are going to try to prove some estimates. We want to prove some inequalities. Then, we mollify and prove it for the mollification. If we mollify a Sobolev function, we get an approximation.

**Theorem 2** (Local Approximation Away from  $\partial\Omega$ ). Assume  $u \in W^{k,p}(\Omega)$  [aka function has  $k$ 'th weak derivatives which are locally integrable in order  $p$ ]. Define  $u_\epsilon = \eta_\epsilon * u$  in  $\Omega_\epsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\}$

Then,  $u_\epsilon \in C^\infty(\Omega_\epsilon)$

And  $u_\epsilon \rightarrow u$  in  $W_{loc}^{k,p}(\Omega)$

*Proof.* Note infinite derivative we already have by property of mollification.

Fix the multi-index  $\alpha$  such that  $|\alpha| \leq k$ .

Claim 1:  $D^\alpha u_\epsilon = \eta_\epsilon * D^\alpha u$

In other words, mollification and derivatives commute.

To see this, consider  $x \in \Omega_\epsilon$

$$\begin{aligned} D^\alpha u_\epsilon &= D^\alpha \int_{\Omega} \eta_\epsilon(x-y) u(y) dy \\ &= \int_{\Omega} D_x^\alpha \eta_\epsilon(x-y) u(y) dy \\ &= (-1)^{|\alpha|} \int_{\Omega} D_y^\alpha \eta_\epsilon(x-y) u(y) dy \\ &= (-1)^{|\alpha|} (-1)^{|\alpha|} \int_{\Omega} \eta_\epsilon(x-y) D^\alpha u(y) dy \\ &= \eta_\epsilon * D^\alpha u \end{aligned}$$

proving the claim.

Now, fix  $V \subset \Omega$  with open  $\bar{V} \subset \Omega$  ( $V \subset\subset \Omega$ )

Apply previous theorem, item iv and the claim

$D^\alpha u_\epsilon \rightarrow D^\alpha u$  in  $L^p(V)$  for all  $\alpha$  such that  $|\alpha| \leq k$

□

**Theorem 3** (Global Approximation). Theme:

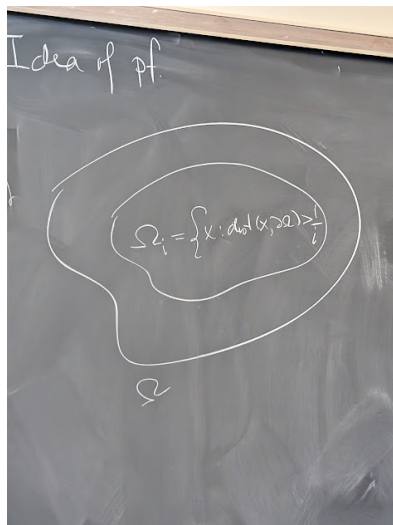
Sobolev can be approximated by smooth sobolev

Let  $\Omega \subset \mathbb{R}^n$  be open, bounded. Assume  $u \in W^{k,p}(\Omega)$  for  $1 \leq p < \infty$

Then,  $\exists \{u_m\} \subset C^\infty(\Omega) \cap W^{k,p}(\Omega)$  [infinitely smooth AND sobolev] such that  $u_m \rightarrow u$  in  $W^{k,p}(\Omega)$ . Meaning:

$$\lim_{m \rightarrow \infty} \|u_m - u\|_{W^{k,p}(\Omega)} \rightarrow 0$$

Idea of Proof:



We have a bunch of  $\Omega_i$  with  $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$   
 Use a partition of unity  $\{\phi_i\}$  so that  $\sum \phi_i(x) = 1$ , mollify.  
 Proof is on Evans.

## Wednesday, 9/4/2024

### Traces

We are going to solve PDEs using Sobolev spaces. We typically specify boundary conditions in PDEs. But in first glance, since Sobolev functions are defined upto a set of measure zero, it seems ill suited to dealing with boundary conditions [since boundary  $\partial\Omega$  has measure 0].

How to define boundary values for a Sobolev functions?

We are going to establish an inequality for smooth function.

**Theorem 4.** Assume  $\partial\Omega$  is  $C^1$  surface<sup>1</sup>. Then  $\exists$  a bounded linear operator  $T : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$  for  $1 \leq p < \infty$  such that:

- i)  $Tu = u|_{\partial\Omega}$  if  $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$
- ii)  $\|Tu\|_{L^p(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}$  for some  $C = C(p, \Omega)$

*Proof.* Outline:

- 1) Fix  $x_0 \in \partial\Omega$ . Suppose  $\partial\Omega$  is ‘flat’ near  $x_0$ . In fact, define ball  $B$  centered at  $x_0$  such that inside  $B$ ,  $\partial\Omega$  is flat.

[insert picture]

Then we can define  $x' = (x_1, \dots, x_{n-1})$ . If the last coordinate is positive, we are inside  $\Omega$ .

Inside  $B$ ,  $\partial\Omega$  is flat. we can define  $B' \subset B$  centered at  $x_0$ .

Let  $\zeta \in C_0^\infty(B)$  so that  $\zeta \equiv 1$  on  $B'$  and  $\zeta \geq 0$ .

Define  $\Gamma := \partial\Omega \cap B'$ .

Assume  $u \in C^1(\overline{\Omega})$ .

$$\begin{aligned} \int_{\Gamma} |u|^p dx' &\leq \int_{\{x_n=0\} \cap B} \zeta |u|^p dx' \\ &= - \int_{\{x_n=0\} \cap B} (0, \dots, 0, \zeta |u|^p) \cdot (0, \dots, 0, -1) dx' \end{aligned}$$

We are almost set up for divergence theorem. Since  $\zeta = 0$  on the boundary of  $B$ , we actually have the whole boundary! Applying the divergence theorem, we get:

$$\begin{aligned} &= - \int_{B \cap \Omega} \frac{\partial}{\partial x_n} [\zeta(x) |u(x)|^p] dx \\ &= - \int_{B \cap \Omega} \left[ \zeta_{x_n} |u|^p + \zeta p |u|^{p-1} \frac{u}{|u|} u_{x_n} \right] dx \\ &= - \int_{B \cap \Omega} [\zeta_{x_n} |u|^p + \zeta p |u|^{p-1} \operatorname{sgn}(u) u_{x_n}] dx \end{aligned}$$

Now we use Young’s Inequality.

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<sup>1</sup>meaning  $\partial\Omega$  can be described as the graph of a  $C^1$  function.

**Theorem 5** (Young's Inequality). For any  $a, b > 0$  and  $p, q$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ , we have:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

We are estimating absolute value, so we can take absolute value. Using Young's inequality on  $a = |u_{x_n}|$  and  $b = |u|^{p-1}$ ,

$$\begin{aligned} & \int_{\Gamma} |u|^p dx' \\ & \leq \int_B^\infty \left[ |\zeta_{x_n}| |u|^p + p |\zeta| \left[ \frac{(|u|^{p-1})^{p/(p-1)}}{p/(p-1)} + \frac{|u_{x_n}|^p}{p} \right] \right] dx \\ & \leq C \int_{B \cap \Omega} [|u|^p + |\nabla u|^p] dx \leq C \int_{\Omega} [|u|^p + |\nabla u|^p] dx \end{aligned}$$

Now,  $\partial\Omega \in C^1$  means, centered at any point  $x_0 \in \partial\Omega$   $\partial\Omega$  can be written as a graph  $x_n = f(x')$  where  $|f|_{C_1}$  is bounded.<sup>2</sup>

Therefore, Ball inside of which  $\partial\Omega$  is a graph has radius  $R$  depending on  $C^1$  norm.

- 2) For the next step, we flatten the boundary of  $\Omega$  by a change of variables.

[insert picture]

[insert picture 2]

Change variables:  $y = (y', y_n)$ .

Set  $y' = x'$

$y_n := x_n - f(x')$

$y = G(x)$ .

So,  $y_n = 0$  means  $x_n = f(x')$ , which means we're on the graph.

We can think of this in terms of the Inverse Function Theorem. What is the Jacobian of  $G$ ?

$$\text{Jac}(G) = \det \nabla G = \det \begin{pmatrix} I & 0 \\ -f_{x_1} - f_{x_2} - \cdots - f_{x_{n-1}} & 1 \end{pmatrix} = 1$$

So,  $G^{-1}$  is  $C^1$ .

Then,  $|\nabla G| \leq C(\partial\Omega)$  and  $|\nabla(G^{-1})| \leq C(\partial\Omega)$ .

$u$  is a given function. We can define a new function that has a flat boundary which we can use instead of  $u$ .

Define  $\tilde{u}(y) := u(G^{-1}(y))$ .

Then,  $u(x) = \tilde{u}(G(x))$

Suppose  $G(\Gamma) = \tilde{G}$ .

Then, Step 1 implies,

$$\int_{\tilde{\Gamma}} |\tilde{u}|^p dy' \leq C \int_{\text{shaded region}} [|\tilde{u}|^p + |\nabla \tilde{u}|^p] dy$$

Where the shaded region is  $\{x \mid x_n > f(x'), x \in B_R\}$ . Now,

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<sup>2</sup> $|f|_{C_1} = \sup |f| + \sup |\nabla f|$



$$|\nabla \tilde{u}(y)| \leq |Du \cdot \nabla(G^{-1})| \leq C |\nabla u|$$

Continuing,

$$\begin{aligned} \int_{\tilde{\Gamma}} |\tilde{u}|^p dy' &\leq C \int_{\text{shaded region}} [|\tilde{u}|^p + |\nabla \tilde{u}|^p] dy \\ \implies \int_{\Gamma} |u|^p dx' &\leq C \int_{\Omega \cap B_R} [|u|^p + |\nabla u|^p] dx \leq C \int_{\Omega} [|u|^p + |\nabla u|^p] dx \end{aligned}$$

Which finishes Step 2.

3) Decompose  $\partial\Omega$  into  $N_R$  pieces, and add them up using Step 2.

Since  $\partial\Omega$  is compact,  $N_R < N(\partial\Omega)$ .

[insert picture]

We have:  $\forall u \in C^1(\bar{\Omega})$ ,

$$\int_{\partial\Omega} |u|^p dS \leq C(p, \Omega) \int_{\Omega} [|u|^p + |\nabla u|^p] dx$$

Now, suppose  $u \in W^{1,p}(\Omega)$ . Approximate  $u$  in  $W^{1,p}$  norm by  $\{u_m\} \subset W^{1,p}(\Omega) \cap C^1(\bar{\Omega})$ . Then,

$$\underbrace{\|u_m - u_l\|_{L^p(\partial\Omega)}}_{\text{cauchy sequence}} \leq C \|u_m - u_l\|_{W^{1,p}}^p$$

$RHS \rightarrow 0$ , so  $LHS \rightarrow 0$  for  $m, l \gg 1$ .

Therefore,  $\exists Tu \in L^p(\partial\Omega)$  which is the limit of this cauchy sequence.

□

## Friday, 9/6/2024

A few comments about traces.

Last time: If  $u \in W^{1,p}(\Omega)$  for  $1 \leq p < \infty$  and  $\partial\Omega \in C^1$ , we can define the trace of  $u$  on  $\partial\Omega$ ,  $Tu \in L^p(\partial\Omega)$  and  $\exists C = C(p, \Omega)$  such that,

$$\|Tu\|_{L^p(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}$$

Recall:  $W_0^{k,p}(\Omega) = \text{closure of } C_c^\infty(\Omega) \text{ in the } W^{k,p} \text{ norm.}$

Suppose  $k = 1, u \in W_0^{1,p}(\Omega)$ . Then,  $Tu = 0$  on  $\partial\Omega$

Also, if  $k > 1$  and  $u \in w_0^{k,p}(\Omega)$ , we have  $Tu = 0$ .

In fact,  $T(D^\alpha u) = 0 \forall \alpha$  such that  $|\alpha| \leq k - 1$ .

## Calculus of Variations

Our objects of interest are functionals.

Consider an integral functional of the form:

$$E(u) = \int_{\Omega} L(x, u, \nabla u) dx$$

We use  $E$  because this often denotes Energy. It comes from physics often.

In that case, we call  $L$  an energy density. It is also often called the energy density.

A fundamental problem is:

Determine the infimum of  $E$  among all functions  $u$  in an admissible set  $\mathcal{A}$ .

We define:

$$m := \inf_{u \in \mathcal{A}} E(u)$$

“First Variation of  $E$ ”

This is the ‘calculus of variations’ version of a derivative.

If  $\Omega \subset \mathbb{R}^n$  and  $u : \Omega \rightarrow \mathbb{R}$ ,

$$L = L(x, z, p)$$

where:  $x$  is a point in  $\Omega \subset \mathbb{R}^n$

$z \in \mathbb{R}$  [we put  $u$  here]

$p \in \mathbb{R}^n$  [we put  $\nabla u$  here]

We put  $u$  in  $E$ , but we perturb it a little bit.

Suppose  $u \in \mathcal{A}$ , and  $v$  such that  $u + tv \in \mathcal{A}$  for  $|t|$  small.

$E(u + tv)$

Then,  $t \mapsto E(u + tv)$  is a function  $\mathbb{R} \rightarrow \mathbb{R}$ . We take derivative and set  $t = 0$

$$\left. \frac{d}{dt} \right|_{t=0} E(u + tv) = \delta E(u; v)$$

Note: if  $u$  is a minimum of  $E$ , i.e.  $E(u) = m$ , then the first variation  $\delta E(u; v) = 0$  for all  $v$  such that  $u + tv \in \mathcal{A}$ .

**Definition 5.**  $u$  is called a critical point of  $E$  in  $\mathcal{A}$  if  $\delta E(u; v) = 0 \forall v$  such that  $u + tv \in \mathcal{A}$ .

Question: What is true about a critical point  $u$  of an integral functional of form  $(*)$ ?

$$\begin{aligned} 0 &= \left. \frac{d}{dt} \right|_{t=0} E(u + tv) \\ &= \left. \frac{d}{dt} \right|_{t=0} \int_{\Omega} L(x, u + tv, \nabla u + t \nabla v) dx \\ &= \int_{\Omega} \left[ \frac{\partial L}{\partial z}(x, u, \nabla u) v + \sum_{j=1}^n \frac{\partial L}{\partial p_j}(x, u, \nabla u) v_{x_j} \right] dx \\ &= \int_{\Omega} \left[ \frac{\partial L}{\partial z}(x, u, \nabla u) v + \nabla_p L \cdot \nabla_x v \right] dx \end{aligned}$$

Applying IBP,

$$\begin{aligned} &= \int_{\Omega} \left[ \frac{\partial L}{\partial z}(x, u, \nabla u) - \sum_{j=1}^n \frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial p_j} L(x, u, \nabla u) \right) \right] v dx \\ &\quad + \int_{\partial \Omega} v \nabla_p L(x, u, \nabla u) \cdot \nu dS \end{aligned}$$

Now, suppose all allowable  $v$ ’s are included in  $C_c^\infty(\Omega)$  functions. That would make the boundary term 0. Then, since we can choose  $v$  however we want, the big integral is 0.

$$\sum_{j=1}^n \frac{\partial}{\partial x_j} \left[ \frac{\partial}{\partial p_j} L(x, u, \nabla u) \right] = \frac{\partial L}{\partial z}(x, u, \nabla u)$$

This is a 2nd order PDE. This is called the Euler-Lagrange equation for  $E$ .

Example: Take  $E(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + W(x, u) dx$

Here  $L(x, z, p) = \frac{1}{2} |p|^2 + W(x, z)$

Then, we find the Euler-Lagrangian.

$$\frac{\partial L}{\partial p_j}(x, z, p) = \frac{\partial}{\partial p_j} \left[ \frac{1}{2} |p|^2 + W(x, u) \right] = p_j$$

$$\frac{\partial L}{\partial p_j}(x, u, \nabla u) = u_{x_j}$$

Also,

$$\frac{\partial L}{\partial z}(x, z, p) = \frac{\partial W}{\partial z}(x, z)$$

So, the Euler-Lagrangian equation gives us:

$$\sum_{j=1}^n \frac{\partial}{\partial x_j}(u_{x_j}) = \frac{\partial W}{\partial z}(x, u)$$

$$\Delta u = \frac{\partial W}{\partial z}(x, u)$$

This is a Nonlinear Poisson Equation.

Question: What should we choose  $\mathcal{A}$  to be to make our life easiest?

We want integration by parts to be justified, but we want our topology to be weak enough so that finding minimizers is easy.

In the previous example, we have integral of  $|\nabla u|^2$ . So, ideally we want it to be  $L^2$ .

So, in the example, best choice is  $H^1(\Omega)$ .

How does one find a minimizer?

Direct Method: Our problem is, we want to find  $u \in \mathcal{A}$  such that:

$$m := \inf_{u \in \mathcal{A}} E(u)$$

Idea: We find a sequence  $\{u_j\}$  [called a minimizing sequence]: so that  $\{u_j\} \subset \mathcal{A}$  and  $E(u_j) \rightarrow m$ .

Step 1: Try to get a convergent subsequence of  $u_{j_k} \rightarrow u_* \in \mathcal{A}$ . [Compactness]

In this step, if we choose  $\mathcal{A}$  to be ‘too strong’, we lose. Ideally we want  $\mathcal{A}$  to be sequentially compact.

Step 2:  $E$  might not be continuous in this topology, then we don’t have  $E(u_*) = m$ .

So, we want lower semi-continuity.

$$\liminf_{k \rightarrow \infty} E(u_{j_k}) \geq E(u_*)$$

Then we have  $E(u_*) \leq m$ .

This tells us that  $u_*$  is a minimum.

## Monday, 9/9/2024

**Theorem 6** (Extension of Sobolev Functions). Assume  $\Omega \subset \mathbb{R}^n$ , open, bounded with  $\partial\Omega \in C^1$ . Then,  $\exists C = C(p, \Omega)$  and an extension  $E : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^n)$  such that:

- i)  $Eu = u$  in  $\Omega$
- ii)  $Eu$  has compact support
- iii)  $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)}$

Idea is, we can let a function down to zero ‘gently’ and extend that way.

**Theorem 7** (Gagliardo - Nirenberg - Sobolev Inequality). Assume  $1 \leq p < n$ . Then, there exists  $C = C(p, n)$  such that for every  $u \in C_0^1(\mathbb{R}^n)$  [compactly supported continuously differentiable function] one has the following inequality:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

Where  $p^* = \frac{np}{n-p}$  is the critical sobolev exponent.

Note that  $\frac{np}{n-p} = \frac{np-p^2+p^2}{n-p} = p + \frac{p^2}{n-p} > p$ .

Question: Where does  $p^*$  come from?

Given  $u \in C_0^1(\mathbb{R}^n)$ , define:

$$u_\lambda(x) := u(\lambda x)$$

for  $\lambda > 0$ .

Suppose  $|u|_{L^q(\mathbb{R}^n)} \leq C |\nabla u|_{L^p(\mathbb{R}^n)}$

For which  $q$  could this be true? Consider the norm of the scaled function:

$$\begin{aligned} \|u_\lambda\|_{L^q} &= \left[ \int_{\mathbb{R}^n} |u(\lambda x)|^q dx \right]^{\frac{1}{q}} = \frac{1}{\lambda^{n/q}} \left[ \int_{\mathbb{R}^n} |u(y)|^q dy \right]^{\frac{1}{q}} \\ \|\nabla u_\lambda\|_{L^p} &= \left[ \int_{\mathbb{R}^n} |\nabla_x u(\lambda x)|^p dx \right]^{\frac{1}{p}} = \underbrace{\frac{1}{\lambda^{\frac{n}{p}}} (\lambda^p)^{\frac{1}{p}}}_{=\lambda^{1-\frac{n}{p}}} \left[ \int_{\mathbb{R}^n} |\nabla_y u(y)|^p dy \right]^{\frac{1}{p}} \end{aligned}$$

Applying the same inequality,

$$\begin{aligned} \frac{1}{\lambda^{\frac{n}{q}}} \left[ \int_{\mathbb{R}^n} |u(y)|^q dy \right]^{\frac{1}{q}} &\leq C \lambda^{1-\frac{n}{p}} \left[ \int_{\mathbb{R}^n} |\nabla u(y)|^p dy \right]^{\frac{1}{p}} \\ \|u\|_{L^q} &\leq C \lambda^{1-\frac{n}{p}+\frac{n}{q}} \|\nabla u\|_{L^p} \quad \forall \lambda > 0 \end{aligned}$$

This means  $1 - \frac{n}{p} + \frac{n}{q}$  must be 0.

Therefore,  $q = p^*$

So, if the theorem is true we must have the  $q$ .

Now we prove the theorem.

*Proof.* First take  $n = 3$ .

Take  $p = 1$ .

$$\begin{aligned} u(x) &= \int_{-\infty}^{x_1} u_{x_1}(y_1, x_2, x_3) dy_1 \\ u(x) &= \int_{-\infty}^{x_2} u_{x_2}(x_1, y_2, x_3) dy_2 \\ u(x) &= \int_{-\infty}^{x_3} u_{x_3}(x_1, x_2, y_3) dy_3 \end{aligned}$$

Therefore,

$$\begin{aligned} |u(x)| &\leq \int_{-\infty}^{x_1} |\nabla u(y_1, x_2, x_3)| dy_1 \\ |u(x)| &\leq \int_{-\infty}^{x_2} |\nabla u(x_1, y_2, x_3)| dy_2 \\ |u(x)| &\leq \int_{-\infty}^{x_3} |\nabla u(x_1, x_2, y_3)| dy_3 \end{aligned}$$

Multiplying,  $|u(x)|^3$

$$\leq \left[ \int_{-\infty}^{x_1} |\nabla u(y_1, x_2, x_3)| dy_1 \right] \left[ \int_{-\infty}^{x_2} |\nabla u(x_1, y_2, x_3)| dy_2 \right] \left[ \int_{-\infty}^{x_3} |\nabla u(x_1, x_2, y_3)| dy_3 \right]$$

Therefore,  $|u(x)|^{\frac{3}{2}} \leq$

$$\underbrace{\left[ \int_{-\infty}^{\infty} |\nabla u(y_1, x_2, x_3)| dy_1 \right]^{\frac{1}{2}}}_{=I_1(x_2, x_3)} \underbrace{\left[ \int_{-\infty}^{\infty} |\nabla u(x_1, y_2, x_3)| dy_2 \right]^{\frac{1}{2}}}_{=I_2(x_1, x_3)} \underbrace{\left[ \int_{-\infty}^{\infty} |\nabla u(x_1, x_2, y_3)| dy_3 \right]^{\frac{1}{2}}}_{=I_3(x_1, x_2)}$$

Integrating with respect to  $x_1$  and applying Hölder,

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{3}{2}} dx_1 \leq (I_1(x_2, x_3))^{\frac{1}{2}} \left[ \int_{-\infty}^{\infty} I_2(x_1, x_3) dx_1 \right]^{\frac{1}{2}} \left[ \int_{-\infty}^{\infty} I_3(x_1, x_2) dx_1 \right]^{\frac{1}{2}}$$

repeat for  $x_2$  and  $x_3$  and use the fact:

$$\int |f||g| \leq \left[ \int f^2 \right]^{\frac{1}{2}} \left[ \int g^2 \right]^{\frac{1}{2}}$$

We see that,

$$\iiint_{\mathbb{R}^3} |u(x)|^{\frac{3}{2}} dx \leq \left[ \iiint_{\mathbb{R}^3} |\nabla u(x)| dx \right]^{\frac{3}{2}}$$

Therefore,  $\|u\|_{L^{\frac{3}{2}}} \leq \|\nabla u\|_{L^1}$

For  $p = 1, n \geq 3$ , same proof, but use ‘generalized Hölder inequality’ given by:

$$\int |u_1 \cdots u_m| \leq \|u_1\|_{L^{p_1}} \cdots \|u_m\|_{L^{p_m}}$$

provided  $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$

□

Now, for any  $1 < p < n$ , let  $v(x) = |u(x)|^\gamma$  for some  $\gamma > 1$

$v \in C_0^1$  since  $u \in C_0^1$ . Use previous case  $[p = 1]$ . Note that, in that case,  $p^* = \frac{n}{n-1}$ .

Also note that,  $\nabla v = \gamma |u|^{\gamma-1} \text{sgn}(u) \nabla u$

Therefore  $|\nabla v| \leq \gamma |u|^{\gamma-1} |\nabla u|$

Then,

$$\begin{aligned} \left[ \int |v(x)|^{\frac{n}{n-1}} \right] &\leq \int |\nabla v(x)| \\ \left[ \int |u|^{\frac{\gamma n}{n-1}} \right]^{\frac{n-1}{n}} &\leq \gamma \int |u|^{\gamma-1} |\nabla u| \stackrel{\text{Hölder}}{\leq} \gamma \left[ \int |u|^{(\gamma-1)\frac{p}{p-1}} \right]^{\frac{p-1}{p}} \left[ \int |\nabla u|^p \right]^{\frac{1}{p}} \end{aligned}$$

We pick  $\gamma = \frac{p(n-1)}{n-p} > 1$ . This gives us,

$$\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1} \implies \frac{\gamma n}{n-1} = p^*$$

So we get:

$$\begin{aligned} \left[ \int |u|^{p^*} \right]^{\frac{n-1}{n} - \frac{p-1}{p}} &\leq \gamma \|\nabla u\|_{L^p} \\ \left[ \int |u|^{p^*} \right]^{\frac{n-p}{np}} &\leq \gamma \|\nabla u\|_{L^p} \\ \|u\|_{L^{p^*}} &\leq \gamma \|\nabla u\|_{L^p} \end{aligned}$$

## Wednesday, 9/11/2024

**Theorem 8.** Assume  $u \in W_0^{1,p}(\Omega)$  for  $\Omega \subset \mathbb{R}^n$  open, bounded,  $\partial\Omega \in C^1$ . Then, for  $1 \leq p < n$  there exists  $C = C(p, n, \Omega)$  such that,

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p(\Omega)}$$

for  $p^* = \frac{np}{n-p}$

[Note that before we had condition  $u \in C_0^1(\mathbb{R}^n)$ , so this is a stronger theorem]

*Proof.* Extend  $u$  to be  $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$  so that it is compactly supported  
[insert picture here]  
Then we approximate  $\tilde{u}$  by smooth functions  $\{u_m\} \subset C_0^\infty(\mathbb{R}^n)$ .  
Apply previous result to get:

$$\|u_m - u_j\|_{L^{p^*}} \leq C \|\nabla u_m - \nabla u_j\|_{L^p} \rightarrow 0$$

Therefore,  $\{u_m\}$  is cauchy in  $L^{p^*}(\Omega)$ .

So we have  $u_m \xrightarrow{L^{p^*}(\Omega)} u$

To prove the inequality, we let  $m \rightarrow \infty$  in the previous result. □

Corollary: For  $1 \leq q \leq p^*$  and  $1 \leq p < n$ ,  $\exists C = C(p, n, \Omega)$  such that,

$$\|u\|_{L^q(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}$$

*Proof.* By Hölder,

$$\|v\|_{L^q} \leq C' \|v\|_{L^{p^*}} \quad \forall q < p^*$$

□

This is only half the story since we have  $p < n$ . What if  $p > n$ ? What about  $p = n$ ?

**$p = n$  case**

Here  $u \in W_0^{1,n}(\Omega)$

Note that  $p^* = \frac{np}{n-p}$  which is undefined. So we have problem. For that, we do:

Suppose  $\epsilon > 0$ . Apply GNS inequality for  $p_\epsilon = n - \epsilon$ .

Then  $p_\epsilon^* = \frac{np_\epsilon}{n-p_\epsilon} = \frac{n(n-\epsilon)}{\epsilon}$ . We have:

$$\|u\|_{L^{p_\epsilon^*}(\Omega)} < C_\epsilon \|\nabla u\|_{L^n(\Omega)}$$

Such an inequality exists for every  $\epsilon$ . So,  $u$  is in every  $L_q(\Omega)$  space for  $1 \leq q < \infty$ .

Note that  $q = \infty$  not necessary, there are counterexamples.

Corollary[Poincaré Inequality]: For  $\Omega \subset \mathbb{R}^n$  open, bounded and for  $1 \leq p < \infty$  there exists  $C = C(p, n, \Omega)$  such that

$$\|u\|_{L^p(\Omega)} \leq C(p, n, \Omega) \|\nabla u\|_{L^p(\Omega)}$$

This is true for all  $u \in W_0^{1,p}(\Omega)$ .

This is weaker than previous inequalities, since  $p < p^*$ .

**What if  $u \in W^{1,p}$  or  $u \in W_0^{1,p}$  and  $\Omega \subset \mathbb{R}^n$  and  $p > n$ ?**

These functions are even better!

But first we need to talk about Hölder Continuous Functions.

**Definition 6** (Hölder Quotient). Let  $0 < \alpha < 1$ . Define Hölder quotient:

$$[u]_{C^{0,\alpha}(\bar{\Omega})} = \sup_{x,y \in \bar{\Omega}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

**Definition 7** (Hölder Space).  $u \in C^{0,\alpha}(\bar{\Omega})$  if

$$\|u\|_{C^{0,\alpha}(\bar{\Omega})} := \sup_{x \in \bar{\Omega}} \left[ |u(x)| + [u]_{C^{0,\alpha}(\bar{\Omega})} \right] < \infty$$

This is a Banach Space.

Note: if  $u \in C^{0,\alpha}(\overline{\Omega})$  then  $u$  is uniformly continuous.

Example: Suppose  $\Omega = [0, 1]$  and  $u(x) = x^\beta$  for any  $\beta \in (0, \alpha]$ .

[graph]

It's derivative goes to  $\infty$  but it is still Hölder continuous.

**Theorem 9** (Morrey's Inequality). Assume  $n < p \leq \infty$ . Then  $\exists C = C(p, n)$  such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all  $u \in C_0^1(\mathbb{R}^n)$  where  $\gamma = 1 - \frac{n}{p}$ .

We can approximate so this is also true for  $u \in W_0^{1,p}(\Omega)$ .

Thus, when  $u \in W_0^{1,p}(\Omega)$  we can say that  $u$  is in fact Hölder continuous.

*Proof.* We have to prove that both sup and the Hölder quotient are controlled by the Sobolev norm.

Step 1: Claim: There exists  $C = C(n)$  such that  $\forall x \in \mathbb{R}^n$  such that  $r > 0$ ,

$$\int_{B(x,r)} |u(y) - u(x)| \, dy \leq C \int_{B(x,r)} \frac{|\nabla u(y)|}{|y - x|^{n-1}} \, dy \quad \forall u \in C^1(\mathbb{R}^n)$$

Proof of Claim: Fix  $x$ . Fix  $r$ . Fix  $w \in \partial B(0, 1)$ . Then,

$$|u(x + sw) - u(x)| = \left| \int_0^s \frac{d}{dt} u(x + tw) \, dt \right| \leq \int_0^s |\nabla_x u(x + tw)| \cdot \underbrace{|w|}_{=1} \, dt$$

Integrating over all  $w \in \partial B(0, 1)$  we get

$$\begin{aligned} \int_{w \in \partial B(0,1)} |u(x + sw) - u(x)| \, dS_w &\leq \int_{w \in \partial B(0,1)} \int_0^s |\nabla_x u(x + tw)| \, dt \, dS_w \\ &= \int_0^s \int_{w \in \partial B(0,1)} |\nabla_x u(x + tw)| \underbrace{t^{n-1} dS_w}_{=dS \text{ on } \partial B(0,t)} \, dt \end{aligned}$$

Let  $y = x + tw \implies t = |y - x|$

$$\begin{aligned} &= \int_{B(x,s)} \frac{|\nabla_y u(y)|}{|y - x|^{n-1}} \, dy \\ &\leq \int_{B(x,r)} \frac{|\nabla_y u(y)|}{|y - x|^{n-1}} \, dy \end{aligned}$$

TBC

## Friday, 9/13/2024

Continuing Morrey's Inequality.

We had:

$$\int_{\partial B(0,1)} |u(x + sw) - u(x)| \, dS_w \leq \int_{B(x,r)} \frac{|\nabla u(y)|}{|y - x|^{n-1}} \, dy$$

Multiplying by  $s^{n-1}$ ,

$$\int_{\partial B(0,1)} |u(x + sw) - u(x)| s^{n-1} \, dS_w \leq \int_{B(x,r)} \frac{|\nabla u(y)| s^{n-1}}{|y - x|^{n-1}} \, dy$$

Integrating over  $s$  from 0 to  $r$ ,

$$\int_0^r \int_{\partial B(0,1)} |u(x + sw) - u(x)| s^{n-1} \, dS_w \, ds \leq \int_0^r \int_{B(x,r)} \frac{|\nabla u(y)| s^{n-1}}{|y - x|^{n-1}} \, dy \, ds$$

$$\begin{aligned}
\int_0^r \int_{\partial B(x,s)} |u(y) - u(x)| \, dS_y \, ds &\leq \int_0^r \int_{B(x,r)} \frac{|\nabla u(y)| s^{n-1}}{|y-x|^{n-1}} \, dy \, ds \\
\int_{B(x,r)} |u(y) - u(x)| \, dy &\leq \frac{r^n}{n} \int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} \, dy \\
\frac{1}{\alpha(n)r^n} \int_{B(x,r)} |u(y) - u(x)| \, dy &\leq \frac{1}{n\alpha(n)} \int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} \, dy
\end{aligned}$$

which is the claim.

Step 2: (Bounding  $\sup |u|$ )

Fix  $x \in \mathbb{R}^n$ .

$$\begin{aligned}
|u(x)| &= \int_{B(x,1)} |u(x)| \, dy \\
&\leq \int_{B(x,1)} |u(x) - u(y)| \, dy + \int_{B(x,1)} |u(y)| \, dy
\end{aligned}$$

First term: apply step 1. Second term: apply Young's inequality

$$|u(x)| \leq C \int_{B(x,1)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} \, dy + C \left( \int 1^{\frac{p}{p-1}} \, dy \right)^{\frac{p-1}{p}} \left( \int_{B(x,1)} |u(y)|^p \, dy \right)^{\frac{1}{p}} \quad (*)$$

Now apply Hölder's inequality on the first part:

$$\int_{B(x,1)} \frac{1}{|x-y|^{n-1}} |\nabla u(y)| \, dy \leq \left[ \int_{B(x,1)} \left[ \frac{1}{|x-y|^{n-1}} \right]^{\frac{p}{p-1}} \, dy \right]^{\frac{p-1}{p}}$$

We simplify:

$$\begin{aligned}
\left[ \int_{B(x,R)} \left[ \frac{1}{|y-x|^{n-1}} \right]^{\frac{p}{p-1}} \, dy \right]^{\frac{p-1}{p}} &= \left[ \int_0^R \int_{\partial B(0,s)} s^{(n-1)\frac{p}{p-1}} \, dS \, ds \right]^{\frac{p-1}{p}} \\
&= \left[ \int_0^R s^{(n-1)\frac{p}{p-1}} \omega(n) s^{n-1} \, ds \right]^{\frac{p-1}{p}} = C \left[ \int_0^R s^{n-1-(n-1)\frac{p}{p-1}} \, ds \right]^{\frac{p-1}{p}} \\
&= C \left[ R^{n-(n-1)\frac{p}{p-1}} \right]^{\frac{p-1}{p}} = CR^{n\frac{p-1}{p}-(n-1)} = CR^{1-\frac{n}{p}} \quad (**)
\end{aligned}$$

Applying  $R = 1$  on  $(**)$  and substituting to  $(*)$ ,

$$|u(x)| \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

Step 3: Bounding the Hölder quotient  $[u]_{C^{0,\gamma}}$ :

Fix  $x, y$  in  $\mathbb{R}^n$  and let  $r = |x - y|$ .

Define  $W = B(x, r) \cap B(y, r)$ .

[insert picture]

$$\begin{aligned}
|u(x) - u(y)| &= \int_{z \in W} |u(x) - u(y)| \, dz \\
&\leq \int_W |u(x) - u(z)| \, dz + \int_W |u(y) - u(z)| \, dz
\end{aligned}$$

Now,

$$\begin{aligned}
\int |u(x) - u(z)| \, dz &\leq \frac{1}{|W|} \int_{B(x,r)} |u(x) - u(z)| \, dz \\
&\leq C(n) \int_{B(x,r)} |u(x) - u(z)| \, dz
\end{aligned}$$



Applying the claim,

$$\leq C \int_{B(x,r)} \frac{|\nabla u(z)|}{|x-z|^{n-1}} dz$$

Applying Hölder and (\*),

$$\leq C \|\nabla u\|_{L^p(\mathbb{R}^n)} r^{1-\frac{n}{p}}$$

So we're done. □

**Theorem 10.** Let  $\Omega \subset \mathbb{R}^n$  be bounded, open with  $\partial\Omega \in C^1$ . Assume  $n < p < \infty$ . Then,  $\exists C(n, p, \Omega)$  such that: for every  $u \in W^{1,p}(\Omega)$  one has  $u \in C^{0,\gamma}(\Omega)$  where  $\gamma = 1 - \frac{n}{p}$  and:

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} \leq C \|u\|_{W^{1,p}(\Omega)}$$

So, if our Sobolev  $L^p$  space is better than our dimension, our function must be continuous, or even better, Hölder Continuous!

Actually, Sobolev Functions are defined upto a set of measure 0. So, we actually have a continuous representative.

*Proof.* Extension Theorem □

If we have  $W^{k,p}$  we can have better statements.

Morrey and G-N-S inequalities can be concatenated.

[See Evans]

Example: Suppose  $u \in W^{2,2}(\Omega)$ . Suppose  $\Omega \subset \mathbb{R}^3$  and bounded.

$u \in H^2(\Omega)$  therefore  $u_{x_j} \in W^{1,2}(\Omega)$ .

$p = 2, n = 3$ . Use GNS.  $2^* = \frac{3 \times 2}{3-2} = 6$ .

Thus,  $u_{x_j} \in L^{2^*} = L^6$ .

Since also  $u \in W^{1,2}$ ,  $u \in L^6$ .

Therefore,  $u \in W^{1,6}$ . We have jumped above the dimension, since  $p = 6, n = 3$ .

$n < p$  so we can use Morrey.

$\implies u \in C^{0,\gamma}(\Omega)$  where  $\gamma = 1 - \frac{n}{p} = 1 - \frac{3}{6} = \frac{1}{2}$ .

So,  $u \in W^{2,2} \implies u \in C^{0,\frac{1}{2}}(\Omega)$ .

## Monday, 9/16/2024

**Definition 8** (continuously / compactly embedded). Given two Banach spaces  $X, Y$  with  $X \subseteq Y$  we say  $X$  is continuously embedded in  $Y$  if  $\exists C > 0$  such that:

$$\|x\|_Y \leq C \|x\|_X \forall x \in X$$

$X$  is compactly embedded in  $Y$  if it is continuously embedded and every bounded sequence in  $X$  is pre-compact in  $Y$ .

In other words, if every bounded sequence in  $X$  has a  $Y$ -convergent subsequence.

Typically people use  $X \subset\subset Y$  to denote this.

**Theorem 11** (Rellich-Kondrachov Compactness). Let  $\Omega \subset \mathbb{R}^n$  open, bounded with  $\partial\Omega \in C^1$ . If  $1 \leq p < n$  then  $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$  for all  $q \in [1, p^*)$

[here  $p^* = \frac{np}{n-p}$ ]

Remark: We already have continuous embedding for  $q \in [1, p^*]$  by GNS inequality.

**Theorem 12** (Hölder Generalization). Assume  $1 \leq s < r < t \leq \infty$  and  $1 = \frac{r\theta}{s} + \frac{r(1-\theta)}{t}$  for some  $\theta \in (0, 1)$ .

Then, if  $u \in L^s(\Omega) \cap L^t(\Omega)$  then  $u \in L^r(\Omega)$  and:

$$\|u\|_{L^r} \leq \|u\|_{L^s}^\theta \|u\|_{L^t}^{1-\theta}$$

*Proof.*

$$\begin{aligned} \int_{\Omega} |u|^r &= \int_{\Omega} |u|^{r\theta} |u|^{(1-\theta)r} \leq \left[ \int_{\Omega} |u|^{r\theta \frac{s}{r\theta}} \right]^{\frac{r\theta}{s}} \left[ \int_{\Omega} |u|^{(1-\theta)r \frac{t}{r(1-\theta)}} \right]^{\frac{r(1-\theta)}{t}} \\ &= \left[ \int_{\Omega} |u|^s \right]^{\frac{r\theta}{s}} \left[ \int_{\Omega} |u|^t \right]^{\frac{r(1-\theta)}{t}} \end{aligned}$$

□

*Proof of Rellich-Kondrachov.* Assume  $\|u_m\|_{W^{1,p}(\Omega)} < C_0$

Step 1: Mollify  $u_m \rightsquigarrow u_m^\epsilon$  and argue that  $u_m^\epsilon$  approximates  $u_m$  [uniformly in  $m$ ] in  $L^q$ .

Step 2: Apply Arzela-Ascoli to  $\{u_m^\epsilon\}$  for each  $\epsilon$  fixed.

Step 3: Use diagonalization argument. We control things when  $m$  is fixed, we control things when  $\epsilon$  is fixed so we can vary both.

First extend  $u_m$  to be compactly supported in  $V' \supset \Omega$ . Still have  $\|u_m\|_{W^{1,p}(V')} < \tilde{C}_0$ . Now Mollify:  $u_m^\epsilon := \eta_\epsilon * u_m$ .  $\eta_\epsilon(x) = \epsilon^{-n} \eta(\frac{x}{\epsilon})$  with  $\eta \in C_0^\infty$  and  $\text{supp}(\eta) \subset B(0,1)$  with  $\int \eta = 1$

Note  $u_m^\epsilon$  is compactly supported. Say  $\text{supp}(u_m^\epsilon) \subset V$ .  $V' \subset V$ .

Claim:  $\{u_m^\epsilon\}$  converges as  $\epsilon \rightarrow 0$  to  $u_m$  in  $L^q(\Omega)$  uniformly in  $m$ . That is, we claim:  $\forall \delta > 0$  there exists  $\epsilon_0 > 0$  such that  $\forall \epsilon < \epsilon_0$

$$\|u_m^\epsilon - u_m\|_{L^q(V)} < \delta$$

$\forall m$ .

To see this, first assume  $u_m$  is smooth.

$$u_m^\epsilon(x) - u_m(x) = \int_{B(0,\epsilon)} \eta_\epsilon(y) (u_m(x-y) - u_m(x)) \, dy$$

Let  $z = \frac{y}{\epsilon}$

$$\begin{aligned} &= \int_{B(0,1)} \eta(z) [u_m(x - \epsilon z) - u_m(x)] \, dz \\ &= \int_{B(0,1)} \eta(z) \int_0^1 \frac{d}{dt} u_m(x - \epsilon t z) \, dt \, dz \\ &= -\epsilon \int_{B(0,1)} \eta(z) \int_0^1 \nabla u_m(x - \epsilon t z) \cdot z \, dt \, dz \\ &\leq \epsilon \int_V \int_{B(0,1)} \int_0^1 \eta(z) |\nabla u_m(x - \epsilon t z)| \, dt \, dz \, dx \end{aligned}$$

Let  $y' = x - \epsilon t z$

$$\begin{aligned} &= \epsilon \int_{\tilde{V}} \int_{B(0,1)} \int_0^1 \eta(z) |u_m(y')| \, dt \, dz \, dy' \\ &= \epsilon \int_{\tilde{V}} |\nabla u_m(y')| \, dy' \end{aligned}$$

Apply Hölder:

$$\leq \epsilon C \|\nabla u_m\|_{L^p} \leq \epsilon \tilde{C}_0$$

If  $u_m$  is not smooth then approximate  $u_m$  by  $\tilde{u}_m$  smooth such that  $\|\tilde{u}_m - u_m\|_{W^{1,p}} < \delta$  for all  $m$ . Then,

$$\begin{aligned} &\int_V |u_m^\epsilon(x) - u_m(x)| \, dx \leq \\ &\int_V |u_m^\epsilon(x) - \tilde{u}_m(x)| \, dx + \int_V |\tilde{u}_m(x) - u_m(x)| \, dx < \delta + \delta \end{aligned}$$

Now apply interpolation:

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \|u_m^\epsilon - \tilde{u}_m\|_{L^1}^\theta \|\tilde{u}_m - u_m\|_{L^{p^*}}^{1-\theta}$$

Here  $s = 1, t = p^*$

$$\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$$

Recall GNS gave us  $\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \tilde{C}_0$

## Wednesday, 9/18/2024

$$\begin{aligned} |u_m^\epsilon| &\leq \int_V \epsilon^{-n} \eta \left( \frac{x-y}{\epsilon} \right) |u_m(y)| \, dy \\ &\leq \epsilon^{-n} \int_V |u_m(y)| \, dy \end{aligned}$$

$$\leq \epsilon^{-n} \|u_m\|_{W^{1,p}(V)} < \text{constant}$$

Note that

$$\begin{aligned} \nabla u_m^\epsilon(x) &= \int_V \epsilon^{-n} \nabla \eta \left( \frac{x-y}{\epsilon} \right) \frac{1}{\epsilon} u_m(y) \, dy \\ |\nabla u_m^\epsilon(x)| &\leq \int_V \epsilon^{-n} \left| \nabla \eta \left( \frac{x-y}{\epsilon} \right) \right| \frac{1}{\epsilon} |u_m(y)| \, dy \\ &\leq \epsilon^{-(n+1)} \sup |\nabla \eta| \int |u_m(y)| \, dy \\ &\leq \epsilon^{n+1} \sup |\nabla \eta| \|u_m\|_{W^{1,p}(V)} \leq \text{constant} \end{aligned}$$

Therefore,  $u_m^\epsilon$  is equi-Lipschitz and thus equicontinuous.

Step 3:

Claim: We can find a subsequence  $\{u_{m_j}\}$  of the  $u_m$  so that:

$$\limsup_{j,k \rightarrow \infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} < \delta$$

Use step 1 to find  $\epsilon$  such that  $\|u_m^\epsilon - u_m\|_{L^q(V)} < \frac{\delta}{2}$  for all  $m$ .

Apply Arzela-Ascoli for that  $\epsilon$  to get subsequence  $u_{m_j}$  such that:

$$\begin{aligned} \limsup_{j,k \rightarrow \infty} \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\| &\rightarrow 0 \\ \implies \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\|_{L^q(V)} &= 0 \end{aligned}$$

Therefore,

$$\|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \|u_{m_j} - u_{m_j}^\epsilon\|_{L^q} + \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\|_{L^q} + \|u_{m_k}^\epsilon - u_{m_k}\|_{L^q}$$

Take  $\limsup_{j,k \rightarrow \infty}$  to see  $< \frac{\delta}{2} + 0 + \frac{\delta}{2} = \delta$

Then take  $\delta = 1, \frac{1}{2}, \frac{1}{4}, \dots$  [subsequences of subsequences] to get  $u_{m_{i_k}} \xrightarrow{L^q} \bar{u}$ .

□

Remark: What if  $\|u_m\|_{W^{1,p}(\Omega)} < C_0$  with  $n < p$ ?

Morrey's inequality gives us:

$$\|u_m\|_{C^{0,\alpha}} \leq \text{constant}$$

$$|u_m(x) - u_m(y)| \leq C|x - y|^\alpha$$

Thus we also have equicontinuity.

## Elliptic PDE Definition

Suppose  $u$  = density, amount/volume,  $u = u(x, t)$ ,  $x \in \mathbb{R}^3$ .

Let  $\Omega \subset \mathbb{R}^3$  be any domain where the whole process is taking place.

Suppose we want to know the amount of stuff in  $\Omega$  in time  $t$ . It is:

$$\int_{\Omega} u(x, t) \, dx$$

Now we ask: how does it change w.r.t. time?

$$\frac{d}{dt} \int_{\Omega} u(x, t) \, dx$$

This depends on the stuff entering or exiting [= flux accross  $\partial\Omega$ ] + sources/sinks.

What is flux? It must be a vector  $\vec{Q}(x, t)$ . So, total contribution of flux is:

$$- \int_{\partial\Omega} \vec{Q}(x, t) \cdot \nu \, dS$$

Plus sources/sinks density.

$$+ \int_{\Omega} F(x, t) \, dx$$

Thus we have,

$$\frac{d}{dt} \int_{\Omega} u(x, t) \, dx = - \int_{\partial\Omega} \vec{Q}(x, t) \cdot \nu \, dS + \int_{\Omega} F(x, t) \, dx$$

The choice of flux  $\vec{Q}$  distinguishes different physical settings.

Let's rewrite our equation as 1 volume integral.

$$\int_{\Omega} \left[ u_t(x, t) + \operatorname{div} \vec{Q}(x, t) - F(x, t) \right] \, dx = 0$$

This is true for arbitrary  $\Omega$  so we have:

$$u_t = -\operatorname{div}(\vec{Q}) + F$$

Naturally, we pick  $Q$  to model a diffusion process.

If there's a lot of stuff 'inside' then stuff will go outside and vice versa. So, we can take,

$$\vec{Q} = -k \nabla u$$

Or more generally,

$$\vec{Q} = -A(x) \nabla u$$

where  $A$  is positive definite  $3 \times 3$  matrix.

For  $Q = -k \nabla u$  we have:

$$u_t = \operatorname{div}(k \nabla u) + F = k \Delta u + F$$

which is the heat equation.

For  $Q = -A(x) \nabla u$ ,

$$u_t = \operatorname{div}(A(x) \nabla u) + F$$

This is a model where stuff 'smooths out / settles down'.

As  $t \rightarrow \infty$  we expect  $u(x, t) \rightarrow \tilde{u}(x)$ .

In that case,  $-\operatorname{div}(A(x) \nabla u) = F$ .

If  $A$  has entries  $a_{ij}(x)$ , then,

$$\operatorname{div}(A(x)\nabla u) = \sum_{i,j} \frac{\partial}{\partial x_i} (a_{ij}(x)u_{x_j})$$

This is an elliptic operator [if  $a_{ij} = a_{ji}$ ,  $A$  positive definite].  
This is the divergence form.

## Friday, 9/20/2024

Assume  $A$  is symmetric. We define the linear elliptic operator  $L$  by:

- i)  $Lu = -\frac{\partial}{\partial x_j} (a_{ij}u_{x_i}) + b_i(x)u_{x_i} + c(x)u$  [divergence form elliptic operator]
- ii)  $Lu = -a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i} + c(x)u$  [non divergence form] where  $b_i : \Omega \rightarrow \mathbb{R}$ ,  $i = 1, \dots, n$ ,  $c : \Omega \rightarrow \mathbb{R}$ ,  $\Omega \subset \mathbb{R}^n$ . If  $a_{ij}$  are  $C^1$  then  $a_{ij}(x)u_{x_i x_j} = \frac{\partial}{\partial x_j} (a_{ij}(x)u_{x_i}) - \left(\frac{\partial}{\partial x_j} a_{ij}(x)\right)u_{x_i}$

Why elliptic?  $A = (a_{ij})$  and  $a_{ij}(x)\xi_i \xi_j \geq \theta|\xi|^2 \quad \forall \xi \in \mathbb{R}^n$  for some  $\theta > 0$ .

**Definition 9.** We say  $u \in H_0^1(\Omega)$  [same as  $W_0^{1,2}$ ] is a weak solution to the equation  $Lu = f$  for some  $f \in L^2(\Omega)$  and  $u = 0$  on  $\partial\Omega$  [homogeneous Dirichlet condition] if:

$$\int_{\Omega} A\nabla u \cdot \nabla v \, dx + \int_{\Omega} b_i(x)u_{x_i}v \, dx + \int_{\Omega} c(x)uv \, dx = \int_{\Omega} vf \, dx \quad (*)$$

for all  $v \in H_0^1(\Omega)$ .

**Definition 10.** We say  $u \in H^1(\Omega)$  is a weak solution to

$$\begin{cases} Lu = f, & \text{in } \Omega; \\ A\nabla u \cdot \nu_{\partial\Omega} = 0, & \text{on } \partial\Omega \end{cases}$$

[Neumann boundary condition]

if  $(*)$  holds for all  $v \in H^1(\Omega)$ .

Suppose

$$\int_{\Omega} a_{ij}(x)u_{x_i}v_j + b_i(x)u_{x_i}v + cuv \, dx + \int_{\partial\Omega} vA\nabla u \cdot \nu \, dx = \int_{\Omega} fv \, dx \quad (**)$$

why? Suppose  $u$  was a classical solution. Then integrating by part yields the second equation.

## Functional Analysis

### Background on Hilbert Spaces

Recall: a Hilbert Space  $H$  is a Banach space [normed and complete] that posses an inner product  $(\cdot, \cdot)_H$  such that  $\|\cdot\|$  is inherited from the inner product.

Basically: complete space with inner product.

Example:  $\mathbb{R}^n$  with dot product,  $L^2(\Omega)$  with  $(u, v)_{L^2} = \int_{\Omega} uv \, dx$

Or,  $H^1(\Omega)$  with  $(u, v)_H = \int_{\Omega} uv \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx$

Or, most importantly, in  $H_0^1(\Omega)$  we have  $(u, v)_{H_0^1} = \int_{\Omega} \nabla u \cdot \nabla v \, dx$

By Poincare, then  $C_1\|u\|_{H^1}^2 \leq \|u\|_{H_0^1}^2 = \int_{\Omega} |\nabla u|^2 \, dx \leq C_2\|u\|^2$

Given  $u, v \in H$  we'll say  $u$  is orthogonal to  $v$  if  $(u, v) = 0$ .

Given a subspace  $M \subset H$ , we'll write  $M^{\perp} := \{v \in H : (u, v) = 0 \forall u \in M\}$

**Proposition 1.** Let  $M \subset H$  be a closed subspace of  $H$ . Then  $\forall x \in H, \exists y \in M, z \in M^{\perp}$  such that  $x = y + z$ .

*Proof.* Idea:  $y$  is the point closest to  $x$  in  $M$ .  
Consider  $x \notin M$ . Define

$$d = \inf_{x' \in M} \|x - x'\|$$

Then there must be a sequence  $\{y_n\} \subset M$  such that  $\|x - y_n\| \rightarrow d$ .  
Recall:

$$\|X - Y\|^2 + \|X + Y\|^2 = 2\|X\|^2 + 2\|Y\|^2$$

Pick  $X = x - y_n, Y = x - y_m$ . Then,

$$\|y_m - y_n\|^2 + 4 \left\| x - \frac{1}{2}(y_n + y_m) \right\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2$$

Therefore,

$$4d^2 + \|y_n - y_m\|^2 \leq 2\|x - y_n\|^2 + 2\|x - y_m\|^2$$

$$\|y_n - y_m\|^2 \leq 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4d^2$$

Taking limsup we see that  $\{y_n\}$  is cauchy.  
Define  $z = x - y$ . It is easy to see that  $z \in M^\perp$ .

□

## Monday, 9/23/2024

### Divergence from Linear Elliptic Operator

$$L(u) = -\frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) + b_i(x)u_{x_i} + c(x)u$$

Uniform ellipticity:

$a_{ij} = a_{ji}$  symmetric

$a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \forall \xi \in \mathbb{R}^n \forall x \in \Omega$  for some  $\theta > 0$ .

Meaning minimum eigenvalue is uniformly bigger than some  $\theta$

Review:  $u \in H_0^1(\Omega)$  is a weak solution to  $Lu = f$  in  $\Omega$  for  $f \in L^2(\Omega)$  and  $u = 0$  on  $\partial\Omega$  if:

$$B(u, v) := \int_{\Omega} [a_{ij}(x)u_{x_i}v_{x_j} + b_i(x)u_{x_i}v + c(x)uv] \, dx = \int_{\Omega} f v \, dx \, \forall v \in H_0^1(\Omega)$$

$u$  is a weak solution to  $Lu = f$  in  $\Omega$  and  $A\nabla u \cdot \nu_{\partial\Omega} = 0$  on  $\partial\Omega$  if

$$B(u, v) = \int_{\Omega} f v \, dx \, \forall v \in H^1(\Omega)$$

Note that  $B$  is a bilinear form.

**Definition 11.** A bounded linear operator  $L : X \rightarrow Y$  that is linear that satisfies:

$$\|L\| := \sup_{\|x\|_X \leq 1} \|L(x)\|_Y < \infty$$

For linear operators, boundedness is the same as continuity.

**Definition 12.** A bounded linear functional on  $X$  is a bounded linear operator  $L : X \rightarrow \mathbb{R}$

Notation: If  $u^*$  is a bounded linear functional we'll often write  $\langle u^*, x \rangle$  for the evaluation of  $u^*$  at  $x$ .

$$\|u^*\| = \sup_{\|x\| \leq 1} \langle u^*, x \rangle$$

**Definition 13.** The dual of a Banach Space  $X$  is the set of bounded linear functionals on  $X$ . Notation:  $X^*$

**Theorem 13** (Riesz Representation Theorem). Assume  $H$  is a Hilbert space with inner product  $(\cdot, \cdot)$ .

For every bounded linear functional  $u^* : H \rightarrow \mathbb{R}$  there exists a unique  $u \in H$  such that  $(u^*, v) = (u, v) \forall v \in H$  and  $\|u^*\| = \|u\|$

*Proof.* Suppose  $u^* \in H^*$ .

Let  $N$  be the nullspace of  $u^*$ :

$$N = \{v \in H : \langle u^*, v \rangle = 0\}$$

If  $N = H$  then  $\langle u^*, - \rangle = 0$  so  $u$  must be 0. Assume  $N$  is a proper subspace.  $N$  must be closed since  $u^*$  is continuous.

Fix  $z \in N^\perp$ ,  $\langle u^*, z \rangle \neq 0$ .

Then  $\forall x \in H$  we have:

$$\left\langle u^*, x - \frac{\langle u^*, x \rangle}{\langle u^*, z \rangle} z \right\rangle = \langle u^*, x \rangle - \langle u^*, x \rangle = 0$$

Thus,  $x - \frac{\langle u^*, x \rangle}{\langle u^*, z \rangle} z \in N$ .

$$\begin{aligned} \left( x - \frac{\langle u^*, x \rangle}{\langle u^*, z \rangle} z, z \right) &= 0 \\ \implies (x, z) &= \frac{\langle u^*, x \rangle}{\langle u^*, z \rangle} \|z\|^2 \\ \implies \langle u^*, x \rangle &= \left( \frac{\langle u^*, z \rangle}{\|z\|^2} z, x \right) \end{aligned}$$

We just take  $u := \frac{\langle u^*, z \rangle}{\|z\|^2} z$

Uniqueness: if  $\langle u^*, x \rangle = (u_1, x) = (u_2, x)$  then  $(u_1 - u_2, x) = 0$ , choose  $x = u_1 - u_2$  to deduce that  $u_1 = u_2$ .

$$\|u^*\| = \sup_{\|x\| \leq 1} \langle u^*, x \rangle = \sup_{\|x\| \leq 1} (u, x) \leq \sup_{\|x\| \leq 1} \|u\|_H \|x\|_H \leq \|u\|_H$$

On the other hand,

$$\|u\| = \frac{(u, u)}{\|u\|} = \left\langle u^*, \frac{u}{\|u\|} \right\rangle \leq \|u^*\|$$

Thus  $\|u\| = \|u^*\|$

□

## Solving a PDE [finally]

Poisson: Find a weak solution to  $-\Delta u = f$  where  $f \in L^2(\Omega)$  in  $\Omega$  and  $u = 0$  on  $\partial\Omega$   
We seek  $u \in H_0^1(\Omega)$  satisfying

$$B(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \forall v \in H_0^1(\Omega)$$

Recall: By Poincaré's inequality, an inner product for  $H_0^1$  can be taken as:

$$(u, v)_{H_0^1} = \int_{\Omega} \nabla u \cdot \nabla v \, dx$$

We seek  $u$  such that:

$$(u, v)_{H_0^1} = \int_{\Omega} f v \, dx \forall v \in H_0^1(\Omega)$$

Existence of unique  $u$  follows from Riesz Representation. We just need to show that  $v \mapsto \int_{\Omega} f v \, dx$  is a bounded linear functional on  $H_0^1(\Omega)$ . It is obviously linear.

$$\sup_{\|v\|_{H_0^1} \leq 1} \int_{\Omega} f v \, dx \leq \sup_{\|v\|_{H_0^1} \leq 1} \|f\|_{L^2} \|v\|_{L^2} \leq c_p \cdot 1 \|f\|_{L^2}$$

Where  $c_p$  is the Poincaré constant:  $\int u^2 \leq c_p \int |\nabla u|^2 \forall u \in H_0^1$

## Wednesday, 9/25/2024

Last time, we used Riesz Representation Theorem to get existence of a weak solution to  $Lu = f$  in  $\Omega$  and  $u = 0$  on  $\partial\Omega$

for  $Lu = -\frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) = f$

$a_{ij}$  elliptic.

Weak formulation: seek  $u \in H_0^1(\Omega)$  such that:

$$B[u, v] := \int_{\Omega} a_{ij} u_{x_i} v_{x_j} \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H_0^1(\Omega)$$

For  $-\Delta u = f$  we seek  $u$  such that  $B[u, v] = \int_{\Omega} \nabla u \cdot \nabla v$  where  $B[u, v] = (u, v)_{H_0^1(\Omega)}$

How do we obtain a weak solution for

$$Lu := -\frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) + b_i(x)u_{x_i} + c(x)u = f$$

Consider:

$$B[u, v] = \int_{\Omega} a_{ij} u_{x_i} v_{x_j} + b_i(x)u_{x_i}v + c(x)uv \, dx$$

Not symmetric so not an inner product but that's not the only problem. Consider 1-dim.

$$-u'' - u = 1 \quad \Omega = (0, \pi)$$

$$u(0) = u(\pi) = 0.$$

Suppose  $u$  solves this.

Multiply the ODE by  $\sin x$  and integrate.

$$\int_0^{\pi} -\sin x \cdot u'' - \sin x \cdot u \, dx = \int_0^{\pi} \sin x \, dx = 2$$

Note

$$\int_0^{\pi} -\sin x \cdot u'' \, dx = [-\sin x \cdot u']_0^{\pi} - \int_0^{\pi} \cos x \cdot u' \, dx = [-\cos x \cdot u]_0^{\pi} + \int_0^{\pi} \sin x \cdot u \, dx$$

Therefore, by plugging it into the original argument,

$$0 = 2$$

Thus we don't have solutions!

Today: Lax-Milgram Lemma

**Theorem 14** (Lax-Milgram Lemma). Assume  $B : H \times H \rightarrow \mathbb{R}$  is a bilinear form on a Hilbert space  $H$ . Suppose,

$$\text{i) } \exists \alpha > 0 \text{ such that } B(u, v) \leq \alpha \|u\| \cdot \|v\| \forall u, v \in H$$

$$\text{ii) } \exists \beta > 0 \text{ such that } B[u, u] \geq \beta \|u\|^2$$

Then we have the same conclusion as Riesz:  $\forall f \in H^* \exists! u \in H$  such that  $B[u, v] = \langle f, v \rangle \forall v \in H$



*Proof.* For any fixed  $u \in H$  consider the map  $v \mapsto B[u, v]$ .

It is a bounded linear functional in  $H$ .

Apply Riesz Representation: for each fixed  $u$  there exists unique  $w \in H$  such that  $B[u, v] = (w, v) \forall v$ .

We write  $Au = w$

Then  $B[u, v] = (Au, v)$

Claim:  $A : H \rightarrow H$  is linear, bounded, 1-1

Linearity:  $B[c_1 u_1 + c_2 u_2, v] = (A(c_1 u_1 + c_2 u_2), v)$

$= c_1 B[u_1, v] + c_2 B[u_2, v] = c_1 A(u_1, v) + c_2 A(u_2, v)$ .

$A$  is bounded since  $\|Au\|^2 = (Au, Au) = B[u, Au] \leq \alpha \|u\| \|Au\|$

$A$  is 1-1 since  $\beta \|u\|^2 \leq B[u, u] = (Au, u) \leq \|Au\| \|u\|$

So  $\beta \|u\| \leq \|Au\|$

So, if  $u \neq 0$  we have  $Au \neq 0$  so 0 is the only element in the kernel, so it is 1-1.

Claim: range of  $A, R(A)$  is closed.

Consider  $\{w_j\} \subset R(A)$ . Suppose  $w_j \rightarrow w$ .  $\exists \{u_j\} \subset H$  so that  $Au_j = w_j$

Since  $w_j$  are cauchy,  $u_j$  are cauchy:

$\|w_j - w_k\| = \|Au_j - Au_k\| \geq \beta \|u_j - u_k\|$

$\implies u_j$  are cauchy. So  $u_j \rightarrow u \in H$  and by continuity  $Au_j \rightarrow Au = w$  and thus  $R(A)$  is closed.

Claim:  $R(A) = H$ .

If not, apply projection lemma.  $\exists w \in R(A)^\perp$  so that  $w \neq 0$ .

$$\beta \|u\|^2 \leq B[w, w] = (Aw, w)$$

Since  $Aw \in R(A)$  we have the inner product is 0 and thus  $w = 0$ .

Now, let  $f \in H^*$  be any bounded linear functional. Apply Riesz to show that there exists unique  $w \in H$  such that  $\langle f, v \rangle = (w, v) = (Au, v) = B[u, v]$  for unique  $u$ . □

**Theorem 15.** For  $Lu = -\frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) + b_i(x)u_{x_i} + c(x)u$  with  $a_{ij}$  elliptic and with  $a_{ij}, b_i, c \in L^\infty(\Omega)$ .

Then there exists a number  $\gamma > 0$  such that  $\forall \mu \geq \gamma$  :

a weak  $H_0^1(\Omega)$  solution exists to:

$$Lu + \mu u = f \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

$$\forall f \in L^2(\Omega)$$

*Proof.* We just apply Lax-Milgrim

We will prove:

$$\text{i) } |B[u, v]| \leq \alpha \|u\|_{H_0^1} \|v\|_{H_0^1}$$

$$\text{ii) } B[u, u] + \gamma \|u\|_{L^2}^2 \geq \beta \|u\|_{H_0^1}^2$$

First Condition: Need to check  $B[u, v] \leq \alpha \|u\|_{H_0^1} \|v\|_{H_0^1}$  for some  $\alpha$

$$\begin{aligned} B[u, v] &= \int_{\Omega} a_{ij}(x)u_{x_i}v_{x_j}b_i(x)u_{x_i}v + c(x)uv \, dx \\ &\leq B[u, v] \leq \int_{\Omega} n^2 M |\nabla u| |\nabla v| + nM |\nabla u| |v| + M |u| |v| \, dx \\ &\leq n^2 M \|u\|_{H_0^1} \|v\|_{H_0^1} + nM \|u\|_{H_0^1} \|v\|_{L^2} + M \|u\|_{L^2} \|v\|_{L^2} \end{aligned}$$

Apply Poincaré

$$\leq \alpha \|u\|_{H_0^1} \|v\|_{H_0^1}$$

So we have the first condition.

**Friday, 9/27/2024**

Now we check the second condition.

$$B[u, u] = \int_{\Omega} a_{ij}(x) u_{x_i} u_{x_j} + b_i(x) u_{x_i} u + c(x) u^2 \, dx$$

Use the fact that  $a_{ij}$  is elliptic, meaning  $a_{ij}(x) \zeta_i \zeta_j \geq \theta |\zeta|^2$

$$\geq \theta \int_{\Omega} |\nabla u|^2 - nM |\nabla u| |u| - M |u|^2 \, dx$$

We don't have to worry about the  $-M|u|^2$  because we can choose  $\gamma$ . We need to deal with  $-nM |\nabla u| |u|$ .

Recall that AM-GM implies:

$$\begin{aligned} \varepsilon^2 a^2 + \frac{1}{4\varepsilon^2} b^2 &\geq ab \\ &\geq \theta \int_{\Omega} |\nabla u|^2 - nM \left( \varepsilon |\nabla u|^2 + \frac{1}{4\varepsilon^2} |u|^2 \right) - M |u|^2 \, dx \\ &\geq \theta \int_{\Omega} (1 - nM\varepsilon) |\nabla u|^2 - \left( M + \frac{nM}{4\varepsilon^2} \right) |u|^2 \, dx \end{aligned}$$

We can choose  $\varepsilon$  so that  $nM\varepsilon = \frac{\theta}{2}$  and then choose appropriate  $\gamma$ . So we're done.  $\square$

**Theorem 16.** For  $L$  as defined,  $\forall \mu \geq \gamma \forall f \in L^2(\Omega)$  there exists a unique weak solution to  $L_{\mu} u = f, u = 0$  on  $\partial\Omega$  where  $L_{\mu} := L + \mu$

*Proof.* Lax-Milgrim:  $B_{\mu}[u, v] := B[u, v] + \mu \int uv$  so  $L_{\mu}$  has inverse.  $\square$

## Functional Analysis

**Definition 14** (Adjoint). Given a bounded linear operator on Hilbert spaces

$$A : H \rightarrow H \quad H \text{ is a Hilbert space}$$

The adjoint of  $A$  is the bounded linear operator  $A^* : H \rightarrow H$  defined by:

$$(x, A^* y) = (Ax, y) \quad \forall x, y \in H$$

If  $A = A^*$  we say  $A$  is self-adjoint.

Example: If  $H = \mathbb{R}^n$  then  $A$  is a matrix and  $A^*$  is the transpose.

**Definition 15** (Compact bounded linear operator). Let  $X, Y$  be Banach spaces. A bounded linear operator  $K : X \rightarrow Y$  is compact if for every sequence  $\{x_j\} \subset X$  such that  $\|x_j\|_X$  is uniformly bounded then there exists a subsequence and  $y \in Y$  such that:

$$Kx_{j_i} \rightarrow y$$

Example: Suppose  $X = C([0, 1])$  and  $Y = C^1([0, 1])$ .  $\forall f \in X$  define  $Kf = u$  provided  $u$  solves  $u'' = f$  on  $(0, 1)$  and  $u(0) = 0, u'(0) = 0$ . We have a formula:

$$u(x) = \int_0^x \int_0^y f(s) \, ds \, dy$$

Suppose  $\|f_j\|_{C([0,1])} \leq M$ . Then,

$$u_j(x) = \int_0^x \int_0^y f_j(s) \, ds \, dy$$

$$|u_j| \leq M$$

$$|u'_j| \leq \left| \int_0^x f_j(s) ds \right| \leq M$$

$$\|u''_j\| = \|f_j\| \leq M$$

Apply Arzela Ascoli to find:

$$u_{j_k} \xrightarrow{C^1} u_0$$

Thus  $K$  must be compact.

**Theorem 17** (Fredhome Alternative). Assume  $K : H \rightarrow H$  is a compact operator on a Hilbert space. Then either:

- i) The homogeneous equation  $x - Kx = 0$  has a non-trivial solution. OR
- ii)  $\forall y \in H, \exists! x \in H$  such that  $x - Kx = y$

*Proof.* There are 4 steps.

Step 1: Instead of  $x - Kx$  we write  $S := I - K$ .  $N(S)$  denotes the nullspace of  $S$ .

Claim:  $\exists C$  such that:

$$\text{dist}(x, N(S)) \leq C\|Sx\| \quad \forall x \in H$$

Proof: Suppose we cannot find such a  $C$ . Then we can find a sequence  $\{\tilde{x}_k\} \subset H$  such that:

$$\text{dist}(\tilde{x}_k, N(S)) \geq k\|S\tilde{x}_k\|$$

Replace with  $x_k = \frac{\tilde{x}_k}{\|S\tilde{x}_k\|}$ . Then  $\|Sx_k\| = 1$  and

$$d_k := \text{dist}(x_k, N(S)) \rightarrow \infty$$

Thus, we can find  $\{y_k\} \subset N(S)$  such that:

$$d_k \leq \|x_k - y_k\| \leq 2d_k$$

define  $z_k := \frac{x_k - y_k}{\|x_k - y_k\|}$  and so  $\|z_k\| = 1$ .

But  $\|Sz_k\| = \frac{1}{d_k}\|Sx_k - Sy_k\| = \frac{1}{d_k}\|Sx_k\| = \frac{1}{d_k} \rightarrow 0$

Therefore  $Sz_k \rightarrow 0$

$K$  is compact, so we have subsequence  $Kz_{k_j} \rightarrow y_0 \in H$ .

$Sz_{k_j} \rightarrow 0 \implies z_{k_j} - Kz_{k_j} \rightarrow 0$  and since  $Kz_{k_j} \rightarrow y_0$  we have  $z_{k_j} \rightarrow y_0$ .

Since  $Sz_{k_j} = 0$  we have  $Sy_0 = 0$  by continuity. Thus  $y_0 \in N(S)$

However,  $\text{dist}(z_{k_j}, N(S)) = \inf_{y \in N(S)} \|z_{k_j} - y\| = \inf_{y \in N(S)} \left\| \frac{x_{k_j} - y_{k_j}}{\|x_{k_j} - y_{k_j}\|} - y \right\|$

$$= \frac{1}{\|x_{k_j} - y_{k_j}\|} \inf_{y \in N(S)} \|x_{k_j} - \underbrace{(y\|x_{k_j} - y_{k_j}\| + y_k)}_{\in N(S)}\|$$

$$\geq \frac{1}{\|x_{k_j} - y_{k_j}\|} d(x_{k_j}, N(S)) \geq \frac{d_{k_j}}{2d_{k_j}} = \frac{1}{2}$$

$z_{k_j}$  converges to something in  $N(S)$  but is a set distance away from  $N(S)$ , which is impossible. Thus we have proved the claim.

## Monday, 9/30/2024

Step 2: Claim: Let  $R(S) = \text{range of } S$ . Then,  $R(S)$  is a closed subspae of  $H$ .

Proof: Consider a sequence  $\{x_k\} \subset H$  so that  $Sx_k \rightarrow y$  for some  $y$ . We must show that  $y \in R(S)$ .

From step 1, define  $d_k := \text{dist}(x_k, N(S)) \leq C\|Sx_k\| \rightarrow \|y\|$ .

$d_k$  is uniformly bounded.

By projection theorem, we can find:

$$d_k = \underbrace{\|x_k - y_k\|}_{w_k} \quad y_k = \text{closest point to } x_k \text{ in } N(S).$$

Then  $\|w_k\| \leq \text{const.}$

Since  $K$  is compact,  $Kw_{k_j} \rightarrow w_0 \in H$

Thus,  $Sx_{k_j} = x_{k_j} - Kx_{k_j} \rightarrow y$

Thus,  $x_{k_j} \rightarrow y + w_0$

Since  $S$  is continuous,  $S(y + w_0) = y$

So,  $y \in R(S)$ .

Step 3: If  $N(S) = \{0\}$  then  $R(S) = H$ .

Let  $R_j = \text{range of } S^j [= S(S(\cdots S(H)))]$ .

By Step 2,  $\{R_j\}$  is a sequence of closed subspaces. Furthermore, it is non-increasing.

We claim that it eventually stabilizes.

Suppose, for contradiction, the sequence keeps decreasing.

By projection theorem,  $\forall j \exists y_j \in R_j$  such that  $\|y_j\| = 1$  and  $\text{dist}(y_j, R_{j+1}) \geq \frac{1}{2}$ .

Let  $n > m$ . We look at  $Ky_m - Ky_n$ .

$$Ky_m - Ky_n = (I - S)y_m - (I - S)y_n = y_m - \left( \underbrace{Sy_m}_{\in R_{m+1}} - \underbrace{Sy_n}_{\in R_{n+1} \subset R_{m+1}} + \underbrace{y_n}_{\in R_n \subset R_{m+1}} \right)$$

Thus,  $\|Ky_m - Ky_n\| \geq \frac{1}{2}$ .

This contradicts the compactness of  $K$ .

Therefore,  $\exists k$  such that  $R_j = R_k \forall j > k$ .

So, assume  $N(S) = \{0\}$ . Let  $y \in H$ .

We have  $S^k y \in R_k = R_{k+1}$ .

Therefore,  $S^k y = S^{k+1} x$  for some  $x$ .

Therefore,  $S^k(y - Sx) = 0$ .

Since  $N(S) = \{0\}$  we have  $y = Sx$ .

Step 4. If  $R(S) = H$  then  $N(S) = \{0\}$ .

Let  $N_j := \text{nullspace of } S^j$ . Now,  $N_j$  is a non-decreasing sequence of closed subspaces.

Claim:  $\exists k$  such that  $N_j = N_k \forall j > k$ . Argue by contradiction as before.

Assume  $R(S) = H$ .

$\forall y \in N_k, S^k y = 0$  and furthermore  $\exists x_1$  such that  $y = Sx_1$ . Repeating,  $y = S^k x$ .

So,  $S^{2k} x = 0$ . Since null space stabilizes after  $k$  we have  $S^k x = 0$ . Therefore  $y = 0$ .

Thus  $N_k = \{0\} \implies N(S) = \{0\}$ .

In case ii why is  $(I - K)^{-1}$  bounded?

Use Step 1:  $\text{dist}(x, N(S)) \leq C\|Sx\| = C\|(I - K)x\|$

For ii we have  $N(S) = \{0\}$  so:

$$\|x\| \leq C\|Sx\|$$

Writing  $Sx = y$ ,

$$\|(I - K)^{-1}y\| \leq C\|y\|$$

□

Given a bounded linear operator  $T : H \rightarrow H$  [could be a normed linear space as well],

**Definition 16.** The resolvent set of  $T$  is:

$$\rho(T) = \{\lambda \in \mathbb{R} : T - \lambda I \text{ is bijective}\}$$

The spectrum of  $T$  is:

$$\sigma(T) = \mathbb{R} \setminus \rho(T)$$

We can substitute real for complex.

**Definition 17.**  $\lambda \in \sigma(T)$  is called an eigenvalue of  $T$  if  $\exists x \in H$  such that  $Tx = \lambda x$ .

Note:  $(\sigma(T) \setminus \{\text{eigenvalues}\})$  is called the continuous spectrum.

Wednesday, 10/2/2024

$$Lu := -(a^{ij}u_{x_i})_{x_j} + b^i(x)u_{x_i} + c(x)u$$

$$a^{ij} = a^{ji}, a^{ij}(x)\zeta_i\zeta_j \geq \theta|\zeta|^2, \theta > 0$$

$$a^{ij}, b^i, c \in L^\infty$$

Recall that adjoint is defined by

$$(Ax, y) = (x, A^*y)$$

Can we find the adjoint of  $L$ ?

Formal Adjoint of  $L : L^*$

$$(Lu, v)_{L^2} = (u, L^*v)_{L^2}$$

$$(Lu, v)_L^2 = \int_{\Omega} -(a^{ij}u_{x_i})_{x_j}v + b^i u_{x_i}v + cuv \, dx$$

$$= \int_{\Omega} a^{ij}u_{x_i}u_{x_j} - \operatorname{div}(v\vec{b})u + cuv \, dx$$

$$= \int_{\Omega} -(a^{ij}v_{x_i})_{x_j}u - b^i v_{x_i}cu \, dx = \int_{\Omega} -b^i v_{x_i} - b_{x_i}^i v + cv \, dx$$

Define that to be  $L^*v$

Lemma:  $R(I + K) = N(I + K^*)$

*Proof.*  $w \in N(I + K^*) \iff$

$$(x, (I + K^*)w) = 0 \, \forall x \in H$$

$$\iff ((I + K)x, w) = 0 \, \forall x \in H$$

$$\iff w \in R(I + K)^\perp.$$

□

**Theorem 18.** For  $L$  as defined, for  $\Omega \subset \mathbb{R}^n$  bounded,  $\partial\Omega \in C^1$ , then either:

i)  $\exists$  weak  $H_0^1(\Omega)$  solution to  $Lu = 0, u = 0$  on  $\partial\Omega$ .

ii)  $\forall f \in L^2(\Omega) \exists!$  weak  $H_0^1(\Omega)$  solution to  $Lu = f, u = 0$  on  $\partial\Omega$

Furthermore if i holds thereen  $Lu = f, u = 0$  on  $\partial\Omega$  has a weak solution  $\iff (f, v)_{L^2} = 0 \, \forall v$  such that  $L^*v = 0, v = 0$  in  $\partial\Omega$ .

Example:  $-u'' - u = \sin x$  and  $u(0) = u(\pi) = 0$ .

Here  $Lu = -u'' - u$ . We found  $\nexists$  solution.

Note that  $L$  has non-trivial nullspace.  $\sin x$  is in the nullspace.

$$\int \sin x \cdot \sin x \neq 0$$

Theorem not applicable.

*Proof.* Recall  $\exists \gamma > 0$  such that

$$B_\gamma[u, v] = B[u, v] + \gamma \int_{\Omega} uv$$

where:

$$B(u, v) := \int_{\Omega} (a^{ij}u_{x_i}v_{x_j} + b^i u_{x_i}v + cuv) \, dx$$

We can apply Lax-Milgrim to obtain a unique weak solution to  $L_\gamma u = f$  where  $L_\gamma u = Lu + \gamma u$ . ch that: So  $B_\gamma[u, v] = (f, v) \, \forall v \in H_0^1(\Omega)$

We say  $L_\gamma^{-1} = u$  if this holds.

We seek a function  $u \in H_0^1(\Omega)$  such that:

$$B_\gamma[u, v] = (f, v)_{L^2} + \gamma(u, v)_{L^2}$$

We want  $u$  such that:

$$u = L_\gamma^{-1}(f + \gamma u) = \underbrace{L_\gamma^{-1}(f)}_{=h} + \gamma L_\gamma^{-1}(u)$$

Note:  $L_\gamma^{-1} : L^2 \rightarrow H_0^1$  or  $L_\gamma^{-1} : L^2 \rightarrow L^2$

Let  $h := L_\gamma^{-1}(f)$ .

So we're trying to solve  $u - \gamma L_\gamma^{-1}(u) = h$ .

Define  $K := \gamma L_\gamma^{-1}$

So we want to solve  $(I - K)u = h$

$K : L^2 \rightarrow L^2$ .

We want to use Lax Milgrim. Why must  $K$  be bounded?

Let  $g \in L^2(\Omega)$ ,  $\frac{1}{\gamma}K(g) = L_\gamma^{-1}(g) =: u$

$\|K(g)\|_{H_0^1} = \|\gamma u\|_{H_0^1}$ .  $B_\gamma[u, v] = (g, v) \forall v$ .

Pick  $v = u$  then,

$$B_\gamma[u, u] \geq \beta \|u\|_{H_0^1}^2$$

$$\beta \|u\|^2 \leq B_\gamma[u, u] = (g, u) \leq \|g\|_{L^2} \|u\|_{L^2} \leq C_p \|g\|_{L^2} \|u\|_{H_0^1}$$

$$\gamma \beta \|u\|_{H_0^1} \leq \gamma C_p \|g\|_{L^2}$$

$$\|K(g)\|_{L^2} \leq C \|K(g)\|_{H_0^1} \leq \frac{\gamma C_p}{p} \|g\|_{L^2}$$

Claim:  $K : L^2 \rightarrow L^2$  is compact.

Let  $\{g_k\} \subset L^2$   $\|g_k\| \leq C$ .

$$\|K(g_k)\|_{H_0^1} \leq \tilde{C} \|g_k\|_{L^2} \leq \text{const}$$

By Rellich-Kondrachov,

$$K(g_{k_j}) \xrightarrow{L^2} W$$

Now, apply Fredholme alternative. Then,  $\forall h \in L^2 \exists!$  solution  $u$  to  $(I - K)u = h$  or else the nullspace of  $(I - K^*)$  is non-trivial.

$$(I - K)u = h \iff u \text{ weakly solution to } Lu = f$$

Then apply the Lemma.

□

## Friday, 10/4/2024

Recall: Given a bounded linear operator  $T : H \rightarrow H$ , the resolvent set:

$$\rho(T) := \{\lambda \in \mathbb{R} : T - \lambda I \text{ is bijective}\}$$

Then the spectrum of  $T$  is:

$$\sigma(T) = \mathbb{R} \setminus \rho(T)$$

$\lambda \in \sigma(T)$  is an eigenvalue if  $\exists x \in H$  such that  $(T - \lambda I)x = 0$ .

$\{\lambda \in \sigma(T) : \lambda \text{ is not an eigenvalue}\}$  is called the continuous spectrum.

Now, what if  $T = K$  compact, linear operator?

We have seen that  $I - K$  has a nontrivial nullspace or else it is invertible, and  $(I - K)^{-1}$  is bounded.

Further, we have seen that for  $\lambda \neq 0$  either  $I - \frac{1}{\lambda}K$  has a nontrivial nullspace or it is invertible.

Then,  $(I - \frac{1}{\lambda}K)^{-1}$  is bounded.

Nontrivial nullspace implies  $\lambda$  is an eigenvalue of  $K$ .

Thus,  $K$  compact  $\implies$  no continuous spectrum [except perhaps 0].

**Theorem 19.** A compact operator  $K : H \rightarrow H$  possesses at most a countable set of eigenvalues having no limit point except possibly 0.

Furthermore, Each eigenvalue has a finite multiplicity [ $\dim N(K - \lambda I)$  is finite].

*Proof.* Suppose not. Then we have an accumulation point  $\exists \{\lambda_n\}$  of eigenvalues such that  $\lambda_n \rightarrow \lambda \in (\mathbb{R} \cup \{\pm\infty\}) \setminus \{0\}$  and a sequence of linearly independent eigenvectors  $x_n$ .

Let  $M_n = \text{span}\{x_1, \dots, x_n\}$ .

$M_n$  is a closed subspace.

Projection Lemma  $\implies \exists \{y_n\} \subset M_n \setminus M_{n-1}$  so that  $\|y_n\| = 1, \text{dist}(y_n, M_{n-1}) \geq \frac{1}{2}$ .

Let  $S_\lambda := \lambda I - K$ .

For  $n > m$ : we have:

$$\lambda_n^{-1} K y_n - \lambda_m^{-1} K y_m = y_n - \lambda_n^{-1} S_{\lambda_n} y_n - y_m + \lambda_m^{-1} S_{\lambda_m} y_m = y_n + z$$

Claim:  $z \in M_{n-1}$ . To prove this, note that  $z$  is sum of elements of  $M_{n-1}$ .

$y_m \in M_m \subseteq M_{n-1}$ .

Write  $y_n = \sum_{j=1}^n c_j x_j$

$$S_{\lambda_n} y_n = (\lambda_n I - K) \left( \sum_{j=1}^n c_j x_j \right) = \sum_{j=1}^n (\lambda_n c_j x_j - c_j \lambda_j x_j) = \sum_{j=1}^{n-1} (\lambda_n c_j x_j - c_j \lambda_j x_j)$$

Thus  $S_{\lambda_n} y_n \in M_{n-1}$

Also  $S_{\lambda_m} y_m \in M_{m-1} \subseteq M_{n-1}$ .

Now,  $y_n = \lambda_n^{-1} K y_n$ .

Therefore,  $\|\lambda_n^{-1} K y_n - \lambda_m^{-1} K y_m\| \geq \frac{1}{2}$ .

$$\lambda_n^{-1} \|K y_n - \frac{\lambda_n}{\lambda_m} K y_m\| \geq \frac{1}{2}.$$

If  $\lambda_m$  approaches finite value, by taking  $m, n$  large enough we get  $\|K y_n - K y_m\| \geq \frac{\lambda}{4}$ .

Contradiction.

If we have an infinite limit then LHS approaches 0 which is also not possible.

□

**Theorem 20.** i)  $\exists$  an at most countable set  $\Sigma \subset \mathbb{R}$  such that  $Lu = \lambda u + f$  in  $\Omega$  and  $u = 0$  on  $\partial\Omega$  has a solution  $\forall f \in L^2$  provided  $\lambda \neq \Sigma$ .

ii) If  $\Sigma$  is infinite then writing  $\Sigma = \{\lambda_n\}, \lambda_1 \leq \lambda_2 \leq \dots$  then  $\lambda_n \rightarrow \infty$ .

iii) If  $\lambda \in \Sigma$  then  $Lu = \lambda u + f$  for  $f \in L^2$  with  $u = 0$  on  $\partial\Omega$  is solvable  $\iff (f, v)_{L^2} = 0 \forall v \in N(L^* - \lambda I)$ .

Recall:  $u$  is a weak solution to  $Lu = \lambda u$  in  $\Omega, u = 0$  on  $\partial\Omega \iff B[u, v] = \lambda(u, v) \forall v \in H_0^1(\Omega)$ .

Where  $B[u, v] = \int_{\Omega} a_{ij}(x) u_{x_i} v_{x_j} + b_i(x) u_{x_i} v + c(x) uv \, dx$ .

Now,  $B[u, v] = \lambda(u, v) \iff B_{\gamma}[u, v] := B[u, v] + \gamma \int uv = (\gamma + \lambda)(u, v)$ .

Where  $\gamma$  is sufficiently large to make  $B_{\gamma}[u, v] \geq \beta \|u\|_{H_0^1}^2$

So  $u$  solves  $(P) \iff$

$$u = L_{\gamma}^{-1} ((\lambda + \gamma)u)$$

$$\iff u = \gamma L_{\gamma}^{-1} \left( \frac{\lambda + \gamma}{\gamma} u \right) = \frac{\lambda + \gamma}{\gamma} K u \text{ where } K = \gamma L_{\gamma}^{-1}$$

Thus,  $u$  solves  $P \iff \left( K - \frac{\gamma}{\gamma + \lambda} I \right) u = 0$ .

Then the only possible limit point of eigenvalues of  $K$  is 0  $\iff$  only possible limit point of eigenvalues of  $L = \infty$ .

Monday, 10/7/2024

## Weak Convergence

Given a Banach space  $X$  let  $X^*$  be the space of bounded linear functionals.  
Given  $x^* \in X^*$ :

$$\|x^*\|_{X^*} := \sup_{\|x\|_X \leq 1} \langle x^*, x \rangle$$

**Definition 18.** We say  $\{x_n\} \subset X$  converges weakly to  $x \in X$  if:

$$\forall x^* \in X^*, \langle x^*, x_n \rangle \rightarrow \langle x^*, x \rangle$$

Notation:  $x_n \xrightarrow{X} x$

Example: for  $1 < p < \infty$  If  $X = L^p(\Omega)$  then when  $\frac{1}{p} + \frac{1}{q} = 1$  we have  $X^* \cong L^q(\Omega)$ .

$u_n \xrightarrow{L^p} u$  means:

$$\forall v \in L^q, \int_{\Omega} u_n v \, dx \rightarrow \int_{\Omega} u v \, dx$$

Facts:

- $x_n \rightarrow x \implies x_n \rightharpoonup x$
- $x_n \rightharpoonup x$  and  $\|x_n\| \rightarrow \|x\| \implies x_n \rightarrow x$
- $x_n \rightharpoonup x \implies \|x_n\| \leq M$  for some  $M$ .

**Definition 19.** A reflexive Banach space  $X$  is one such that:

$$(X^*)^* = X$$

Example:  $L^p(\Omega)$  for  $1 < p < \infty$

If  $X$  is reflexive and  $\|x_n\| \leq M$  then  $\exists \{x_{n_j}\}, x \in X$  such that  $x_{n_j} \rightharpoonup x$ .  
If  $x_n \rightharpoonup x$  then  $\liminf_{n \rightarrow \infty} \|x_n\| \geq \|x\|$ .

*Proof.* If  $x^* \in X^*, \|x^*\| \leq 1$  then,

$$\liminf_{n \rightarrow \infty} \|x_n\| \geq \liminf_{n \rightarrow \infty} \langle x^*, x_n \rangle = \langle x^*, x \rangle$$

We see our answer by taking sup over all  $\|x^*\| \leq 1$ . □

Basic Question:

Suppose  $u \in H^1(\Omega)$  is a weak solution to  $Lu = f$  in  $\Omega$  where  $L$  is an elliptic operator and  $f \in L^2$  [eg  $-\Delta u = f$ ]. Can one argue that  $u$  is better than  $H^1(\Omega)$ ?

A (formal) calculation suggesting that this isn't a ridiculous question:

Suppose  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  is smooth and compactly supported and solving  $-\Delta u = f$  for  $f \in L^2$ .

$$\begin{aligned} \infty &> \int_{\mathbb{R}^n} f^2 \, dx = \int_{\mathbb{R}^n} (\Delta u)^2 \, dx = \int_{\mathbb{R}^n} \nabla \cdot (\nabla u) \Delta u \, dx \\ &= - \int_{\mathbb{R}^n} \nabla u \cdot \nabla (\Delta u) \, dx = - \int_{\mathbb{R}^n} u_{x_j} \frac{\partial}{\partial x_j} (u_{x_i x_i}) \, dx \\ &= \int_{\mathbb{R}^n} u_{x_i x_j} u_{x_i x_j} \, dx = \int_{\mathbb{R}^n} |D^2 u|^2 \, dx \end{aligned}$$

So we have control over all second derivatives of  $u$ .



## Difference Quotients

**Definition 20.** Given  $u : \Omega \rightarrow \mathbb{R}$  given  $h \in \mathbb{R}$  given  $k \in 1, \dots, n$ , define:

$$D_k^h := \frac{u(x + he_k) - u(x)}{h}$$

We want to bound the difference quotient of the derivative to control the second derivaive.

Lemma 1: For  $1 < p < \infty$ , let  $u \in W^{1,p}(\Omega)$  for  $\Omega \subset \mathbb{R}^n$ . Then  $\forall \Omega' \subset\subset \Omega$  and for  $h$  such that  $0 < |h| < \text{dist}(\Omega', \partial\Omega)$  we have:

$$\|D_k^h u\|_{L^p(\Omega')} \leq \|u_{x_k}\|_{L^p(\Omega)}$$

*Proof.* First assume  $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$ .

$$\begin{aligned} D_k^h u(x) &= \frac{1}{h} \int_0^h u_{x_k} u(x_1, \dots, x_k + s, \dots, x_n) \, ds \\ |D_k^h u(x)|^p &\stackrel{\text{H\"older}}{\leq} \frac{1}{h^p} \left( \int_0^h 1^{\frac{p}{p-1}} \, dx \right)^{p-1} \left( \int_0^h |u_{x_k}(\dots, x_k + s, \dots)|^p \, dx \right) \\ \int_{\Omega'} |D_k^h u(x)|^p \, dx &\leq \frac{1}{h} \int_{\{\text{dist}(x, \partial\Omega) > h\}} \int_0^h |u_{x_k}(\dots, x_k + s, \dots)|^p \, ds \, dx \\ &\stackrel{\text{Fubini}}{\leq} \frac{1}{h} \int_0^h \int_{\Omega} |u_{x_k}(x)|^p \, dx \, ds = \|u_{x_k}\|_{L^p(\Omega)}^p \end{aligned}$$

Now we approximabe by  $C^1$  functions. □

Lemma 2: Let  $u \in L^p(\Omega)$  for  $1 < p < \infty$  and suppose for some  $k \in \{1, \dots, n\}$ ,  $\exists M$  such that

$$\|D_k^h u\|_{L^p(\Omega')} \leq M$$

$\forall \Omega' \subset\subset \Omega, \forall h$  such that  $\text{dist}(\Omega', \partial\Omega) > h$ . Then,

$$\|u_{x_k}\|_{L^p(\Omega)} \leq M$$

*Proof.*  $\exists \{h_j\} \rightarrow 0, \exists v \in L^p(\Omega)$  such that:

$$D_k^{h_j} u \xrightarrow{L^p(\Omega)} v$$

[We are using diagonalization. We cannot directly go near  $\partial\Omega$  but by making  $h_j \rightarrow 0$  and taking subsequences, we get the convergence for whole  $\Omega$ ].

Furthermore, by lower semicontinuity of  $\|\cdot\|_{L^p}$  under weak convergence:  $\|v\|_{L^p(\Omega)} \leq M$ .

Must still sho  $v$  is the weak  $k$ 'th derivative of  $u$ .

Fix  $\phi \in C_0^1(\Omega)$ .

We have:

$$\lim_{h_j \rightarrow 0} \int_{\Omega} \phi D_k^{h_j} u \, dx = \int_{\Omega} \phi v \, dx$$

Now we do integration by parts on difference quotient. We also have:

$$\int_{\Omega} \phi D_k^{h_j} u \, dx = \int_{\Omega} \phi(x) \left( \frac{u(x + h_j e_k) - u(x)}{h_j} \right) \, dx$$

For first term, let  $y = x + h_j e_k$ .

$$= \int_{\text{supp}(\phi + h_j e_k)} \frac{\phi(y - h_j e_k) u(y)}{h_j} \, dy - \int_{\text{supp}(\phi)} \frac{\phi(x) u(x)}{h_j} \, dx$$

$$= - \int_{\Omega} \frac{\phi(y - h_j e_k) - \phi(y)}{-h_j} u(y) \, dy = - \int_{\Omega} D_k^{-h_j} \phi(y) u(y) \, dy$$

Now let  $h_j \rightarrow 0$

$$\implies \lim_{h_j \rightarrow 0} \int_{\Omega} \phi D_k^{h_j} u \, dx = - \int_{\Omega} \phi_{x_k} u \, dx$$

Thus  $v$  is the weak  $x_k$  derivative of  $u$ .

□

## Wednesday, 10/9/2024

Recall:

$$D_k^h u(x) := \frac{u(x + h e_k) - u(x)}{h}$$

$$\int D_k^h u_1(x) u_2(x) \, dx = - \int D_k^{-h} u_2(x) u_1(x) \, dx$$

**Lemma 1:** Suppose  $1 < p < \infty$ . Let  $u \in W^{1,p}(\Omega)$ . Then,  $\forall \Omega' \subset\subset \Omega$  such that  $\text{dist}(\Omega', \partial\Omega) > |h|$  we have:

$$\|D_k^h u\|_{L^p(\Omega)} \leq \|u_{x_k}\|_{L^p(\Omega)}$$

**Lemma 2:** Let  $u \in L^p(\Omega)$ ,  $1 < p < \infty$ . Assume  $\exists M > 0$  such that  $D_k^h u \in L^p(\Omega')$  and  $\|D_k^h u\|_{L^p(\Omega')} \leq M \, \forall k, \forall \Omega' \subset\subset \Omega$  such that  $\text{dist}(\Omega', \partial\Omega) \geq |h|$ .

Then  $u_{x_k}$  exists  $\forall k$  and  $\|u_{x_k}\|_{L^p(\Omega')} \leq M$ .

Suppose  $Lu := -\frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) + b_i(x)u_{x_i} + c(x)u$

Assume  $a_{ij}(x)\zeta_i\zeta_j \geq \theta|\zeta|^2$ ,  $a_{ij} = a_{ji}$

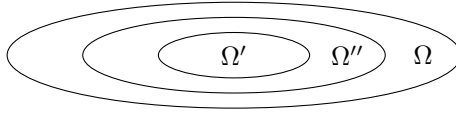
Assume  $a_{ij} \in C^1(\Omega)$ ,  $b_i, c \in L^\infty(\Omega)$

**Theorem 21** (Interior Regularity). Assume  $u \in H^1(\Omega)$  weak solution to  $Lu = f$  in  $\Omega$  for  $f \in L^2(\Omega)$ . Then  $u \in H_{loc}^2(\Omega)$  with:

$$\|u\|_{H^2(\Omega')} \leq C(\Omega', \Omega, \theta, \dots) \left( \|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

$\forall \Omega' \subset\subset \Omega$ .

*Proof.* Consider  $\Omega' \subset\subset \Omega'' \subset\subset \Omega$



Let  $\zeta$  be a smooth cutoff function:

$\zeta \equiv 1$  in  $\Omega'$

$\zeta \equiv 0$  on  $\Omega \setminus \Omega''$

$0 \leq \zeta \leq 1$

We have:

$$\int_{\Omega} a_{ij}(x)u_{x_i}v_{x_j} + b_i(x)u_{x_i}v + c(x)uv \, dx = \int_{\Omega} f v \, dx$$

$\forall v \in H_0^1(\Omega)$

Note that,

$$\int_{\Omega} a_{ij}(x)u_{x_i}v_{x_j} + \boxed{b_i u_{x_i}} v + \boxed{cu} v \, dx = \int_{\Omega} f v \, dx$$

Choose  $v = -D_k^{-h}(\zeta^2 D_k^h u)$

in  $\Omega'$  where  $\zeta \equiv 1$  we have:

$$\begin{aligned} v &= - \left( D_k^{-h} \left( \frac{u(x + he_k) - u(x)}{h} \right) \right) \\ &= - \left( \frac{u(x - he_k + he_k) - u(x - he_k) - u(x + he_k) + u(x)}{-h^2} \right) \\ &= - \left( \frac{u(x + he_k) + u(x - he_k) - 2u(x)}{h^2} \right) \end{aligned}$$

Using this  $v$ , we get:

$$\begin{aligned} I &= \int_{\Omega} -a_{ij}u_{x_i}D_k^{-h}(\zeta^2 D_k^h u)_{x_i} - b_i u_{x_i} D_k^{-h}(\zeta^2) - cu D_k^{-h}(\zeta^2 D_k^h u) \, dx \\ &= - \int_{\Omega} f D_k^{-h}(\zeta^2 D_k^h u) \, dx \\ I &= \int_{\Omega} D_k^h(a_{ij}u_{x_i}) (\zeta^2 D_k^h u_{x_i} + 2\zeta \varphi_{x_j} D_k^h u) \, dx \\ &= \int_{\Omega} (a_{ij}(x + he_k) D_k^h(u_{x_i}) + D_k^h(a_{ij})u_{x_i}) (\zeta^2 D_k^h u_{x_j} + 2\zeta \varphi_{x_j} D_k^h u) \, dx \\ &= \int_{\Omega} \zeta^2 a_{ij}(x + he_k) D_k^h(u_{x_i}) D_k^h(u_{x_j}) + \text{others} \, dx \\ &\geq \theta \int_{\Omega} \zeta^2 |D_k^h(\nabla u)|^2 + \text{others} \, dx \end{aligned}$$

Write  $\tilde{f} := f - b_i u_{x_i} - cu \in L^2$ .

Weak form becomes:

$$\int_{\Omega} a_{ij}(x) u_{x_i} v_{x_j} \, dx = \int_{\Omega} \tilde{f} v \, dx$$

We have:

$$\begin{aligned} \theta \int_{\Omega} \zeta^2 |D_k^h(\nabla u)|^2 \, dx &\leq \int_{\Omega} \tilde{f} D_k^{-h}(\zeta^2 D_k^h u) - a_{ij}(x + he_k) 2\zeta \zeta_{x_j} D_k^h(u_{x_i}) D_k^h u \\ &\quad - D_k^h(a_{ij})u_{x_i} \zeta^2 D_k^h u_{x_j} - D_k^h(a_{ij})u_{x_i} 2\zeta \zeta_{x_j} D_k^h u \, dx \end{aligned}$$

Use  $ab \leq \varepsilon^2 a^2 + \frac{1}{4\varepsilon^2} b^2$  to estimate terms 2,3,4.

$$\leq C \int_{\Omega''} (|D_k^h(\nabla u)| |D_k^h u| + |D_k^h(\nabla u)| |\nabla u| + |\nabla u| |D_k^h u|) \zeta \, dx$$

Note:  $C \rightarrow \infty$  as  $\Omega'' \rightarrow \Omega$  since it involves the derivative of the cutoff function.

## Monday, 10/14/2024

Recall: weak formulation:

$$\begin{aligned} \int_{\Omega} a_{ij} u_{x_i} v_{x_j} \, dx &= \int_{\Omega} f v - b_i u_{x_i} - cu v \, dx \\ &=: \int_{\Omega} \tilde{f} v \, dx, \tilde{f} \in L^2(\Omega) \end{aligned}$$

Choose:  $v = -D_k^{-h}(\zeta^2 D_k^h u)$

Start with LHS.

We arrived at:

$$\begin{aligned}
LHS &= \int_{\Omega} a_{ij}(x + he_k) D_k^h D_k^h(u_{x_j}) \zeta^2 dx \\
&+ \int_{\Omega} \left[ a_{ij}(x + he_k) 2\zeta \zeta_{x_i} D_k^h u D_k^h(u_{x_i}) + D_k^h(a_{ij}) 2\zeta \zeta_{x_j} D_k^h(u) u_{x_i} \right. \\
&\quad \left. + D_k^h(a_{ij}) u_{x_i} D_k^h(u_{x_j}) \zeta^2 \right] dx \\
&\implies LHS \geq \theta \int_{\Omega} |D^h(\nabla u)|^2 \zeta^2 dx \\
&\quad - C \left\{ \int_{\Omega} |D_k^h u| |D_k^h(\nabla u)| + |D_k^h u| |\nabla u| + |D_k^h(\nabla u)| |\nabla u| dx \right\}
\end{aligned}$$

Now use  $(\varepsilon a - \frac{1}{2\varepsilon} b)^2 \geq 0 \implies ab \leq \varepsilon^2 a^2 + \frac{1}{4\varepsilon^2} b^2$

Now use our lemmas.

$$LHS \geq \theta \int_{\Omega} |D^h(\nabla u)|^2 - \varepsilon^2 |D_k^h(\nabla u)|^2 - C_{\varepsilon} |\nabla u|^2$$

Pick  $\varepsilon^2 = \frac{\theta}{2}$

$$LHS \geq \frac{\theta}{2} \int_{\Omega} |D_k^h(\nabla u)|^2 \zeta^2 dx - C_{\varepsilon} \int_{\Omega} |\nabla u|^2 dx$$

Now we estimate RHS with a little of  $L^2$  norm  $v$  + a lot of  $L^2$  norm  $\tilde{f}$   
By lemma 1:

$$\begin{aligned}
\int v^2 &\leq \int |\nabla(\zeta^2 D_k^h u)|^2 \\
&\leq C \int |D_k^h u|^2 + |D_k^h(\nabla u)|^2 dx \\
v \tilde{f} dx &\leq \varepsilon^2 \int v^2 + C_{\varepsilon} \int \tilde{f}^2
\end{aligned}$$

Pick  $\varepsilon^2 = \frac{\theta}{4}$

$$\begin{aligned}
\implies \frac{\theta}{4} \int_{\Omega'} |D_k^h(\nabla u)|^2 dx &\leq C \int_{\Omega} (f^2 + |\nabla u|^2 + u^2) dx \\
&\leq C \int_{\Omega} f^2 + \|u\|_{H^1(\Omega)}^2 dx
\end{aligned}$$

Apply Lemma 2

$u \in H_{loc}^2(\Omega)$  and  $\forall \Omega' \subset \subset \Omega$

$$\int_{\Omega'} |D^2 u|^2 dx \leq C \left( \|f\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 \right)$$

Finally, we need to replace  $\|u\|_{H^1}^2$  with  $\|u\|_{L^2}^2$  on RHS.

Using a new cut-off function  $\zeta$  such that  $\zeta \equiv 0$  on  $\Omega \setminus \Omega''$ , we have:

$$\|u\|_{H^2(\Omega')} \leq C \left( \|f\|_{L^2(\Omega'')}^2 + \|u\|_{H^1(\Omega'')}^2 \right)$$

Now go back to (\*) weak formulation:

Choose  $v = \zeta^2 u$

$$\int_{\Omega} a_{ij} u_{x_i} u_{x_j} \zeta^2 + a_{ij} u_{x_i} 2\zeta \zeta_{x_i} dx = \int \tilde{f} \zeta^2 u$$

Again, by ellipticity and  $ab \leq \varepsilon a^2 + C_{\varepsilon} b^2$ ,

Choosing  $\varepsilon$  small in terms of  $\theta$ ,

$$\int_{\Omega'} |\nabla u|^2 \, dx \leq C \int_{\Omega''} f^2 + u^2 \, dx$$

so we're done!

□