Group Representations MATH 607

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Texts: Lang, Algebra, Revised Third Edition, Chapter 17 (sections 1-5) and 18 (sections 1-8)
Serre, Linear Representations of Finite Groups, Parts II and III

Monday, 8/26/2024

Today:
History
Modular
Quotients
Matrices
Lang XVII, Section 1

(Fake) History

History of Groups
Most notions (let's say what is a vector spce, what is a group) were vague.
Originally, groups were seen as:

- Symmetry Groups S_n
- $GL_n(\mathbb{R})$ aka $n \times n$ invertible matrices
- Subgroups of the above
- Representations of the above

For representation, consider G and a homomorphism $G \to S_n$ [which is a group action $G \curvearrowright \{1, 2, \dots, n\}$] or a homomorphism $G \to GL_n$ [which is a group action on vector space].
Part I of this course will be Ring Theory.

Part I: Ring Theory

Module

Convention: $R =$ Ring with unity

Definition (Left Module)**.** Left Module is an abelian group M with a function $R \times M \to M$ so that $(r, m) \mapsto rm$ such that $R \times M \to M$ is $\mathbb{Z}$-billinear.
Meaning, we have:
$(r + r')m = rm + r'm$
$r(m + m') = rm + rm'$
Also $(rr')m = r(r'm)$
And finally $1m = m$

By default, module = left module (since Jim doesn't want Trump to get reelected, he prefers left module)
module / field [module over field] = vector space
We can have submodules $M' \triangleleft M$
We have quotients M/M'
We have the short exact sequence:

$$0 \to M' \to M \to M/M' \to 0$$

which means in each homomorphism, im = ker
So, $M' \to M$ is injective and $M \to M/M'$ is surjective.
Also, kernel of $M \to M/M'$ is M'

Remark. Note that R is itself an R-module.
Convention: Submodule M of R = left ideal of R.
Left ideals are not enough to take quotients (like how we need normal subgroup for group quotients).

So we need two sided ideals.

Definition (Two Sided Ideals)**.** $I \subset R$ is 2-sided ideal if I is abelian subgroup and $ri \in I, ir \in I$ aka "closed".

Example. Consider a homomoprhism $f : R \to R'$. Then $\ker f$ is a 2-sided ideal of R.

For ring homomorphism we need:
$f(r + r') = f(r) + f(r')$
$f(rr') = f(r)f(r')$
$f(1) = 1$
If $I \subset R$ is 2-sided then R/I is a quotient ring.
For example, $M_2(\mathbb{R})$ has no proper 2-sided ideal. But there exists left ideals!
$\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$ is a left ideal
Matrices are a good 'source' of non-commutative rings.
Given any ring R we can consider ring $M_n(R)$ of $n \times n$ matrices.
Given R-module M we can get $\mathrm{End}_R(M) = \{f : M \to M, f \text{ is } R\text{-module map}\}$
We have $(f + g)m = f(m) + g(m), (fg)m = f(g(m))$.
This is a 'coordinate free approach' to matrices.

Remark. $M_n(R)$ and $\mathrm{End}_R(R^n)$ often looks the same, but in general $M_n(R) \not\cong \mathrm{End}_R(R^n)$.

Let's first take $n = 1$. Let $r_0 \in R$.
Consider $R \to R$ map $r \mapsto r_0 r$
We don't like this because this is not a left module map!!!
So this is not even in $\mathrm{End}_R(R)$
What if we consider $r \mapsto r r_0$?
This is a left module map, aka $\in \mathrm{End}_R(R)$
But $R \to \mathrm{End}_R(R)$ is not a ring homomorphism.
So we are going to take the opposite ring.
Fix 1:
Given ring R, we can look into the mirror and find opposite ring R^{op}
Elements of R^{op} = elements of R.
$0, 1, +$ remain the same
But multiplication is reversed: define $r \cdot_{op} r' = r'r$
Alternate notation, we write *op* on elements.
Then $r^{op}(r')^{op} = (r'r)^{op}$
Then we have isomorphism $R^{op} \cong \mathrm{End}_R(R)$ which is a ring homomoprhism!

Exercise.
1) $R \cong R^{op} \iff \exists$ antiautomorphism $\alpha : R \to R$

   Antiautomorphism means α preserves $0, 1, +$ but reverses mutliplication

2) R commutative, then $(M_n R) \cong (M_n R)^{op}$

3) Real quaternions $\mathbb{H} \cong \mathbb{H}^{op}$

Remark. If you take right modules, you don't need op.

There is a <u>contravariant endofunctor</u> in the category of rings which takes objects of rings to their opposite.
$\text{Ring}^{op} \to \text{Ring}$ [opposite category, not the same thing]
$R \mapsto R^{op}$
<u>Fix 2</u>: [From Lang]
Suppose we have module homomorphism $\phi : E = E_1 \oplus \cdots \oplus E_n \to F_1 \oplus \cdots \oplus F_m = F$
Then we have $E_j \to E \xrightarrow{\phi} F \to F_i$ which we define to be $E_j \xrightarrow{\phi_{ij}} F_i$
Then we have a matrix $M(\phi)$ so that $M(\phi) = (\phi)_{ij}$
Then for $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in E_1 \oplus \cdots \oplus E_n$
Then $\phi(x) = (\phi_{ij}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$
So, if we have $E^n = E \oplus \cdots \oplus E$ [n times]
Lang says, there is a ring isomorphism

$$\text{End}_R(E^n) \overset{\cong}{\to} M_n(\text{End}_R E)$$

$$\phi \mapsto (\phi_{ij})$$

If $E = R$ as left module, then $\text{End}_R R \cong R^{op}$
By combining these, $\text{End}_R(R^n) \cong M_n(R^{op})$

Wednesday, 8/28/2024

Today:
Group ring
Category
Simple modules
Question: The course is about 'group representations'. So why study rings?
Answer: A group representation [homomorphism $G \to GL_n(\mathbb{R})$] is exactly the same as a module over the ring $\mathbb{R}G$.
So knowing everything about modules would tell us everything about representation.
Abelian Category!
Suppose we have a ring R and a group G. We can get a ring out of G

Definition (Group Ring RG)**.** As an abelian group, this is the free R-module with basis the elements of G.
Elements are symbols of the form $r_1 g_1 + \cdots + r_n g_n$ [finite linear combination].
0 is the trivial linear combination. So $0 = 0$
$1 = 1e = 1_R e_G$
Multiplication is defined in the obvious way.
$(\sum_i r_i g_i)(\sum_j r'_j g'_j) = \sum_{i,j} r_i r'_j g_i g'_j$

Suppose V is a R-module.
Then a homomorphism $\rho : G \to \text{Aut}_R(V) \leftrightarrow V$ is RG-module.
$\rho \mapsto (\sum_i r_i g_i)v := \sum_i r_i \rho(g_i)v$
$g \mapsto (v \to gv) \leftarrow V$ RG module.

Example. $C_2 = \{1, t\}$
Then we have $\mathbb{Z}C_2 = \{a + bt \mid a, b \in \mathbb{Z}, t^2 = 0\} = \mathbb{Z}[t]/(t^2)$
Note that $(1 + t)(1 - t) = 1 - t^2 = 0$ so we have zero divisors.
Take $C_\infty = \langle t \rangle$
Then $\mathbb{Z}C_\infty = \mathbb{Z}[t, t^{-1}]$ the laurent polynomial ring.
$\mathbb{Q}C_\infty = \mathbb{Q}[t, t^{-1}]$ is a PID [since it is a euclidean ring]

Now we see categories.
If we fix R then we have a functor Group $\to$ Ring given by $G \mapsto RG$
Or we could say we have a functor Ring $\times$ Group $\to$ Ring given by $(R, G) \to RG$

Definition. A category $\mathcal{C}$ consists of:

- objects Ob $\mathcal{C}$
- morphism $C(X, Y)$ for $X, Y \in$ Ob $\mathcal{C}$
- compositions $C(X, Y) \times C(Y, Z) \to C(X, Z)$ given by $(g, f) \mapsto f \circ g$
- identity $\mathrm{Id}_X \in C(X, X) \forall X \in \mathrm{Ob}\mathcal{C}$

Such that we have:

- associativity: $(f \circ g) \circ h = f \circ (g \circ h)$
- composition with identity: $\mathrm{Id}_Y \circ f = f = f \circ \mathrm{Id}_X$ for $f \in C(X, Y)$

For example in the cateogry of groups, we have objects groups and morphisms homomorphism.
Morphism notations: $f : X \to Y$ or $X \xrightarrow{f} Y$ for $f \in C(X, Y)$

Definition. $f : X \to Y$ is isomorphism if $\exists g : Y \to X$ such that $f \circ g = \mathrm{Id}, g \circ f = \mathrm{Id}$.
Thehen we say X and Y are isomorphic and write $X \cong Y$.

Example. Example of Categories:

- Set
- Ring
- Group
- Ab (Abelian Groups)
- R-modules (objects are modules, morphisms are homomorphisms $h(rm) = rh(m)$)
- Given a group G we can get a category BG such that:

  Ob $BG = \{*\}$ and $BG(*, *) = G$

  In this category, there is only one object $*$. The elements of the group are morphisms.

Definition. Functor $F : \mathcal{C} \to \mathcal{D}$ is F : Ob $\mathcal{C} \to$ Ob $\mathcal{D}$ given by $X \mapsto F(X)$
And $F : \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ such that
$X \xrightarrow{f} Y$ gives us $F(X) \xrightarrow{F(f)} F(Y)$
such that $F(f \circ g) = F(f) \circ F(g)$ and $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$

Example. Unit Functor Ring $\to$ Group given by $R \mapsto R^\times = \{r \in R \mid \exists s \in R, rs = 1 = sr\}$
For example, $\mathbb{Q}^\times \cong C_2 \oplus \mathbb{Z}^\infty [= \pm p_1^{e_1} p_2^{e_2} \cdots]$
$\mathbb{Z}^\times \cong \{\pm 1\} = C_2$
$(\mathbb{Z}C_2)^\times \cong \{\pm 1, \pm t\} \cong C_2 \times C_2$

Definition. R is a division ring (= skew field) if $1 \neq 0$ and $R^\times = R - 0$.

Definition. Quaternions
$\mathbb{H} = \{a + bi + cj + dh \mid a, b, c, d, \in \mathbb{R}\}$
Where $i^2 = j^2 = k^2 = -1$
$ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$
This is a division ring since we can write down inverses.
$\alpha = a + bi + cj + dk$ gives us $\overline{\alpha} = a - bi - cj - dk$
So, $\mathrm{norm}(\alpha) = \alpha\overline{\alpha} = a^2 + b^2 + c^2 + d^2$
So, $\alpha^{-1} = \frac{\overline{\alpha}}{\mathrm{norm}(\alpha)}$

Remark. Note that the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is a subgroup of $\mathbb{H}^\times = GL_1(\mathbb{H})$.
So, $\mathbb{H}$ is a $\mathbb{R}Q_8$ module.

Theorem 1 (Weddenburn's Little Theorem)**.** a. A finite commutative domain is a field [easy]
b. A finite skew field is a field [aka commutative]

a is easy: suppose F is finite commutative domain. For $0 \neq f \in F$, consider multiplication by f as a map $F \to F$. It is injective, and finiteness implies surjective. So, it is bijective, and there exsits inverse.
eg $\mathbb{Z}/p$ is a field.

Simple Modules

These are like primes. We also have some analogue of prime factorization.

Definition. R-module E is simple if:
$E \neq 0$
No proper submodules, aka $M \triangleleft E \implies M = 0$ or E
In other words, E is a simple module if it only has two submodules: 0 and E.

eg simple $\mathbb{R}$-modules are 1 dim vector spaces, aka $\mathbb{R}$

Exercise. a) $\mathbb{R}^2$ is a simple $M_2(\mathbb{R})$-module

b) Express $M_2(\mathbb{R})$ as direct sum of simple modules.

Friday, 8/30/2024

Exercise. Suppose finite $G \neq 1$ and $R \neq 0$ Prove that RG has zero divisors.

Definition. Direct product of rings $R \times S$, addition and multiplication is done componentwise.
It is a product in the category of rings. aka:

$$\begin{tikzcd} & T \arrow[dl, "f_1"'] \arrow[d, dotted, "f"] \arrow[dr, "f_2"] & \\ R & R \times S \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & S \end{tikzcd}$$

for any pair of ring homomorphisms $T \xrightarrow{f_1} R$ and $T \xrightarrow{f_2} S$ we have a unique ring homomorphism $f : T \xrightarrow{f} R \times S$ so that the diagram commutes.

Definition. $e \in R$ is an idempotent if $e^2 = e$.

$0, 1$ are trivial idempotents.
$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is an idempotent in $M_2(\mathbb{R})$
$(0, 1)$ is an idempotent in $\mathbb{R} \times \mathbb{R}$
If e is an idempotent so is $1 - e$

Definition. Idempotent $e \in R$ is central if $\forall r$ we have $er = re$

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is not central, but $(0, 1)$ is.

Exercise. A ring can be written as a product ring, aka $R \cong R_1 \times R_2$ with $R_i \neq 0$ if and only if there exists a nontrivial central idempotent.

Semisimiple Modules

Definition. E is a simple R-module if it doesn't have any nontrivial submodules.
If $E \neq 0$ and $M \triangleleft E$ then $M \neq 0$ or $M = E$

Example. R^2 is a simple $M_2\mathbb{R}$-module.
$\mathbb{R} \times 0$ is a simple $\mathbb{R} \times \mathbb{R}$ module.
$\mathbb{Z}/p\mathbb{Z}$ is a simple $\mathbb{Z}$-module

Lemma 2. [Schur's Lemma]: Let E, F be simple R-modules. Then any nonzero homomoprhism $f : E \to F$ is an isomorphism.

Proof. $f \neq 0$ means $\ker f \neq E$ and im $f \neq 0$.
Since they are submodules, $\ker f = 0$ and im $f = F$
So f is bijective. $\square$

Corollary 3. If E is simple, then $\mathrm{End}_R E$ is a skew field [any non-zero element is invertible]

Example. Commutative example: $\mathrm{End}_{M_2\mathbb{R}}(\mathbb{R}^2)$ is a skew field.
In fact, $\mathrm{End}_{M_2\mathbb{R}}(\mathbb{R}^2) \cong \mathbb{R}$

Definition (Direct Sum)**.** Suppose $M_i \triangleleft M$ for $i \in I$
Then, $M = \bigoplus_{i\in I} M_i$ means, $\forall m \in M_i$ we have $m = \sum_{i\in I} m_i$ with $m_i \in M_i$ uniquely.
There are notions of internal and external direct sums. The above is an internal direct sum.
External direct sum: given $\{M_i\}_{i\in I}$ we can construct $\bigoplus_{i\in I} M_i$

Proposition 4 (Universal Property)**.** Given a collection of homomorphisms $\{t_i : M_i \to N\}_{i\in I}$, it extends directly to a homomorphism $\bigoplus M_i \to N$. We denote this by $\bigoplus f_i$

Remark. Note: Maps to product are easy, maps from direct sum are easy.

Proposition 5 (1.2, Lang XVII)**.** Suppose we have isomorphism $E_1^{n_1} \oplus \cdots \oplus E_r^{n_r} \overset{\cong}{\to} F_1^{m_1} \oplus \cdots \oplus F_s^{m_s}$ with E_i and F_j simple and non-isomorphic [ie for all $k \neq i, E_k \ncong E_i$ and $k \neq j, F_k \ncong F_j$]
Then $r = s$ and there exists a permutatation $\sigma \in S_r$ so that $E_j \cong F_{\sigma(j)}$ and $n_j = m_{\sigma(j)}$

Corollary: If E is a finite direct sum of simple modules, then the isomorphism class of simple components of E and multiplicities are well-defined.

Proof. We use Schur's Lemma.
We write ϕ as a matrix $(\phi_{ji} : E_i^{n_i} \to F_j^{m_j})$
Since ϕ is injective, for all i there exists a j such that $\phi_{ji} \neq 0$
Then, $E_i \cong F_j$ by Schur's Lemma
Note that F_j are isomorphic. So, for all i, the j such that $\phi_{ji} \neq 0$ is unique!
We also get $\sigma : \{1, \dots, r\} \to \{1, \dots, s\}$ so that $\sigma(i) = j$
Since σ^{-1} exists σ^{-1} exists, and thus $r = s$
Since ϕ is an isomorphism, individual $\phi_{ji} : E_i^{n_i} \to F_{\sigma(i)}^{m_{\sigma(i)}}$ are isomorphisms.
To complete the proof, we need a lemma
Lemma: Let E be simple. If $E^n \cong E^m$ then $n = m$
Proof of lemma; Let $D = \mathrm{End}_R E$. By Schur's Lemma, D is a division ring.
Since $E^n \cong E^m$, we have $\mathrm{End}_R(E^n) \cong \mathrm{End}_R(E^m)$
So, $M_n(D) \cong M_m(D)$
Also, isomorphism not just as rings, but also as D-modules.
Every module over a skew field is free, and the number of dimensions is the same.
So, $n^2 = m^2 \implies n = m$
This finishes the proof.

$\square$

Lang XVII section 2

Theorem 6. Let E be an R-module. Then TFAE:
SS1: E is a sum of simple modules [so, we can write $m \in E$ as sum of m_i but it is not unique]
SS2: E is a direct sum of simple modules [we can write as a sum, and it's unique]
SS3: Every submodule of E is a summand.
$F \triangleleft E \iff$ we can find F' so that $E = F \oplus F'$
SS3$'$: any monomorphism $F \to E$ 'splits'
SS3$''$ Short exact sequence

$$0 \to F \to E \to H \to 0$$

splits.

This leads us to:

Definition. E is semisimple if it satisfies one of the above.

Davies: SS2 is best
eg: $R = \mathbb{R} \times \mathbb{R}$
$E = \mathbb{R} \times \mathbb{R}$ is semisimple but not simple.
Because: $E = \mathbb{R} \times 0 \oplus 0 \times \mathbb{R}$

Wednesday, 9/4/2024

Recap: Semisimple modules.

Lemma 7. If $E = \sum_{i \in I} E_i$ with E_i simple. Then, $\exists J \subset I$ such that $E = \bigoplus_{j \in J} E_j$

Corollary 8. SS1 $\implies$ SS2

Proof. Let $J \subset I$ be maximal such that $\sum_{j \in J} E_j = \bigoplus_{j \in J} E_j$
This exists by Zorn's lemma.
$\forall i \in I - J$, we have $E_i \cap \bigoplus_{j \in J} E_j \neq \varnothing$ by maximality.
Since E_i is simple, $E_i \subset \bigoplus_{j \in J} E_j$. Therefore, $E = \bigoplus_{j \in J} E_j$.

□

<u>True of False</u>? Every module has a maximal proper submodule.
False!!! Exercise.

Exercise.
a) If $M \triangleleft F$ proper and M maximal, then F/M is simple.
b) Find a ring R, module M which does not have proper maximal submodules.
c) If F is a finitely generated R-module, then it is contained in a proper maximal submodule.

Proof of SS2 $\implies$ SS3. Suppose $F \triangleleft E = \bigoplus_{i \in I} E_i$ with E_i simple. Let $J \subset I$ be maximal such that:

$$F + \bigoplus_{j \in J} E_j = F \oplus \bigoplus_{j \in J} E_j$$

Take any $i \in I - J$. Then, $E_i \cap \left[F \oplus \bigoplus_{j \in J} E_j\right] \neq 0$ by maximality of J.
Since E_i is simple, $E_i \subset F \oplus \bigoplus_{j \in J} E_j$.
Therefore, $E = F \oplus \underbrace{\bigoplus_{j \in J} E_j}_{F'}$.
We have found F', which proves SS3.

□

Proof of SS3 $\implies$ SS1.

Lemma 9. $0 \neq F \triangleleft E$ and E satisfies SS3. Then, there exists simple finitely generated $S \triangleleft F$.
Plan: $M \underset{\neq}{\triangleleft} \underset{\text{f.g.}}{F_0} \triangleleft F \triangleleft E$.
Then, choose $0 \neq v \in F$. Let $F_0 = Rv$.

Exercise. M exists. [Zorn's Lemma]

Let $E = \sum_{\text{simple } S \triangleleft E} S$.
Then, by SS3, $E = E_0 \oplus E_0'$.
Lemma and definition of E_0 implies: $E_0' = 0$. So, E is indeed a sum of simple R-modules. We're done!

□

Proposition 10 (2.2)**.** Every quotient module and submodule of a semisimple modules is semisimple.

Proof. Quotients: Suppose $M = E/N$. We have surjective $f : E \to M$ with E semisimple.
SS1 implies $E = \sum_{i \in I} S_i$ with S_i simple.
Then, $M = \sum_{i \in I} f(S_i)$
Schur's lemma implies $f(S_i)$ is either 0 or simple, so M satisfies SS1.
Submodules: Suppose $F \triangleleft E$ with E semisimple. SS3 implies $E = F \oplus F'$. Thus $E \cong E/F'$, so it is semisimple by the quotient result.

□

Preview:

Definition. A ring R is semisimple if and only if all R-modules are semisimple.
Lang defines semisimple differently: A ring R is semisimple if it is semisimple as an R-module.

Theorem 11 (Artin-Weddenburn Theorem)**.** A ring is semisimple if and only if it is isomorphic to a finite product of matrix rings over division algebras:

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$$

$\mathbb{C}G, \mathbb{R}G$ are semisimple. We also have the result:

Theorem 12 (Maschke's Theorem)**.** The group ring kG is semisimple if G is finite and k is a field of characteristic prime to G.
This also works with $\operatorname{char} k = 0$. It is in fact an if and only if.

So $\mathbb{F}_p G$ is also semisimple given $p \nmid |G|$

Proof. Outline: let $|G| = n$. We will verify SS3.
Let $F \triangleleft E$ be kG modules.
k is a field, so there exists a k-linear projection $\pi : E \to F$ such that $\pi(f) = f$ for $f \in F$ [take a basis of F as a k-vector space, complete it to a basis of E].
Now, define an 'average'.

$$\pi'(e) = \frac{\sum_{g \in G} g\pi(g^{-1}e)}{n}$$

Then, $\pi' : E \to F$ is a kG-linear projection, meaning $\pi'(ge) = g\pi'(e)$.
Then $E = \underset{F}{\operatorname{im}} \pi' \oplus \underset{F'}{\ker} \pi'$

□

Friday, 9/6/2024

Lang XVII, Sectiion 3

"Density Theorem"
Suppose R is a ring and E is a R-module. Then we have maps $R \times E \to E$ by mutliplication on the left.

Definition (Commutant)**.** $R' = R'(E) = \mathrm{End}_R(E)$ is a ring.
$\phi \in R' \iff \phi : E \to E$ such that $\phi(re) = r\phi(e)$. It 'commutes with E'.
Note that E is also an R'-module, with $R' \times E \to E$ given by $(\phi, e) = \phi(e)$.

Definition (Double Commutant)**.** We can iterate on the previous definition.

$$R'' = R'(R'E) = \mathrm{End}_{R'}(E)$$

Therefore,

$$R'' = \mathrm{End}_{R'}(E) = \mathrm{End}_{\mathrm{End}\ R(E)}(E)$$

This means, $f \in R'' \iff f : E \to E, \forall \phi \in R', f \circ \phi = \phi \circ f$. So, things in R'':

<u>commute</u> with things which commute with $r \in R$.

Example. Suppose $R = \mathbb{R}$ and $E = \mathbb{R}^n$. Then,

$$\mathbb{R}' = \mathrm{End}_{\mathbb{R}}(\mathbb{R}^n) = M_n(\mathbb{R})$$

$$\mathbb{R}'' = \underset{rI}{\mathrm{End}_{M_n(\mathbb{R})}(\mathbb{R}^n)} \underset{\leftarrow}{=} \underset{r}{\mathbb{R}}$$

Suppose V = vector space.
$V^* = \mathrm{Hom}(V, \mathbb{R})$
Then we have evaluation map $ev : V \to V^*$ given by $v \mapsto (\phi \mapsto \phi(v))$.
ev is 1-1.
ev is onto iff $\dim V < \infty$.
With inspiration from this, we define,

Definition (Evaluation map)**.** $ev : R \to R''$ given b $r \mapsto (e \mapsto re)$
We define $f_r : E \to E$ given by $f_r = ev(r)$

Proposition 13. a) $f_r \in R''$

b) ev is a ring homomorphism.

Proof. a) $f_r(\phi(e)) = r\phi(e) = \phi(re)\phi(f_r(e))$

b) $ev(r + r') = ev(r) + ev(r'), ev(1) = 1$.
$(ev(r))(ev(r'))e = ev(r)(r'e) = rr'e = ev(rr')e$

□

Lemma 14 (3.1)**.** Suppose E is semisimple over R, $e \in E$ and $f \in R''$
Then $\exists r \in R$ such that $re = f(e)$ [i.e. $f(e) = ev(r)(e)$]

Proof. E is semisimple, and Re is a submodule. Therefore, we can write $E = Re \oplus F$.
Define $\pi : E \to E$ be projection to Re.
Then $\pi \in E' \implies f \circ \phi = \pi \circ f \implies f(e) = f(\pi(e)) = \pi(f(e)) = re$ for some $r \in R$. □

We will prove a stronger version of this lemma called the <u>Jacobson Density Theorem</u>.

Theorem 15 (3.2, Jacobson Density Theorem)**.** Suppose E is semisimple over R
$e_1, \cdots e_n \in E$
$f \in R''$
Then, $\exists r \in R$ such that $re_i = f(e_i) \forall i$.
Therefoe, if E is finitely generated over R', then $R \to R''$ is onto.

Proof. We use a diagonal trick.
Special Case: E is simple.
Idea: Apply the lemma on E with $\underline{e} = (e_1, \cdots, e_n)$ and $f^n : E^n \to E^n$ such that $f(y_1, \cdots, y_n) = (f(y_1), \cdots, f(y_n))$.
We need to check that $f \in R'(R'(E))$ to apply it.
This would imply that $f^n \in R'(M_n R) \underset{E \text{ simple}}{=} R'(R'(E^n))$
Therefore, $\exists r$ such that $r\underline{e} = f^n(\underline{e})$. This finishes the proof.
For E semisimple, key idea is $f^n \in R'(R'(E))$ as above. [Complicated for infinite sums. We avoid.]

□

Application:

Theorem 16 (Burnside's Theorem)**.** Suppose k is an algebraically closed field.
Take subring R such that $k \subset R \subset M_n(k)$
If $k^n(= E)$ is a simple R-module, then prove that:

$$R = M_n(k)$$

Exercise. Suppose D_{2n} is the dihedral group of order $2n$, aka

$$D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, srs^{-1} = r^{-1} \rangle$$

Let $\zeta_n = e^{2\pi i/n} \in \mathbb{C}$
Then we can define a homomorphism $D_{2n} \to GL_2(\mathbb{C})$ given by:

$$r \mapsto \begin{bmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{bmatrix}$$

$$s \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This gives us a ring map $\pi : \mathbb{C}D_{2n} \to M_2\mathbb{C}$
Prove the following:

a) Prove that $\mathbb{C}^2$ is a simple $\mathbb{C}D_{2n}$ module [can be done without technology]

b) Use Burnside's theorem to show that π is onto.

Note that Burnside's theorem doesn't work if k is not algebraically closed.
We have:

$$\mathbb{R} \subset \mathbb{C} \subset M_2\mathbb{R}$$

since we can embed $\mathbb{C}$ into $M_2\mathbb{R}$.
$\mathbb{C}$ is a simple R module, but $\mathbb{C} \neq M_2\mathbb{R}$

Proof of Burnside's Theorem. Step 1: We show that $\mathrm{End}_R(E) = k$
Note that, $k \underset{\text{central}}{<} \underset{\text{skew field}}{\mathrm{End}_R(E)} \subset \underset{\text{finite dim}/k}{\mathrm{End}_k(E)}$
$\forall \alpha \in \mathrm{End}_R(E), k(\alpha)$ is a field and finite dimensional $/k$.
Therefore, $k(\alpha) = k$ since k is algebraically closed.
Thus, $\alpha \in k$. This finishes Step 1.
Step 2: We show that $R = \mathrm{End}_k(E)$.
$R \subset \mathrm{End}_k(E)$ by hypothesis.
Suppose $A \in \mathrm{End}_k(E)$. Let $e_1, \cdots, e_n$ be a k-basis for $E = k^n$.
Density theorem implies: $\exists r \in R$ such that $Ae_i = re_i$ for all i.
Therefore, $A = r \in R$.

□

Monday, 9/9/2024

Today:
Density Theorem
Characters determine representation
Artin-Wedderburn Theorem
Homework due Monday 9/16, Exercises 1-7
Recall Jacobson Density Theorem:
If E is semisimple over R, $e_1, \ldots, e_n \in E$ and $f \in R''$ then,

$$\exists r \in R \text{ s.t. } f(e_i) = re_i \forall i$$

Recall that R'' is defined as follows:

$$f \in R'' \iff f : E \to E \text{ s.t. } \forall \phi \in R' = \operatorname{End}_R E, f \circ \phi = \phi \circ f$$

Also recall Burnside's Theorem:
Suppose k is an algebraically closed field, and $k \subset R \subset M_n(k)$ are subrings
If k^n is a simple R-module, then
$R = M_n(k)$

3.7 Existence of Projection Operators

Theorem 17. Suppose $E = V_1 \oplus \cdots \oplus V_m$, simple non-isomorphic R-modules. Then, for any i, there exists $r_i \in R$ such that,

$$r_i v = \begin{cases} v, & \text{if } v \in V_i; \\ 0, & \text{if } v \in V_j, i \neq j \end{cases}$$

So, each projection map is just multiplication.

Proof. This is a consequence of the density theorem.
Choose nonzero $e_k \in V_k$.
Let $f = \pi_i : E \to E$ which is a projection on V_i.
Note that $f \in R''$ since for all $\phi \in R', \phi(V_k) \subset V_k$ [Schur's Lemma, non-isomorphic].
Density theorem $\implies \exists r_i \in R$ such that $r_i e_k = \pi_i(e_k)$.
Note that $V_k = Re_k$ so $\forall v \in V_k, v = re_k$.
So, $r_i v = r_i r e_k = r\pi_i(e_k) = \pi_i(re_k) = \pi_i(v)$
Which is what we wanted.
□

Correction to the Existence of Projection Operators
Suppose k is a field, R is a k-algebra so that R is semisimple. Suppose R-module $E = V \oplus V', \dim_k E < \infty$.
For all simple $L \triangleleft V, \forall L' \triangleleft V'$ then $L \cong L'$
Then, $\exists r \in R$ such that for all $e \in E$,

$$re = \begin{cases} e, & \text{if } e \in V; \\ 0, & \text{if } e \in V'; \end{cases}$$

Proof. We apply density theorem. Since we have finite dimension, we have:

$$\{e_1, \cdots, e_n\} = (k\text{-basis of } V) \cup (k\text{-basis of } V')$$

Let $\pi_V : E \to E$ be the projection on V.
Then, $\pi_V \in R''$ [the second commutant] since $\forall \phi \in R', \phi(v) \subset V, \phi(v') \subset V'$.
Density theorem implies $\exists r$ such that $re_i = \pi_v(e_i)$.
Then $\forall a \in k \subset$ center R,
$r(ae_k) = a(re_k) = a\pi_v(e_k) = \pi_v(ae_k)$
Therefore, $re = \pi_v(re)$. □

Question: What is a k-algebra?
Following Atiyah-McDonald, let k be a commutative ring [often but not always a field]. Then,

$$R \text{ is a } k\text{-algebra} \overset{\text{def}}{\iff} \text{homomorphism } h : k \to R, h(k) \subset \text{center}(R)$$

Example. Any ring is a $\mathbb{Z}$-algebra, homomorphism sends n to $1 + 1 + \cdots + 1$
k field, $R \neq 0 \implies k \hookrightarrow R$
k-algebra $\iff k \subset \text{center}(R)$

Corollary 18 (3.8)**.** Suppose $\operatorname{char} k = 0$, R is a k-algebra, E, F semisimple over R, finite dimensional over k.
For $r \in R$, let:
$f_r^E : E \to E$ be $f_r^E(e) = re$
$f_r^F : F \to F$ be $f_r^F(f) = rf$
If $\operatorname{Tr}(f_r^E) = \operatorname{Tr}(f_r^F)$ for all $r \in R$,
Then $E \cong F$ as R-modules.

Proof. Let V be a simple R-module.
Suppose $E = V^n \oplus$ direct sum of simple R-modules not isomorphic to V
$F = V^m \oplus$ direct sum of simple R-modules not isomorphic to V
We want to show $n = m$
Let $r_v \in R$ be the projection operation from 3.7.
Then, $\operatorname{Tr}(f_{r_v}^E) = \operatorname{Tr}(r_v \cdot : E \to E) = \dim_k V^n = n \dim_k V$
Similarly, $\operatorname{Tr}(f_{r_v}^F) = m \dim_k V \implies n = m$

□

Corollary 19 (Characters determine representations)**.** Suppose k is a field and $\operatorname{char} k = 0$. Let G be a finite group. Suppose:
$\rho : G \to GL_n(k)$
$\rho' : G \to GL_m(k)$
with kG-modules $E = k^n$ over ρ and $F = k^m$ over ρ'
If $\operatorname{Tr}(\rho(g)) = \operatorname{Tr}(\rho'(g))$ for all g,
Then $E \cong F$ as kG-modules.
Note that, substituting $g = 1$ gives us:
$\operatorname{Tr}(\rho(1)) = \operatorname{Tr}(\rho'(1)) \implies \operatorname{Tr}(I) = \operatorname{Tr}(I) \implies n = m$.

Definition ((semi)simple rings)**.** Note that if R is a ring, then R is a left module as well. We write ${}_RR$ when we're considering it as a left module, and ${}_RR_R$ when we are considering a two sided ideal.
R is called a semisimple ring if ${}_RR$ is a semisimple R-module.
R is called a simple ring if R is a semisimple ring, and for all simple $L, L' \lhd_R R \implies L \cong L'$
This means, ${}_RR = \oplus_{i \in I} L_i$ where L_i are simple (left) ideals such that $L_i \cong L_j$ for all i, j.
Recall that an ideal is simple if it has no proper sub-ideals.

Example. $M_2(\mathbb{H})$ is a simple ring. We can write it as direct sum of two ideals

$$\begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$$

Example. $M_2(\mathbb{H}) \times \mathbb{R}$ is semisimple.

$$\begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix} \times 0 \oplus \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix} \times 0 \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \times \mathbb{R}$$

Artin-Wedderburn generalizes this.

Theorem 20 (Artin-Wedderburn Theorem)**.** i) R simple $\iff R \cong M_n(D)$ where D is a skew-field.

ii) R semisimple $\iff R \cong R_1 \times \cdots \times R_s$ simple rings.

Wednesday, 9/11/2024

Today we discuss the Artin-Wedderburn Theorem.
Exercise: $C_2 = \{1, g\}$, prove that $\mathbb{Q}C_2$ is a semisimple ring.
$\mathbb{Q}C_2 = B_1 \oplus B_2$ 2-sided ideals
$\mathbb{Q}C_2 \cong \mathbb{Q} \times \mathbb{Q}$.

Lemma 21. Suppose we have a ring R which is decomposed as a sum of (left) ideals:

$$_RR = \bigoplus_{i \in I} L_i \quad \text{with } L_i \neq 0$$

Then $|I| < \infty$.

Proof. Suppose $_RR = \bigoplus_{j \in J} L_j$ where L_j are ideals. We want to prove that only finitely many are non-zero.
Note that, $1 = \sum_{j \in J} e_j$. We use only finitely many elements here, so $1 = \sum_{i \in I} e_i$ where $e_i \neq 0, I \subset J, |I| < \infty$.
For all $r \in R$ we have $r = r \cdot 1 = r \sum_{i \in I} e_i = \sum_{i \in I} re_i \in \sum_{i \in I} L_i$.
Therefore, $_RR = \bigoplus_{i \in I} L_i$ a finite sum! $\square$

Now we go to the theorem.

Proof of Artin-Wedderburn Theorem Part I. We want to prove: R simple ring $\iff$ $R \cong M_n D$ where D is a skew field.
First, note that $_RR \cong L^n$ where L is a simple ideal [so no proper sub-ideals]. Therefore,

$$R^{op} \cong \operatorname{End}_R(_RR) \cong \operatorname{End}_R(L^n) \cong M_n(\underbrace{\operatorname{End}_R L}_{\text{division ring}})$$

Taking transpose,

$$R \cong M_n(\operatorname{End}_R L)^{op} \cong M_n((\operatorname{End}_R L)^{op}) = M_n(D)$$

So we are done with one direction!
The other direction is a exercise. Here are the steps:

Step 1: $M_nD = \begin{bmatrix} * & 0 & \cdots & 0 \\ * & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 0 & \cdots & * \\ 0 & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{bmatrix}$

Step 2: Each summand is isomorphic to $D^n = \begin{bmatrix} * \\ * \\ \vdots \\ * \end{bmatrix}$

Step 3: D^n is a simple module.

$\square$

Remark. R simple $\iff$ R artinian, R has no proper 2-sided ideals. Some definitions forgo the artinian condition, in this case these are called artinian simple rings.

Lemma 22 (4.2)**.** Suppose L is a simple ideal and M is a simple module so that $L \ncong M$. Then $LM = 0$.

Proof. This is a direct consequence of Schur's lemma. Consider the map $\phi_m : L \to M$ given by $l \mapsto lm$ for $m \in M$. Since this can't be an isomorphism, it must be the zero map. Thus, $lm = 0$. $\square$

Proof of Artin-Wedderburn Theorem Part II. Idea: Decompose R as direct sum of simple ideals. Partition the set of simple ideals so that members of a partition are isomorphic to each other, members of a partition are not isomorphic to members of another partition. Direct sum of each partition gives us one R_j.
Suppose R is semisimple.
Let $L_1, \cdots, L_s$ be a set of pairwise non-isomorphic simple ideals [meaning $L_i \ncong L_j$]
So that, for all simple $L <_R R, L \cong L_i$ for some i.
Let $B_i = \sum_{L \cong L_i} L$.
Claim: B_i is a 2-sided ideal.
Proof of Claim:

$$B_i R \underset{4.2}{=} B_i B_i \subset R B_i \underset{B_i \text{ is a left ideal}}{=} B_i$$

Thus the claim is proven.
Claim: We have a 'block decomposition of R', meaning,
Proof of Claim:

$$_R R_R = B_1 \oplus \cdots \oplus B_s$$

Subclaim: $B_i \cap \sum_{j \neq i} B_j = 0$
Proof of Subclaim: Every $r \in R$, we have that $r \in L$ where L is simple. $L \subset B_i \implies L \cong L_i$. $L \subset \sum_{j \neq i} B_j \implies L \cong B_j$ for some $j \neq i$ which is not possible.
Now, we go back to the main proof.
We can write $1 = e_1 + \cdots + e_s$.
Then, $R_i \coloneqq (B_i, e_i)$ is a ring!
We have $R \cong (R_1, e_1) \times \cdots \times (R_s, e_s)$, so we're done.
The other direction is an exercise.

□

Friday, 9/13/2024

Key idea:

$$_R R = L^n \implies \operatorname{End}_R R \cong M_n(\operatorname{End}_R L)$$

Note that $R^{op} \cong \operatorname{End}_R R$ [function composition is written in the opposite direction].
Suppose $L_1, \cdots, L_s$ are non-isomorphic simple R-ideals.
L simple $\implies L \cong L_i$.
Define $B = \sum_{\text{simple } L \cong L_i} L \triangleleft_R R_R$.
We can prove that it is a two sided ideals.
Then we can write $R \cong R_1 \times \cdots R_s$ simple, where
$R_i = (B_i, e_i)$ [e_i is the identity in B_i].

Theorem 23 (4.4)**.** Suppose E is a R-module.

$$E_i \coloneqq \sum_{\substack{\text{simple } M \triangleleft E \\ M \cong L_i}} M$$

Then, $E = \bigoplus_{i=1}^s E_i$
$E_i = e_i E = B_i M$.

Corollary 24 (4.5)**.** If R is semisimple, M a simple R-module, then $M \cong L_i$ for some i.

Corollary 25 (4.6)**.** All simple modules of a simple ring are isomorphic.

$$M \cong \oplus L$$

External Product vs. Internal Product

Definition (External Product)**.** If we have [finite] rings $R_1, \cdots, R_s$ we can construct the ring:

$$R_1 \times R_2 \times \cdots \times R_s$$

Definition (Internal Product)**.** 'Block Decomposition': If we have a ring R and we can write it as sum of 2 sided ideals:

$${}_RR_R = B_1 \oplus \cdots \oplus B_s$$

Then we have $e_j \in B_j$ so that:

$$1 = e_1 + \cdots + e_s$$

Then, each B_j has a ring structure with e_j as identity. Then,

$$R \cong (B_1, e_1) \times \cdots \times (B_s, e_s)$$

Just for clarity:

Definition (Direct Sum of Ideals)**.**

$${}_RR_R = B_1 \oplus \cdots \oplus B_s$$

If and only if for every $r \in R$,

$$r = b_1 + \cdots + b_s$$

where $b_j \in B_j$ and the expression is unique.

<u>Jim's Rant</u>: A subring has to have the same identity. So, (B_j, e_j) is <u>not a subring</u>.
Block Decomposition is <u>not a direct sum of rings</u>!
This is because in category theory, sum refers to the co-product.

Lemma 26. Let k be a field, and let D be a skew-field which is a k-algebra such that $\dim_k D < \infty$. Then,

a) $\forall \alpha \in D$ we have $k[\alpha]$ is a field.

b) k algebraically closed $\implies D = k$.

Example. If $k \in \mathbb{R}, D = \mathbb{H}, \alpha \in \mathbb{H} - \mathbb{R}$ then $k[\alpha] \cong \mathbb{C}$.

It is not completely obvious since $k[i + j] \cong \mathbb{C}$ as well.

Proof. a) D is a k-algebra. Therefore, $k[\alpha]$ is commutative. We just need to find inverse.

Let $0 \neq \beta \in k[\alpha]$. It is enough to prove that for $\beta \in k[\alpha]$, multiplication map $\cdot\beta : k[\alpha] \to k[\alpha]$ is bijective.

$\cdot\beta$ is a finite dimensional linear transformation so those are true.

b) For all $\alpha \in D$ we have: $k[\alpha] = k$ since k is closed. So, $\alpha \in K$. Thus $D = k$. □

Corollary 27. Suppose G is finite. Then,

$$\mathbb{C}G \cong \prod_{i=1}^{s} M_{n_i}(\mathbb{C})$$

Proof. Artin-Wedderburn Theorem plus the previous lemma. □

Example. Suppose $C_n = \langle g \rangle$ cyclic and $\zeta_n = e^{2\pi i/n}$. Then,
$\mathbb{Q}C_2 \cong \mathbb{Q}_+ \times \mathbb{Q}_-$ where $g \mapsto (1, -1)$.
If p is prime we can write:
$\mathbb{Q}(C_p) \cong \mathbb{Q} \times \mathbb{Q}(\zeta_p)$ where $g \mapsto (1, \zeta_p)$.
$\mathbb{C}[C_n] \cong \mathbb{C}^n$ where:
$g \mapsto (1, \zeta_n, \cdots, \zeta_n^{n-1})$
$\mathbb{Q}[C_2 \times C_2] \cong \mathbb{Q}^4$ where:

$$(1, g) \mapsto (1, 1, -1, -1)$$

$$(g, 1) \mapsto (1, -1, 1, -1)$$

$\mathbb{R}[Q_8] \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{H}$ where $\mathbb{R}[Q_8] \rightarrowtail \mathbb{R}[C_2 \times C_2]$
Some other examples: $\mathbb{Q}[C_n], \mathbb{C}[Q_8], \mathbb{Q}[D_{2n}], \mathbb{R}[D_{2n}], \mathbb{C}[D_{2n}]$

Representation Theory

Here, G is a finite group and k is a field.

Representations	Modules over kG	Characters
$\rho : G \to GL(V)$ where V is a finite dimensional vector space	V is a kG module	$\chi : G \to k, \chi_\rho(g) \operatorname{Tr} \rho(g)$

Table 1: Representations, Modules and Characters

Monday, 9/16/2024

We have:

$$\text{representation} \iff \text{modules over } kG \implies [\impliedby \text{ only if } \operatorname{char} k = 0] \text{ characters.}$$

rep $\to$ kG-module
$\rho \mapsto V_\rho$ by $(\sum_g a_g g)v \coloneqq \sum_g a_g \rho(g) v$
$\rho_v \leftarrow V$
$\rho_V(g)v \coloneqq gv$
Recall the definition of character:
We have the trace map:

$$\operatorname{Tr} : M_n k \to k$$

Where $\operatorname{Tr}(a_{ij}) = \sum_j a_{jj}$ [or the sum of eigenvalues]
We have $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ which implies $\operatorname{Tr}(PAP^{-1}) = \operatorname{Tr}(A)$.
So, Tr is basis independent. Thus,

$$\operatorname{Tr} : \operatorname{End}_k V \to k$$

Definition (character)**.** Trace is an endomorphism map. This gives us:

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[rr, bend right, "\chi_p"'] & GL(V) \arrow[r, "\operatorname{Tr}"] & k \end{tikzcd}$$

This is called the <u>character of p</u>

There's a correspondence between kG modules and Representations concepts:

Repesentations	Modules over kG
irreducible	simple
	isomorphism
	direct sum
	Hom
	dual
	tensor product

Table 2: Rep and kG-mod

Irreducible vs Simple
We say irreducible representation, when we on the other hand say simple modules. Same concept!
Isomorphism
Suppose we have two representations:
$\rho : G \to GL(V)$
$\rho' : G \to GL(V')$
We say two representations are isomorphic when:

$$\rho \cong \rho' \overset{def}{\iff} V_\rho \overset{\phi}{\cong} V_\rho \overset{\phi}{\cong} V'_{\rho'} \underset{\phi(gv)=g\phi(v)}{\iff} \exists k \text{ isomorphism s.t.}$$

$\phi : V \to V'$ s.t. $\forall g \in G$ we have the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ V' & \xrightarrow{\rho'(g)} & V' \end{array}$$

ϕ is called the intertwining map.

Corollary 28. $\rho \cong \rho' \implies \chi_\rho = \chi_{\rho'}$

Direct Sum
Suppose $V \oplus W$ is a kG-module.

$$\rho_{V\oplus W} : G \to GL(V \oplus W)$$

is given by:

$$\rho_{V\oplus W} = \begin{bmatrix} \rho_V & 0 \\ 0 & \rho_W \end{bmatrix}$$

We also have $\chi_{V\oplus W} = \chi_V + \chi_W$.
Two Representations

Definition (Trivial Representations)**.**

$$\rho : G \to GL(k)$$

$$g \mapsto 1$$

Is the trivial representation. Also, $\chi_\rho \equiv 1$.

Definition (Regular Representation)**.** Consider the kG-module ${}_{kG}kG$. We have:

$$\rho_{kG} : G \to GL(kG)$$

This is injective.

Note that $G \curvearrowright G$ by multiplication, this is a free action. For finite group G with $|G| = n$,
$G \rightarrowtail$ Bijection(G, G) so G is a subgroup of S_n. So we have:

$$\begin{tikzcd} G \arrow[r] \arrow[rr, bend left, dotted, "\text{regular rep.}"] & S_n \arrow[r] & GL(k^n) \end{tikzcd}$$

With the action of 'permuting the standard basis'.
Exercise: Compute character of Regular Representation.
We have, in line of the previous theorem:

Theorem 29 (Maschke's Theorem)**.** If $V \subset W$ as kG-modules and char $k \nmid |G|$ then $\exists V'$ such that $W = V \oplus V'$

Proof. First, find a k-linear map $\pi : W \to V$ such that $\pi(v) = v$ for all $v \in V$.
We average it to make it kG-linear:
$\pi' : W \to V$ given by:

$$\pi'(w) := \frac{\sum_g g\pi(g^{-1}w)}{|G|}$$

We have: π' is kG-linear and $\pi'(v) = v$
We can take $V' := \ker \pi$ □

Thus, for $w \in W$ we can write $w = \pi'(w) + (w - \pi'(w))$.
Note that Maschke's theorem implies kG is semisimple. Artin Wedderburn implies semisimple kG module is a direct sum of irreducible modules.

$$V \cong \bigoplus_i n_i V_i$$

$$\chi_V = \sum_i n_i \chi_i$$

Homomorphisms:
Suppose V, W are kG-modules, "representations". Then,
$\mathrm{Hom}_{kG}(V, W)$ is a k-vector space.
$\mathrm{Hom}_k(V, W)$ is a kG-module.
we define: $(gf)v := gf(g^{-1}v)$
i.e. $((\sum_g a_g g)f)v = \sum_g a_g(gf(g^{-1}v))$
The g^{-1} inside is needed for associativity: $(g'g)f = g'(gf)$
Officially this is a functor.
$\mathrm{Hom}_k(-,-) : (kG\text{-mod})^{op} \times kG\text{-mod} \to kG\text{-mod}$
Special case:
Dual Representation: $W = k$. Then,
$V^* = \mathrm{Hom}_k(V, k)$.
So, $(gf)(v) = gf(g^{-1}v) = f(g^{-1}v)$
Exercise: $\chi_{V^*} =?$

Wednesday, 9/18/2024

Tensor Products

Motivation:
Product Structure: $- \otimes -$: kG-mod $\times kG$-mod $\to kG$-mod given by $V \otimes_k W$.
Group action works diagonally, $g(x \otimes y) = (gx) \otimes (gy)$, extended linearly.
Extension of scalars:

$$\mathbb{R}G \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}G$$

Product of Groups: $k[G \times H] = kG \otimes_k kH$

When for k a field then modules are vector spaces k^m and k^n which are easy:

$$k^n \otimes_k k^m = k^{nm}$$

$$\dim(k^n \otimes_k k^m) = mn$$

$\{e_i\}$ a basis for k^n
$\{f_j\}$ a basis for k^m
Then $\{e_i \otimes f_j\}$ is a basis for $k^n \otimes k^m$.
However, tensor product consists of more than 'pure' tensors.

Definition (Tensor Product)**.** Let R be a commutative ring. Tensor product is a functor:

$$- \otimes_R - : R - \mathrm{mod} \times R - \mathrm{mod} \to R - \mathrm{mod}$$

$$(A, B) \mapsto A \otimes_R B$$

[Functor meaning if we have homomorphism on the left we will have homomorphisms on the right]
Construction:
Let $F(A \times B)$ be the free R-module with basis $A \times B$. Then a typical element of the basis is $(a, b) \in A \times B$.
Let S be the sub-module generated by the following:

1) $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$
2) $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$
3) $r(a, b) - (ra, b)$
4) $r(a, b) - (a, rb)$

Then, we define:

$$A \otimes_R B := \frac{F(A \times B)}{S}$$

and write $a \otimes b$ for the image of (a, b).
This means, a typical element of $A \otimes_R B$ is:

$$\sum_{i=1}^{n} a_i \otimes b_i \in A \otimes_R B$$

We also have the following relations:
$(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \times b$
$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$
$r(a \otimes b) = (a \otimes rb) = (ra \otimes b)$

Exercise. $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/3 = 0$

Proposition 30. Suppose A, B, M are R-modules, and

$$\phi : A \times B \to M \text{ is } R\text{-billinear}$$

Meaning,

1) $\phi(a_1 + a_2, b) = \phi(a_1, b) + \phi(a_2, b)$
2) $\phi(a, b_1 + b_2) = \phi(a, b_1) + \phi(a, b_2)$
3) $r\phi(a, b) = \phi(ra, b) = \phi(a, rb)$

Then, by definition,

$$\pi : A \times B \to A \otimes_R B$$

is R-bilinear.

Proposition 31 (Universal Property of Tensor Product)**.** π is initial in the category of bilinear maps with domain $A \times B$. Meaning, every bilinear map from $A \times B$ factors through π.

$$\begin{tikzcd} A \times B \arrow[r, "\forall\phi \text{ bilinear}"] \arrow[d, "\pi"] & M \\ A \otimes_R B \arrow[ur, dotted, "\exists!\overline{\phi}"'] & \end{tikzcd}$$

This diagram commutes

Proof. For uniqueness, note that, $\overline{\phi}(a \otimes b) = \overline{\phi}(\pi(a, b)) = \phi(a, b)$
For existence, define $\hat{\phi}(a, b) = \phi(a, b)$ where $\hat{\phi} : F(A \times B) \to M$. Then $\overline{\hat{\phi}}(S) = 0$ so $\overline{\phi} : A \otimes_R B \to M$ exists. □

Proposition 32 (Rephrasing Universal Property in Terms of Adjoint Functors)**.**

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$$

Proof.

$$f \mapsto (a \mapsto (b \mapsto f(a \otimes b)))$$

$$(a \otimes b \mapsto g(a)b) \leftarrow g$$

$$\begin{tikzcd} R\text{-mod} \arrow[r, bend right, "\mathrm{Hom}(A, \mathrm{Hom}(-, C))"'] & R\text{-mod} \arrow[l, bend right, "\mathrm{Hom}(A \otimes -, C)"'] \end{tikzcd}$$

□

Proposition 33. 1) Commutative $A \otimes_R B \cong B \otimes_R A$

2) Identity $R \otimes_R B \cong B$

3) Assocative $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$

4) Distributive $(\bigoplus_\alpha A_\alpha) \otimes B \cong \bigoplus_\alpha (A_\alpha \otimes B)$

5) Functorial $\begin{pmatrix} f : A \to A' \\ g : B \to B' \end{pmatrix} \implies \quad f \otimes g : A \otimes B \to A' \otimes B'$

6) Exactness Short Exact Sequence $0 \to A \xrightarrow{f} B \to C \to 0 \implies$ Short Exact Sequence $0 \to A \otimes M \xrightarrow{f \otimes 1_M} B \otimes M \to C \otimes M \to 0$

7) Right Exactness M R-mod,$0 \to A \to B \to C \to 0 \implies$ Exact Sequence $A \otimes M \to B \otimes M \to C \otimes M \to 0$

Friday, 9/20/2024

Lang Section 2

Tensor Product of Representation
Suppose V, W are k-vector spaces, then we have $V \otimes_k W$ is also a k-vector space. But they all are kG-modules as well:

$$g(v \otimes w) = gv \otimes gw$$

Proposition 34. The character is multiplicative:

$$\chi_{v \otimes w} = \chi_v \chi_w$$

Proof. Let $\{e_i\}$ be a basis for V and $\{f_j\}$ a basis for w.
Suppose $ge_i = \sum_k a_{ki} e_k$
And $gf_j = \sum_l b_{lj} f_l$
Then, $g(e_i \times f_j) = ge_i \times gf_j = \sum_{k,l} e_{ki} b_{lj} e_k \times f_l$
Take $(k, l) = (i, j)$.
Then, $\chi_{v \times w}(g) = \sum_{i,j} a_{ii} b_{jj} = \chi_v(g) \chi_w(g)$ $\square$

Consider $f : G \to k$. We have:
$\{$1d chars$\} \subset \{$simple chars$\} \subset \{$chars$\} \subset \{$virtual chars$\} \subset \{$class functions$\}$
We explain these later.

Definition. f is a character if $\exists \rho : G \to GL_k(V)$ such that $f = \chi_\rho = \operatorname{Tr} \circ \rho$

Definition. f is a class function if $\forall g, h \in G$ we have $f(hgh^{-1}) = f(g)$

Definition. f is a virtual character if $\exists \rho, \rho'$ such that $f = \chi_\rho - \chi_{\rho'}$

Definition. f is simple (=irreducible) character if $f = \chi_V$ where V is a simple kG-module.

Definition. f is 1-dimensional character if $f : G \to k^\times$ is a homomorphism. eg trivial character $\chi_1(g) \equiv 1$.

Proposition 35. Class Functions are k-algebras. Virtual characters are a commutative ring.

Now, suppose $\operatorname{char} k \nmid |G|$. Then,

$$kG \cong M_{n_1}(D_1) \times \cdots \times M_{n_s}(D_s)$$

Assume $M_{n_1}(D_{n_1}) = k$. Then we have the trivial representation: $ga = a$.
If $L_i = D_i^{n_i}$ is a simple kG-module, then
$\chi_i = \chi_{L_i}$ is a simple characteristics.
We have $1 = e_1 + \cdots + e_s$ [central non-trivial idempotents].
$\chi_i(e) = \operatorname{Tr}(\operatorname{Id}_{L_i}) = \dim_k L_i = n_i \dim_k D_i$.

Example. Consider $Q_8 \hookrightarrow \mathbb{H}^\times$. Then,

$$\chi_{\mathbb{H}}(e) = 4$$

Now, consider ${}_{kG}kG \cong \bigoplus_i n_i L_i$, the 'regular representation'. $e_j L_i = 0$ for $i \neq j$. Then,

$$\chi_i(e_i) = \chi_i(1) = \chi_i(e) = \dim_k L_i$$

So, char $\chi : G \to k$ extends to $\chi : kG \to k$ by $\sum a_g g \mapsto \sum a_g \chi(g)$.
If V is a finitely generated kG-module, we have

$$V \cong m_1 L_1 \oplus \cdots \oplus m_s L_s$$

where $m_i \geq 0$.

Theorem 36 (2.2, 2.3)**.** $\chi_v = \sum_i m_i \chi_i : G \to k$ with m_i uniquely determined if $\operatorname{char} k = 0$.

Theorem 37 (2.3)**.** Characters Determine Representations: suppose $\operatorname{char} k = 0$. Then,

$$V \cong V' \iff \chi_V = \chi_{V'}$$

Proof. $\Longrightarrow$: Trace is independent of basis, so this is easy.
$\Longleftarrow$: We already gave a proof using projection operators. Second Proof:
Assume $\chi_V = \chi_{V'}$. We decompose:

$$V \cong \oplus m_i L_i, V' \cong m_i' L_i$$

Note that we have $\chi_V(e_i) = m_i \dim_k L_i = m_i' \dim_k L_i = \chi_{V'}(e_i)$
Thus we must have $m_i = m_i'$. □

Representation Ring

$R_k(G) = (\text{virtual char}, +, \times) \cong (\text{virtual rep}, \oplus, \otimes)$.
Example: $R_{\mathbb{Q}}[C_2] \cong \frac{\mathbb{Z}[X]}{(X^2-1)}$

Monday, 9/23/2024

Dual Characters

Consider $\rho : G \to GL_k(V)$
Dual $V^* = \operatorname{Hom}_k(V, k)$ is also a representation.

$$(g\phi)(v) = \phi(g^{-1}v)$$

Inverse because we want it to be a left module.
Claim: $\rho : G \to GL(V) \to \rho^* : G \to GL(V^*)$
$\rho^*(g) = (\rho(g)^{-1})^T$

Proof. $\rho^*(g) = (\rho(g^{-1}))^* = \rho(g^{-1})^T$ □

Corollary 38. a) $\chi_{V^*}(g) = \chi_v(g^{-1})$

b) $\chi_{\operatorname{Hom}(V,W)}(g) = \chi_V(g^{-1})\chi_W(g)$

Proof. a follows from the claim.
b: Consider the slant homomorphism:

$$V^* \otimes W \to \operatorname{Hom}(V, W)$$

$$\sum_i \phi_i \otimes w_i \mapsto \left(v \mapsto \sum_i \phi_i(v) w_i \right)$$

It is an isomorphism since V, W are both finite dimensional.

$$\chi_{\operatorname{Hom}(V,W)}(g) = \chi_{V^* \otimes W}(g) = \chi_{V^*}(g)\chi_W(g) = \chi_V(g^{-1})\chi_W(g)$$

□

1 Dimensional Characters

Definition. 1 D representation is a homomorphism $\rho : G \to k^\times$

$$\begin{tikzcd} G \arrow[rr] \arrow[dr] & & k^\times \\ & G^{ab} \arrow[ur] & \end{tikzcd}$$

Question: What are the 1d representations for D_6?
$D_6 \cong \mathbb{Z}/3 \rtimes \mathbb{Z}/2$
So, $D_6^{ab} \cong \mathbb{Z}/2$
So, we have k_T, k_-
$r \mapsto 1$
$s \mapsto -1$
Exercise: Trivial Representation / Idempotent

$$e_T = \frac{\sum_{g \in G} g}{|G|} \in kG$$

$$e_T^2 = e_T$$

$$g e_T = e_T = e_T g$$

$$e_T \in Z(kG)$$

$$kG = (kG)e_T \oplus (kG)(1 - e_T)$$

$$kG \cong k \times \frac{kG}{\langle e_T \rangle}$$

Lemma 39 (2)**.** Any finite subgroup of $k^\times$ is cyclic.

Proof. Key Fact: $x^e - 1 \in k[x]$ has at most e roots [proof: long division].
Note: $x^2 - 1 \in \mathbb{Z}/8[x]$ has 4 roots. This implies $\mathbb{Z}/8$ is not a field.
Consider finite abelian $A < k^\times$
Consider $e = \text{exponent } A = \inf\{m \geq 1 \mid \forall a \in A, a^m = e\}$
Then, $\forall a \in A, a^e - 1 = 0$. From the key fact, $|A| \leq e \leq |A|$
Thus, $e = |A|$ □

Corollary 40. $\forall \text{hom} \quad \rho : G \to k^\times, \exists$ Cyclic C such that:

$$\begin{tikzcd} G \arrow[rr, "\rho"] \arrow[dr, two heads] & & k^\times \\ & C \arrow[ur, tail] & \end{tikzcd}$$

Recall only finite subgroup of $\mathbb{Q}$ is ± 1.
$1 - d$ $\mathbb{Q}$ reps of $G \leftrightarrow$ trivial representation + index 2 subgroups
Now we suppose k is algebraically closed, eg $k = \mathbb{C}$. Then,

$$kG \cong \prod_i M_{n_i}(k)$$

If G is abelian, then,

$$kG \cong k \times \cdots \times k$$

Corollary 41 (3)**.** k is algebraically closed and G is abelian $\iff$ all irreducible representations are 1-dimensional.

Corollary 42. Let $|G| = n, k = \mathbb{C}$.

a) $\forall V, \chi_V(G) \subset \mathbb{Q}(\zeta_n)$

b) $\forall V, \chi_{V^*}(g) = \overline{\chi_V(g)}$

c) $\forall V, W, \chi_{\mathrm{Hom}(V,W)}(g) = \overline{\chi_V(g)}\chi_W(g)$

Proof. a) True for 1d representation from the lemma.
$\implies$ True for G abelian (corollary 3)
$\implies$ True for cyclic G
$\implies$ always true: $g \in G \implies \langle g \rangle$ cyclic.

$$\chi_\rho(g) = \chi_{\rho|_{\langle g \rangle}}(g)$$

Then, $\rho : G \to GL(V)$, consider $g \in G$.
Then $\rho(g)^n = I \implies \mathrm{Tr}(\rho_V(g)) \in \mathbb{Q}(\zeta_n)$.

b) Same as (a).
$\rho^*(g) = (\rho(g)^{-1})^t$
For 1-dim, $\rho^* = \overline{\rho}$.

c) $\chi_{\mathrm{Hom}(V,W)}(g) = \chi_V(g^{-1})\chi_W(g) = \overline{\chi_V(g)}\chi_W(g)$

□

Two Bases for center kG

Definition. $g \in G$ is conjugate to $\sigma \in G$ if $\exists \tau$ such that,

$$\tau g \tau^{-1} = \sigma$$

Write $g \sim \sigma$

$G = \coprod_{G/\sim} [g]$
$[g] = \{\sigma \in G \mid g \sim \sigma\}$ conjugacy classes

Proposition 43. $\{\sum_{\sigma \in [G]} \sigma\}_{[g] \in G/\sim}$ is a k-basis for center of kG.

Proof. Clearly these are linearly independent.
$\alpha = \sum_{\sigma \in G} a_\sigma \sigma \in$ center
$\iff \tau\alpha = \alpha\tau \iff \tau\alpha\tau^{-1} = \alpha$
$\sigma a_\sigma \tau\sigma\tau^{-1} = \sum a_\sigma \sigma \implies (g \sim \sigma \implies a_g = a_\sigma)$ □

Wednesday, 9/25/2024

Lang XVIII, 4
Two bases for $Z(kG)$
conjugacy classes
primitive cental idempotents [k algebraically closed]

Exercise. $G \twoheadrightarrow Q$, prove that $kG \cong kQ \times R$

Proposition 44 (4.1)**.** Suppose $\{\sum_{\sigma \in [g]}\}_{[g] \in G/\sim}$ form a $\{{k \atop \mathbb{Z}}\}$-basis for $\{{Z(kG) \atop Z(\mathbb{Z}G)}\}$

Consider a ring R.

Definition. $e \in R$ is a primitive central idempotent if:
e is a central idempotent $[e^2 = e, e \in Z(R)]$
$e = e' + e''$ with e', e'' central idempotent $\implies \{e', e''\} = \{0, e\}$

Then, $kG \ni 1 = e_1 + \cdots + e_s, kG \cong \prod M_{d_i}(D_i)$
$e_i \to (0, \cdots, 0, 1, 0, \cdots, 0)$
Now suppose $n = |G|$
We have irreducible representations $L_1, \cdots, L_s$ and degrees $d_1, \cdots, d_s$ then $L_i \cong D_i^{d_i}$. We have irreducible characteristics $\chi_1, \cdots, \chi_s$ and primitive central idempotents (p.c.i.) $e_1, \cdots, e_s$
Facts: $(*)$: ${}_{kG}kG = \bigoplus_i d_i L_i$
$(**)$: $\alpha \in kG, i \neq j$ then $\chi_j(e_i\alpha) = 0$ since $e_i L_j = 0, \chi_i(e_i\alpha) = \chi_i(1\alpha) = \chi_i(\alpha)$
We have: $\chi_{\text{reg}} = \sum_i d_i \chi_i$

Proposition 45 (4.3)**.** $\chi_{\text{reg}}(g) = \begin{cases} n, & \text{if } g = e; \\ 0, & \text{if } g \neq e \end{cases}$

Proof. $\chi_{\text{reg}}(g) = \text{Tr}(\cdot g : kG \to kG)$
Thus, $\chi_{\text{reg}}(e) = \text{Tr}(I) = n$
If $g \neq e$ note that G has $\{\sigma_1, \cdots, \sigma_n\}$ and $\rho_{\text{reg}}(g)(\sigma_j) = g\sigma_j \neq \sigma_j$ for all j. So, there is nothing in the diagonal matrix and trace is 0. $\square$

Motivation for k algebraically closed:
Consider $\mathbb{Q}C_3 \cong \mathbb{Q} \times \mathbb{Q}(\zeta_3)$. We only have primitive central idempotents, $1 = e_1 + e_2$.
But the center has dimension 3: $\dim_{\mathbb{Q}}(Z(\mathbb{Q}C_3)) = 3$.
Assume k is algebraically closed.
Claim: k algebriacally closed, D skew field, $k < Z(D)$, $\dim_k D < \infty$ implies $k = D$
Now, $kG \neq \prod M_{d_i}(k)$
Consider primitimve central idempotents $e_1, \cdots, e_s$ for a basis.
$n = \sum_{i=1}^s d_i^2$
e.g. $S_3 = D_6$. $s = ?$ $d_1, d_2, d_3 = ?$
We have represantatives of conjugacy classes: $(1), (12), (123)$.
$s = 3, 6 = 1^2 + 1^2 + 2^2$
Char. Table:

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Table 3: characteristic table

We have $\mathbb{C}S_3 = \mathbb{C}_+ \times \mathbb{C}_- \times M_2\mathbb{C}$
Our representatives are $(1), (12), (123), (1234), (12)(34)$
$d_i = 1, 1, 2, 3, 3$
Goal: Express the p.c.i basis in terms of conjugacy class basis.

Corollary 46 (4.2)**.** If k is algebraically closed,
the number of conjugacy classes $= \dim_k Z(G) =$ number of irreducible representation $= s$

Proposition 47 (4.4)**.** k algebraically closed, then

$$e_i = \frac{d_i}{n} \sum_{\tau \in G} \chi_i(\tau^{-1})\tau$$

Proof. Let $e_i = \sum_{\tau \in G} a_\tau \tau$.
We compute $\chi_{\text{reg}}(e_i\tau^{-1})$ in two ways.
1: $\chi_{\text{reg}}(e_i\tau^{-1}) = \chi_{\text{reg}}(\sum a_\sigma \sigma\tau^{-1}) = \sum a_\sigma \chi_{\text{reg}}(\sigma\tau^{-1}) = a_\tau n$
2: $\chi_{\text{reg}}(e_i\tau^{-1}) \overset{(*)}{=} \sum_j d_j\chi_j(e_i\tau^{-1}) \overset{(**)}{=} d_i\chi_i(e_i\tau^{-1}) = d_i\chi_i(\tau^{-1})$
Thus, $a_\tau n = d_i\chi_i(\tau^{-1}) \implies a_\tau = \frac{d_i}{n}\chi_i(\tau^{-1})$ $\square$

Recall that $\exp G$ is the smallest positive integer m such that $g^m = \text{id}$ for all g.

Corollary 48 (4.5)**.** Let $m = \exp G$. Then,

$$e_i \in \frac{1}{n}\left[\mathbb{Z}[\zeta_m]G\right] \subset \frac{1}{n}\left[\mathbb{Z}[\zeta_n]G\right]$$

Corollary 49 (4.6)**.** $\operatorname{char} k \nmid d_i$

Proof. If not, $\operatorname{char} k \mid d_i$ then $e_i = 0$ which is a contradiction. □

Corollary 50 (4.7)**.** $\chi_1, \cdots, \chi_s$ are linearly independent over k. In fact they form a basis for the class functions $f : G \to k$.

Proof. Suppose $0 = \sum a_i \chi_i$.
Then $0 = \sum a_i \chi_i(e_j) = a_j \chi_j(e_j) = a_j d_j \implies a_j = 0$ □

Then $\dim_k$(class functions) = number of conjugacy classes = s.

Friday, 9/27/2024

Review:

$$e_i = \frac{d_i}{n} \sum_{\sigma \in G} \chi_i(\sigma)\sigma^{-1} \in kG \quad (*)$$

Is a primitive central idempotent.

$$\chi_{\text{reg}} = \chi_{kG} = \sum_i d_i \chi_i$$

$\sigma = 1, n = \sum_i d_i^2$
$d_i \mid n$

$$\sum_{\sigma \in G} \chi_i(\sigma)\chi_j(\sigma^{-1}) = n\delta_{ij}$$

$$\sum_{i=1}^{s} \chi_i(\sigma)\chi_i(\tau^{-1}) = \begin{cases} \frac{n}{|\sigma|}, & \text{if } \tau = \sigma; \\ 0,` & \text{otherwise.} \end{cases}$$

If $G = S_3$ then:

	(1)	(12)	(123)	
χ_1	1	1	1	6
χ_2	1	-1	1	6
χ_3	2	0	-1	6
	6	2	3	

Table 4: Characeristic Table of S_3

$0 = \chi_{\text{reg}}(123) = 1\chi_1(123) + 1\chi_2(123) + 2\chi_3(123)$
$k = \mathbb{C}, \chi(\sigma^{-1}) = \overline{\chi(\sigma)}$
End of review
$X(G) = \{\text{class functions } f : G \to k\}$ so that $f(\tau\sigma\tau^{-1}) = f(\sigma)$.

Definition (Perfect Pairing)**.** A perfect pairing of k vector space is a k-bilinear map $\beta : V \times W \to k$ such that $\exists$ basis $\{v_i\}, \{w_j\}$ such that

$$\beta(v_i, w_j) = \delta_{ij}$$

$$\iff \operatorname{Ad}_b : V \to W^*$$

$$v \mapsto (w \mapsto \beta(v, w))$$

Theorem 51 (4.9)**.**
$$X(G) \times Z(kG) \to k$$

$$(f, \alpha) \mapsto f(\alpha)$$

is a perfect pairing.

Proof. Dual basis: $\left\{\frac{1}{d_i}\chi_i\right\}, \{e_j\}$
$$\frac{1}{d_i}\chi_i(e_j) = \delta_{ij}$$
$\square$

Corollary 52 (4.8)**.** Suppose k is algebraically closed, $\operatorname{char} k = 0$. Then $d_i = \dim_K L_i \mid n$

We need integrality theory (M502)
See Lang p 334.
A subring of B, $\alpha \in B$.
α is integral over A if $\exists$ monic $f(x) \in A[x]$ such that $f(\alpha) = 0$.
$\alpha \in \mathbb{Q} \implies \alpha \text{ int}/\mathbb{Z} \iff \alpha \in \mathbb{Z}$
Condition $(**)$: α being integral is equivalent to the existence of a faithful $A[\alpha]$-module M which is finitely generated as A-module.
Faithful means: $\forall \beta \in A[\alpha], \beta M = 0 \iff \beta = 0$.
In other words, $A[\alpha] \hookrightarrow \operatorname{End}_{A[\alpha]}(M)$.
Condition $(**) \iff \alpha \text{ int}/A$. This is proved by a determinant trick.
Applying $(**)$ on $A = \mathbb{Z}$, $\frac{n}{d_i} \in \mathbb{Q}$,
Multiplying $e_i = \frac{d_i}{n}\sum_{\sigma \in G}\chi_i(\sigma)\sigma^{-1} \in kG$ with e_i,
$$e_i = e_i^2 = \frac{d_i}{n}\sum_\sigma \chi_i(\sigma)\sigma^{-1}e_i$$
$$\frac{n}{d_i}e_i = \sum_\sigma \chi_i(\sigma)\sigma^{-1}e_i$$
$$M = \mathbb{Z}\langle \zeta_n^j \sigma e_i\rangle_{j, \sigma \in G} \text{ is a } \mathbb{Z}\left[\frac{n}{d_i}\right]\text{-module}$$
We are done by $(**)$. $d_i \mid n$.

Orthogonality, Lang XVIII, 5, Serre 2.3

Theorem 53. Suppose we have $\langle , \rangle : X(G) \times X(G) \to k$ by:
$$\langle f, g\rangle = \frac{1}{n}\sum_{\sigma \in G} f(\sigma)g(\sigma^{-1})$$
is a nonsingular symmetric form and $\{\chi_1, \cdots, \chi_s\}$ forms an orthonormal basis.

Proof. Symmetric form, k-bilinear $\langle f, g\rangle = \langle g, f\rangle$
Apply χ_j to $(*)$
$$d_i\delta_{ij} = \chi_j(e_i) = \frac{d_i}{n}\sum_\sigma \chi_i(\sigma)\chi_j(\sigma^{-1})$$
$\square$

Remark: Irreducibility criterion: $\langle \chi, \chi\rangle = 1 \iff \chi$ irreducible.
$(\sum_i a_i\chi_i, \sum_i a_i\chi_i) = \sum_i a_i^2$

Proposition 54 (I.7, Serre p20)**.** a) $\sum_{i=1}^s \chi_i(\sigma)\chi_i(\sigma^{-1}) = \frac{n}{|[\sigma]|}$

b) $[\sigma] \neq [\tau] \implies \sum_{i=1}^s \chi_i(\sigma)\chi_i(\tau^{-1}) = 0$

Proof. Consier the characteristic function for $[\sigma]$:
$f_\sigma = 1$ on $[\sigma]$ and 0 everywhere else.
$f_\sigma = \sum_i \lambda_i \chi_i$.
$\lambda_j = \langle f_\sigma, \chi_j \rangle = \frac{1}{n} \sum_{\tau \in G} f_\sigma(\tau)\chi_j(\tau^{-1}) = \frac{|[\sigma]|}{n}\chi_j(\sigma^{-1})$
$f_\sigma(-) = \sum_i \frac{|[\sigma]|}{n}\chi_i(\sigma^{-1})\chi_i(-)$ □

This finishes the proof.

Monday, 9/30/2024

Serre Ch 4

What about representations of infinite groups?

$$\begin{tikzcd}
 & G \arrow[dl] \arrow[d] \\
\text{compact groups } S^1 \subset \mathbb{C}^\times & \text{discrete groups } \mathbb{Z}(\cong C_\infty) \arrow[dl] \arrow[d] \\
C^*G \ C^*\text{-algebras} & \mathcal{N}(G) \text{ von Neumann Algebra}
\end{tikzcd}$$

Definition (Topological Group)**.** Topological Group is a group $(G, \cdot)$ such that G has a topology so that:

$$G \times G \to G$$

$$(g, h) \mapsto gh^{-1}$$

is continuous.

Definition (Lie Group)**.** Lie Group is a topological lie group G where G is a smooth manifold and $(g, h) \mapsto gh^{-1}$ is smooth.

Compact Lie Groups:
Torus $T^r = S^1 \times \cdots \times S^1$
$O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = I\}$
$U(n) = \{A \in M_n(\mathbb{C}) \mid AA^* = I\}$
Exceptional: G_2, F_4, E_6, E_7, E_8
We also have compact groups are not lie groups¿
$(\mathbb{Z}/p)^\infty = \prod \mathbb{Z}/p\mathbb{Z}$
p-adic $\mathbb{Z}_p = \lim \mathbb{Z}/p^n\mathbb{Z}$
Serre Ch 4 says that:

Representation of compact groups is almost the same as finite group!

We need Haar Measure.

Proposition 55. For locally compact Hausdorff topological group G there exists a unique Haar Measure:

$$\begin{array}{rcl}
\mathrm{d}t : \{\text{Borel Subsets of } G\} & \to & [0,1] \\
B & \mapsto & \int_B \mathrm{d}t = \int_G \chi_B(t)\mathrm{d}t
\end{array}$$

So that $\int_G \mathrm{d}t = 1$ and $\mathrm{d}t$ is translation invariant:

$$\int_G f(t)\,\mathrm{d}t = \int_G f(gt)\,\mathrm{d}t = \int_G f(tg)\,\mathrm{d}t$$

Example. If G is finite:

$$\int_G f \,\mathrm{d}t = \frac{1}{|G|} \sum_{g \in G} f(g)$$

$G = S^1$

$$\int_{S^1} \mathrm{d}t = 1 \qquad \int_{\text{quarter circle}} \mathrm{d}t = \frac{1}{4}$$

Theorem 56 (Maschke's Theorem, Peter-Weyl Theorem)**.** Let G be a compact group, $k = \mathbb{C}$. Let $W \subset V$ be a subrepresentation of $\rho : G \to GL(V)$. Then $\exists$ subrepresentation W' such that $V = W \oplus W'$.

Proof. Let $\langle , \rangle' : V \times V \to \mathbb{C}$ be any inner product.
We define a new inner product by averaging this inner product.

$$\langle v, w \rangle = \int_G \langle \rho(t)v, \rho(t)w \rangle' \,\mathrm{d}t$$

This gives us a G-invariant inner product.
We take W' to be orthogonal to W w.r.t. this inner product. $\square$

Corollary 57. Any representation is the direct sum of irreducible representation (unique upto multiplicity).

Consider the regular representation $L^2(G) \cong$ "$\bigoplus_i$"$d_i L_i$.
We don't have characteristic of regular representation
We don't have a group ring
Suppose $G = S^1, n \in \mathbb{Z}$
$\chi_n : S^1 \to \mathbb{C}^\times$
$\chi_n(z) = z^n$ gives us $\mathbb{C}_n$
$L^2(S^1) =$ "$\oplus$"$\mathbb{C}_n$
Representation Ring: $R(S^1) \ni \rho - \rho'$
$R(S^1) = \mathbb{Z}[\chi_1, \chi_1^{-1}], \chi_n = \chi_1 \otimes_G \cdots \otimes_G \chi_1$
Then, $R(S^1 \times \cdots \times S^1) = \mathbb{Z}[\alpha_1, \alpha_1^{-1}, \cdots, \alpha_r, \alpha_r^{-1}]$ where:

$$\begin{tikzcd} S^1 \times \cdots \times S^1 \arrow[r, "\text{proj}"] \arrow[rr, bend right] & S^1 \arrow[r, hook] & \mathbb{C}^\times \end{tikzcd}$$

Consider $T^n \subset U(n)$
$\Sigma_n = U(n)/T^n$
$R(U(n)) \hookrightarrow R(T^n)$.
image $\mathbb{Z}[\sigma_1, \cdots, \sigma_{n-1}, \sigma_n, \sigma_n^{-1}]$ where
σ_i is the i-th elementary symmetric function in $\alpha_1, \cdots, \alpha_n$.

Infinite Discrete Groups

$C_\infty = \langle x \rangle$
$\mathbb{Z}C_\infty = \mathbb{Z}[x, x^{-1}]$ the Laurent Polynomial Ring.
We can think of it like the localization of $\mathbb{Z}[x]$ at x [aka $x^{-1}\mathbb{Z}[x]$] or $\mathbb{Z}[x, x^{-1}] \subset \mathbb{Q}(x)$ the rational function field.
This is not a super well behaved domain since it has dimension 2.
$\mathbb{Q}[x, x^{-1}]$ is a Euclidean domain and hence a PID. But not $\mathbb{Z}[x, x^{-1}]$.

Some Conjectures about Torsion-Free Groups

Torsion free: If $g \in G - \{e\}, n > 0$ then $g^n \neq e$.

Proposition 58 (Farrell-Jones Conjecture)**.** for $R = \mathbb{Z}$ or a field, all finitely generated projective $\mathbb{R}G$-modules are stably-free.

Projective means it's a summand of a free module.
P is stably free if $P \oplus$ free is free.
It has been proved for the torsion-free groups we care about, but not generally.

Proposition 59 (Kaplansky Idempotent Conjecture)**.** Suppose R is an integral domain. Then the only idempotents in RG are 0 and 1.

Proposition 60 (Zero Divisor Conjecture)**.** Suppose R is an integral domain. Then RG has no zero divisor.

Proposition 61 (Embedding Conjecture)**.** Suppose R is an integral domain. Then RG is a subring of a skew field.

We have Embedding Conjecture $\implies$ Zero Divisor Conjecture $\implies$ Kaplansky Idempotent Conjecture

Proposition 62 (Unit Conjecture)**.** Suppose k is a field. Then,

$$(kG)^\times = \langle k^\times, G \rangle$$

Wednesday, 10/2/2024

Serre Chapter 5
Examples
$k = \mathbb{C}$: Use characters.
5.1: $C_n = \langle r \rangle, \zeta_n = e^{2\pi i/n}$.
$n = \#$conjugacy classes $\implies n = s$ irreducible representations.
C_n is abelian $\implies$ all irreducible representation (=char) is one dimensional.

$$\chi : C_n \to \mathbb{C}^\times$$

$$\chi(r)^n = \chi(r^n) = \chi(e) = 1$$

Irreducible representation $\chi_h(r) = \zeta_n^h$. We have characters $\chi_0, \chi_1, \cdots, \chi_{n-1}$.
$\chi_h \chi_{h'} = \chi_{h+h' \pmod n}$
Representation Ring $\mathbb{Z}[\text{characters}] = \mathbb{Z}[\chi_1] \cong \mathbb{Z}[x]/(x^n - 1)$.
Trivial character is 1 in $R(G)$.

$$\begin{array}{rccc} \phi : & \mathbb{C}[C_n] & \to & \mathbb{C} \times \cdots \times \mathbb{C} \\ & r & \mapsto & (\rho^0, \rho^1, \cdots, \rho^{n-1}) \end{array}$$

$$\Phi : \mathbb{Q}[C_n] \to \prod_{d|n} \mathbb{Q}(\zeta_d)$$

a
Question: How to justify that ϕ and Φ are isomorhisms?
Answer: CRT
For a non-abelian group G, recall that:
\# of 1d rep $= |G^{ab}| = |G/[G, G]|$
\# of irreducible rep = \# of conjugacy classes.
Suppose $d_i = \dim_{\mathbb{C}} L_i$ then $n = d_1^2 + \cdots + d_s^2$ and $d_i \mid |G|$.
5.1 Dihedral Group D_{2n} (order $2n$)
Recal,

$$D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, srs = r^{-1} \rangle$$

isometries of a regular n-gon.
Here, $(srs^{-1})^k = sr^ks^{-1}$ so $sr^ks^{-1} = r^{-k}$. Also, $r^ksr^{-k} = r^{2k}s$.
Conjugacy classes are given by the following:

$$\begin{array}{cc} \{e\} & \{s\} \\ \{r, r^{-1}\} & \{r^2s\} \\ \{r^2, r^{-2}\} & \{r^4s\} \\ & \{r^6s\} \end{array}$$

We have split based on whether n is even or odd.

$$\begin{array}{cc} n \text{ odd} & n \text{ even} \\ \{e\} & \{e\} \\ \{r, r^{-1}\} & \{r, r^{-1}\} \\ \vdots & \vdots \\ \{r^{\frac{n-1}{2}}, r^{-\frac{n-1}{2}}\} & \{r^{\frac{n-2}{2}}, r^{-\frac{n-2}{2}}\} \\ \{s, rs, r^2s, \cdots, r^{n-1}s\} & \{r^{\frac{n}{2}}\} \\ & \{r, r^2s, \cdots, r^{n-1}s\} \\ & \{rs, r^3s, \cdots, r^{n-2}s\} \end{array}$$

So, for n odd:
\# of conjugacy class is $\frac{n+3}{2}$
$D_{2n}^{ab} = \{1, \overline{s}\} \cong C_2$
$Z(D_{2n}) = \{e\}$
For n even,
\# of conjugacy classes is $\frac{n+6}{2}$
$D_{2n}^{ab} = \{1, \overline{s}, \overline{r}, \overline{rs}\} \cong C_2 \times C_2$
1-dim representations:
n odd implies we have representations $\mathbb{C}_+, \mathbb{C}_-$
$\chi_\pm(r) = 1, \chi_\pm(s) = \pm 1$
n even implies we have representations $\mathbb{C}_{++}, \mathbb{C}_{+-}, \mathbb{C}_{-+}, \mathbb{C}_{--}$
$\varepsilon_r = \pm 1, \varepsilon_s = \pm 1$
$\chi_{\varepsilon_r\varepsilon_s}(r) = \varepsilon_r$ and $\chi_{\varepsilon_r\varepsilon_s} = \varepsilon_s$
2-dim representations:

$$\rho^h : D_{2n} \to GL_2(\mathbb{C})$$

$$\rho^h(r) = \begin{bmatrix} \zeta_n^h & 0 \\ 0 & \zeta_n^{-h} \end{bmatrix}$$

$$\rho^h(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

[Induced from C_n-representation $\mathbb{C}_h$ later]
For $0 < h < \frac{n}{2}$ it is irreducible [homework].
$\chi_h(r^k) = e^{2\pi ihk/n} + e^{-2\pi ihk/n} = 2\cos\frac{2\pi hk}{n}$
$\chi_h(r^ks) = 0$
Since characters determine representation, we have $\rho_h \cong \rho_{-h} = \rho_{n-h}$.
Also, for $0 < h < \frac{n}{2}$ the repesentations are distinct.
We have all irreducible 2-dim representations.
<u>Remark</u>: $\exists$ real representations $D_{2n} \to GL_2(\mathbb{R})$ [isometries in $\mathbb{R}^2$]. Then,

$$\hat{\rho}^h(r) = \begin{bmatrix} \cos\frac{2\pi h}{n} & -\sin\frac{2\pi h}{n} \\ \sin\frac{2\pi h}{n} & \cos\frac{2\pi h}{n} \end{bmatrix}$$

$$\hat{\rho}(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We have $\chi_h = \hat{\chi}_h$ and thus $\rho_h \cong \hat{\rho}_h$

Friday, 10/4/2024

Serre 5.4
Suppose $G = D_{2n} \times C_2$.
Then, $\mathbb{C}G = \mathbb{C}D_{2n} \otimes_{\mathbb{C}} \mathbb{C}C_2 = (\mathbb{C}D_{2n})_+ \times (\mathbb{C}D_{2n})_-$.
Twice as many irreducible representation as D_{2n}. 5.7 and 5.8
We have the following exact sequence:

$$1 \to A_4 \to S_4 \overset{\text{sign}}{\to} \{\pm 1\} \to 1$$

We have $|S_4| = 24 = 4!, |A_4| = 12$.
$\begin{Bmatrix} S_4 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} \\ \text{o.p} \end{Bmatrix}$ isometries of a tetrahedron.
Conjugacy classes (c.c.) in $\begin{Bmatrix} S_4 \\ A_4 \end{Bmatrix}$ are $\begin{matrix} (1), (12), (12)(34), (123), (1234) & s = 5 \\ (1), (12)(34), (123), (213) & s = 4 \end{matrix}$
Interestingly, not all 3-cycles are conjugates in A_4. For example, $(123) \not\sim (124)$. Intuition: we need to swap 3 and 4, but in A_4 we need something else because swapping 3 and 4 is odd.
Also: A_4 is not simple [even though A_5, A_6 etc are].
$S_4 = C_2 \times C_2 \rtimes S_3$
$A_4 = C_2 \times C_2 \rtimes C_3$.
Also: $S_4^{ab} = C_2$
$A_4^{ab} = C_3$
Then, $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$
$12 = 1^2 + 1^2 + 1^2 + 3^2$
$\mathbb{C}[A_4] = \underbrace{\mathbb{C} \times \mathbb{C} \times \mathbb{C}}_{C_3\text{-quotient}} \times \underbrace{M_3(\mathbb{C})}_{\text{geometry}}$

$$\mathbb{C}[S_4] = \overbrace{\underbrace{\mathbb{C} \times \mathbb{C}}_{C_2\text{-quotient}} \times M_2\mathbb{C}}^{D_6\text{-quotient}} \times \underbrace{M_3\mathbb{C}}_{\text{geometry}} \times \underbrace{M_3\mathbb{C}}_{\text{geom}\otimes_{\mathbb{C}}\mathbb{C}_{\text{sign}}}$$

Chapter 6
Suppose we have a finite group G and $(\operatorname{char} k, |G|) = 1$. Then kG is semisimple.

Proposition 63 (10)**.** Let A be semisimple ring. Suppose $L_1, \cdots, L_s$ are simple, non-isomorphic kG-modules such that $\forall$ simple L we have $L \cong L_i$ for some i. Then,

$$A \underset{\text{mul}}{\longrightarrow} \prod \operatorname{End}_A L_i$$

Corollary: $t < s$ implies:

$$A \to \prod_{i=1}^{t} \operatorname{End}_A L_i$$

is onto.
6.5:
Review: Corollary 2: if k is algebraically closed and $\operatorname{char} k = 0$ and $d = \dim_k L$ where L is a simple kG module, then

$$d \mid |G|$$

We strengthen this.

Proposition 64 (17)**.** Let $Z = Z(G)$ be the center of G. Then,

$$d \mid \frac{|G|}{|Z|}$$

Proof. Let $\rho : G \to GL(L)$ be an irreducible representation and $d = \dim$. Define homomorphism $\lambda : Z \to k^\times$ such that:

$$\rho(s) = \lambda(s)\,\text{id}$$

$\forall m \geq 1$ let $\rho^m : G \times \cdots \times G \to GL(L \otimes \cdots \otimes L)$ which is irreducible.
Then we have $\lambda^m : Z \times \cdots \times Z \to k^\times$ with:

$$(s_1, \cdots, s_m) \mapsto \lambda(s_1 \cdots s_m)$$

Let $H = \{(s_i) \in Z^m \mid s_1 \cdots s_m = 1\} < Z^m < G^m$.
$H \cong Z^{m-1}$ and $H \subset \ker \rho^m$.
Then $\overline{\rho^m} : G^m/H \to GL(L \otimes \cdots \otimes L)$ irreducible.
Therefore, $\forall m, d^m \mid |\frac{G^m}{H}| = \frac{|G|^m}{|Z|^{m-1}}$ which implies by taking m big enough that $d \mid \frac{|G|}{|Z|}$. □

Tensor Product for Non-Commutative Rings

Suppose R is a non-commutative ring. Then, tensor product is a functor

$$- \otimes_R - : \underset{\text{right mod}}{\operatorname{mod} R} \times \underset{\text{left mod}}{R \operatorname{mod}} \to \text{Ab}$$

$$\begin{aligned} A_R \otimes_R {}_R B \quad \ni \quad & a_1 \otimes b_1 + \cdots + a_k \otimes b_k \\ & (a + a') \otimes b = a \otimes b + a' \otimes b \\ & a \otimes (b + b') = a \otimes b + a \otimes b' \\ & ar \otimes b = a \otimes rb \end{aligned}$$

Exercise. Formulate adjoint proposition:

$$\operatorname{Hom}_?(\overset{?}{A} \otimes \overset{?}{B}, \overset{?}{C}) \cong \operatorname{Hom}_?(A, \operatorname{Hom}_?(B, C))$$

Definition (Induced module)**.** : Suppose k is a field and $H < G$. Then,

$$\operatorname{Ind}_H^G : kH\text{-mod} \to kG\text{-mod}$$

$$\operatorname{Ind}_H^G W = kG \otimes_{kH} W$$

eg. Suppose $H = C_n = \langle r | r^n = 1\rangle$ and $G = D_{2n} = \langle r, s \mid r^n = 1 = s^2; srs = r^{-1}\rangle$.
If $W = \mathbb{C}$ we have $H \to \mathbb{C}^\times$ by $r \mapsto \zeta_n$.

$$V = \mathbb{C}D_{2n} \otimes_{\mathbb{C}[C_n]} \mathbb{C}_1 = (\mathbb{C}[C_n] \oplus s\mathbb{C}[C_n]) \otimes_{\mathbb{C}[C_n]} \mathbb{C}_1$$

$\mathbb{C}$-basis of V is $1 \otimes 1, s \otimes 1$.
Recall $r \mapsto \begin{bmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{bmatrix}, s \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
$s(1 \otimes 1) = s \otimes 1$
$s(s \otimes 1) = s^2 \otimes 1 = 1 \otimes 1$
$r(1 \otimes 1) = r1 \otimes 1 = \zeta_n \otimes 1 = \zeta_n(1 \otimes 1)$
$r(s \otimes 1) = rs \otimes 1 = sr^{-1} \otimes 1 = s \otimes \zeta_n^{-1} 1 = \zeta_n^{-1}(s \otimes 1)$

Monday, 10/7/2024

Exercise. Work out the representation theory of $G = C_7 \underset{\cdot 2}{\rtimes} C_3 = \langle r, s \mid r^7 = 1, s^3 = 1, srs^{-1} = r^2\rangle$.
Meaning: find an isomorphism $\mathbb{C}G \overset{\cong}{\to} M_{d_i}\mathbb{C}$

Suppose we have a (most likely non-commutative) ring R and
A tensor product functor $- \otimes_R - : \text{mod-}R \times R\text{-mod} \to \text{Ab}$

Proposition 65 (Universal Property)**.** Suppose A is a right R-module and B is a left R-module and G is an abelian group.
$\pi : A \times B \to G$ is <u>R-balanced</u>. Meaning: π is $\mathbb{Z}$-bilinear and $\pi(ar, b) = \pi(a, rb)$.
There exists an R-balanced $\pi : A \times B \to A \otimes_R B$ which is <u>initial</u>.

$$
\begin{tikzcd}
A \times B \arrow[d, "\pi"] \arrow[dr, "\forall R\text{-balanced}"] & \\
A \otimes_R B \arrow[r, dotted, "\exists ! \mathbb{Z}\text{-hom}"'] & G
\end{tikzcd}
$$

Construction:

$$A \otimes_R B := \frac{F(A \times B)}{T}$$

Where $F(A \times B)$ is the free abelian group with basis of $A \times B$. We write $F(A \times B) = \mathbb{Z}[A \times B]$.
T is the subgroup generated by $(a + a', b) - (a, b) - (a', b), (a, b + b') - (a, b) - (a, b'), (ar, b) - (a, rb)$.
Main thing to remember:

$$\boxed{ar \otimes b = a \otimes rb}$$

Proposition 66. Suppose we have a ring homomorphism $f : R \to S$ of possibly non-commutative rings. We preserve addition, multiplicationand identity.
We then have the restriction functor

$$f^* : S\text{-mod} \to R\text{-mod}$$

$$f^* M = M \text{ (as abelian group)}$$

$$\begin{array}{ccc} R \times f^* M & \to & f^* M \\ (r, m) & \mapsto & f(r)m \end{array}$$

If we have inclusion $\mathrm{inc} : kH \to kG$ then we have:

$$\mathrm{inc}^* = \mathrm{Res}^G_H : kG\text{-mod} \to kH\text{-mod}$$

We also have the left adjoint of f^*.

$$f_* : R\text{-mod} \to S\text{-mod} \text{ “base change”}$$

$$f_* M = S \otimes_R M$$

S is a right R-module. We have $S \times R \to S$ given by $(s, r) \mapsto sf(r)$ which trns S to a (S, R) -bimodule: ${}_S S_R$. So we can take the tensor product.

Proposition 67.

$$\boxed{\mathrm{Hom}_S(f_* M, N) \cong \mathrm{Hom}_R(M, f^* N)}$$

is an isomorphism of abelian groups.

So we can go back and forth between S-modules and R-modules.

$$
\begin{tikzcd}
S\text{-mod} \arrow[rr, bend left, "f^*"] \arrow[r, phantom, ""] & \perp & R\text{-mod} \arrow[ll, bend left, "f_*"]
\end{tikzcd}
$$

f_* is left adjoint.
f^* is right adjoint.
Adjoint of $\mathrm{Id}_{f^* N}$: $\boxed{f_* f^* N \to N}$ is the counit.
Adjoint of $\mathrm{Id}_{f_* M}$: $\boxed{M \to f^* f_* M}$ is the unit.
We also have:

$$\mathrm{inc}_* = \mathrm{Ind}_H^G : kH\text{-mod} \to kG\text{-mod}$$

Which gives us:

$$\mathrm{Hom}_{kG}(\mathrm{Ind}_H^G W, V) \cong \mathrm{Hom}_{kH}(W, \mathrm{Res}_H^G V)$$

Remark: If we have a module, how do we know it is induced?

Proposition 68. If $V = \bigoplus_{i \in I} W_i$ and G permutes summands transitively and $\exists W = W_{i_0}$ and $H = \{g \in G \mid gW = W\}$ then V is induced.

Example: $\mathbb{C}D_{2n} \otimes_{\mathbb{C}C_n} \mathbb{C}_1 = 1\mathbb{C}C_n \otimes \mathbb{C}_1 + s\mathbb{C}G_n \otimes \mathbb{C}_1$.

Proposition 69. V is induced if $\exists W < V$ invariant under H:

$$V = \bigoplus_{r \in R} rW$$

R is a set of left coset representation for H in G.

Character of Induced representation

Theorem 70 (12, p30)**.** $V = \mathrm{Ind}_H^G W$.

$$\chi_V(u) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_W(r^{-1}ur) = \frac{1}{|H|} \sum_{\substack{g \in G \\ g^{-1}ug \in H}} \chi_W(g^{-1}ug)$$

Proof. Write $V = \bigoplus_{r \in R} rW$. We care about when $urW = rW$, since otherwise we have non-diagonal terms so they don't contribute to the trace.

$$urW = rW \iff r^{-1}urW = W \iff r^{-1}ur \in H$$

$$\chi_V(u) = \mathrm{Tr}(u\cdot : V \to V) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \mathrm{Tr}(u\cdot : rW \to rW)$$

$$= \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \mathrm{Tr}\big(r^{-1}ur\cdot : rW \to rW\big) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_W(r^{-1}ur) = \frac{1}{|H|} \sum_{\substack{g \in G \\ g^{-1}ug \in H}} \chi_W(g^{-1}ug)$$

□

Frobenius Reciprocity

$$\langle \mathrm{Ind}\,\psi, \phi \rangle_G = \langle \psi, \mathrm{Res}\,\phi \rangle_H$$

Wednesday, 10/9/2024

Recall: If

$$V = \mathrm{Ind}_H^G W$$

Then V as a k-vector space can be written as direct sum of k-vector spaces:

$$V = \bigoplus_{g \in G/H} gW$$

And action of H permutes the summands.

$$\mathrm{Stab}(W) \coloneqq \{g \in W \mid gW = W\} = H$$

Also recall Class Functions:

$$\mathrm{Cl}(G) = \{f : G \to k \mid f(g\sigma g^{-1}) = f(\sigma)\}$$

The charcters χ_V are a basis of the vector space of class functions.
For $H < G$ we have restriction:

$$\begin{array}{rccc} \mathrm{Res}: & \mathrm{Cl}(G) & \to & \mathrm{Cl}(H) \\ & f & \mapsto & f|_H \end{array}$$

We also have induction:

$$\mathrm{Ind} : \mathrm{Cl}(H) \to \mathrm{Cl}(G)$$

$$(\mathrm{Ind}\, f)(\sigma) \coloneqq \frac{1}{|H|} \sum_{\substack{g \in G \\ g^{-1}\sigma g \in H}} f(g^{-1}\sigma g)$$

Last time we did:

$$\chi_{\mathrm{Ind}\, W} = \mathrm{Ind}\, \chi_W$$

Also we had the following:

$$\mathrm{Hom}_{kG}(\mathrm{Ind}\, W, V) \cong \mathrm{Hom}_{kH}(W, \mathrm{Res}\, V)$$

Today we give a character version of this.

Frobenius Reciprocity

Theorem 71 (Frobenius Reciprocity)**.** Suppose k is algebraically closed. Then:

$$\langle \mathrm{Ind}\, \psi, \phi \rangle_G = \langle \psi, \mathrm{Res}\, \phi \rangle_H$$

where $\psi \in \mathrm{Cl}(H)$ and $\phi \in \mathrm{Cl}(G)$ with $H < G$.
Also, for review: if $\alpha, \beta \in \mathrm{Cl}(G)$ then,

$$\langle \alpha, \beta \rangle_G = \sum_{g \in G} \alpha(g)\beta(g^{-1}) \in k$$

And irreducible characters are an orthonormal basis w.r.t. this inner product.

$$\langle \chi_i, \chi_j \rangle_G = \delta_{ij}$$

Proof. Suppose

$$V \cong \bigoplus_i m_i V_i$$

where $V_1, \cdots, V_s$ are irreducible. We define multiplicity: $m^V_{V_i} \coloneqq m_i$. Then,

$$\langle \chi_V, \chi_{V'} \rangle = \sum_{i=1}^{s} m^V_{V_i} m^{V'}_{V_i} \underset{\text{Schur}}{=} \dim_k \mathrm{Hom}_{kG}(V, V')$$

We finally start the proof.

$$\mathrm{Cl}(G) = \mathrm{span}\{\chi_i\}$$

WLOG assume ψ, ϕ ae characters of W and V.

$$\dim_k \mathrm{Hom}_{kG}(\mathrm{Ind}\, W, V) = \dim_k(\mathrm{Hom}_{kH}(W, \mathrm{Res}\, V))$$

$$\implies \langle \mathrm{Ind}(\chi_W), \chi_V \rangle_G = \langle \chi_W . \mathrm{Res}\, \chi_V \rangle_H$$

Since this is true for basis, it is true for general character.

$\square$

Mackey's Double Coset Formula

Suppose G is a group with subgroups H, K. aka $H, K < G$. Let W be a kH-module.
Question: What is $\operatorname{Res}_K^G \operatorname{Ind}_H^G W$ as a kK-module?
Let $s = [K\backslash G/H]$ be the double coset representation. Meaning:

$$G = \coprod_{s \in S} KsH$$

i.e.

$$\begin{tikzcd} G \arrow[r, "\pi"'] & K\backslash G/H \arrow[l, bend right, dotted] \end{tikzcd}$$

The above dotted map is $[\,]$. Then,

$$\pi \circ [\,] = \mathrm{Id}$$

We have:

$$H_s := sHs^{-1} \cap K < K$$

$$\rho : H \to \mathrm{GL}(W)$$

We thus have the twisted representation:

$$\rho^s : H_s \to \mathrm{GL}(W)$$

$$\rho^s(x) = \rho_W(s^{-1}xs)$$

$W_s = W_{\rho^s}$ is a kHs-module.

Proposition 72 (Mackey's Double Coset Formula, MDCF)**.**

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W \cong \bigoplus_{s \in [K\backslash G/H]} \operatorname{Ind}_{H_s}^K W_s$$

Proof. Suppose $V := \operatorname{Ind}_H^G W$. Then, from the definition of $\operatorname{Ind} W$,

$$V = \bigoplus_{x \in G/H} xW$$

Where $\mathrm{Stab}(W) = H$.

$$V = \bigoplus_{x \in G/H} xW$$

Then, as hK-module,

$$V = \bigoplus_{s \in [K\backslash G/H]} KsW$$

Note that, since $\mathrm{Stab}^K(sW) = H_s$,

$$KsW = \bigoplus_{x \in K/H_s} xsW$$

$$= \operatorname{Ind}_{H_s}^K sW$$

$$= \operatorname{Ind}_{H_s}^K W_s$$

Since

$$W_s \cong sW$$
$$w \mapsto sw$$

So we're done.

$\square$

Mackey's Irreducibility Criterion, MIC

Suppowe $W = W_\rho$ is kH-module. TFAE:

1) $V = \operatorname{Ind}_H^G W$ is irreducible
2) a) W irreducible
   b) $\forall s \in G \setminus H$, ρ^s and $\operatorname{Res}_{H_s} \rho$ are disjoint.

Recall: V, V' are disjoint if $\operatorname{Hom}_{kG}(V, V') = 0$.

Proof. We asssume k is algebraically closed.

$$V \text{ irreducible} \iff \langle \chi_V, \chi_V \rangle_G = 1$$

$$\langle \chi_V, \chi_V \rangle_G = \langle \operatorname{Ind} \chi_W, \operatorname{Ind} \chi_W \rangle_G$$

$$= \langle \chi_W, \operatorname{Res} \operatorname{Ind} \chi_W \rangle_H \, [FR]$$

$$= \langle W, \bigoplus_{s \in [K \setminus G/H]} \operatorname{Ind}_{H_s}^H (\rho_s) \rangle_H \, [MDCF]$$

$$= \sum_s \langle \operatorname{Res}_{H_s} \rho, \rho^s \rangle_{H_s} \, [FR]$$

$$= \sum_s d_s$$

$$d_s = \langle \operatorname{Res} \rho, \rho^s \rangle_{H_s}$$

$$d_1 = \langle \rho_W, \rho_W \rangle \geq 1$$

Thus,

$$1 = \langle V, V \rangle_G \iff \begin{matrix} d_1 = 1 \\ d_s = 0 \end{matrix}$$

So we're done.

□

Example: Suppose $G = H \times K$ where $H = C_3, G = D_6 = S_3, K = C_2$.
Then,

$$\mathbb{C}[C_3] = \mathbb{C}_0 \times \mathbb{C}_1 \times \mathbb{C}_2$$

$$\mathbb{C}[D_6] = \mathbb{C}_+ \times \mathbb{C}_- \times M_2\mathbb{C}$$

$$\operatorname{Res} \mathbb{C}_+ = \mathbb{C}_0$$

$$\operatorname{Res} \mathbb{C}_- = \mathbb{C}_0$$

$$\operatorname{Res} \mathbb{C}^2 \stackrel{?}{=} \mathbb{C}_1 \times \mathbb{C}_2$$

Monday, 10/14/2024

Exercises 8-13 due Friday
Wed, Chapter 9
Suppose $K, H < G$ and $\rho : H \to GL(W)$.
For $s \in G$ consider $H_s = sHs^{-1} \cap K < K$
Then $\rho^s : H_s \to GL(W)$
$\rho^s(x) \coloneqq \rho(s^{-1}xs)$
MDCT:

$$\operatorname{Res}^G_K \operatorname{Ind}^G_H \rho \cong \sum_{s \in [K \backslash G/H]} \operatorname{Ind}^K_{H_s} \rho^s$$

Take $K = H$.
MIC:

$$\operatorname{Ind}^G_H \rho \text{ is irreducible}$$

$\iff$

a) ρ irredicuble

b) $\forall s \in G - H, \rho^s$ and $\rho\big|_{H_s}$ are disjoint.

Now take $H = K \triangleleft G$ normal.
Corollary: $\operatorname{Ind} \rho$ is irreducible $\iff$ ρ irredcible and $\forall s \notin H$ ρ is not isomorpic to conjugate ρ^s.
e.g. $H = C_3 = \langle r \rangle$
$G = D_6 = S_3 = \langle r, s \rangle$
$\mathbb{C}H \cong \mathbb{C}_0 \times \mathbb{C}_1 \times \mathbb{C}_2$
$r \mapsto (1, \zeta_3, \zeta_3^2)$
$\mathbb{C}G \cong \mathbb{C}_+ \times \mathbb{C}_- \times M_2\mathbb{C}$
Only two dimensional irredicuble reps are $\mathbb{C}_+ \times \mathbb{C}_-$ and $\mathbb{C}^2$
$\operatorname{Ind}^G_H \mathbb{C}_0 \cong \mathbb{C}_+ \times \mathbb{C}_-$
$\operatorname{Ind}^G_H \mathbb{C}_1 \cong \mathbb{C}^2$
Corollary?: $\operatorname{Ind} \mathbb{C}_0$ is real since $\rho \cong \rho^s, \rho^s = \rho(s^{-1}xs)$
$\operatorname{Ind} \mathbb{C}_1$ is [], $(\rho : H \to \mathbb{C})$, $\underset{\mathbb{C}_1}{\rho} \ncong \underset{\mathbb{C}_2}{\rho^s}$.
More on MCDF "Mackey Functors"
Review
Ring $f : R \to S$

$$\begin{tikzcd} S\text{-mod} \arrow[r, bend right, "f^*"'] & R\text{-mod} \arrow[l, bend right, "f_*"'] \end{tikzcd}$$

"Res" $f^* N = N$
"Ind" $f_* M = S \otimes_R M$
MDCF: $H, K < G$
$K^s = s^{-1}Ks$
${}^sH = sHs^{-1}$
$c_s : \underset{g}{K^s} \underset{\mapsto}{\to} \underset{sgs^{-1}}{K}$
$(\operatorname{Ind} c_s)M = kK \otimes_{kK^s} M$

$$\boxed{\operatorname{Res}^G_K \operatorname{Ind}^G_H = \sum_{s \in [K \backslash G/H]} \operatorname{Ind}^K_{K \cap {}^sH} \operatorname{Ind} c_s \operatorname{Res}^H_{K^s \cap H}}$$

Definition. A Mackey Functor M is:

$$M : \{\text{subgroups of } G\} \to \text{Ab}$$

$\forall H \leq K \leq G$, we have:
Induction map $I_H^K : M(H) \to M(K)$
Restriction map $R_K^H M(K) \to M(H)$
Conjugation map $\forall g \in K, c_g : M(K^s) \to M(K)$
Satisfies 6 axioms. Key one is MDCF.

$$H, K \leq J \leq G$$

$$R_K^J I_H^J = \sum_{K \backslash J/H} \cdots$$

<u>Examples of Mackey Functors</u>
$M(H) = R_K(H)$ representations.
Homology groups $M(H) H_n(H; -)$
Cohomology groups $M(H) = H^n(H; -)$
Stable Homotopy theory: $M(H)$ equals X based G-space $\Pi_n^H X$
Number theory: if we have $K/_{\text{finite galois}} L/_{\text{finite}} \mathbb{Q}$,

$$M(H) = \mathrm{Cl}(\mathcal{O}(K^H))$$

Wednesday, 10/16/2024

No class Friday
Homework due monday, 8-13

Representation Ring

Representation $R(G) = \mathbb{Z}[\chi_1, \cdots, \chi_n] \subset \mathrm{Cl}(G) = \{f : G \to \mathbb{C} : f(\sigma\tau\sigma^{-1}) = f(\tau)\}$ where $\chi_1, \cdots, \chi_h$ are irreducible $\mathbb{C}$-rep.

- $(R(G), +) \cong \mathbb{Z}^n$
- $R(G) \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Cl}(G)$

A basis of $\mathbb{C}G$ can be found the following way: Fix σ. Then $\sum_{\tau \sim \sigma} \tau$ gives us the basis where $\sim$ means they are in the same conjugacy class.
Another basis are $\chi_1, \cdots, \chi_h$. So, $h =$ the number of conjugacy classes.

Theorem 73 (Artin Induction Theorem)**.**

$$\mathrm{Ind} : \mathbb{Q} \otimes \bigoplus_{\text{cyclic } C < G} R(C) \twoheadrightarrow \mathbb{Q} \otimes R(G)$$

<u>Exercise</u>: Let χ_T be the trivial characteristic of D_6 Express $a\chi_T$ as a subrepresentationm of characters $a > 0$ induced from cyclic subgroups.

Proof.

$$\mathrm{Res} : R(G) \rightarrowtail \bigoplus_C R(C)$$

$$\mathrm{Res} : R(G) \otimes \mathbb{C} \rightarrowtail \bigoplus_C R(C) \otimes \mathbb{C} \, \text{injective}$$

$$\underset{\text{Frob. Reciprocity}}{\Longrightarrow} \mathrm{Ind} : \bigoplus_C R(C) \otimes \mathbb{C} \twoheadrightarrow R(G) \otimes \mathbb{C} \, \text{surjective}$$

Why? in matrix terms, we can think of the matrices being transposed, A injective implies A^T is surjective. We can also think of dual maps, $V \rightarrowtail W \iff W^* \twoheadrightarrow V^*$

$$\Longrightarrow \mathrm{Ind} : \bigoplus_C R(C) \otimes \mathbb{Q} \twoheadrightarrow R(G) \otimes \mathbb{Q}$$

$\square$

Another view of $R(G)$

Let V be a representation, $[V]$ be its isomorphism class. Then,

$$R(G) \in [V] - [V']$$

"virtual representation"

0.1 Grothendieck Construction

Define the category CMon, commutative monoids.

$$(M, + : M \times M \to M)$$

$$\text{commutative, associative, identity}$$

The morphisms are homomorphism [preserves unity].

$$\begin{tikzcd} \mathrm{CMon} \arrow[r, bend left, "\mathrm{Gr}"] & \mathrm{Ab} \arrow[l, bend left, "F \text{ Forgetful}"] \end{tikzcd}$$

$$\mathrm{Ab}(\mathrm{Gr}\, M, A) \cong \mathrm{CMon}(M, FA)$$

$\iff$ universal property:

$$\begin{tikzcd} M \arrow[r] \arrow[dr, "\forall \text{monoid map}"'] & \mathrm{Gr}\, M \arrow[d, dashed, "\exists!"] \\ & \underset{\text{ab}}{A} \end{tikzcd}$$

[Take $A = \mathrm{Gr}\, M$]
Note: $\mathrm{Gr}(\mathbb{Z}_{\geq 0}, +) = (\mathbb{Z}, +)$
$\mathrm{Gr}(\mathbb{Z}_{>0}, \cdot) = (\mathbb{Q}^{\times}_{>0}, \cdot)$
$\mathrm{Gr}(\mathbb{Z}_{\neq 0}, \cdot) = (\mathbb{Q}^{\times}, \cdot)$
Consider a field k and a group G.
$\mathrm{Iso}(k, G)$ = isomorphism class of finite dimensional k-representations $\rho : G \to GL(V)$ with $\dim_k V < \infty$.
We define $R_k(G) \coloneqq \mathrm{Gr}(\mathrm{Iso}(k, G), \oplus)$
Is is a group. We can make this a ring by defining the product as:

$$[V][W] \coloneqq V \otimes_k W$$

the diagonal k-action.
Suppose X is a set of subgroups of G.

Definition. $R_k G$ is $\begin{Bmatrix} \text{detected} \\ \text{generated} \end{Bmatrix}$ by X if:
$\begin{Bmatrix} \mathrm{Res} : R(G) \to \bigoplus_{H \in X} R(H) \\ \mathrm{Ind} : \bigoplus_{H \in X} R(H) \to R(G) \end{Bmatrix}$ is $\begin{Bmatrix} \text{injective} \\ \text{surjective} \end{Bmatrix}$

e.g. $R(G)$ is detected by cyclics
$R(G) \otimes \mathbb{Q}$ is generated by cyclics.
Consider:

$$\text{hom } f : H \to G$$

$$\mathrm{Res} : R_k G \to R_k H \text{ is a ring hom}$$

$$\mathrm{Ind} : R_k H \to R_k G \text{ is a } R_k G\text{-module map}$$

$1 = [k] \in R(G)$.

$$\operatorname{Res} W \otimes_k f_* V \cong f_*(W \otimes_k V)$$

$$W \otimes_k (kG \otimes_{kH} V) \cong kG \otimes_{kH} (W \otimes_k V)$$

$$w \otimes (\alpha \otimes v) \underset{?}{\hookleftarrow} \alpha \otimes (w \otimes v)$$

Note: Consider $f : X \to Y$. Then $f^* : H^*Y \to H^*X$ is a ring map, $f_* : H_*X \to H_*Y$ is a module map.

Monday, 10/21/2024

Brauer Induction Theorem

Let p be a prime.

Definition. H is p-elementary if

$$H \cong P \times C$$

where P is a p-group and C is a cyclic group with order prime to p.

Definition. H is elementary if H is p-elementary for some p.

Example. $Q_8 \times C_3$ is 2-elementary.

Theorem 74 (Brauer Induction Theorem)**.** $R(G)$ is generated by elementary subgroups. i.e.:

$$\operatorname{Ind} : \bigoplus_{\text{elem } E<G} R(E) \twoheadrightarrow R(G)$$

in other words,

$$\forall \rho : G \to GL(V); \chi_\rho = \sum_i a_i \operatorname{Ind}_{E_i}^G \rho_i$$

where E_i are elementary.

Example. Consider $D_6 = C_3 \rtimes C_2$. Elementary subgroups are $1, C_3, C_2$.
For p odd prime, D_{2p} has elementary subgroups $1, C_2, C_p$.

Remark. We can't always choose $a_i \geq 0$ in χ_ρ.

Theorem 75 ($18'$)**.** Let $|G| = p^k l$ with $(l, p) = 1$.
$[\mathbb{C}^l] = l[\mathbb{C}] = l$ is induced by p-elementary subgroups.

$$l = \sum_{E_i, p \text{ elem}} a_i \operatorname{Ind}_{E_i}^G \rho_i$$

Note: Theorem $18' \implies$ Brauer Induction Theorem. Let $|G| = p_1^{e_1} \cdots p_r^{e_r}$. Then $\gcd\left(\frac{|G|}{p_1^{e_1}}, \cdots, \frac{|G|}{p_r^{e_r}}\right) \in \operatorname{image} \operatorname{Ind}\left(\bigoplus_{E<G} R(E)\right) \implies \forall x \in R(G), x \in \text{image}$ [Ind is $R(G)$-module map] $\implies$ Brauer Induction Theorem.
Proof of theorem $18'$ is ommitted.

Applications of Brauer Induction Theorem

Definition. A representation $\rho : G \to GL(V)$ is a monomial if

$$\rho = \operatorname{Ind}_H^G \hat{\rho}$$

where $\hat{\rho} : H \to \mathbb{C}^\times$ is a 1-dim representation.
In other words, "ρ is induced by irreducible representation of G^{ab}."

Application (Brauer): Artin L-functions are meromorphic (on $\mathbb{C}$).

Chapter 8

Goal:

Theorem 76 (20)**.** Every $\chi \in R(G)$ is a $\mathbb{Z}$-linear combination of monomial characters. This is stronger than Brauer Induction Theorem.

Why does Brauer induction theorem imply this?
We want to show: Every character of an elementary group is a monomial.

Definition. G is supersolvable if:

$$\exists 1 = G_0 \subset G_1 \subset \cdots \subset G_n = G$$

such that $G_i \triangleleft G$ and G_i/G_{i-1} is cyclic.
Sylow theorem $\implies$ p-groups are super solvable.
Hence elementary subgroups are super-solvable.

Remark. p-group $\implies$ nilpotent $\implies$ super-solvable $\implies$ solvable.

Definition. R-module

Our goal changes to proving: every character of super-solvable group is monomial.

Definition. R-module M is isotypic if M is a direct sum of simple, isomorphic submodules.

$$M \cong S \oplus \cdots \oplus S$$

Proposition 77 (24)**.** Suppose $(\operatorname{char} k, |G|) = 1$. Suppose V is an irreducible kG-module and $A \triangleleft G$. Then either:

a) $\exists$ proper $H < G$ such that $A < H$ and there eixsts an irreducible kH-module W such that $V \cong \operatorname{Ind}_H^G W$

b) $\operatorname{Res}\big|_A V$ is isotypic.

Proof. $V = \bigoplus_{i=1}^h V_i$
V_i isotypic and $i \neq j \implies V_i$ and V_j are disjoint.
$\forall s \in G$,

$$sV_i = sAV_i \underset{A \triangleleft G}{=} AsV_i$$

Thus, $sV_i = V_j$ for some j.
Thus, $s : V \to V$ permutes V_i transitively [since W is irreducible].
Case b: $V = V_1$.
Case a: $H = \operatorname{Stab}(V_1) = \{s \in G \mid sV_1 = V_1\} < G$ proper $\implies W = \operatorname{Ind}_H^G V_1$.

Remark. If A is abelian and $k = \mathbb{C}$ then Case b $\iff \rho(a) = \alpha I \, \forall a \in A$.

$\square$

Wednesday, 10/23/2024

Goal: Theorem 20: $R(G)$ is generated by monomial characters
Recall: R-module M is isotypic if:

$$M \cong S \oplus \cdots \oplus S$$

where S is simple.
We also have proposition 24: Suppose we are in the Maschke case $(\operatorname{char} k, G) = 1$ and V is an irreducible kG-module and $A \triangleleft G$.
Then either:

a) $\exists$ proper $H < G$ containing A and irreducile kH-module W such that $V \cong \mathrm{Ind}_H^G W$ or:

b) $\mathrm{Res}_A V$ is isotypic.

Proof. $\mathrm{Res}_A V = V_1 \oplus \cdots \oplus V_n$ isotypic, nonzero, disjoint (meaning no common irreducible subrepresentation).
Then $\forall s \in G, sV_i = V_j$ [use A normal $\implies sV_i$ is isotypic]
V irreducible $\implies G$ permutes V_i transitively.
Let $H = \{s \in G \mid sV_1 = V_1\}$. Let $W = V_1$.
Then $V = \mathrm{Ind}_H W$.
$n > 1$ puts us in case a, $n = 1$ gives us case b. $\square$

Remark. If V is a $\mathbb{C}A$ module and A is abelian, $\rho : G \to GL(V)$
Then V is isotypic $\iff \forall a \in A, \exists \alpha \in \mathbb{C}^\times$ such that $\rho(a) = \alpha I$.
Why $\mathbb{C}$? Then representation is 1-dimensional since A is abelian.

Corollary 78. Consider abelian $A \lhd G$. Let V be a simple $\mathbb{C}G$ module and $d = \dim_{\mathbb{C}} V$.
Then $d \mid (G : A) = \frac{|G|}{|A|}$.
eg $C_p \lhd D_{2p} \implies d = 1, 2$.
In $C_7 \rtimes C_3$ since C_7 is normal $d \mid \frac{21}{7} = 3$ so $d = 1, 3$.

Proof. Recall $d \mid |G|$ [on page 52].
We also have $d \mid (G : Z(G))$ [on page 53].
We use the second result to prove this. We use induction on $|G|$.
We use Proposition 24/77:
Case a:

$$d \underset{\text{induction hypothesis}}{\mid} (H : A) \mid (G : A)$$

Case b: $\mathrm{Res}_A \rho$ is isotypic.
$\rho : G \to GL(V), G' = \rho(G), A' = \rho(A)$.
$G/A \underset{\rho}{\twoheadrightarrow} G'/A'$

Remark. $A' \subset Z(G')$

$$d \underset{p.53}{\mid} [G' : Z(G')] \mid [G' : A'] \mid [G : A]$$

$\square$

Recall irreducible $\mathbb{C}G$-module V is monomial if it is induced from a 1-dim representation.

Definition. G is $\left\{\begin{matrix}\text{supersolvable} \\ \text{solvable}\end{matrix}\right\}$ if $\exists 1 = G_0 \subset G_1 \subset \cdots \subset G_n = G$ such that $\left\{\begin{matrix}G_i \lhd G \\ G_i \lhd G_{i+1}\end{matrix}\right\}$ and G_i/G_{i-1} is $\left\{\begin{matrix}\text{cyclic} \\ \text{abelian}\end{matrix}\right\}$

Theorem 79. Evey irreducible representation of a semsimple group is <u>monomial</u>.

Lemma 80 (4)**.** Let G be a non-abelian supersolvable group. Then $\exists$ abelian $A \lhd G$ such that $A \not\subset Z(G)$.

Proof. $H = G/Z(G)$ is supersolvable. $\implies \exists$ cyclic normal $1 \neq H_1 \lhd H$.
Let $A = \pi^{-1} H_1$ where $\pi : G \to G/Z(G)$.
Claim:

$$1 \to \underset{\text{central}}{A} \to B \to \underset{\text{cyclic}}{C} \to 1 \implies B \text{ abelian}$$

choose $b \in B$ such that $\langle \mathrm{im} b \rangle = C$.
Every element of B looks like ab^i:
$ab^i \underline{ab^j} = \underline{ab^j} ab^i$.
$a \in Z(B)$. $\square$

Proof of theorem 16. induction on $|G|.\rho : G \to GL(V)$, irreducible, G supersolvable.
Case 1: ρ not injective. $\overline{\rho} : G/\ker\rho \to GL(V)$.
$\overline{\rho} = \mathrm{Ind}_{\overline{H}}^{\rho(G)}$ (1-dim) by induction hypothesis so $\rho = \mathrm{Ind}_{\rho^{-1}\overline{H}}$ is 1 dim.
Case 2: G abelian then we're done.
Case 3: irreducible $\rho : G \rightarrowtail GL(V)$ and G not abelian.
Lemma 4 $\implies \exists$ abelian $A \triangleleft G, A \not\subset Z(G) \implies \rho(A) \not\subset Z(\rho(G)) \implies \exists a \in A$ such that $\rho(a) \not\subset Z(\rho(G)) \implies$ remarkin case a. $\square$

Corollary 81. Every irreducible representation of elementary group is monomial.

Corollary 82 (using BIT)**.** Theorem 20

Friday, 10/25/2024

3 Applications of rep theory to group theory:
Exercise 8.6:

Theorem 83 (Burnside's Theorem)**.** Let $\#G = p^a q^b$ where p, q are primes. Then G is not simple ($\exists 1 < N \triangleleft G$), all proper.

Frobenius I (Exercise 7.3)
If $G \curvearrowright X$ effectively, transitvely, $\forall g \in G \setminus e, X^g$ is a point or empty. Then,

$$G \cong H \rtimes K$$

$H = \mathrm{Stab}(x_0)$ for some $x_0 \in X$.
For example, $D_6 \curvearrowright \triangle$ so $D_6 = C_2 \rtimes C_3$.
Frobenius II (Corollary 2, page 83)
Suppose $n \mid \#G$. Then,

$$n \mid \#\{x \in G \mid x^n - 1\}$$

Suggestion
Look at exercises for Chapter 12.

Chapter 12 Rationality

$\mathbb{Q}C_2 \cong \mathbb{Q}_+ \times \mathbb{Q}_-$
$\mathbb{Q}C_p \cong \mathbb{Q} \times \mathbb{Q}(\zeta_p)$
D_{2p} has C_p inside of it.

$$\mathbb{Q}D_{2p} \cong \underbrace{\mathbb{Q}_+}_{r\mapsto 1, s\mapsto 1} \times \underbrace{\mathbb{Q}_-}_{r\mapsto 1, s\mapsto -1} \times M_2(\mathbb{Q}[\lambda_p])$$

$$\mathbb{Q}Q_8 \cong \mathbb{Q}_{++} \times \mathbb{Q}_{+-} \times \mathbb{Q}_{-+} \times \mathbb{Q}_{--} \times \mathbb{Q}[i, j, k]$$

$\mathbb{R}C_2 \cong \mathbb{R}_+ \times \mathbb{R}_-$
$\mathbb{R}C_p \cong \mathbb{R} \times \mathbb{C}^{\frac{p-1}{2}} = \mathbb{R} \times \mathbb{C}_1 \times \cdots \times \mathbb{C}_{\frac{p-1}{2}}$

$$\mathbb{R}D_{2p} \cong \mathbb{R}_+ \times \mathbb{R}_- \times M_2(\mathbb{R})^{\frac{p-1}{2}}$$

$$\mathbb{R}Q_8 \cong \mathbb{R}^4 \times \mathbb{H}$$

$$\mathbb{H} = \mathbb{R}(i, j, k)$$

$\mathbb{C}C_2 \cong \mathbb{C}_+ \times \mathbb{C}_-$
$\mathbb{C}C_p \cong \mathbb{C}_0 \times \mathbb{C}_1 \times \cdots \times \mathbb{C}_{p-1}$
Where we map to ζ_p^k at $\mathbb{C}_k$.
$\mathbb{C}_1 \cong \mathbb{C}_{p-1}$ as $\mathbb{R}C_p$ modules $[z \mapsto \overline{z}]$
$\mathbb{C}_1 \not\cong \mathbb{C}_{p-1}$ as $\mathbb{C}C_p$-modules.

$$\mathbb{C}D_{2p} \cong \mathbb{C}_+ \times \mathbb{C}_- \times M_2(\mathbb{C})^{\frac{p-1}{2}}$$

$$\mathbb{C}Q_8 \cong \mathbb{C}^4 \times M_2(\mathbb{C})$$

$$D_{2p} \to GL(\mathbb{C}^2)$$

$$r \mapsto \begin{bmatrix} \zeta_p & 0 \\ 0 & \zeta_p^{-1} \end{bmatrix}$$

$$s \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$D_{2p} \to GL(\mathbb{R}^2)$$

$$r \mapsto \begin{bmatrix} 0 & -1 \\ 1 & \lambda_p \end{bmatrix}$$

$$s \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Note that the matrices that map from r are conjugate over $\mathbb{C}$. Both have the same characteristic polynomial: $x^2 - \lambda_p x + 1$.

12.1

Suppose K is a subfield of $\mathbb{C}$.

$$\{kG\text{-mod}\} \to \{\mathbb{C}G\text{-mod}\}$$

$$V \mapsto V_{\mathbb{C}} = \mathbb{C}G \otimes_{KG} V = \mathbb{C} \otimes_K V$$

$$\left\{\begin{matrix} \text{central} \\ \text{idempotents of} \\ KG \end{matrix}\right\} \to \left\{\begin{matrix} \text{central} \\ \text{idempotents of} \\ \mathbb{C}G \end{matrix}\right\}$$

<u>Question</u>: What about irreducible representation?
V irreducible $\overset{?}{\Longrightarrow}$ $V_{\mathbb{C}}$ irreducible?
W irreducible over $\mathbb{C}G$ $\overset{?}{\Longrightarrow}$ $W \cong V_{\mathbb{C}}$ for some V.
<u>Question</u>: What about primitive central idempotents?

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[rr, bend right, "\rho_{\mathbb{C}}"'] & GL_K(V) \arrow[r, "\mathrm{Id}\otimes -"] & GL_{\mathbb{C}}V_{\mathbb{C}} \end{tikzcd}$$

$\chi_p = \mathrm{Tr}(\rho) = \mathrm{Tr}(\rho_{\mathbb{C}}) = G \to K$.

Definition. $\mathbb{C}G$-module W is <u>realizable</u> over K if $W \cong V_{\mathbb{C}}$ for some kG-mod V.

Consider the <u>Representation Ring</u> $RG = R_{\mathbb{C}}G$.
$R_K G$ = subring of class function $f : G \to K$, generated by the characters of K-representation.
$R_K G$ is a subring of RG.

$$= \mathrm{Gr}(\mathrm{Isom}(\text{f.g. } KG\text{-mod}), \oplus)$$

"virtual representations"
Let $\chi_1, \cdots, \chi_n$ be distinct irreducible character of KG.
$R_K(G) = \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n$ additively.

$\{\chi_i\}$ are orthogonal [but not orthonormal] under the usual bilinear form:

$$\langle f, g\rangle = \frac{1}{\#G}\sum f(\sigma)g(\sigma^{-1})$$

Theorem 84 (12.3)**.** Every $\mathbb{C}$-rep of G is realizable over $\mathbb{Q}(\zeta_{|G|})$.
In fact let $m = \text{l.c.m}\{\text{order}(g) \mid g \in G\} \mid \#G$.
Every $\mathbb{C}$ representation of G is realizable over $\mathbb{Q}(\zeta_m)$.

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Proof. Special case: G abelian.
Follows since irreducible rep $G \to \mathbb{C}^\times$.
General case: Let $\chi \in R(G)$.
Monomial representations generate $R(G)$.

$$\chi = \sum_i n_i \operatorname{Ind}_{H_i}^G(\phi_i) \quad \phi_i \text{ 1-dim.}$$

Then $\phi_i : H \to \mathbb{C}^\times$
$\phi_i(H) \subset \mathbb{Q}(\zeta_m)$
Thus $\operatorname{Ind}_{H_i}^G(\phi_i) \subset \mathbb{Q}(\zeta_m)$.
Therefore $\chi \in R_{\mathbb{Q}(\zeta_m)}G$. □

12.2 Brauer Groups

Definition. A central simple algebra over K is:
A simple ring A.
$K = Z(A)$.
$(A : K) < \infty$.

Example. $\mathbb{H}$ is a CSA over $\mathbb{R}$.

Recall that a simple ring is simply a matrix ring over a division algebra.
Artin Wedderbern $\implies$ $A \cong M_n(D)$ where D is a central simple division algebra over K.
Facts:

1) A, B csa $/K \implies A \otimes_K B$ is csa $/K$.
2) K subfield of L and A case $/K \implies L \otimes_K A$ is csa $/L$.
3) K alg. closed and A csa $/K \implies A \cong M_n(K)$.

Definition. L is a splitting field for csa A if

$$L \otimes_K A \cong M_n L$$

Facts $\implies$ Algebraically closed is splitting field for A.
3 $\implies$ $(A : K) = m^2$ since $(A : K) = (A_L : L)$ where L is splitting field which has dimension m^2 since it is isomorphic to $M_m L$. $m = \sqrt{A : K}$ is the Schur Index
Harder Fact: maximal subfield of A is splitting field for A.
e.g. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2\mathbb{C}$.
If D is a skew field CSA $/K$ then $(D : K) = m^2$ where $m =$ Schur index of D.
A case $/K$ so schur index of A is divisible by schur index of D.

Definition (Brauer Group)**.** Let K be a field.

$$\operatorname{Br}(K) = \left(\frac{\text{csa}/K}{M_n(D) \sim D}\right), \otimes_K$$

eg $\operatorname{Br}\mathbb{C} = 1$
$\operatorname{Br}\mathbb{R} = C_2 = \langle \mathbb{H} \rangle$. $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R})$
$\operatorname{Br}(K) = H^2(\operatorname{Gal}(\overline{K}/K); \mathbb{Z}/2)$

12.2 Schur Indices

$$\mathbb{R}Q_8 \cong \mathbb{R}^4 \times \mathbb{H}$$

$$\mathbb{C}Q_8 \cong \mathbb{C}^4 \times M_2\mathbb{C}$$

$$i \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

Consider $\mathbb{R}Q_8$ module $V = \mathbb{H}$ and $\mathbb{C}Q_8$ module $W = \mathbb{C}^2$ not realizable over $\mathbb{R}$.
$\chi_V(\pm 1) = \pm 4$
$\chi_V(\pm i, \pm j, \pm k) = 0$
$\chi_W(\pm 1) = \pm 2, \chi_W(\pm i, \pm j, \pm k) = 0$
We have:

$$kG \cong \prod M_{n_i}(D_i)$$

$$K_i = \text{center} D_i$$

$$\text{schur index} m_i = \sqrt{(D_i : K_i)}$$

eg $G = Q_8, K = \mathbb{R}, m_5 = 2$.

Definition. $R_K(G) \subset \overline{R}_K G = \{f \in R(G) \mid f(G) \subset K\} \subset R(G)$

eg $\chi_W = \chi_{\mathbb{C}^2} \in \overline{R}_{\mathbb{R}}(Q_8) - R_{\mathbb{R}}(Q_8)$

Proposition 85 (35)**.** $\chi_1, \cdots, \chi_h$ are the irreducible characters of KG. Then they are $\mathbb{Z}$ basis for $R_K G$. Then,
$\frac{\chi_1}{m_1}, \cdots, \frac{\chi_h}{m_h}$ are a $\mathbb{Z}$-basis for $\overline{R}_K G$.

Corollary 86. $R_K(G) \subset \overline{R}_K(G)$ finite index with equality iff all D_i are fields.

Wednesday, 10/30/2024

12.4 Rank $R_K G$

$\mathbb{C}C_p \cong \mathbb{C}^p$
$\mathbb{Q}C_p \cong \mathbb{Q} \times \mathbb{Q}(\zeta_p)$
$\zeta = \zeta_m = e^{2\pi i/m}$ where m is multiple of $lcm(\text{ord}(g))$ e.g. $m = |G|$.

$$\begin{tikzcd} & & \mathbb{C} \\ & L = K(\zeta) \arrow[ur, dash] & \\ K \arrow[ur, dash] & & \end{tikzcd}$$

$LG \cong \prod M_{n_i}(L)$

$$\begin{aligned} \text{rank}\, RG &= \#\text{ of irreducible } \mathbb{C}G\text{-modules} \\ &= \#\text{ of irreducible } LG\text{-modules} \\ &= \#\text{ of conjugacy classes of } G \end{aligned}$$

What about # of irreducible KG-reps?

$$\Gamma = \Gamma_K := \{t \in (\mathbb{Z}/m)^\times \mid \exists \sigma \in \text{Gal}(L/K) \text{s.t. } \sigma(\zeta) = \zeta^t\} < (\mathbb{Z}/m)^\times$$

$$\Gamma = \text{image}(\underset{\sigma_t}{\text{Gal}(L/K)} \underset{\mapsto}{\rightarrowtail} \underset{t}{(\mathbb{Z}/m)^\times})$$

where $\sigma_t(\zeta) = \zeta^t$.
eg $\Gamma_\mathbb{Q} = \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m)^\times$
$\Gamma_\mathbb{C} = 1$
$\Gamma_\mathbb{R} = \begin{cases} 1, & \text{if } m \text{ odd;} \\ \pm 1, & \text{if } m \text{ even.} \end{cases}$

Definition. $s, s' \in G$ are Γ_K-conjugate if $\exists \tau \in G, t \in \Gamma_K$ such that:

$$\tau s' \tau^{-1} = s^t$$

we write $s' \underset{K}{\sim} s$

Corollary 87 (page 96)**.** rank $R_K G = \#$ of Γ_K conjugacy classes.

If $G = C_p$ then Γ_Q conjugacy classes are $\{1\}, \{r^t\}_{t \not\equiv o(p)}$
Recall that $\mathbb{R}C_p \cong \mathbb{R} \times \mathbb{C}^{\frac{p-1}{2}}$
$G = C_p$ then $\Gamma_\mathbb{R}$ conjugacy classes are $\{1\}, \{r, r^{-1}\}, \{r^2, r^{-2}\}, \cdots, \{r^{\frac{p-1}{2}}, r^{\frac{p-1}{2}}\}$
We have:

$$RG \to \text{Cl}_L G = \{f : G \to L \mid f(\tau s \tau^{-1}) = f(s)\}$$

We can take K linear combinations of this.

$$K \otimes_\mathbb{Z} RG \hookrightarrow \text{Cl}_L G = \{f : G \to L \mid f(\tau s \tau^{-1}) = f(s)\}$$

Theorem 88 (25)**.** Let $f \in \text{Cl}_L G$. TFAE:

a) $f \in K \otimes_\mathbb{Z} RG$

b) $\forall t \in \Gamma, \forall s \in G$ we have $\sigma_t(f(s)) = f(s^t)$

Proof. a $\implies$ b: It is enough to show it for characters. We want to show for χ_ρ where $\rho : G \to GL(\mathbb{C}^n)$. Let $\lambda_1, \cdots, \lambda_n$ be eigenvalues of $\rho(s)$. They must all be roots of unity. Then $\chi_\rho(s) = \sum_i \lambda_i$.

$$\sigma_t(\chi_\rho(s)) = \sigma_t\left(\sum_i \lambda_i\right) = \sum_i \lambda_i^t = \chi_\rho(s^t)$$

b $\implies$ a: Let $f \in \text{Cl}_L$.
Irreducible characters form an orthonormal basis.

$$f = \sum_{\chi \text{ irr}} \langle f, \chi \rangle \chi$$

$\forall t \in \Gamma_K$ we have:

$$\begin{aligned} \langle f, \chi \rangle &= \frac{1}{|G|} \sum_{s \in G} f(s)\chi(s^{-1}) \underset{\text{reindex}}{=} \frac{1}{|G|} \sum_{s \in G} f(s^t)\chi(s^{-t}) \\ &= \frac{1}{|G|} \sum_{s \in G} \sigma_t(f(s))\sigma_t(\chi(s^{-1})) = \sigma_t(\langle f, \chi \rangle) \end{aligned}$$

Thus, $\langle f, \chi \rangle$ are invariant under Galois therefore $\langle f, \chi \rangle \in K$ which is what we wanted to prove. □

Corollary 89 (1)**.** Let $f \in \text{Cl}_K$.
$f \in K \otimes R_K G \iff f$ is constant on Γ_K conjugacy classes.

Proof. $\implies$: WLOG $f = \chi_\rho$ where $\rho : G \to GL(K^n)$.
$\tau s' \tau^{-1} = s^t$
$\implies \chi_\rho(s') = \chi_\rho(s^t) \underset{25b}{=} \sigma_t \chi_\rho(s) \underset{\chi_\rho(s) \in K}{=} \chi_\rho(s)$.
$\impliedby$: $f : G \to K$ is constant on Γ_K conjugacy classes.
Thus, 25b holds for f.
Thus, 25a holds for f.
Thus, $f \in K \otimes_{\mathbb{Z}} RG$.

$$f = \sum_{\text{irr } \chi \in R_L(G)} \langle f, \chi \rangle \chi$$

We need to take L representations to K representations.

$$f = \frac{1}{|\Gamma_K|} \sum_{\text{irr } \chi \in R_L(G), t \in \Gamma_K} \langle f, \sigma_t \circ \chi \rangle (\sigma_t \circ \chi)$$

$$f = \frac{1}{|\Gamma_K|} \sum_{\text{irr } \chi \in R_L(G), t \in \Gamma_K} \langle \sigma_{t^{-1}} \circ f, \chi \rangle (\sigma_t \circ \chi)$$

$$f = \frac{1}{|\Gamma_K|} \sum_{\text{irr } \chi \in R_L(G), t \in \Gamma_K} \underbrace{\langle f, \chi \rangle}_{\in K} (\sigma_t \circ \chi)$$

$$f = \frac{1}{|\Gamma_K|} \sum_{\text{irr } \chi \in R_L(G)} \langle f, \chi \rangle \sum_t (\sigma_t \circ \chi)$$

$$f = \frac{1}{|\Gamma_K|} \sum_{\text{irr } \chi \in R_L(G)} \langle f, \chi \rangle (\operatorname{Tr} \chi)$$

Last equality is due to the fact:
$G \xrightarrow{\rho} GL_L(L^n) \xrightarrow{\operatorname{Tr}} GL_K(L^n)$
$\chi_{\operatorname{Tr} \circ \rho} = \sum \sigma_t \circ \chi_\rho$ ∎

Friday, 11/1/2024

Recap:

$$\begin{tikzcd}[column sep=small] & & \mathbb{C} \\ & L = K(\zeta_m) \arrow[ur, dash] & & m = \exp G \\ K \arrow[ur, dash] \end{tikzcd}$$

$$R_L G \xrightarrow{\cong} RG := R_{\mathbb{C}} G$$

$$\Gamma_K = \operatorname{image}\left(\operatorname{Gal}(L/K) \to (\mathbb{Z}/m)^\times\right)$$

$$\sigma_t \mapsto t$$

$$\sigma_t(\zeta_m) = \zeta_m^t$$

$$s' \underset{K}{\sim} s \, (s' \text{ is } K\text{-conjugate to } s)$$

If $\exists \tau \in G, t \in \Gamma_K$ such that:

$$\tau s' \tau^{-1} = s^t$$

Corollary 2, page 96: rank $R_K G = \#$ of K-conj classes.
13.1: $K = \mathbb{Q}$. Then,

$$\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \overset{\cong}{\to} (\mathbb{Z}/m)^\times$$

Thus,

$$s' \underset{\mathbb{Q}}{\sim} s \iff \exists \tau \in G \text{ s.t. } \tau\langle s'\rangle\tau^{-1} = \langle s\rangle$$

Corollary 1: $\#$ of $\mathbb{Q}G$-reps $= \#$ of conjugacy classes of cyclic subgroups.
Corollary 2: G finite, following TFAE:

i) $\langle s\rangle = \langle s'\rangle \implies s$ is conjugate to s'.

ii) $\#$ of conjugacy classes $= \#$ of conjugacy classes of cyclic subgroups.

iii) $\#$ of p.c.i in $\mathbb{Q}G = \#$ of p.c.i in $\mathbb{C}G$

iv) $\forall \rho : G \to GL(\mathbb{C}^n), \forall s \in G, \chi_\rho(s) \in \mathbb{Q}$ [characters are rational valued].

v) $\forall \rho : G \to GL(\mathbb{C}^n), \forall s \in G, \chi_\rho(s) \in \mathbb{Z}$.

Proof. "Think about it" $\square$

eg Symmetric grouo S_n satisfies (i).
Fact [stronger than this] $\mathbb{Q}S_n \cong \prod M_{n_i}(\mathbb{Q})$
eg $\mathbb{Q}S_3 = \mathbb{Q}D_6 \cong \mathbb{Q} \times \mathbb{Q} \times M_2(\mathbb{Q}[\lambda_3]) = \mathbb{Q} \times \mathbb{Q} \times M_2(\mathbb{Q})$.
All $\mathbb{C}$-rep of S_n are realizable over $\mathbb{Q}$.
"Young diagrams".
$G = Q_8$ also satisfies (i).
$\mathbb{Q}Q_8 \cong \mathbb{Q}^4 \times \mathbb{H}_\mathbb{Q}$
$\mathbb{C}Q_8 \cong \mathbb{C}^4 \times M_2(\mathbb{C})$
But irreducible representation $\mathbb{C}^2$ not realizable over $\mathbb{Q}$.

12.5

$$\begin{array}{ccc} & & \mathbb{C} \\ & / & \\ K & & \end{array}$$

Theorem 90 (Artin's Theorem)**.**

$$\bigoplus_{\text{cyclic } C<G} R_K C \otimes \mathbb{Q} \twoheadrightarrow R_K G \otimes \mathbb{Q}$$

Same proof as for $K = \mathbb{C}$.
Characters are determined by cyclics.

Theorem 91 (Brauer's Theorem)**.**

$$\bigoplus_{\text{elem } E<G} RE \twoheadrightarrow RG$$

Definition. E is <u>elementary</u> if $E = P \times C$ where P is p-group, C is cyclic, $(|P|, |C|) = 1$

Theorem 92 (Brauer's Theorem)**.**

$$\bigoplus_{\Gamma_K\text{-elem } E<G} R_K E \twoheadrightarrow R_K G$$

Definition. E is Γ_K-elementary if $E = C \rtimes_\phi P, P$ p-group, C cyclic, $(|P|, |C|) = 1$
If ϕ factos as

$$\begin{tikzcd} P \arrow[r] \arrow[rrr, bend right, "\phi"] & \Gamma_K \arrow[r, hook] & (\mathbb{Z}/m)^\times \arrow[r, two heads] & \mathrm{Aut}(C) \end{tikzcd}$$

13.2 $K = \mathbb{R}$

Fact: Only finite dimensional division algebras $/\mathbb{R}$ are $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$.
"Proof": $\operatorname{Br}\mathbb{R} = H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}); \mathbb{Z}/2) = \{\mathbb{R}, \mathbb{H}\}$.
$\mathbb{C}$ alg closed
$/$ $\deg 2$
$\mathbb{R}$
$\operatorname{Br}\mathbb{C} = 1$.
Thus only $\mathbb{R}, \mathbb{C}, \mathbb{H}$ are possible.
We achieve all:
$\mathbb{R}C_2 \cong \mathbb{R} \times \mathbb{R}$.
$\mathbb{R}C_3 \cong \mathbb{R} \times \mathbb{C}$
$\mathbb{R}Q_8 \cong \mathbb{R}^4 \times \mathbb{H}$
3 types of finite dimensional simple reps over $\mathbb{R}$.
3 types of irreducible $\mathbb{R}G$ reps
3 types of irreducible $\mathbb{Q}G$ reps
Let χ_0 be char of irreducible $\mathbb{R}G$ module.
$\chi =$ char of irreduible $\mathbb{C}G$ module
such that χ is a component of $\mathbb{C} \otimes_{\mathbb{R}} V_0 \iff \chi_0$ is a component of $\operatorname{res}\chi$.
Type O: $\chi = \chi_0$. Complexification gives you the same representation.
$\mathbb{R} = \operatorname{Hom}_{\mathbb{R}G}(V_0, V_0)$ by Schur.
Type U: $\chi \neq \overline{\chi}$. Then $\chi_0 = \chi + \overline{\chi}$.
$\mathbb{C} = \operatorname{Hom}_{\mathbb{R}G}(V_0, V_0)$
Type S_P: $\chi = \overline{\chi}, \chi = 2\chi_0$.
$\mathbb{H} = \operatorname{Hom}_{\mathbb{R}G}(V_0, V_0)$

Exercise. G odd order $\implies$ all nontrivial irreducible representation have type U.

Monday, 11/4/2024

$K = \mathbb{R}$

$\mathbb{R}C_3 = \mathbb{R} \times \mathbb{C}$
$\mathbb{R}Q_8 = \mathbb{R}^4 \times \mathbb{H}$
$\mathbb{C}C_3 = \underset{O}{\mathbb{C}_0} \times \underset{U}{\mathbb{C}_1} \times \underset{U}{\mathbb{C}_2}$
$\mathbb{C}Q_8 = \underset{O}{\mathbb{C}^4} \times \underset{S_P}{M_2(\mathbb{C})}$
χ type O if χ is realizeable over $\mathbb{R}$.
χ is type U if $\chi \neq \overline{\chi}$
χ is type S_P if $\chi = \overline{\chi}$ and χ is not realizable $/\mathbb{R}$.
Let $i = \mathbb{R}G \hookrightarrow \mathbb{C}G$.
Let χ_0 be irreducible component of $i^*x[= x \circ i]$.
χ type $O \iff \chi = \chi_0$
χ type $U \iff \chi_0 = \chi + \overline{\chi}$
χ type $S_P \iff \chi_0 = 2\chi$
Goal: Propoistion 39:

$$\sum_{g \in G} \chi(g^2) = \begin{cases} |G|, & \text{if } \chi \text{ has type } O; \\ 0, & \text{if } \chi \text{ has type } U; \\ -|G|, & \text{if } \chi \text{ has type } S_P. \end{cases}$$

Let V be finite dimensional vecto space over F.
A bilinear $B : U \times V \to F$ is nonsingular if:

$$\operatorname{Ad} B : V \overset{\cong}{\to} V^*$$

given by

$$x \mapsto (y \mapsto B(x, y))$$

$\iff \forall$ basis $\{e_i\}$ for V,

$$\det(B(e_i, e_j)) \neq 0$$

V is a FG-module, so is $V^* = \mathrm{Hom}_F(V, F)$. Action is like:

$$(g\phi)(v) = \phi(g^{-1}v)$$

$F = \mathbb{C}$ then,

$$\chi^*(g) = \overline{\chi(g)} = \chi(g^{-1})$$

Theorem 93 (31, FS)**.** $\rho : G \to GL_{\mathbb{C}}V, \chi = \chi_\rho : G \to \mathbb{C}$.

i) $\chi = \overline{\chi} \iff \exists$ nonsingular G-invariant form $B : V \times V \to \mathbb{C}$.

ii) χ realizable over $\mathbb{R} \iff \exists$ nonsingular symmetric G-invariant $B : V \times V \to \mathbb{C}$.

Proof. i) $\chi = \overline{\chi}(= \chi^*) \iff V \cong V^* \iff \exists G$-invariant nonsingular bilinear $V \times V \to \mathbb{C}$

ii) $\implies$: Let V real / $\mathbb{R}$. $V = \mathbb{C} \otimes_{\mathbb{R}} V_0$ where V_0 is $\mathbb{R}G$ module.
$\exists$ symmetric, positive definite $B : V_0 \times V_0 \to \mathbb{R}$.
$\implies$ symmetric, positive definite, G-invariant $B_1 : V_0 \to V_0$:

$$B_1(x, y) = \frac{1}{|G|} \sum_{g \in G} B(gx, gy)$$

Extension of scalars: Define $B_{\mathbb{C}} : V \times V \to \mathbb{C}$ by:

$$B_{\mathbb{C}}(z \otimes v, z', z' \otimes v') = zz' B_{\mathbb{C}}(v, v')$$

$\impliedby$: (outline)
Suppose we have nonsingular symmetric G-invariant $B : V \times V \to \mathbb{C}$.
Step 1: Choose G-invariant inner product:

$$\langle -, - \rangle : V \times V \to \mathbb{C}$$

[average any inner product]
Step 2: Definea bijection $\varphi : V \to V$:

$$B(x, y) = \overline{\langle \varphi(x), y \rangle}$$

φ is conjugate linear.
Step 3: $\varphi^2 : V \to V$ is $\mathbb{C}$-linear, hermitian w.r.t. $\langle -, - \rangle$ and has positive eigenvales.

$$\langle \varphi^2 x, y \rangle = \langle x, \varphi^2 y \rangle$$

Then φ^2 has positive eigenvalues.
Step 4: Spectral theorem $\implies \exists!$ square root $v : V \to V$ of φ^2.
$v : V \to V$ of φ^2.
v is $\mathbb{C}$-linear, and $v^2 = \varphi^2$ where v is hermitian, positive eigenvalues.
Step 5: Let $\sigma = \varphi \circ v^{-1}$.
$\sigma : V \to V$ is the conjugate linear with $\sigma^2 = \mathrm{Id}$.
Step 6: σ eigenvalues are 1 and -1. So we split into two eigenspaces: $V = V_+ \oplus V_-$.
$iV_+ = V_- \implies V = \mathbb{C} \otimes_{\mathbb{R}} V_+$ (since $V_+ = V_-$).

□

Corollary 94. Let V be an irreducible $\mathbb{C}G$-module.

a) If $\nexists$ non-zero G-invariant bilinear form $V \times V \to \mathbb{C}$ then V has type U.

b) A non-zero G-invariant bilinear form $V \times V \to G$ is unique up to a multiple.
   B symmetric $\iff$ V has type O.
   B alternating $[B(x,y) = -B(y,x)\ ]$ $\iff$ V has type S_P.

Proof. Note that in irreducible, by Schur, nonsingular iff nonzero. This also gives us the uniqueness upto a multiple in ii.
a $\iff$ i: Contrapositive.
ii: $B(x,y) = \frac{B(x,y)+B(y,x)}{2} + \frac{B(x,y)-B(y,x)}{2} = B_+ + B_-$.
Uniqueness $\implies$ $B_+ = 0$ or $B_- = 0$.
B symmetric $\iff$ V type O.
V type S_P $\iff$ not type O on V $\iff$ B alternates.

□

Wednesday, 11/6/2024

Proposition 95 (39)**.** Let $\chi = \chi_V$ be irreducible $/\mathbb{C}G$.

$$\sum_{g\in G} \chi(g^2) = \begin{cases} |G| & \text{if } \chi \text{ has type } O \\ 0 & \text{if } \chi \text{ has type } U \\ -|G| & \text{if } \chi \text{ has type } S_P \end{cases}$$

Proof. Use sym and alt squares 1.6, 2.1, 13.2.

$$sw : V \otimes_{\mathbb{C}} V \to V \otimes_{\mathbb{C}} V$$

$$a \otimes b \mapsto b \otimes a$$

$$sw^2 = \mathrm{id}$$

We know that $V \otimes_{\mathbb{C}} V = S(V) \oplus \Lambda(V) = V_\sigma \oplus V_a$
$S(V)$ is symmetric, $+1$ eigenspace containing $a \otimes a$ and $a \otimes b + b \otimes a$.
$\Lambda(V)$ is altrnating, -1 eigenspae containing $a \otimes b - b \otimes a$.
Then $(V_\sigma)^* = G$-invariant symmetric $V \times V \to \mathbb{C}$.
$(V_a)^* = G$-invariant alternating $V \times V \to \mathbb{C}$.

$$\begin{array}{cccc} \text{type} & & \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}, V_\sigma^*) & \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}, V_a^*) \\ O & \overset{Thm35}{\iff} & 1 & 0 \\ U & & 0 & 0 \\ S_P & & 0 & 1 \end{array} \quad (*)$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}, V_\sigma^*) = \langle 1, \overline{\chi}_\sigma \rangle = \frac{1}{|G|} \sum_{g\in G} \overline{\chi}_\sigma(g^{-1}) = \frac{1}{|G|} \sum_{g\in G} \chi_\sigma(g)$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}, V_a^*) = \frac{1}{|G|} \sum_{g\in G} \chi_a(g)$$

□

Proposition 96 (3)**.** $\chi_\sigma(g) = \frac{\chi(g)^2 + \chi(g^2)}{2}, \chi_a(g) = \frac{\chi(g)^2 - \chi(g^2)}{2}$.

Proof. $\rho_v(g)$ is diagonalizable with eigenvalue $\lambda_i \implies \chi_v(g) = \sum_i \lambda_i$ with eigenvector e_i.
V_σ has eigenvectors $e_i \otimes e_j + e_j \otimes e_i \; i \leq j$.
V_a has eigenvectors $e_i \otimes e_j - e_j \otimes e_i \; i < j$.

$$\chi_\sigma(g) = \sum_{i \leq j} \lambda_i \lambda_j = \frac{\left(\sum_i \lambda_i\right)^2 + \sum_i \lambda_i^2}{2} = \frac{\chi(g)^2 + \chi(g^2)}{2}$$

$$\chi_a(g) = \frac{\chi(g)^2 - \chi(g^2)}{2}$$

Proposition 3 + Table $(*)$ + $(**)$ implies Proposition 39.

$$\chi_{V \otimes V}(g) = \chi^2(g) = \chi_\sigma(g) + \chi_a(g)$$

□

Research Project?

Consider ring R and nonzero divisor $\Delta = \Delta_R = \left\{ r \in R \mid \forall r' \in R - 0, {}^{rr' \neq 0}_{r'r \neq 0} \right\}$.

Definition (Ore)**.** A left classical ring of quotient (q.r. = quotient ring) of R is a ring homomorphism $i : R \to A$:
$\forall a \in A, \exists r \in R, \exists \delta \in \Delta$ such that $a = i(\delta)^{-1} i(r)$.
We write:

$$A = \Delta^{-1} R$$

eg if R is a commutative domain then $\Delta^{-1} R = \mathrm{Frac}(R)$.
Question: What rings have q.r.?
Question: For what group G does $\mathbb{Z}G$ have a q.r.?
R commutative ring $\implies \exists$ q.r. by localization.
G finite $\implies \mathbb{Z}G$ has quotient ring, $\Delta^{-1}\mathbb{Z}G = \mathbb{Q}G$.
We don't know a lot about infinite groups.
$\mathbb{F}_2\langle x, y\rangle$ non-commutative polynmials and $\mathbb{Z}[F(2)]$ have no q.r.s.

Proposition 97.

$$R \text{ has q.r.} \iff \text{“Ore Conditions hold”} :$$

$\forall r \in R, \forall \delta \in \Delta$,

$$\Delta r \cap R\delta \neq \varnothing$$

Definition. G is virtually abelian if $\exists$:

$$1 \to \underset{abel}{A} \to G \to \underset{finite}{F} \to 1$$

G virtually abelian $\implies$ q.r. for G.

$$\Delta^{-1}_{\mathbb{Z}G} G = \left(\Delta^{-1}_{\mathbb{Z}A} \mathbb{Z}A\right) \otimes_{\mathbb{Z}A} \mathbb{Z}F$$

Now assume $A = \mathbb{Z}^n$.

$$1 \to \mathbb{Z}^n \to G \to \underset{finite}{F} \to 1$$

Remark. G is classified by 2 invariants.
$F \to GL_n(\mathbb{Z})$
and an extension class $\in H^2(F; \mathbb{Z}^n)$.

Theorem 98. $\Delta^{-1}\mathbb{Z}G$ is semisimple.

$$\Delta^{-1}\mathbb{Z}G \cong M_{d_i}(D_i)$$

Research project: Redo Parts I and II of Serre. h =? divisibility for d_i? types? Splitting fields? $\mathbb{Q}(\zeta_{|F|}) \otimes_{\mathbb{Z}} \Delta^{-1}\mathbb{Z}G \overset{?}{=} \prod M_j$ (fields)? induction theorem?
Warm up: $G = \mathbb{Z}^n \rtimes S_n$.
Q: $\Delta^{-1}\mathbb{Z}G =$??

Friday, 11/8/2024

Modular Representation Theory

Recall Maschke's theorem:
kG semisimple $\iff$ $(\operatorname{char} k, |G|) = 1$.
We ask the question: what happens if $\operatorname{char} k \mid |G|$?
eg $\mathbb{F}_p G$ where $p \mid |G|$.
It is not semisimple, but it is not BAD. For example, they're Artinian.
Motivation:
1. (Jim) study $\mathbb{Z}G$ modules.
$G \curvearrowright \widetilde{X} \to X$, $\pi_1 X = G$.
$H_n\widetilde{X}, \pi_n\widetilde{X}$ are $\mathbb{Z}G$ modules.
We can consider:

$$\begin{array}{ccc} \mathcal{O}_K & \text{———} & K \\ \Big| & & \Big|\,\text{Galois} \\ \mathbb{Z} & \text{———} & \mathbb{Q} \end{array}$$

$\mathcal{O}_K$ is $\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]$.
2. Classification of (simple) groups.
3. Algebraic K-theory: $K_*(\mathbb{F}_p)$. eg $G = GL_2(\mathbb{F}_p)$.
4. Non-abelian class field theory: Gal $\to GL_n(\mathbb{Z}_p)$. Here we want to deal with $\mathbb{Z}_pG$-modules.
Technique: Use p-adic integers $\mathbb{Z}_p$ to interpolate between $\mathbb{Q}$ and $\mathbb{F}_p$.
Now we start studying $\mathbb{F}_pG$.

Example. Exercise: Let p, q be distinct primes. Then,

$$\mathbb{F}_pC_q = \prod_{i=1}^{h} \mathbb{F}_{p^{f_i}}$$

What is h and f_i?
eg $\mathbb{F}_pC_2 \cong$ trivial rep and sign rep $\cong \mathbb{F}_p \times \mathbb{F}_p$
$\mathbb{F}_2C_q =$?
Hint: Multiplicative group of a finite field ($\mathbb{F}_p^\times$) is cyclic. $\mathbb{F}_2 \times C_3 \cong \mathbb{F}_2 \times \mathbb{F}_4$ since $\mathbb{F}_4^\times \cong \mathbb{Z}/(4-1) = \mathbb{Z}/3$.
It is given by $r \mapsto (1, \zeta_3)$.
$\mathbb{F}_2C_5 =$?
We have $\zeta_5 \in \mathbb{F}_{16}^\times \cong \mathbb{Z}/15$ so:
$\mathbb{F}_2C_5 \cong \mathbb{F}_2 \times \mathbb{F}_{16}$.
Actually we can say $\mathbb{F}_2C_5 = \mathbb{F}_2 \oplus \mathbb{F}_{16}$.
$\mathbb{F}_2C_7 \cong \mathbb{F}_2 \times \mathbb{F}_8 \times \mathbb{F}_8$.
$r \mapsto (1, \zeta_7, \zeta_7^3)$ or $r \mapsto (1, \zeta_7, \zeta^{-1})$
Minimal polynomial: $\Phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
$\Phi_7(x) = f(x)g(x) \in\in \mathbb{F}_2[x]$.
$\mathbb{F}_2C_7 = \frac{\mathbb{F}_2[x]}{(x^7-1)} = \frac{\mathbb{F}_2[x]}{(x-1)f(x)g(x)} \cong \frac{\mathbb{F}_2(x)}{x-1} \times \frac{\mathbb{F}_2[x]}{f(x)} \times \frac{\mathbb{F}_2[x]}{g(x)} \cong \mathbb{F}_2 \times \mathbb{F}_8 \times \mathbb{F}_8$.
Now, we deal with $p \neq 3$ and $\mathbb{F}_pC_3$.

$$\mathbb{F}_p C_3 = \begin{cases} \mathbb{F}_p \times \mathbb{F}_p \times \mathbb{F}_p, & \text{if } p \equiv 1(3); \\ \mathbb{F}_p \times \mathbb{F}_{p^2}, & \text{if } p \not\equiv 1(3). \end{cases}$$

How do we know $\mathbb{F}_2 C_7 \cong \mathbb{F}_2 \times \mathbb{F}_8 \times \mathbb{F}_8$ and not $\mathbb{F}_2 \times \mathbb{F}_{64}$?
The image of r lies in $\mathbb{F}_8$ so it is actually in $\mathbb{F}_2 \times \mathbb{F}_8$!
We look for the minimal field where the cyclotimic polynomial splits.

Modular Case

Complete list of ideals in $\mathbb{F}_2 C_2$.
$O \subset \langle 1-r \rangle \subset \mathbb{F}_2 C_2$.
$\langle 1-r \rangle$ is isomorphic to $\mathbb{F}_2$, simple, not projective [not summand of free modules].
Why is it not projective?
Consider the augmentation map:

$$\varepsilon : \underset{\sum_i r_i g_i}{RG} \underset{\mapsto}{\rightarrow} \underset{\sum_i r_i}{R}$$

It is a ring map.
Augmentation ideal $I = \ker(\varepsilon) \subset RG$.
We have Norm element $N = \sum_{g \in G} g \in RG$.
If G is a p-group then $N \in \ker(\varepsilon : \mathbb{F}_p G \to \mathbb{F}_p)$.
Aug map $\varepsilon : \mathbb{F}_2 C_2 \to \mathbb{F}_2$ as $\mathbb{F}_2 C_2$ module.
Therefore $\mathbb{F}_2$ is not projective over $\mathbb{F}_2 C_2$.
Complete list of finitely generated $\mathbb{F}_2 C_2$-modules (up to isomorphism):

$$(\mathbb{F}_2)^a \oplus (\mathbb{F}_2 C_2)^b$$

Complete list of $\mathbb{F}_p C_p$-ideals:

$$0 \subset \underset{\langle N \rangle}{\langle 1-r \rangle^{p-1}} \subset \cdots \subset \underset{\ker \varepsilon}{\langle 1-r \rangle} \subset \mathbb{F}_p C_p$$

Thus $\mathbb{F}_p C_p$ is local.
It is simple, not projective.
Complete list of finitely generated $\mathbb{F}_p C_p$-modules up to isomorphism: direct sum of ideals.

Definition. Ring R is semilocal if $R/J(R)$ is semisimple.

eg kG is always semilocal.
Serre p 163

Definition (Artinian Ring)**.** R is artinian if:

i) Every decreasing sequence of ideals is stationary.

ii) $\iff$ every f.g. R-module has finite length.

eg $\mathbb{Z}$ is not artinian, but kG is artinian.
This is because f.d. k-algebra is artinian.

Remark. If R is artinian then every finitely generated module has a minimal submodule and hence simple.

Theorem 99. If R is artinian then $\exists$ unique minimal 2-sided ideal $J(R)$ so that $R/J(R)$ is semisimple.

Here, $R/J(R)$ is the maximal semisimple quotient. $J(\mathbb{F}_p C_p) = \langle 1-r \rangle$ since the quotient is $\mathbb{F}_p$.
For a general ring R we have:

$$J(R) = \bigcup_{\substack{\text{max left} \\ \text{ideals}}} M$$

Despite having a one-sided definition it is a two sided ideal.
Then, $J(R)S = 0$ when S is a simple module.
R artinian:
Simple modules over $R \leftrightarrow$ simple modules over $R/J(R)$.

Monday, 11/11/2024

Simple vs Indecomposable

Simple and Indecomposable are not the same thing.
We have Jordan-Hölder Theorem and Krull-Schmidt Theorem.
Let R be a ring and M be a module. Then,
$l(M) = n$ if chain $0 = M_0 \subsetneq M_1 \subsetneq \cdot \subsetneq M_n = M$ and n is maximal.

Definition. Composition series for M is maximal chain $\iff$ all the quotient modules M_i/M_{i-1} are simple.

Definition. Module M is indecomposable if $M = A \oplus B \implies A = 0$ or $B = 0$.

Let M be of finite length.

Theorem 100 (Jordan-Hölder Theorem)**.** If M has finite length, then M has a composition series. Any two composition series have the same simple quotients.

Theorem 101 (Krull-Schmidt Theorem)**.** If M has finite length then $M = I_1 \oplus \cdots \oplus I_k$ with I_j indecomposable and if $M = I_1' \oplus \cdots \oplus I_{k'}'$ with I_j' independent then $k = k'$ and $I_j = I_{\sigma(j)}'$ for $\sigma \in S_k$.

Works for abelian categories, works for groups.
Group Ring where the ring is a field has finite length.
Consider $S_3 \cong D_6 = \langle r, s \mid r^3 = 1, s^2 = 1, srs = r^{-1} \rangle = C_3 \rtimes C_2$.
$\mathbb{Q}D_6 = \mathbb{Q} \oplus \mathbb{Q} \oplus M_2(\mathbb{Q})$

$$r \mapsto \left(1, 1, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\right)$$

$$s \mapsto \left(1, -1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

$\mathbb{F}_2 D_6 = ?$
We have: $\frac{1}{3}(1 + r + r^2)$ a central idempotent.

$$\mathbb{F}_2 D_6 = \mathbb{F}_2 C_2 \oplus M_2 \mathbb{F}_2$$

$\mathbb{F}_2 C_2$ is projective, not simple.

$$\mathbb{F}_2 D_6 = \mathbb{F}_2 C_2 \oplus \begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$$

$JH \implies \mathbb{F}_2, \mathbb{F}_2, (\mathbb{F}_2)^2, (\mathbb{F}_2)^2$.
Maximal semisimple quotient $\mathbb{F}_3 D_6/J = \mathbb{F}_3 C_2 = \mathbb{F}_3 \times \mathbb{F}_3$.
Jacobson Radical $J = \langle 1 - r \rangle$.
We have a (not central) idempotent: $e = \frac{1+s}{2}$. So we don't have block decomposition.
$\mathbb{F}_3 D_6 = \frac{\mathbb{F}_3 D_6}{1-e} \oplus \frac{\mathbb{F}_3 D_6}{e}$ not block decomposition.
Now we go back to Serre.
Let R be semisimple. Then Projective $\iff \oplus$ simple.
If R is Artinian, which is better? Both

Serre 14.1 Simple

The abelian group R_kG is $\mathbb{Z}[T]/R$ with generator set T where:
$T =$ isomorphism classes of finitely generated kG-modules $[M]$.
We have following relations R:
$[M] = [M'] + [M'']$ if there exists a short exact sequence:

$$0 \to M' \to M \to M'' \to 0$$

In the Maschke case the short exact sequence splits and so $M = M' \oplus M''$.
Ring with $- \otimes_k -$.
$S_k = S_kG =$ isomorphism classes of simple kG-modules.
$(R_{\mathbb{F}_2}D_6, +) \cong \mathbb{Z}^2, [\mathbb{F}_2 D_6] = [S_1] + [S_1] + [S_2] + [S_2]$.
$S_1 = \mathbb{F}_2, S_2 = \begin{bmatrix} * \\ * \end{bmatrix}$
$(R_{\mathbb{F}_3}D_6, +) \cong \mathbb{Z}^2$.
$[\mathbb{F}_3 D_6] = S_1' + S_1' + S_1' + S_2' + S_2' + S_2'$
We want to prove proposition 40:

Proposition 102 (Serre 40)**.** *S_k is $\mathbb{Z}$-basis for the representation ring $R_k(G)$ additively. $[s] \mapsto [s]$.*

Proof.

$$\mathbb{Z}[S_k] \leftrightarrow R_kG$$

$$\sum [M_i/M_{i-1}] \leftarrow\!\shortmid M$$

□

Projective Module Review

Let R be a ring.

Lemma 103. *R-module P. TFAE:*

i) $\exists Q$ such that $P + Q =$ free [has a basis].

ii) We have the following:

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dotted, "\exists"] \\ M \arrow[r, two heads] & N \end{tikzcd}$$

iii) A surjection to P splits.

$$\begin{tikzcd} M \arrow[r, two heads] & P \arrow[l, dotted, bend right] \end{tikzcd}$$

iv) SES

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & N \arrow[r] & P \arrow[r] \arrow[l, dotted, bend right] & 0 \end{tikzcd}$$

splits.

v) P isimage of projection.

$$\exists \pi \circ \pi = \pi : R^s \to R^s \,\text{s.t.} P \cong \pi(R^s)$$

eg $R = \mathbb{R} \times M_2\mathbb{R}$, $\begin{pmatrix} * \\ * \end{pmatrix} \cong \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$ is projective, not free.

Let R be a ring.
$K_0 R = \mathrm{Gr}(\text{iso chlass of f.g. projective } R\text{-modules}, \oplus)$.
Serre writes $P_A(G) = K_0(AG)$ for ring A.
$K_0(kG)$ is module over $R_k G$. [Not ring since we don't have identity].
Key point: $M \otimes_k kG \cong i^* M \otimes_k kG$ where $i : k \hookrightarrow kG$ is free.
$m \otimes g \mapsto g^{-1} m \otimes g$.
Note that $M \otimes_k$ proj is proj.

Wednesday, 11/13/2024

Serre 14.3

We are looking at kG, character possibly dividing $\#G$.

$$\begin{array}{ccc} \text{indecomposable} & & \text{simple} \\ K_0(kG) & & R_k G \\ P & \mapsto & P/J(R)P \\ P_S & \leftarrow\!\shortmid & S \\ \text{projective cover} & & \end{array}$$

Definition. $f : M \to M'$ is <u>essential</u> if:

- f onto.
- $\forall M'' \subsetneq M'$, $f\big|_{M''}$ not onto.

The idea is f is essential if it is 'barely onto'.

Definition. $f : P \to M$ where P is projective and f is essential is a <u>projective cover</u>.
Note: P is the projective cover of M.

Proposition 104 (4.1)**.** If $l(M) < \infty$ there exists projective cover, unique upto isomorphism.
If P is projective and E is maximal semisimple quotient, then $P \to E$ is a projective cover.
eg if R is <u>artinian</u>, then $l(M) < \infty \iff M$ finitely generated.
P projective implies $P \to P/JP$ is projective cover. P/JP is semisimple.
eg $\mathbb{F}_2 C_2 \to \mathbb{F}_2$ is a projective cover.
$e = \frac{1+s}{2}$, $\mathbb{F}_3 D_6 e \to \mathbb{F}_3$ is a projective cover.
$\overset{\text{proj}}{\mathbb{F}_3 D_6} \underset{\text{essential}}{\twoheadrightarrow} \overset{\text{s.s.}}{\mathbb{F}_3 C_2}$.

Proof. Existence:

- Choose SES (choice in blue):

$$0 \to R \to \overset{\text{proj}}{L} \to M \to 0$$

- Choose $N \subset R$ minimal such that:

$$L/N \overset{\text{ess}}{\to} M$$

  Let $P \coloneqq L/N$.

- Let $Q \subset L$ minimal such that:

$$\begin{tikzcd} & L \arrow[d, two heads] \\ Q \arrow[ur, hook] \arrow[r, two heads, "\text{onto}"] & P \end{tikzcd}$$

- Choose lift

$$\begin{tikzcd} & L \arrow[d, two heads] \arrow[dl, dotted, "q"'] \\ Q \arrow[r] & P \end{tikzcd}$$

2nd choice and 3rd choice implies:

$$0 \to N \to L \xrightarrow{q} Q \to 0$$

SES $\implies P \cong Q$.

3rd choice and 4th choice $\implies$ $\begin{tikzcd} L \arrow[r, "q"'] & Q \arrow[l, bend right, dashed, "i"'] \end{tikzcd}$ split.
$L \cong N \oplus Q \cong N \oplus P$, P projective.
Uniqueness:

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, "\text{lift } q \cong"'] \\ P' \arrow[r, two heads] & M \end{tikzcd}$$

$P' \to M$ essential so q onto.
$P \to M$ essential so q is 1-1.

$\square$

Suppose R is artinian eg $R = kG$.

Corollary 105 (1)**.**

$$\text{proj. indecomposable } R\text{-mod} \leftrightarrow \text{simple} R\text{-mod}$$

$$P \mapsto P/JP$$

$$P_E \leftarrow\!\shortmid E$$

Corollary 106. Let \$ be isomorphism classes of simple R-modules.
$\{P_E\}_{E \in \$}$ form a basis of $K_0 R$.

Corollary 107. f.g. projective R-modules P and P', $[P] = [P'] \in K_0 R \iff P \cong P'$.
No stabilization required!

Proof. ?: Suppose $[P] = [P'] \in K_0(kG)$.
$\iff [s] = [s'] \in R_k G$ $[s = P/JP]$
$\iff s \cong s'$
$\iff P \cong P'$. $\square$

Setting of Chapter 14, p-adics

Consider $((K, \nu), A, \mathfrak{m}, k)$.
Example: $(\mathbb{Q}_p, \nu_p), \mathbb{Z}_p, p\mathbb{Z}_p, \mathbb{F}_p$.

Definition (p164)**.** A discrete valuation (K, v) is a field K and a homomorphism $\nu : K^\times \twoheadrightarrow \mathbb{Z}$ such that $\nu(x + y) \geq \min(\nu(x), \nu(y))$.

Basic example: $K = \mathbb{Q}$ then ν_p is the power of p in the factorization.
Generalize: If A is a PID and we have prime $P \lhd A$ we have a discrete valuation $(\text{Frac}(A), \nu_P)$.
Let (K, ν) bea discrete valuation.

Definition. Valuation ring of (K, ν) is:

$$A = \nu^{-1}\mathbb{Z}_{\geq 0}$$

This is a DVR (discrete valuation ring) (= PID with unique maximal ideal).
Maximal ideal is
$\mathfrak{m} = \nu^{-1}\mathbb{Z}_{>0}$
eg for $(\mathbb{Q}, \nu_P)$ we have $A = \mathbb{Z}_{(P)}$.
For (K, ν) we have an absolute value on K which gives us a metric on K.
$|x| = e^{-\nu(x)}$.
metric: $d(x, y) = |x - y|$.

Fact: Completion of K (use Cauchy sequences) $\widehat{K}_\nu$ is also a field with discrete valuation ν.
K is complete if $K = \widehat{K}_\nu$.

Friday, 11/15/2024

Basic plan for learning p-adic: Suppose we want to study $\mathbb{F}_pG$. If $p \mid |G|$ then Maschke doesn't work. So we mod out the Jacobson RadicaL $\mathbb{F}_pG/$.
Our setting:

$$(\underbrace{(K, \nu)}_{\text{complete D.V}}, \underbrace{A}_{\text{valuation ring}}, \underbrace{\mathfrak{m}}_{\text{maximal}}, \underbrace{k}_{\text{residue field}})$$

eg $((\mathbb{Q}_p, \nu_p), \mathbb{Z}_p, p\mathbb{Z}_p, \mathbb{F}_p)$
In $\mathbb{Q}, \nu_p$, $\nu_p(p^n \frac{a}{b}) = n$.
Renormalize: $\|x\| = p^{-\nu(x)}$

$$\lim_{n \to \infty} p^n = 0$$

$\mathbb{Q}_p$ is completion of $\mathbb{Q}$ under $\|x - y\|_p$

$$\mathbb{Q}_p = \left\{ \sum_{i=-k}^{\infty} a_i p^i \mid 0 \leq a_i \leq p - 1 \right\}$$

$$\mathbb{Z}_p = \left\{ \sum_{i=0}^{\infty} a_i p^i \mid 0 \leq a_i \leq p - 1 \right\}$$

$$\mathfrak{m} = \left\{ \sum_{i=1}^{\infty} a_i p^i \mid 0 \leq a_i \leq p - 1 \right\}$$

Better Approach

We use the inverse limit to define it.

$$\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n = \left\{ (b_n) \in \prod \mathbb{Z}/p^n \mid b_{n+1} \equiv b_n \pmod{p^n} \right\}$$

Compact by Tychonoff.
$\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$.

The case $p = 2$

$p = 2$ consider binary expansion.
In $\mathbb{Z}$, 11011 is finite.
In $\mathbb{R}$ we can have $\underbrace{11011}_{\text{finite}}.\underbrace{101110110\cdots}_{\text{infinite}}$
In $\mathbb{Q}_2$ we ca have $\underbrace{\cdots 1011011}_{\text{infinite}}.\underbrace{01101}_{\text{finite}}$
Thus we can have algorithms for adding and other stuff.

Serre 14.4

Lemma 108 (Lemma 20)**.** Let Λ be a commutative ring and P be a ΛG-module. P projective $/\Lambda G \implies P$ projective $/\Lambda$ and $\exists \Lambda$-map $u : P \to P$ so that:

$$\sum_{s \in G} su(s^{-1}x) = x \,\forall x \in P$$

Serre writes it as:

$$\sum_{s \in S} sus^{-1} = 1$$

Proof. Ommitted. Just computation □

Lemma 109 (Lemma 21)**.** Let Λ be local ring, $k = \Lambda/\mathfrak{m}$.

a) Let P be a ΛG-module free $/\Lambda$

$$P \text{ proj.}/\Lambda G \iff \overline{P} = P \otimes_\Gamma k \text{ proj}/kG$$

b) Projectives P, P' implies $P \cong P' \iff \overline{P} \cong \overline{P'}^k$

Proof. Idea: the maps are matrices, we show their determinants are invertible. Local means we need to show dets are not in max ideal.

a) $\implies$ part is clear. We do $\impliedby$:
$\overline{P}$ projective. Lemma 20 implies $\exists \overline{u} : \overline{P} \to \overline{P}$ k-map so that:

$$\sum s\overline{u}s^{-1} = 1$$

We "lift $\overline{u}$".

$$\begin{tikzcd}
& P \arrow[d] \arrow[ddl, dashed, "u"'] \\
& \overline{P} \arrow[d, "\overline{u}"] \\
P \arrow[r, two heads] & \overline{P}
\end{tikzcd}$$

Then $u' = \sum sus^{-1} \equiv 1 \mod \mathfrak{m}$.
Thus u' is ΛG-map, $\det u' \notin \mathfrak{m} \implies \det u' \in \Gamma^\times \implies u'$ invertible.

$$\sum su(u')^{-1}s^{-1} = u'(u')^{-1} = 1 \overset{L20}{\implies} P \text{ proj}$$

b) Let $\overline{w} : \overline{P} \overset{\cong}{\to} \overline{P}'$. Lif $w : P \to P'$. Then $\det w \notin \mathfrak{m} \implies w$ is invertible and thus is isomorphism.

□

Proposition 110 (42)**.** Let A be a complete local ring.

a) E is AG-module. Then E proj / $AG \iff E$ free $/A$ and $\overline{E}$ projective $/kG$.

b) If F is projective kG-module, $\exists!$ projective P/AG such that $\overline{P} \cong F$.

Corollary 111. There exists bijection:

$$\begin{array}{ccccc}
\text{proj indecom} & & \text{proj. indecom} & & \text{simple} \\
AG\text{-mod} & \to & kG\text{-mod} & \to & kG/J\text{-mod} \\
------ & & ------ & & ------ \\
\text{iso} & & \text{iso} & & \text{iso}
\end{array}$$

Now we go back to proposition 42.

Proof of Lemma 21. Lemma 21 $\implies$ a and uniqueness.Question: existence?
F projective kG-module.

$$A = \lim A/\mathfrak{m}^n$$

$(A/\mathfrak{m}^n)G$ is Artinian.
$\exists$ projective cover $P_n \to F$ of $(A/\mathfrak{m}^n)G$-modules.

$$\begin{tikzcd} & P_{n+1} \arrow[dl, dashed] \arrow[d] \\ P_n \arrow[r] & F \end{tikzcd}$$

We have $\cdots \to P_3 \to P_2 \to P_1 \to P_0$
Let $P = \lim_{\leftarrow} P_n$, detailed ommitted. P projective AG-module, $\overline{P} = P \otimes_A k$. $\square$

Monday, 11/18/2024

14.3 and 14.4 Review

In (A, k) [eg $\mathbb{Z}_p, \mathbb{F}_p$] we say A is a complete local ring where valuation ring is complete (K, ν). $k = A/\mathfrak{m}$ is the residue field.
Suppose we have our finite group G. We have the 'reduction mod $\mathfrak{m}$' homomorphism:

$$AG \xrightarrow{\pi} \mathbb{F}_pG$$

Then we have:

$$AG \xrightarrow{\pi} \mathbb{F}_pG \xrightarrow{p} \mathbb{F}_pG/J(\mathbb{F}_pG)$$

J indicates the Jacobson Radical.
We have bijections.

$$\begin{array}{ccccc} \text{basis } K_0(AG) & & \text{basis } K_0(\mathbb{F}_pG) & & \text{basis } R_kG \\ & & & & \\ \text{proj indecom} & & \text{proj. indecom} & & \text{simple} \\ AG\text{-mod} & \to & kG\text{-mod} & \to & kG/J\text{-mod} \\ ------ & \pi_* & ------ & p_* & ------ \\ \text{iso} & & \text{iso} & & \text{iso} \end{array}$$

If M is an AG-module then $\pi_* M = \mathbb{F}_pG \otimes_{AG} M$.
We have $\underset{\text{essential}}{P_E \to E} \hookleftarrow E$
Recall that essential maps are maps that are 'barely surjective'.
We have $P = \lim_{\leftarrow} P_n \hookleftarrow \overline{P}$
$P_n \to \overline{P}$ projective cover of $(A/\mathfrak{m}^n)G$-modules.
Now we deal with the case $\operatorname{char} K = 0, \operatorname{char} k = p$. Recall that K has a valuation ring A with unique maximal ideal $\mathfrak{m}$ and $k = A/\mathfrak{m}$.

Definition. $\begin{Bmatrix} K \\ k \end{Bmatrix}$ is a splitting field for G if:

$$KG \cong \prod M_{n_i} K$$

$$kG/J \cong \prod M_{l_i}(k)$$

Definition. $\begin{Bmatrix} K \\ k \end{Bmatrix}$ is sufficiently large if $\begin{Bmatrix} K \\ k \end{Bmatrix}$ contains all $\begin{Bmatrix} m \\ m' \end{Bmatrix}$.
Where $m = \operatorname{lcm}\{\operatorname{ord}(G) \mid g \in G\} = \exp G$ where $m' = m/p^a$ where $(p, m') = 1$.

Fact: sufficiently large $\implies$ splitting fields.
K due to Brauer, k see remark in 14.5.

Example. $\mathbb{F}_5[C_3] \cong \mathbb{F}_5 \times \mathbb{F}_{25}$. So $\mathbb{F}_5$ is not splitting field.
$\mathbb{F}_{25}[C_3] \cong \mathbb{F}_{25}^3$ so $\mathbb{F}_{25}$ is splitting field for C_3.

Definition. E is absolutely simple if $\dim_{\begin{Bmatrix} K \\ k \end{Bmatrix}} \mathrm{Hom}_{\begin{Bmatrix} KG \\ kG \end{Bmatrix}}(E, E) = 1$.

14.5 Dualities

Suppose $\operatorname{char} K = 0$.
If E, F are KG-modules, we can define:

$$\langle E, F\rangle = \dim_K \mathrm{Hom}_{KG}(E, F) = \langle E, F\rangle = \langle \chi_E, \chi_F\rangle$$

We thus have bilinear $\langle , \rangle : R_k G \times R_k G \to \mathbb{Z}$.
Simples $[E]$ are orthogonal basis.
Orthonormal iff K is a splitting field for G.
Now suppose $\operatorname{char} k = p \mid \#G$.
$\langle , \rangle : R_k G \times R_k G \to \mathbb{Z}$ is <u>not bilinear</u>! This is because SES don't split.
Take $0 \to \mathbb{F}_2 \to \mathbb{F}_2 C_2 \to \mathbb{F}_2 \to 0$. But if we take $\mathrm{Hom}_{\mathbb{F}_2 C_2}(\mathbb{F}_2 C_2, \mathbb{F}_2)$ but $\langle \mathbb{F}_2 C_2, \mathbb{F}_2\rangle \neq \langle \mathbb{F}_2, \mathbb{F}_2\rangle + \langle \mathbb{F}_2, \mathbb{F}_2\rangle$.
But the following is bilinear:

$$\langle , \rangle : K_0(kG) \times R_k G \to \mathbb{Z}$$

If k is a splitting field then $\{P_E\}$ and $\{E\}$ are dual bases.
$\mathrm{Hom}_{kG}(P_E, E') \cong \mathrm{Hom}_{kG}(E, E')$ for E, E' simple.

14.6

Consider K'/K. Then we have $R_K G \hookrightarrow R_{K'} G$.
This is an injection.
This is infact a <u>split injection</u> [so there's a map backwards] iff $\forall$ simple $E, \langle E, E\rangle = 1$ [so the schur index $= 1$].
Isomorphism $\iff$ K is a splitting field.
All follow from KG semisimple:

$$M_n(D) \otimes_K K' = M_n(D \otimes_K K')$$

Example. $R_{\mathbb{R}}(Q_8) \to R_{\mathbb{C}}(Q_8)$:
We have the matrix:

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 2 \end{bmatrix}$$

Since $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$ as rings and $\cong \mathbb{C}^2 \oplus \mathbb{C}^2$ as module and also $\langle \mathbb{H}, \mathbb{H}\rangle_{\mathbb{R}Q_8} = 4$.
So not split injection.

Theorem 112 (Wedderburn)**.** Finite $\begin{Bmatrix} \text{integral domain} \\ \text{skew field} \end{Bmatrix}$ is a field.

Consider $k'/k, R_k(G) \to R_{k'} G, K_0(kG) \to K_0(k'G)$.
These are split injection.
Isomorphism iff k' is spliting field for G.
"Setting":

$$\begin{tikzcd} k' \arrow[d] & A' \arrow[l] \arrow[r] \arrow[d, no head] & K' \arrow[d, no head, "\text{finite}"] \\ k & A \arrow[l] \arrow[r] & K \end{tikzcd}$$

Here A' = integral closure of A in K'
We have:

$$\begin{tikzcd} K_0(AG) \arrow[r] \arrow[d, "\pi_* \cong"] & K_0(A'G) \arrow[d, "\cong"] \\ K_0(kG) \arrow[r] & K_0(k'G) \end{tikzcd}$$

$K_0AG \to K_0A'G$ is splitting.
Isomorphism if K is sufficiently large.

Wednesday, 11/20/2024

CDE Triangle

Recall:
A = completely local ring
K = field of fractions
k = residue field.

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, two heads] & K \\ k & \end{tikzcd}$$

The CDE triangle is the following:

$$\begin{tikzcd} K_0(kG) \arrow[rr, "c", "\text{forgetful}"'] \arrow[dr, "e", "\text{lift}"'] & & R_kG \\ & R_KG \arrow[ur, "d", "\text{lattice}"'] & \end{tikzcd}$$

Each group has a canonical basis.
Therefore, we have matrice C, D, E.

Exercise. Compute C, D, E for $k = \mathbb{F}_2, G = C_6, D_6$.

15.1: $c[P] = [P]$
S = isomorphism classes of simple kG modules.

$$K_0(kG) \xrightarrow{c} R_kG$$

$$\{P_E\}_{E \in S} \qquad\qquad \{E\}_{E \in S}$$

$$C \text{ is square} \qquad\qquad C = (C_{FE})$$

$c[P_E] = \sum_{F \in S} C_{FE}[F]$
$C_{FE} = \#$ of F factor in composition series for P_E.
$d : R_KG \to R_kG$
Let E be finitely generated KG-module.

Definition. A <u>G-lattice</u> in E is a finitely generated AG-submodule of E.

Remark. Existence: If $\{e_1, \cdots, e_n\}$ generates E, then $E_1 = \sum_{i=1}^n AGe_i \subset E$ is G-lattice.

E_1 is G-lattice in E.

$$\overline{E_1} := E_1/\mathfrak{m}E_1 (= k \otimes_A E_1)$$

Define $d[E] = [\overline{E_1}]$
Is d well defined? Proof later!
$e : K_0(kG) \to R_K G$:

$$\begin{tikzcd}
K_0(kG) \arrow[rr, bend left, "e"] & K_0(AG) \arrow[l, "\cong"'] \arrow[r] & K_0(KG) = R_K G \\
\overline{p} \arrow[r, mapsto] & p & KG \otimes_{AG} P \arrow[d, equal] \\
 & & K \otimes_A P
\end{tikzcd}$$

Remark. i) c is defined for any field k.

ii) d is defined when A is a local ring

iii) e is defined wen A is a complete local ring

Remark. The triangle commutes: $c = d \circ e$.

Lemma 113. *d and e are adjoints.*

$$\langle x, d(y) \rangle_k = \langle e(x), y \rangle_K$$

for all $x \in K_0(kG)$ and $y \in R_K G$

Proof. $x = [\overline{X}]$ where X is a projective AG-module.
$y = [K \otimes_A Y]$ where Y is AG-module which is A-free.
$\mathrm{Hom}_{AG}(X, Y)$ is projective A-module. Thus it is a free A-module.
Let r be the rank.
$\langle -, - \rangle_k : K_0(kG) \times R_k G \to \mathbb{Z}$
$\langle A, B \rangle = \dim_k \mathrm{Hom}_{kG}(A, B)$
$\langle x, d(y) \rangle_k = \dim_k \mathrm{Hom}_{kG}(\overline{X}, \overline{Y}) = \dim_k (k \otimes_A \mathrm{Hom}_{AG}(X, Y)) = r$
$\langle e(x), y \rangle_K = \dim_K \mathrm{Hom}_{KG}(K \otimes_A X, K \otimes_A Y) = \dim_K K \otimes_A \mathrm{Hom}_{AG}(X, Y) = r$ ∎

Remark. For K sufficiently large [$\zeta_m \in K, m = \exp(G)$] implies K, k are both splitting fields.
Thus, bases of $K_0(kG)$ and $R_k G$ are duals. Basis of $R_K G$ is orthonormal. So, $\langle -, - \rangle_k$ are perfect parings.
Therefore, $E = D^T$.
Then $C = DE = DD^T \implies C$ is symmetric.

We now prove that d is well-defined.

Friday, 11/22/2024

<u>G-lattice</u> in f.g. KG-module E is f.g. AG-submodule E_1 such that $E = KE_1$.

$$\overline{E}_1 = E_1/\mathfrak{m}E_1$$

$$d[E] = [\overline{E}_1]$$

We want to show this is well defined.

Lemma 114. *If E_1 and E_2 are G-lattices in E, then $[\overline{E}_1] = [\overline{E}_2]$.*

Proof. Recall: $d[E] = [\overline{E_1}]$ where $E_1 \subset E$ is finitely generated AG-submodule and $\overline{E_1} = E_1/\mathfrak{m}E_1$.
Case A: $\mathfrak{m}E_1 \subset E_2 \subset E_1$
Consider:

$$0 \to E_2 \to E_1 \to E_1/E_2 \to 0$$

Third isomorphism theorem:

$$\implies 0 \to E_2/\mathfrak{m}E_1 \to E_1/\mathfrak{m}E_1 \to E_1/E_2 \to 0$$

Thus,

$$(*) 0 \to E_2/\mathfrak{m}E_1 \to \overline{E_1} \to E_1/E_2 \to 0$$

We also have:

$$0 \to \mathfrak{m}E_1 \to E_2 \to E_2/\mathfrak{m}E_1 \to 0$$

Then,

$$0 \to \frac{\mathfrak{m}E_1}{\mathfrak{m}E_2} \to \frac{E_2}{\mathfrak{m}E_2} \to E_2/\mathfrak{m}E_1 \to 0$$

$$\implies (**) 0 \to E_1/E_2 \to \overline{E_2} \to E_2/\mathfrak{m}E_1 \to 0$$

Splicing $(*)$ and $(**)$ we get:

$$\begin{tikzcd}
0 \arrow[r] & E_2/\mathfrak{m}E_1 \arrow[r] & \overline{E_1} \arrow[rr] \arrow[dr, two heads] & & \overline{E_2} \arrow[r] & E_2/\mathfrak{m}E_1 \arrow[r] & 0 \\
& & & E_1\chi E_2 \arrow[ur, tail] & & &
\end{tikzcd}$$

$\implies [\overline{E_1}] = [\overline{E_2}]$
Case B: $E_2 \subset E_1$ $\exists n$ such that $\mathfrak{m}^n E_1 \subset E_2 \subset E_1$.
We show that $[\overline{E_1}] = [\overline{E_2}]$ by induction on n. Case A was our base case.
Let $E_3 = \mathfrak{m}^{n-1}E_1 + E_2$.
$\mathfrak{m}^{n-1}E_1 \subset E_3 \subset E_1$ and $\mathfrak{m}E_3 \subset E_2 \subset E_3$.
Induction hypothesis $\implies [\overline{E_1}] = [\overline{E_3}] = [\overline{E_2}]$.
<u>General Case</u>: G-lattices E_1, E_2 then $\exists l \in A \setminus \{0\}$ such that $lE_2 \subset E_1$.

□

15.5 p' group

i.e. $p \nmid \#G$
$\mathbb{F}_p G$ semisimple.
central idempotents of $\mathbb{Q}G \subset \frac{1}{|G|}\mathbb{Z}G \subset \mathbb{Z}_{(p)}G \subset \mathbb{Z}_p G$

Proposition 115 (43)**.** Premise is as before. Then,

i) All kG-modules are projective.

   All A-free AG-modules are projective.

ii) $\begin{array}{ccc} S_K & \to & S_k \\ E & \mapsto & \overline{E_1} \end{array}$ is bijective.

iii) $C = D = E = I$.

Proof. i) kG semisimple from Maschke.

Let P be an A-free AG-module.

We will prove that any epomorphism to P splits.

Consider $M \overset{\pi}{\twoheadrightarrow} P$

P is A-free, $\exists A$-splitting $M \overset{s}{\leftarrow} P$.

Then we 'average':

$$\widehat{s}(p) = \frac{1}{|G|} \sum_{g \in G} gs(g^{-1}p)$$

$\implies \widehat{s}$ is AG-map.

$\implies \widehat{s}$ is splitting. So we are done.

ii and iii:

$$\begin{tikzcd}
\frac{\text{f.g. } KG\text{-module}}{\cong} & \frac{A\text{-free f.g. } AG\text{-mod}}{\cong} \arrow[l, "\cong"', "K\otimes_A"] \arrow[d, no head, "||"] & \frac{\text{f.g. } kG\text{-mod}}{\cong} \arrow[l, dotted, "\text{proj. cover}"] \\
 & \text{f.g. proj } AG\text{-mod} \arrow[ur, "\cong", "k\otimes_A"']
\end{tikzcd}$$

with $\text{f.g. } KG\text{-module} \xrightarrow{G\text{-lattice}} A\text{-free f.g. } AG\text{-mod}$.

$$\begin{tikzcd}
S_K & S_A \arrow[l, "\cong"'] \arrow[r, "\cong"] & S_k
\end{tikzcd}$$

simple KG-mod proj AG-mod $C = D = E = I$

$\square$

Jacobson Radical

Suppose $\operatorname{char} k = p$

Theorem 116 (Davis Thesis)**.** Suppose we have a p-group $P \lhd G$. $\forall p \in P, p - 1 \in J(kG)$

Corollary 117 (1)**.**

$$1 \to P \to G \to Q \to 1 \implies G = P \rtimes Q.$$

Here Q is a p'-group.
$kG/J(kG) \cong kQ$ is "largest semisimple quotient".

Corollary 118. $1 \to P \to G \to Q \to 1$
$kG/J(kG) \cong kQ/J(kQ)$.

We redefine Jacobson Radical:
Old def: $J(R) = \bigcap_{M \text{ max left}} M$
New Def: $J(R) = \bigcap_{\text{simple } E} \operatorname{Ann}(E)$.
Recall:

$$\operatorname{Ann} E = \{r \in R \mid rE = 0\}$$

$\operatorname{Ann} E$ is 2 sided ideal.
$JE = 0$.

P64 Serre

Theorem 119 (L1)**.** Suppose a p-group $P \curvearrowright X$ finite set.

$$|X^G| \equiv |X| \pmod p$$

Proof. $X - X^G = \sqcup \text{orbits} = \sqcup Gx \cong \sqcup G/G_x$ □

Theorem 120 (L2)**.** If M is f.g. kP-module, then $M^P \neq 0$

Proof. Can assume k finite $\implies \#M$ finite.

$$0 \equiv |M| \equiv |M^p| \pmod p$$

□

Now we prove that $p - 1 \in J(kG)$.

Proof. Let E be a simple kG-module.
$E^p \subset E$ is a kG-submodule (use $P \triangleleft G$).
$L2 \implies 0 \neq E^p \implies E^p =$
Thus, $\forall p \in P, p - 1 \in \operatorname{Ann} E \implies p - 1 \in J(kG)$ □

Monday, 12/2/2024

Recall that we are working on group with characteristic p. Maschke's theorem does not work.
Also recall the CDE triangle:

$$\begin{tikzcd} K_0(kG) \arrow[rr, "c", "\text{forgetful}"'] \arrow[dr, "e", "\text{lift}"'] & & R_k G \\ & R_K G \arrow[ur, "d", "\text{lattice}"'] & \end{tikzcd}$$

The setting of part 3 of Serre is that we have a valuation ring A, fraction field K and residue field k, eg $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{F}_p$.
Recall 15.7:
Serre: $G = P \times Q$ where P is a p group and Q is a p' group.
Davis: $G = P \rtimes Q$.
$\iff \exists$ Split SES:

$$1 \to P \to G \underset{\pi}{\overset{s}{\leftrightarrows}} Q \to 1$$

$\pi \circ s = \mathrm{id}_Q$.
Recall that $\pi : G \to Q$ gives us $\pi : kG \to kQ$ and thus we have π^* and π_*
Recall: if we have $f : R \to S$ we have exactness preserving $f^* : S\text{-mod} \to R\text{-mod}$.
Also, if we have $f_* : R \to S$ we have projectiveness preserving $f_* : R\text{-mod} \to S\text{-mod}$.

Theorem 121. $\exists$ bijections:

a) isomorphism classes of simple kG-modules $\underset{\pi^*}{\overset{s^*}{\rightleftarrows}}$ isomorphism classes of simple kQ-modules.

b) isomorphism classes of projective indecomposable kG modules $\underset{s_*}{\overset{\pi_*}{\rightleftarrows}}$ isomorphism classes of projective indecomposable kQ-modules.

c) isomorphism classes of projective indecomposable AG-modules $\underset{s_*}{\overset{\pi_*}{\rightleftarrows}}$ isomorphism classes of projective indecomposable AQ-modules.

Remark. $E \cong \pi^* F \iff P$ acts trivially on E.
Will prove: π^*, π_*, π_* are bijections, s^*, s_*, s_* are 1-sided inverses $\iff$ 2-sided inverses.
$kG/J(kG) \cong kQ$.

Proof. a and b are general facts:
R artinian means R/J is the maximal semisimple quotient. We have $R \xrightarrow{\pi} R/J$.
Then we have simple R-mod $\underset{\cong}{\overset{\pi^*}{\leftarrow}}$ simple R/J-mod.
Recall $J = \bigcap_{\text{simple } R\text{-mod } E} \operatorname{Ann}(E)$.
p.i R-mod $\underset{\cong}{\overset{\pi^*}{\rightarrow}}$ simple R/J-mod by projective cover.
Thus we are done with a and b.
c:

$$\begin{tikzcd}
\text{p.i } AG\text{-mod} \arrow[r, "\pi_*", "\pi \text{ split}"'] \arrow[d, "p_*", "14.4"'] & \text{p.i } AQ\text{-mod} \arrow[d, "p_* \text{ 15.5 Maschke}", "\cong"'] \\
\text{p.i } kG\text{-mod} \arrow[r, "\pi_*", "\cong (b)"'] & \text{p.i } kQ\text{-mod}
\end{tikzcd}$$

∎

Corollary 122. If $G = P \times Q$ matrix $C = |P| \cdot$ identity.

Proof. Uses a and b.

$$\begin{tikzcd}
K_0kG & R_kQ \arrow[l, "s_*"', "\cong"] \arrow[r, "\pi^*", "\cong"'] & R_kG \arrow[l, bend right, "s^*"']
\end{tikzcd}$$

basis basis basis

$s_*F_1, \cdots, s_*F_t$ $F_1, \cdots, F_t$ $\pi^*F_1, \cdots, \pi^*F_t$

$$s^*Cs_*F_i = s^*(kG \otimes_{kQ} F_i) = s^*(kP \otimes_k F_i) = k^{|P|} \otimes_k F_i = F_i^{|P|}$$

∎

Question: what is C for $P \rtimes Q$?
Next time: First theorem of chapter 16 [theorem 33]: d in the CDE triangle is surjective.

Remark. d is split, since R_kG is free abelian.
d is onto since every k-representation can be lifted to K virtually.

Wednesday, 12/4/2024

Brauer Induction Theorem (BIT)

Definition. E is <u>p-elementary</u> if $E \cong P \times C$ where P is a p-group and C is a cyclic p' group.
E is <u>elementary</u> if it is p-elementary for some p.

Theorem 123 (BIT)**.** Ind : $\bigoplus_{\text{elem } E<G} RE \twoheadrightarrow RG$.

17.1, 17.2: BIT in modular, sufficiently large case:
Suppose char $K = 0, \zeta_m \in K, m = \operatorname{lcm}\{\operatorname{ord}(g) \mid g \in G\}$.
Then <u>BIT</u>: Ind: $\bigoplus_{E<G} R_KE \twoheadrightarrow R_KG$.

Proof. Consider the following isomorphisms:

$$\begin{tikzcd}
R_{\mathbb{Q}(\zeta_m)}(G) \arrow[r, "\cong"] \arrow[d, "\cong"] & R_{\mathbb{C}}G \\
R_K G &
\end{tikzcd}$$

□

BIT $\implies$ the trivial representation is induced by subgroups:
$(*)\,[K] = 1_K = \sum \mathrm{Ind}_E^G(x_E)$.
Setting: $((K, \nu), A, k)$.
BIT: If K is sufficiently large (i.e. $\zeta_m \in K$) then,

$$\mathrm{Ind} : \bigoplus_{E<G} R_k E \twoheadrightarrow R_k G$$

$$\mathrm{Ind} : \bigoplus_{E<G} K_0 kE \to K_0 kG$$

Proof. Apply d [of CDE triangle] to $(*)$:

$$(**) : 1_k = \sum \mathrm{Ind}_E^G(d(x_E))$$

$$\implies \forall y, y = y \cdot 1_k = \sum \mathrm{Ind}_E^G(d(x_E)\,\mathrm{Res}_E^G(y))$$

So we're done. See 17.1 for details (!) □

If K is not sufficiently large, we need Γ_K elementary.

Some more CDE triangle

Recall:
$K_0 kG$ consists of projective modules, $R_k G$ consists of all. Since projective covers are unique, c must be injective.
Our CDE triangle ends up looking like this:

$$\begin{tikzcd}
K_0 kG \arrow[rr, tail, "c"] \arrow[dr, tail, "e"'] & & R_k G \\
& R_K G \arrow[ur, two heads, "d"'] &
\end{tikzcd}$$

We prove this using Brauer induction theorem.

Theorem 124 (33)**.** d is surjective.

Proof. It is true in general. We only prove the case where K is sufficiently large.
Special case: $G =$ elementary, aka $G = P \times C$. We go to the general case using Brauer induction theorem.
Let $\pi : G \to C$ be the projection map.
$y \in R_k G \implies y = \pi^* y'$ where $y' \in R_k C$ by 15.7.
$d_c : R_K C \overset{\cong}{\to} R_k C$ [15.5].
Thus, $\exists y'' \in R_K C$ such that $d(y'') = y'$
Since $d(\pi^* y'') = \pi^*(d(y'')) = \pi^*(y') = y$, we're done in the special case.
For general G: consider $y \in R_k G$ then,
BIT $\implies y = \sum_{E<G} \mathrm{Ind}_E^G(y_E) = \sum_{E<G} \mathrm{Ind}(d(y'_E)) = d\left(\sum_{E<G} \mathrm{Ind}(y'_E)\right)$ so we are done.

□

d is a surjection. Since everything in the CDE triangle is free, it is in fact a split surjection.

Theorem 125 (34)**.** e is a split injection.

Proof. Again suppose K sufficiently large. Then $D = E^t$ where d is a split surjection. Therefore, E is a split injection. $\square$

For general K, we have:

$$\begin{tikzcd} & K' & & k' \\ K \arrow[ur, dash] & & k \arrow[ur, dash] & \end{tikzcd}$$

Where K' is sufficiently large. Then we have:

$$\begin{tikzcd} kG \arrow[rr, tail] \arrow[dr, "e"'] & & K_0k'G \arrow[ll, dotted, bend right, "14.6"'] \arrow[dr, tail] & \\ & R_KG \arrow[rr] & & R_{K'}G \arrow[ul, dotted, bend right] \end{tikzcd}$$

$\implies$ Corollary 1 $K_0kG \to R_{K'}G$ is split injection $\implies$ $K_0kG \to R_KG$ is split injection.

Corollary 126 (7)**.** Let P, P' be f.g. projective AG-modules.
If $K \otimes_A P \cong K \otimes_A P'$ then $P \cong P'$.

Proof. $K \otimes_A P \cong K \otimes_A P' \implies e[\overline{P}] = e[\overline{P'}] \implies [\overline{P}] = [\overline{P'}] \implies \overline{P} \cong \overline{P'} \implies P \cong P'$. $\square$

Theorem 127. Let $p^n \mid\mid |G|$. Then $p^n \operatorname{coker} c = 0$.
i.e. $\forall y \in R_kG, \exists x \in K_0kG$ such that $c(x) = p^n y$.

Proof. Again assume G is sufficiently large.
Special case: $G = P \times C$
15.7 $\implies$ matrix $C = p^n \cdot \mathrm{id}$.
For general G we use BIT:
$y \in R_kG \implies y = \sum_{E<G} \operatorname{Ind}(y_E) \implies p^n y = \sum_{E<G} \operatorname{Ind}(p^n y_E) = \sum_E \operatorname{Ind}(c(x_E))$
$= c\left(\sum_E \operatorname{Ind}(x_E)\right)$ $\square$