

Number Theory Reading Group

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1 Thursday, 9/12/2024, Representation of $\mathfrak{sl}_2(\mathbb{F})$ by Hechi

$$\mathfrak{sl}_2(\mathbb{F}) := \{g \in \mathfrak{gl}_2(\mathbb{F}) \mid \text{Tr}(g) = 0\}$$

We assume $\text{char}(\mathbb{F}) = 0$ and \mathbb{F} is algebraically closed.

Theorem 1.1. $\mathfrak{sl}_2(\mathbb{F})$ is semisimple

Proof. Direct computation of the Killing Form. □

Recall: if \mathfrak{L} is semisimple and $\phi : \mathfrak{L} \rightarrow \mathfrak{gl}(V)$ is a representation.

$\mathfrak{L} \ni x = s + n$ abstract jordan decomposition.

$\implies \phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$ in $\phi(\mathfrak{L})$.

From now on, $\mathfrak{L} = \mathfrak{sl}_2(\mathbb{F}) = \mathfrak{sl}(2, \mathbb{F})$.

(V, ϕ) is a representation.

Basis of \mathfrak{L} :

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus we have $[h, x] = 2x, [h, y] = -2y, [x, y] = h$.

Since h is diagonal, h is semisimple.

$\implies \phi(h)$ is semisimple and thus diagonalizable. $\in \text{End}(V)$.

We can decompose $V = \bigoplus_{\lambda} V_{\lambda}$ where $V_{\lambda} = \{v \in V \mid hv = \lambda v\}$ for all $\lambda \in \mathbb{F}$.

We say V_{λ} is a weight space with λ as its weight.

Lemma 1.2 (7.1). Suppose $v \in V_{\lambda}$. Then,

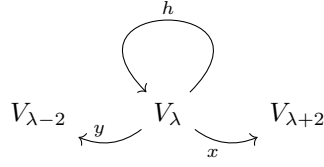
1) $xv \in V_{\lambda+2}$

2) $yv \in V_{\lambda-2}$

Proof. 1) $h(xv) = [h, x]v + x(hv) = 2xv + \lambda xv = (\lambda + 2)xv$

2) $h(yv) = [h, y]v + y(hv) = -2yv + \lambda yv = (\lambda - 2)yv$

□



Note that $\dim V < \infty$

Thus, $\exists v \in V$ such that $x \cdot v = 0$.

Such a v is called a maximal vector.

For now, assume V is irreducible.

Let v_0 be a maximal vector with weight λ .

Definition. For $i > 0$ integer, $v_i = \frac{y^i \cdot v_0}{i!}$

Also, $v_{-1} = 0$.

Lemma 1.3 (7.2). 1) $h \cdot v_i = (\lambda - 2i)v_i$

$$2) y \cdot v_i = (i + 1)v_{i+1}$$

$$3) x \cdot v_i = (\lambda - i + 1)v_{i-1}$$

Proof. 1) We use induction. Base case is clear.

Assume it is true for $i - 1$.

$$v_{i-1} \in V_{\lambda-2(i-1)}$$

$$\text{Thus, } v_i = \frac{1}{i} \cdot yv_{i-1}$$

Lemma 7.1 implies $v_i \in V_{\lambda-2i}$.

$$2) y \cdot v_i = (i + 1)v_{i+1} \text{ by definition of } v_i.$$

$$3) ix \cdot v_i = x(yv_{i-1}) = [x, y]v_{i-1} + yxv_{i-1} = hv_{i-1} + yxv_{i-1} = (\lambda - 2(i - 1))v_{i-1} + (\lambda - i + 2)yv_{i-2} = i(\lambda - i + 1)v_{i-1}$$

□

$\dim V < \infty$ so it must end at some point.

So, at some point, it'll become 0. $v_0, \dots, v_m \neq 0, v_{m+1} = 0$.

Definition. m is the integer so that $v_m \neq 0, v_{m+1} = 0$.

By Lemma 7.2,

$\text{span}\{v_0, \dots, v_m\}$ is a sub-representation of V .

Since V is irreducible,

$$V = \text{span}\{v_0, \dots, v_m\}$$

Note: by 7.2(3),

$$0 = x \cdot v_{m+1} = (\lambda - m)v_m$$

Since $v_m \neq 0$ we have $\lambda = m$.

Thus, $\dim V = m + 1 = \lambda + 1$

Here m is the highest weight.

$$V = V_{-m} \oplus V_{-m+2} \oplus \dots \oplus V_{m-2} \oplus V_m$$

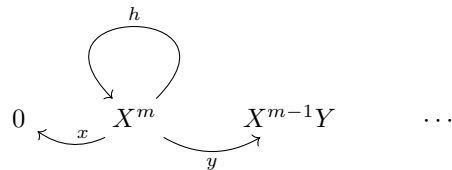
Construction. Suppose $L \hookrightarrow \mathbb{F}[X, Y]$ [as a \mathbb{F} -space].

$$\rho(x) = X \frac{\partial}{\partial Y}$$

$$\rho(y) = Y \frac{\partial}{\partial X}$$

$$\rho(h) = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$$

Consider subrepresentations $\mathbb{F}[X, Y]_m$ [symmetric polynomials of degree m , dimension $m + 1$].



2 Thursday, 9/19/2024, Representation of $\mathfrak{sl}_2(\mathbb{F})$ by Hechi

Root Space Decomposition

Let \mathcal{L} be a non-zero semisimple lie algebra over \mathbb{F} with $\text{char } \mathbb{F} = 0$ and \mathbb{F} algebraically closed.

Definition (Toral Subalgebra). A subalgebra $\mathcal{H} \subseteq \mathcal{L}$ toral if it consists of semisimple elements.

Remark. If every element in \mathcal{L} is ad-nilpotent, then by Engel's Theorem \mathcal{L} is nilpotent. Thus it is not semisimple.

So, there exists a non-zero toral subalgebra.

Fix \mathcal{H} to be the maximal toral subalgebra. A maximal subalgebra exists since \mathcal{L} is finite dimensional.

Lemma 2.1 (8.1). A toral subalgebra \mathcal{T} is abelian.

Proof. Suppose $x \in \mathcal{T}$. We will prove that $\text{ad}_{\mathcal{T}} x = 0$ [as a map].

$\text{ad}_{\mathcal{T}} x$ is diagonalizable. Assume some eigenvalue is non-zero. Then, we can find eigenvector $y \in \mathcal{L}$ with eigenvalue $a \neq 0$. So, $[x, y] = ay$.

Now, $\text{ad}_{\mathcal{T}} y(x) = [y, x] = -ay$. Since $[y, y] = 0$ we see that $-ay$ is an eigenvector of $\text{ad}_{\mathcal{T}} y$ with eigenvalue 0.

$\text{ad}_{\mathcal{T}} y$ is also diagonalizable. Suppose v_1, \dots, v_n is the eigenbasis of $\text{ad}_{\mathcal{T}} y$ with eigenvalues $\lambda_1, \dots, \lambda_n$. Then $x = a_1 v_1 + \dots + a_n v_n$ for $a_i \in \mathbb{F}$.

WLOG, $v_1 = y$.

$$[y, x] = a_1 \lambda_1 v_1 + \dots + a_n \lambda_n v_n = -ay$$

By comparing coefficients, $a_1 \lambda_1 = -a$. But $\lambda_1 = 0$. This is a contradiction. \square

Now, we fix \mathcal{H} to be a maximal toral subalgebra. It is not necessarily unique.

Note that $\text{ad } H$ is a commuting family in $\text{End}(\mathcal{L})$. From linear algebra we know that $\text{ad } H$ is simultaneously diagonalizable.

Definition (Root Space Decomposition). Suppose \mathcal{H}^* is the dual space of \mathcal{H} . We can write:

$$\begin{aligned} \mathcal{L} &= \bigoplus_{\alpha \in \mathcal{H}^*} \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x \forall h \in H\} \\ &= \mathcal{L}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathcal{L}_\alpha \end{aligned}$$

where $\Phi = \{\alpha \in \mathcal{H}^* \setminus \{0\} \mid \mathcal{L}_\alpha \neq 0\}$ and $\mathcal{L}_0 = C_{\mathcal{L}}(\mathcal{H})$ [the centralizer].

This is called the root space decomposition.

Example. $\mathfrak{sl}_2(\mathbb{F})$ has basis:

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then the root space decomposition is:

$$\mathfrak{sl}_2(\mathbb{F}) = \mathcal{H} \oplus \mathcal{L}_{-2} \oplus \mathcal{L}_2$$

\mathcal{L}_{-2} contains the linear form sending h to -2 .

Proposition 2.2 (8.1). Let $\alpha, \beta \in \mathcal{H}^*$. Then,

$$1) [L_\alpha, L_\beta] \subseteq L_{\alpha+\beta} \text{ [by Jacobi Identity]}$$

2) $\alpha \neq 0 \implies \forall x \in L_\alpha$ is nilpotent [by 1]

3) $\alpha + \beta \neq 0 \implies L_\alpha \perp L_\beta$ w.r.t. the Killing Form.

Proof of 3. Find $h \in \mathcal{H}$ such that $(\alpha + \beta)(h) \neq 0$. Then,

$$\kappa([h, x], y) = -\kappa([x, h], y) = -\kappa(x, [h, y])$$

$$\implies (\alpha + \beta)(h)\kappa(x, y) = 0$$

□

In particular, $L_0 \perp L_\alpha$ when $\alpha \in \Phi$.

Corollary 2.3 (8.1). The Killing Form restricted to \mathcal{L}_0 , $\kappa|_{\mathcal{L}_0}$ is non-degenerate.

Proposition 2.4 (8.2). $\mathcal{H} = \mathcal{L}_0 = C_{\mathcal{L}}(\mathcal{H})$.

Proof. Tedious linear algebra

□

Corollary 2.5 (8.2). The Killing Form restricted to \mathcal{H} , $\kappa|_{\mathcal{H}}$ is non-degenerate.

This implies, the map $H \rightarrow H^*$ given by $x \mapsto \kappa(x, -)$ is an isomorphism.

For each $\phi \in \mathcal{H}^*$ we can define $t_\phi \in \mathcal{H}$ to be the pre-image of this isomorphism. So it satisfies

$$\phi(h) = \kappa(t_\phi, h) \quad \forall h \in \mathcal{H}$$

Proposition 2.6 (8.3). 1) Φ spans \mathcal{H}^*

2) If $\alpha \in \Phi$ then $-\alpha \in \Phi$

3) $x \in \mathcal{L}_\alpha, y \in \mathcal{L}_{-\alpha} \implies [x, y] = \kappa(x, y)t_\alpha$

4) $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$

5) $\dim[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] = 1$, spanned by t_α

6) Pick any non-zero $x_\alpha \in L_\alpha \setminus \{0\}$. Then there exists $y_\alpha \in \mathcal{L}_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha := [x_\alpha, y_\alpha]$ spans a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{F})$, with the isomorphism $x_\alpha \mapsto x, y_\alpha \mapsto y, h_\alpha \mapsto h$

7) $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$.

If V is a $\mathfrak{sl}_2(\mathbb{F})$ -module, recalling that $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$$V = \bigoplus_{\lambda \in \mathbb{F}} V_\lambda \text{ eigenspaces of } h$$

Recall that all $\mathfrak{sl}_2(\mathbb{F})$ -module is of the form:

$$\mathfrak{sl}_2(\mathbb{F}) \curvearrowright \mathbb{F}[X, Y]$$

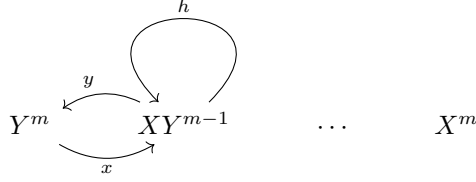
$$\rho(x) = X \frac{d}{dY}, \rho(y) = Y \frac{d}{dX}, \rho(h) = X \frac{d}{dX} - Y \frac{d}{dY}$$

and $V = \mathbb{F}[X, Y]_m$ [homogeneous polynomials of degree m] is irreducible and give us all irreducible representations.

Then we have:

$$V = V_{-m} \oplus V_{-m+2} \oplus \cdots \oplus V_{m-2} \oplus V_m$$

Where V_m is generated by X^m and V_{-m} is generated by Y^m



If m even, $0 \neq V_0 \subseteq V$

If m odd, $0 \neq V_1 \subseteq V$

Corollary 2.7. V is a $\mathfrak{sl}_2(\mathbb{F})$ -module. Then $\dim V_0 + \dim V$ gives the number of summands in the irreducible decomposition of V .

Consider $\mathcal{S}_\alpha = \text{span}\{x_\alpha, y_\alpha, h_\alpha\} \cong \mathfrak{sl}_2(\mathbb{F})$ and its adjoint representation (\mathcal{L} is an \mathcal{S}_α module).

Fix $\alpha \in \Phi$ and let $\mathcal{M} = \mathcal{H} + \sum_{c \in \mathbb{F}^\times} \mathcal{L}_{c\alpha}$.

By proposition 8.1, \mathcal{M} is a submodule of \mathcal{L} [since $[\mathcal{L}_{c_1\alpha}, \mathcal{L}_{c_2\alpha}] \subseteq \mathcal{L}_{(c_1+c_2)\alpha}$].

If $0 \neq x \in \mathcal{L}_{c\alpha}$ we see that $[h_\alpha, x] = c\alpha(h_\alpha) \cdot x = 2cx$

$\implies 2c \in \mathbb{Z}$ and a weight of h_α is 0 or an integer multiple of $\frac{1}{2}$.

Then $\mathcal{M} = \underbrace{\ker \alpha}_{\substack{\text{eigenvalue} \\ \text{vectors of weight 0}}} + \underbrace{\mathbb{F} \cdot h_\alpha}_{\text{weight } 0, \pm 2}$

Therefore, \mathcal{M} contains vectors of weight only 0 or ± 2 .

Therefore, if $\alpha \in \Phi$ we have $c = \pm 1$.

$\mathcal{M} = \mathcal{H} + \mathcal{S}_\alpha$. Suppose h_α^c is the complement of h_α in \mathcal{H} .

Then, $\mathcal{H} + \mathcal{S}_\alpha = \underbrace{h_\alpha^c}_{\text{abelian}} + \underbrace{\mathcal{S}_\alpha}_{\text{irreducible}}$ has $\dim \mathcal{H} - 1 + 1 = \dim \mathcal{H} = \dim \mathcal{M} - 2$ irreducible

summands.

On the other hand, the number of irreducible summands of \mathcal{M} is $\underbrace{\dim \mathcal{M}_0}_{\dim \mathcal{M} - 2} + \underbrace{\dim \mathcal{M}_1}_0$

Therefore, $\mathcal{H} + \mathcal{S}_\alpha \subseteq \mathcal{M}$ must be equal.

Therefore, $\dim \mathcal{L}_\alpha = 1$.

Now, suppose $\beta \neq \pm\alpha \in \Phi$. Then, $\exists r, q$ such that $\beta - r\alpha, \beta - (r-1)\alpha, \dots, \beta + q\alpha$ are roots and outside these, i.e. $\beta - (r+1)\alpha, \beta + (q+1)\alpha$ are not.

To see this, suppose $K = \sum_{i \in \mathbb{Z}} \mathcal{L}_{\beta+i\alpha} \subseteq \mathcal{L}$ is a \mathcal{S}_α -submodule. We know that $\beta + i\alpha \neq 0$.

Weights:

$$\beta(h_\alpha) + i\alpha(h_\alpha) = \beta(h_\alpha) + 2i$$

So, weights are either all even or all odd.

Therefore, K is irreducible.

Consider $\gamma, \delta \in \mathcal{H}^*$.

Define $(\gamma, \delta) = \kappa(t_\gamma, t_\delta)$ on $E_\mathbb{Q} = \text{span}_\mathbb{Q}(\Phi)$ then (\cdot, \cdot) extends to $E = E_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{R}$ is positive definite.

Then E is an Euclidean Space.

(Φ, E) is called a root system.

3 Thursday, 9/26/2024, Root Systems by Zoia

Let E be an euclidean space. Suppose (α, β) is a symmetric bilinear form on E .
 Reflection in E fixes some hyperplane H . If α is perpendicular to H then the reflection sends α to $-\alpha$.
 Consider $\alpha \in E$ and $P_\alpha = \{\beta \in E \mid (\alpha, \beta) = 0\}$ the hyperplane perpendicular to α .
 Suppose σ_α is the reflection w.r.t. this hyperplane. Then,

$$\text{proj}_\alpha(\beta) = \frac{(\beta, \alpha)}{(\alpha, \alpha)} \alpha$$

$$\sigma_\alpha(\beta) = \beta - 2 \text{proj}_\alpha(\beta) = \beta - 2 \frac{(\beta, \alpha)}{(\alpha, \alpha)} \alpha$$

Define:

$$\langle \beta, \alpha \rangle = 2 \frac{(\beta, \alpha)}{(\alpha, \alpha)}$$

Note that $\langle \beta, \alpha \rangle$ is linear only in β . Then,

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$$

Lemma 3.1. Let Φ be a finite subset of E so that Φ spans E . Suppose all reflections $\sigma_\alpha (\alpha \in \Phi)$ leaves Φ invariant. If $\sigma \in \text{GL}(E)$ fixes hyperplane P of E and sends $0 \neq \alpha \in \Phi$ to $-\alpha$, then $\sigma = \sigma_\alpha$ and $P = P_\alpha$.

Proof. Suppose $\tau = \sigma \sigma_\alpha = \sigma \sigma_\alpha^{-1}$.

Then, $\tau(\Phi) = \Phi$, $\tau(\alpha) = \alpha$ and τ acts as id on $\mathbb{R} \cdot \alpha$ and $E/R \cdot \alpha$ eigenvalues are 1.

So we have $(T - 1)^L$ where $L = \dim E$.

$\beta, \tau(\beta), \dots, \tau^k(\beta) \exists k$ that fixes all $\beta \in \Phi$

Φ spans E , so $\tau^k = 1$. So $T^k - 1 = 0$.

If $m(T)$ is the minimal polynomial of τ , then:

$$m(T) \mid T^k - 1$$

$$m(T) \mid (T - 1)^k$$

Therefore, $m(T) = T - 1$.

Therefore, $\tau = \text{id}$.

Thus $\sigma \sigma_\alpha^{-1} = \text{id} \implies \sigma = \sigma_\alpha$ □

Definition (Root Systems). A finite subset Φ of E is a root system in E if:

- 1R) Φ spans E , does not contain 0.
- 2R) If $\alpha \in \Phi$ then only multiples of α in Φ are $\pm\alpha$.
- 3R) If $\alpha \in \Phi$, then σ_α leaves Φ invariant. $[\forall \beta \in \Phi, \sigma_\alpha(\beta) \in \Phi]$
- 4R) If $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in \mathbb{Z}$. $\left[\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \right]$

Definition (Weyl Group). Let Φ be a root system in E . Denote by \mathcal{W} the subgroup of $\text{GL}(E)$ generated by $\sigma_\alpha (\alpha \in \Phi)$.

3R $\implies \mathcal{W}$ is a symmetry group on Φ .

Lemma 3.2. Let Φ be a root system in E with Weyl group \mathcal{W} . If $\sigma \in \text{GL}(E)$ leaves Φ invariant, then $\sigma \sigma_\alpha \sigma^{-1} = \sigma_{\sigma(\alpha)} \forall \alpha \in \Phi$ and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle \forall \alpha, \beta \in \Phi$.

Proof. $\sigma \sigma_\alpha \sigma^{-1}(\sigma(\beta)) = \sigma \sigma_\alpha(\beta) = \sigma(\beta - \langle \beta, \alpha \rangle \alpha) = \sigma(\beta) - \langle \beta, \alpha \rangle \sigma(\alpha)$.

$\sigma(\beta)$ runs over Φ . $\sigma \sigma_\alpha \sigma^{-1}$ fixes $\sigma(P_\alpha)$ pointwise and $\sigma(\alpha) \rightarrow -\sigma(\alpha)$.

Therefore, $\sigma \sigma_\alpha \sigma^{-1} = \sigma_{\sigma(\alpha)}$ by the lemma.

$\sigma_{\sigma(\alpha)}(\sigma(\beta)) = \sigma(\beta) - \langle \sigma(\beta), \sigma(\alpha) \rangle \sigma(\alpha)$

Therefore, we must have $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$. □

Definition (Isomorphisms). Suppose Φ, Φ' are root systems with Euclidean spaces E, E' .

$(\Phi, E) \cong (\Phi', E')$ if there exists map $\varphi : E \rightarrow E'$ such that φ maps Φ to Φ' and $\forall \alpha, \beta \in \Phi$ we have $\langle \varphi(\beta), \varphi(\alpha) \rangle = \langle \beta, \alpha \rangle$.

Note that:

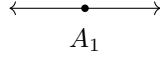
$$\sigma_{\varphi(\alpha)}(\varphi(\beta)) = \varphi(\beta) - \underbrace{\langle \varphi(\beta), \varphi(\alpha) \rangle}_{=\langle \beta, \alpha \rangle} \varphi(\alpha) = \varphi(\beta - \langle \beta, \alpha \rangle \alpha) = \varphi(\sigma_\alpha(\beta))$$

Note that, $\sigma \mapsto \varphi \sigma \varphi^{-1}$ is an isomorphism of Weyl groups.

Thus, \mathcal{W} is a subgroup of $\text{Aut}(\Phi)$.

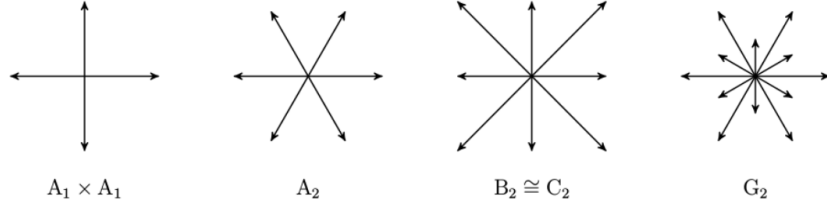
Now we consider root systems of different dimensions. Suppose $L = \dim E$.

$L = 1$: In this case, we have $\alpha, \alpha \in \Phi$ only. This gives us A_1



$$\mathcal{W}(A_1) = \mathbb{Z}_2$$

$L = 2$:



$$\mathcal{W}(A_1 \times A_1) = \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$\mathcal{W}(A_2) = S_3$$

$$\mathcal{W}(B_2) = D_4$$

$$\mathcal{W}(G_2) = D_6$$

These are the only possible cases for $L = 2$, since:

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{2\|\beta\|\|\alpha\| \cos \theta}{\|\alpha\|\|\alpha\|} = \frac{2\|\beta\|}{\|\alpha\|} \cos \theta \in \mathbb{Z}$$

Similarly, $\frac{2\|\alpha\|}{\|\beta\|} \cos \theta \in \mathbb{Z}$. Multiplying, $4 \cos^2 \theta \in \mathbb{Z} \implies 4 \cos^2 \theta = 0, 1, 2, 3, 4$

Thus, $\cos \theta = 0, \pm \frac{1}{2}, \pm \frac{\sqrt{3}}{2} \implies \theta = \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}$.

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\ \beta\ ^2 / \ \alpha\ ^2$
0	0	$\frac{\pi}{2}$	undefined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	2
-1	-2	$\frac{3\pi}{4}$	2
1	3	$\frac{\pi}{6}$	3
-1	-3	$\frac{5\pi}{6}$	3

Table 1: Angle Root System

Lemma 3.3. Suppose α, β are non-proportional root.

If $\langle \alpha, \beta \rangle > 0$ then $\alpha - \beta$ is a root.

If $\langle \alpha, \beta \rangle < 0$ then $\alpha + \beta$ is a root.

Proof. $\langle \alpha, \beta \rangle = 1 \implies \sigma_\beta(\alpha) = \alpha - 1\beta = \alpha - \beta \in \Phi$

If $\langle \beta, \alpha \rangle = 1$ then $\sigma_\alpha(\beta) = \beta - 1\alpha = \beta - \alpha \in \Phi$.

$\sigma_{\beta-\alpha}(\beta - \alpha) = (\beta - \alpha) - \langle \beta - \alpha, \beta - \alpha \rangle(\beta - \alpha) = \alpha - \beta \in \Phi$

□

4 Thursday, 10/3/2024, Simple Roots by Zoia

A root system Φ of rank l , E -Euclidean Space, \mathcal{W} is the Weyl Group.

Definition. A subset Δ of Φ is called a base if:

- B1) Δ is a basis of E [$|\Delta| = l$];
- B2) $\forall \beta \in \Phi, \beta = \sum_{\alpha \in \Delta} k_\alpha \cdot \alpha$, the expression is unique with k_α being integers and k_α are either all non-negative or all non-positive.

Definition. The roots from Δ are simple roots.

Definition. The height of a root β [relative to the base Δ] is:

$$\text{ht}(\beta) = \sum_{\alpha \in \Delta} k_\alpha$$

Definition. We have positive roots Φ^+ and negative roots Φ^- from the sign of k_α . Furthermore $\Phi^- = -\Phi^+$.

Also, we define:

$$\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\}$$

Definition. $\gamma \in E$ is regular if:

$$\gamma \in E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$$

Otherwise it is called singular.

Recall that $P_\alpha = \{\beta \in E \mid (\alpha, \beta) = 0\}$

Definition. $\alpha \in \Phi^+(\gamma)$ is decomposable if $\alpha = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in \Phi^+(\gamma)$. α is indecomposable otherwise.

Definition. We define $\Delta(\gamma)$ to be the set of all indecomposable roots in $\Phi^+(\gamma)$.

Theorem 4.1. Any root system Φ has a base. Let $\gamma \in E$ be a regular.

Then, the set $\Delta(\gamma)$ of all the indecomposable roots in $\Phi^+(\gamma)$ is a base of Φ .

Conversely, every base of Φ is of the form $\Delta(\gamma)$ for some γ .

Proof. We follow the following steps.

Step 1: Each root in $\Phi^+(\gamma)$ is a non-negative \mathbb{Z} -linear combination of $\Delta(\gamma)$.

Step 2: If $\alpha, \beta \in \Delta(\gamma)$ then $(\alpha, \beta) \leq 0$ unless $\alpha = \beta$.

Step 3: $\Delta(\gamma)$ is a linearly independent set.

Step 4: $\Delta(\gamma)$ is a base of Φ .

Step 5: Each base Δ of Φ has the form $\Delta(\gamma)$ for some regular $\gamma \in E$.

Proof of Step 1: Suppose otherwise. Then $\exists \alpha \in \Phi^+(\gamma)$ that cannot be expressed as a non-negative \mathbb{Z} linear combination of $\Delta(\gamma)$.

We can have multiple such α 's. We pick the α with the smallest (γ, α) .

Note that $\alpha \notin \Delta(\gamma)$, since if $\alpha \in \Delta(\gamma)$ then $\alpha = 1 \cdot \alpha$, which violates the assumption.

Thus, α can be written as sum of two elements in $\Phi^+(\gamma)$. Suppose $\alpha = \beta_1 + \beta_2$ so that $\beta_1, \beta_2 \in \Phi^+(\gamma)$. Then, $(\gamma, \alpha) = (\gamma, \beta_1) + (\gamma, \beta_2)$. Due to the minimality of (γ, α) , they are both non-negative \mathbb{Z} -linear combination of $\Delta(\gamma)$ which means so is α , a contradiction.

Proof of Step 2: Suppose otherwise. Then, $(\alpha, \beta) > 0$. β cannot be $-\alpha$, thus $\alpha - \beta$ is a root. Then either $\alpha - \beta$ or $\beta - \alpha$ is in $\Phi^+(\gamma)$. WLOG $\alpha - \beta \in \Phi^+(\gamma)$. Then $\alpha = \beta + (\alpha - \beta)$. Then α is decomposable, which is a contradiction since $\Delta(\gamma)$ consists of all indecomposable roots.

Proof of Step 3: Suppose $\sum_{\alpha \in \Delta(\gamma), r_\alpha \in \mathbb{R}} r_\alpha \cdot \alpha = 0$. r_α can be positive or negative. We redistribute so that both sides have positive coefficient:

$$\varepsilon := \sum_{\alpha} s_\alpha \alpha = \sum_{\beta} t_\beta \beta$$

Then,

$$0 \leq (\varepsilon, \varepsilon) = \sum_{\alpha, \beta} \underbrace{s_\alpha t_\beta}_{\geq 0} \underbrace{(\alpha, \beta)}_{\leq 0} \leq 0$$

Thus, $\varepsilon = 0$. Now,

$$0 = (\gamma, \varepsilon) = \sum_{\alpha} \underbrace{s_\alpha}_{\geq 0} \underbrace{(\gamma, \alpha)}_{> 0} \geq 0$$

Thus, $s_\alpha = 0$ for all $\alpha \in \Delta(\gamma)$. This implies linear independence.

Proof of Step 4: Note that $\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma)$.

B2 is satisfied because of Step 1.

Then $\Delta(\gamma)$ spans E . Step 3 implies $\Delta(\gamma)$ is a basis of E . Thus we have B1.

Proof of Step 5: Given Δ , we select $\gamma \in E : (\alpha, \gamma) > 0 \forall \alpha \in \Delta$. B2 $\implies \gamma$ is regular and $\Phi^+ \subseteq \Phi^+(\gamma)$. Also, $\Phi^- \subseteq -\Phi^+(\gamma)$.

Therefore, $\Phi^+ = \Phi^+(\gamma)$. Δ consists of indecomposable elements, that is $\Delta \subseteq \Delta(\gamma)$.

Coordinates are equal, therefore $\Delta = \Delta(\gamma)$.

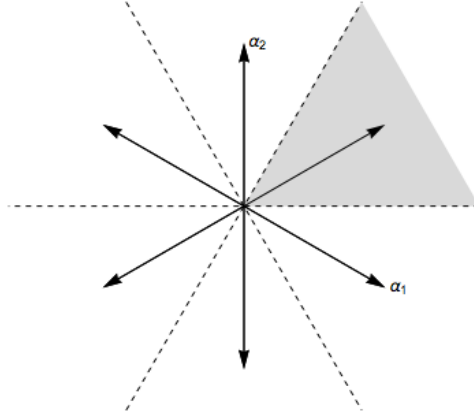
□

Definition (Weyl Chambers). The connected components of $E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$ are called the (open) Weyl Chambers of E .

The fundamental Weyl chamber associated to γ is the open Weyl chamber containing γ . It is denoted by $\mathcal{C}(\gamma)$.

Furthermore, $C(\gamma) = C(\gamma')$ implies γ and γ' are on the same side of each hyperplane P_α . This also means $\Delta(\gamma) = \Delta(\gamma')$, so the Weyl chambers are in 1-1 correspondence with the bases.

For example: here is an open Weyl Chamber for A_2 :



$\mathcal{C}(\Delta)$ -fundamental Weyl chamber relative to the base $\{\alpha_1, \alpha_2\}$.

The Weyl group acts on the Weyl chambers by $\sigma(\mathcal{C}(\gamma)) = \mathcal{C}(\sigma(\gamma))$.

If $\sigma \in \mathcal{W}$ and γ is regular.

Also, \mathcal{W} permutes bases. σ sends Δ to $\sigma(\Delta)$ which is another base.

Since $\sigma(\Delta(\gamma)) = \Delta(\sigma(\gamma))$ because $(\sigma\gamma, \sigma\alpha) = (\gamma, \alpha)$.

5 Thursday, 10/17/2024, Weyl Group by Zoia

Lemma 5.1. Let α be simple. Then σ_α permutes the positive roots other than α .

Corollary 5.2. Set $\delta = \frac{1}{2} \sum_{\beta \prec 0} \beta$. Then,

$$\sigma_\alpha(\delta) = \delta - \alpha \forall \alpha \in \Delta$$

Lemma 5.3. Let $\alpha_1, \dots, \alpha_n \in \Delta$ [not necessarily distinct]. Write $\sigma_i := \sigma_{\alpha_i}$. If $\sigma_1, \dots, \sigma_{t-1}(\alpha_t)$ is negative, then

$$\exists s : 1 \leq s < t : \sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1}$$

Corollary 5.4. If $\sigma = \sigma_1 \cdots \sigma_t$ is an exp for $\sigma \in \mathcal{W}$, t is as small as possible then $\sigma(\alpha_t) \prec 0$.

Proof. Suppose $\sigma(\alpha_t) > 0$. Then,

$$\underbrace{\sigma_1 \cdots \sigma_t}_{t \text{ factors}} = \underbrace{\sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1}}_{t-2 \text{ factors}}$$

which contradicts minimality. \square

The Weyl Group

Definition. \mathcal{W} is the subgroup of $GL(E)$ generated by the reflection $(\sigma_\alpha)_{\alpha \in \Phi}$.

Theorem 5.5. Let Δ be a base of Φ .

- a) If $\gamma \in E$, γ is regular, $\exists \sigma \in \mathcal{W} : (\sigma(\gamma), \alpha) > 0 \forall \alpha \in \Delta$.
- b) If Δ' is another base of Φ , then $\sigma(\Delta') = \Delta$ for some $\sigma \in \mathcal{W}$.
- c) If α is any root $\implies \exists \sigma \in \mathcal{W} : \sigma(\alpha) \in \Delta$.
- d) \mathcal{W} generated by $\sigma_\alpha (\alpha \in \Delta)$.
- e) If $\sigma(\Delta) = \Delta, \sigma \in \mathcal{W}$ then $\sigma = \text{id}$.

Proof. We consider the subgroup \mathcal{W}' generated by $\sigma_\alpha (\alpha \in \Delta)$.

For a, b, c we prove the theorem for \mathcal{W}' and for d, e we prove that $\mathcal{W}' = \mathcal{W}$.

a)

$$\delta := \frac{1}{2} \sum_{\alpha \prec 0} \alpha$$

Choose $\sigma \in \mathcal{W}'$ such that $(\sigma(\gamma), \delta)$ is as big as possible.

If α is simple then $\sigma_\alpha \sigma \in \mathcal{W}' \implies (\sigma(\gamma), \delta) \geq (\sigma_\alpha \sigma(\gamma), \delta) = (\sigma(\gamma), \sigma_\alpha(\delta)) = (\sigma(\gamma), \delta - \alpha) = (\sigma(\gamma), \delta) - (\sigma(\gamma), \alpha)$.

Therefore, $(\sigma(\gamma), \alpha) \geq 0$.

Furthermore, $(\sigma(\gamma), \alpha) \neq 0$ so we have strict inequality. Therefore,

$$\forall \alpha \in \Delta, (\sigma(\gamma), \alpha) > 0$$

Therefore, $\sigma(\gamma)$ is in the fundamental Weyl chamber of Δ and σ sends $\mathfrak{C}(\gamma)$ to $\mathfrak{C}(\Delta)$.

b) Since \mathcal{W}' permutes the Weyl chambers by a , it also permutes the bases of Φ .

c) Hyperplanes $P_\beta (\beta \neq \pm \alpha)$ are distinct from hyperplane $P_\alpha \implies \exists \gamma : \gamma \in P_\alpha, \gamma \notin P_\beta$. Lets choose γ' so that γ' is close to γ such that $(\gamma', \alpha) = \varepsilon > 0$ while $|(\gamma', \beta)| > \varepsilon$ for any $\beta \neq \pm \alpha$.

\square