

# Number Theory Reading Group

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## 1 Thursday, 9/12/2024, Representation of $\mathfrak{sl}_2(\mathbb{F})$ by Hechi

$$\mathfrak{sl}_2(\mathbb{F}) := \{g \in \mathfrak{gl}_2(\mathbb{F}) \mid \text{Tr}(g) = 0\}$$

We assume  $\text{char}(\mathbb{F}) = 0$  and  $\mathbb{F}$  is algebraically closed.

**Theorem 1.1.**  $\mathfrak{sl}_2(\mathbb{F})$  is semisimple

*Proof.* Direct computation of the Killing Form. □

Recall: if  $\mathfrak{L}$  is semisimple and  $\phi : \mathfrak{L} \rightarrow \mathfrak{gl}(V)$  is a representation.

$\mathfrak{L} \ni x = s + n$  abstract jordan decomposition.

$\implies \phi(x) = \phi(s) + \phi(n)$  is the Jordan decomposition of  $\phi(x)$  in  $\phi(\mathfrak{L})$ .

From now on,  $\mathfrak{L} = \mathfrak{sl}_2(\mathbb{F}) = \mathfrak{sl}(2, \mathbb{F})$ .

$(V, \phi)$  is a representation.

Basis of  $\mathfrak{L}$ :

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus we have  $[h, x] = 2x, [h, y] = -2y, [x, y] = h$ .

Since  $h$  is diagonal,  $h$  is semisimple.

$\implies \phi(h)$  is semisimple and thus diagonalizable.  $\in \text{End}(V)$ .

We can decompose  $V = \bigoplus_{\lambda} V_{\lambda}$  where  $V_{\lambda} = \{v \in V \mid hv = \lambda v\}$  for all  $\lambda \in \mathbb{F}$ .

We say  $V_{\lambda}$  is a weight space with  $\lambda$  as its weight.

**Lemma 1.2** (7.1). Suppose  $v \in V_{\lambda}$ . Then,

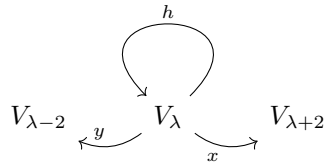
$$1) \quad xv \in V_{\lambda+2}$$

$$2) \quad yv \in V_{\lambda-2}$$

*Proof.* 1)  $h(xv) = [h, x]v + x(hv) = 2xv + \lambda xv = (\lambda + 2)xv$

$$2) \quad h(yv) = [h, y]v + y(hv) = -2yv + \lambda yv = (\lambda - 2)yv$$

□



Note that  $\dim V < \infty$

Thus,  $\exists v \in V$  such that  $x \cdot v = 0$ .

Such a  $v$  is called a maximal vector.

For now, assume  $V$  is irreducible.

Let  $v_0$  be a maximal vector with weight  $\lambda$ .

**Definition.** For  $i > 0$  integer,  $v_i = \frac{y^i \cdot v_0}{i!}$   
Also,  $v_{-1} = 0$ .

**Lemma 1.3** (7.2). 1)  $h \cdot v_i = (\lambda - 2i)v_i$

$$2) y \cdot v_i = (i + 1)v_{i+1}$$

$$3) x \cdot v_i = (\lambda - i + 1)v_{i-1}$$

*Proof.* 1) We use induction. Base case is clear.

Assume it is true for  $i - 1$ .

$$v_{i-1} \in V_{\lambda-2(i-1)}$$

$$\text{Thus, } v_i = \frac{1}{i} \cdot yv_{i-1}$$

Lemma 7.1 implies  $v_i \in V_{\lambda-2i}$ .

$$2) y \cdot v_i = (i + 1)v_{i+1} \text{ by definition of } v_i.$$

$$3) ix \cdot v_i = x(yv_{i-1}) = [x, y]v_{i-1} + yxv_{i-1} = hv_{i-1} + yxv_{i-1} = (\lambda - 2(i - 1))v_{i-1} + (\lambda - i + 2)yv_{i-2} = i(\lambda - i + 1)v_{i-1}$$

□

$\dim V < \infty$  so it must end at some point.

So, at some point, it'll become 0.  $v_0, \dots, v_m \neq 0, v_{m+1} = 0$ .

**Definition.**  $m$  is the integer so that  $v_m \neq 0, v_{m+1} = 0$ .

By Lemma 7.2,

$\text{span}\{v_0, \dots, v_m\}$  is a sub-representation of  $V$ .

Since  $V$  is irreducible,

$$V = \text{span}\{v_0, \dots, v_m\}$$

Note: by 7.2(3),

$$0 = x \cdot v_{m+1} = (\lambda - m)v_m$$

Since  $v_m \neq 0$  we have  $\lambda = m$ .

Thus,  $\dim V = m + 1 = \lambda + 1$

Here  $m$  is the highest weight.

$$V = V_{-m} \oplus V_{-m+2} \oplus \dots \oplus V_{m-2} \oplus V_m$$

**Construction.** Suppose  $L \curvearrowright \mathbb{F}[X, Y]$  [as a  $\mathbb{F}$ -space].

$$\rho(x) = X \frac{\partial}{\partial Y}$$

$$\rho(y) = Y \frac{\partial}{\partial X}$$

$$\rho(h) = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$$

Consider subrepresentations  $\mathbb{F}[X, Y]_m$  [symmetric polynomials of degree  $m$ , dimension  $m + 1$ ].

$$\begin{array}{ccccc} & & \overset{h}{\curvearrowright} & & \\ & & \downarrow & & \\ 0 & \xleftarrow{x} & X^m & \xrightarrow{y} & X^{m-1}Y \quad \dots \end{array}$$