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1. Suppose that X_1, X_2, X_3 are independent with common probability mass function: $P\{X_i = 0\} = .2, P\{X_i = 1\} = .3, P\{X_i = 3\} = .5, i = 1, 2, 3$

- (a) Plot the probability mass function of the average of a sample size of 2. $X_2 = \frac{X_1 + X_2}{2}$

Event	X_2	Probability
0,0	0	0.04
0,1	0.5	0.06
1,1	1	0.09
1,3	2	0.15
3,3	3	0.25
0,3	1.5	0.1

- (b) Determine $E[X_2]$ and $Var(X_2)$

- (c) Plot the probability mass function of the average of a sample size of 3. $X_3 = \frac{X_1 + X_2 + X_3}{3}$

- (d) Determine $E[X_3]$ and $Var(X_3)$

2. If 10 fair dice are rolled, approximate the probability that the sum of the values obtained (which ranges from 10 to 60) is between 30 and 40 inclusive.

Solution:

$$E[X_i] = \sum_{x_i=1}^6 \frac{1}{6}x = \frac{7}{2}$$

$$Var(X_i) = \sum_{x_i=1}^6 \frac{1}{6}(x - \frac{7}{2})^2 = \frac{35}{12}$$

$$\text{Let } X = \sum_{x_i=1}^6 X_i$$

$$E[X_i] = \sum_{x_i=1}^{10} E(X_i) = 10(\frac{7}{2}) = 35$$

$$Var(X_i) = \sum_{x_i=1}^{10} Var[X_i] = 10(\frac{35}{12}) = \frac{350}{12}$$

$$P(30 \leq X \leq 40) = P(29.5 < X < 39.5)$$

$$= P\left(\frac{29.5 - 35}{\sqrt{350/12}} < \frac{X - 35}{\sqrt{350/12}} < \frac{39.5 - 35}{\sqrt{350/12}}\right)$$

$$= P(-1.02 \leq Z \leq 1.02)$$

$$= 2(0.8461) - 1$$

$$= 0.6922$$

3. A highway department has enough salt to handle a total of 80 inches of snowfall. Suppose the daily amount of snow has a mean of 1.5 inches and a standard deviation of .3 inch.

- (a) Approximate the probability that the salt on hand will suffice for the next 50 days.

Solution:

Given:

$$E[X_i] = 1.5 \text{ inches}$$

$$SD[X_i] = 0.3 \text{ inches}$$

$$\begin{aligned}\text{Let } X &= \sum_{x_i=1}^{50} X_i \\ E[X] &= \sum_{x_i=1}^{50} E(X_i) = 50(1.5) = 75 \\ Var(X) &= \sum_{x_i=1}^{50} (SD[X_i])^2 = 50(0.09) = 4.5 \\ P(X \leq 80) &= P\left(\frac{X - 75}{\sqrt{4.5}} < \frac{80 - 75}{\sqrt{4.5}}\right) \\ &= P(\leq Z \leq 2.36) \\ &= 0.9909\end{aligned}$$

(b) What assumption did you make in solving part (a)?

Solution:

It is assumed that X_i are independent and normal distributed random variables

(c) Do you think this assumption is justified? Explain briefly.

Solution:

This assumption is not justified because given snowfall in last few days is dependent on the atmospheric conditions, which would also affect today's rainfall.

4. The lifetime (in hours) of a type of electric bulb has expected value 500 and standard deviation 80. Approximate the probability that the sample mean of n such bulbs is greater than 525 when

Given:

$$E[X_i] = 500 \text{ hours}$$

$$SD[X_i] = 80 \text{ hours}$$

$$\text{Let } \bar{X}_n = \frac{\sum_{x_i=1}^{50} X_i}{n}$$

$$\begin{aligned}P(\bar{X}_n > 525) &= P\left(\frac{\bar{X}_{25} - 500}{80/\sqrt{n}} < \frac{525 - 500}{80/\sqrt{n}}\right) \\ &= P\left(\leq Z > \sqrt{n}\frac{25}{80}\right) \\ &= 1 - P\left(\leq Z = \sqrt{n}\frac{25}{80}\right)\end{aligned}$$

(a) $n = 4$;

$$\begin{aligned}P(\bar{X}_4 > 525) &= 1 - P\left(\leq Z = \sqrt{4}\frac{25}{80}\right) \\ &= 1 - 0.7324 \\ &= 0.2676\end{aligned}$$

(b) $n = 16$;

$$\begin{aligned}P(\bar{X}_{16} > 525) &= 1 - P\left(\leq Z = \sqrt{16}\frac{25}{80}\right) \\ &= 1 - 0.8944 = 0.1056\end{aligned}$$

(c) $n = 36$;

$$\begin{aligned}P(\bar{X}_{36} > 525) &= 1 - P\left(\leq Z = \sqrt{36}\frac{25}{80}\right) \\ &= 1 - 0.9699 \\ &= 0.0301\end{aligned}$$

(d) $n = 64$.

$$\begin{aligned} P(\bar{X}_{64} > 525) &= 1 - P\left(\leq Z = \sqrt{64} \frac{25}{80}\right) \\ &= 1 - 0.9938 \\ &= 0.0062 \end{aligned}$$

5. Fifty-two percent of the residents of a certain city are in favor of teaching evolution in high school. Find or approximate the probability that at least 50 percent of a random sample of size n is in favor of teaching evolution, when

Given:

$$p = 0.52$$

$$X_i = \begin{cases} 1, & \text{if } i^{th} \text{ resident is in favour of teaching evolution,} \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{Let } X_n &= \sum_{i=1}^n X_i \quad P\left(X_n \geq \frac{n}{2}\right) = P\left(X_n > \frac{n}{2} - 0.5\right) \\ &= P\left(\frac{X_n - 0.52n}{\sqrt{0.2496n}} > \frac{n/2 - 0.5 - 0.52n}{\sqrt{0.2496n} \equiv a_n}\right) \\ &= P(Z > a_n) \end{aligned}$$

(a) $n = 10$;

$$\begin{aligned} a_{10} &= \frac{10/2 - 0.5 - 0.52(10)}{\sqrt{0.2496(10)}} \\ &= -0.44 \\ P\left(X_n \geq \frac{n}{2}\right) &= P(Z \leq 0.44) \\ &= 0.67 \end{aligned}$$

(b) $n = 100$;

$$\begin{aligned} a_{100} &= \frac{100/2 - 0.5 - 0.52(100)}{\sqrt{0.2496(100)}} \\ &= -0.5 \\ P\left(X_n \geq \frac{n}{2}\right) &= P(Z \leq 0.5) \\ &= 0.6915 \end{aligned}$$

(c) $n = 1000$;

$$\begin{aligned} a_{1000} &= \frac{1000/2 - 0.5 - 0.52(1000)}{\sqrt{0.2496(1000)}} \\ &= -1.3 \\ P\left(X_n \geq \frac{n}{2}\right) &= P(Z \leq 1.3) \\ &= 0.9023 \end{aligned}$$

(d) $n = 10,000$.

$$\begin{aligned} a_{10000} &= \frac{10000/2 - 0.5 - 0.52(10000)}{\sqrt{0.2496(10000)}} \\ &= -4.01 \\ P\left(X_n \geq \frac{n}{2}\right) &= P(Z \leq 4.01) \\ &= 1 \end{aligned}$$

6. An electric scale gives a reading equal to the true weight plus a random error that is normally distributed with mean 0 and standard deviation $\sigma = .1$ mg. Suppose that the results of five successive weighings of the same object are as follows: 3.142, 3.163, 3.155, 3.150, 3.141. $\bar{x} = \frac{\sum_i x_i}{n}$
 $= 3.1502$

- (a) Determine a 95 percent confidence interval estimate of the true weight.

$$\begin{aligned}\bar{x} - z_{0.025}\sigma/\sqrt{n} &\leq \mu \leq \bar{x} + z_{0.025}\sigma/\sqrt{n} \\ 3.1502 - 1.96(0.1)/\sqrt{5} &\leq \mu \leq 3.1502 + 1.906(0.1)/\sqrt{5} \\ 3.0625 &\leq \mu \leq 3.2379\end{aligned}$$

- (b) Determine a 99 percent confidence interval estimate of the true weight.

$$\begin{aligned}\bar{x} - z_{0.005}\sigma/\sqrt{n} &\leq \mu \leq \bar{x} + z_{0.005}\sigma/\sqrt{n} \\ 3.1502 - 2.58(0.1)/\sqrt{5} &\leq \mu \leq 3.1502 + 2.08(0.1)/\sqrt{5} \\ 3.0348 &\leq \mu \leq 3.2656\end{aligned}$$

7. The following are scores on IQ tests of a random sample of 18 students at a large eastern university.
130, 122, 119, 142, 136, 127, 120, 152, 141, 132, 127, 118, 150, 141, 133, 137, 129, 142

if X_i 's are normally distributed, $t = \frac{\bar{X} - \mu}{S/\sqrt{n}}$ has t distributed with $n - 1$ degrees of freedom.

$$\begin{aligned}\bar{x} &= \frac{\sum_i x_i}{n} \\ &= 133.22 \\ s^2 &= \sqrt{\frac{\sum_i (x_i - \bar{x})^2}{n - 1}} \\ &= 10.2128\end{aligned}$$

- (a) Construct a 95 percent confidence interval estimate of the average IQ score of all students at the university

$$\begin{aligned}\bar{x} - z_{0.025}\sigma/\sqrt{n} &\leq \mu \leq \bar{x} + z_{0.025}\sigma/\sqrt{n} \\ 133.22 - 2.11(10.2128)/\sqrt{18} &\leq \mu \leq 133.22 + 2.11(10.2128)/\sqrt{18} \\ 128.141 &\leq \mu \leq 138.3\end{aligned}$$

- (b) Construct a 95 percent lower confidence interval estimate.

$$\begin{aligned}\mu &\leq \bar{x} + z_{0.005}\sigma/\sqrt{n} \\ \mu &\leq 133.22 + 1.74(10.2128)/\sqrt{18} \\ \mu &\leq 137.41\end{aligned}$$

- (c) Construct a 95 percent upper confidence interval estimate.

$$\begin{aligned}\bar{x} - z_{0.025}\sigma/\sqrt{n} &\leq \mu \\ 133.22 - 1.74(10.2128)/\sqrt{18} &\leq \mu \\ \mu &\leq 129.03\end{aligned}$$

8. The capacities (in ampere-hours) of 10 batteries were recorded as follows:

140, 136, 150, 144, 148, 152, 138, 141, 143, 151

- (a) Estimate the population variance σ^2 .

Solution:

Given:

$$n = 10$$

$$\begin{aligned}s^2 &= \sqrt{\frac{\sum_i (x_i - \bar{x})^2}{n - 1}} \\ s^2 &= 32.23\end{aligned}$$

- (b) Compute a 99 percent two-sided confidence interval for σ^2 .

Solution:

$$\begin{aligned}(n - 1) \frac{s^2}{\chi^2_{0.005}} &\leq \sigma^2 \leq (n - 1) \frac{s^2}{\chi^2_{0.995}} \\ 9 \left(\frac{32.23}{23.59} \right) &\leq \sigma^2 \leq 9 \left(\frac{32.23}{1.73} \right) \\ 12.3 &\leq \sigma^2 \leq 167.67\end{aligned}$$

- (c) Compute a value v that enables us to state, with 90 percent confidence, that σ^2 is less than v .

Solution:

$$0 \leq \sigma^2 \leq (n-1) \frac{s^2}{X_{0.90}^2}$$

$$0 \leq \sigma^2 \leq 9 \left(\frac{32.23}{4.17} \right)$$

$$0 \leq \sigma^2 \leq 69.56$$

9. Independent random samples are taken from the output of two machines on a production line. The weight of each item is of interest. From the first machine, a sample of size 36 is taken, with sample mean weight of 120 grams and a sample variance of 4. From the second machine, a sample of size 64 is taken, with a sample mean weight of 130 grams and a sample variance of 5. It is assumed that the weights of items from the first machine are normally distributed with mean μ_1 and variance σ^2 and that the weights of items from the second machine are normally distributed with mean μ_2 and variance σ^2 (that is, the variances are assumed to be equal). Find a 99 percent confidence interval for $\mu_1 - \mu_2$, the difference in population means.

Solution:

Given:

$$n_1 = 36$$

Sample mean weight of first machine, $\bar{x}_1 = 120$ grams

Sample variance of first machine, $s_1^2 = 4$

$$n_2 = 64$$

Sample mean weight of second machine, $\bar{x}_2 = 130$ grams

Sample variance of second machine, $s_2^2 = 5$

Let \bar{X}_1 and \bar{X}_2 be the random variables corresponding to sample mean of first and second machine respectively, respectively.

$$\frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{S_p \sqrt{1/n_1 + 1/n_2}} \sim t_{n_1+n_2-2} \text{ where}$$

$$S_p^2 = \frac{(n_1-1)(S_1)^2 + (n_2-1)(S_2)^2}{n_1 + n_2 - 2}$$

$$= s_p^2 = \frac{(n_1-1)(s_1)^2 + (n_2-1)(s_2)^2}{n_1 + n_2 - 2}$$

$$= \frac{35(4) + 63(5)}{98}$$

$$= 4.643$$

$$s_p = 2.155$$

$$\left(\bar{x}_1 - \bar{x}_2 - t_{0.005} s_p \sqrt{1/n_1 + 1/n_2}, \bar{x}_1 - \bar{x}_2 + t_{0.005} s_p \sqrt{1/n_1 + 1/n_2} \right)$$

$$(120 - 130 - 2.627(2.155)(1/36 + 1/64), 120 - 130 + 2.627(2.155)(1/36 + 1/64))$$

$$(-11.18, 8.82)$$

10. To estimate p , the proportion of all newborn babies that are male, the gender of 10,000 newborn babies was noted. If 5106 of them were male, determine

Solution:

Given:

$$n = 10,000$$

$$\text{Sample probability for being a male, } \hat{p} = \frac{5106}{10000} = 0.5106$$

(a) a 90 percent and

$$(\hat{p} - z_{0.05} \sqrt{\hat{p}(1 - \hat{p})/n}, \hat{p} + z_{0.05} \sqrt{\hat{p}(1 - \hat{p})/n})$$

$$(0.5106 - 1.645 \sqrt{0.5106(1 - 0.5106)/10000}, 0.5106 + 1.645 \sqrt{0.5106(1 - 0.5106)/10000})$$
$$(0.5023, 0.5188)$$

(b) a 99 percent confidence interval estimate of p .

$$(\hat{p} - z_{0.05} \sqrt{\hat{p}(1 - \hat{p})/n}, \hat{p} + z_{0.05} \sqrt{\hat{p}(1 - \hat{p})/n})$$

$$(0.5106 - 2.58 \sqrt{0.5106(1 - 0.5106)/10000}, 0.5106 + 2.58 \sqrt{0.5106(1 - 0.5106)/10000})$$
$$(0.5977, 0.5235)$$