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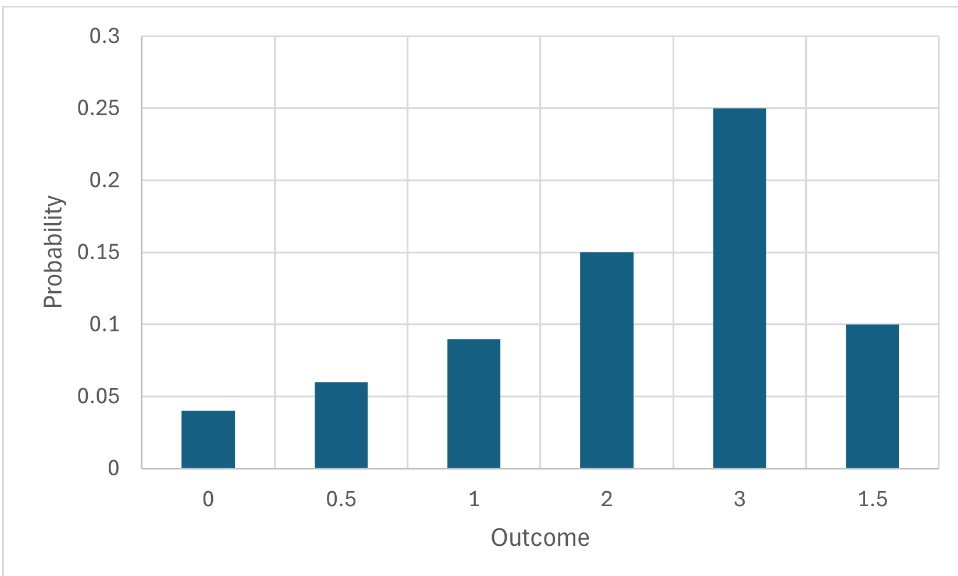
1. Suppose that $X_1, X_2, X_3$ are independent with common probability mass function: $P\{X_i = 0\} = .2, P\{X_i = 1\} = .3, P\{X_i = 3\} = .5, i = 1, 2, 3$

   (a) Plot the probability mass function of the average of a sample size of 2. $X_2 = \frac{X_1+X_2}{2}$

| **Event** | $X_2$ | **Probability** |
|---|---|---|
| 0,0 | 0 | 0.04 |
| 0,1 | 0.5 | 0.06 |
| 1,1 | 1 | 0.09 |
| 1,3 | 2 | 0.15 |
| 3,3 | 3 | 0.25 |
| 0,3 | 1.5 | 0.1 |

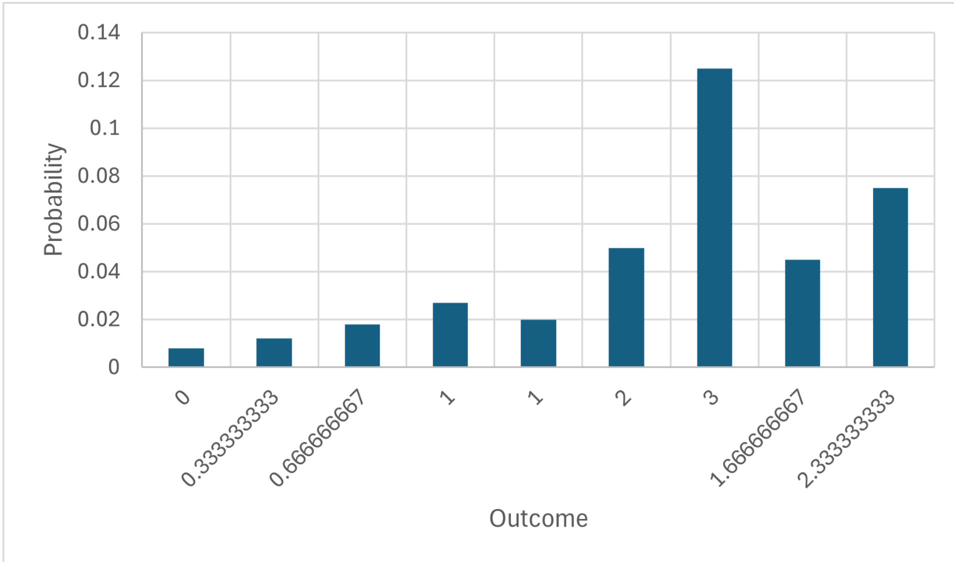



   (b) Determine $E[X_2]$ and $Var(X_2)$

$$E[X_2] = \sum_{i=1}^{n} (x_i)P(x_i)$$
$$= 1.3200$$
$$Var(X_2) = \frac{\sum_{i=1}^{n} (x - \bar{x})^2}{n-1}$$
$$= 1.1667$$

   (c) Plot the probability mass function of the average of a sample size of 3. $X_3 = \frac{X_1+X_2+X_3}{3}$

| **Event** | $X_2$ | **Probability** |
|---|---|---|
| 0,0,0 | 0 | 0.008 |
| 0,0,1 | 0.333333333 | 0.012 |
| 0,1,1 | 0.666666667 | 0.018 |
| 1,1,1 | 1 | 0.027 |
| 0,0,3 | 1 | 0.02 |
| 0,3,3 | 2 | 0.05 |
| 3,3,3 | 3 | 0.125 |
| 1,1,3 | 1.666666667 | 0.045 |
| 1,3,3 | 2.333333333 | 0.075 |




(d) Determine $E[X_3]$ and $Var(X_3)$

$$E[X_3] = \sum_{i=1}^{n} (x_i)P(x_i)$$
$$= 0.7880$$
$$Var(X_3) = \frac{\sum_{i=1}^{n} (x - \bar{x})^2}{n-1}$$
$$= 0.9722$$

2. If 10 fair dice are rolled, approximate the probability that the sum of the values obtained (which ranges from 10 to 60) is between 30 and 40 inclusive.

Solution:

$$E[X_i] = \sum_{x_i=1}^{6} \frac{1}{6}x = \frac{7}{2}$$

$$Var(X_i) = \sum_{x_i=1}^{6} \frac{1}{6}(x - \frac{7}{2})^2 = \frac{35}{12}$$

Let $X = \sum_{x_i=1}^{6} X_i$

$$E[X_i] = \sum_{x_i=1}^{10} E(X_i) = 10(\frac{7}{5}) = 35$$

$$Var(X_i) = \sum_{x_i=1}^{10} Var[X_i] = 10(\frac{35}{12}) = \frac{350}{12}$$

$$P(30 \leq X \leq 40) = P(29.5 < X < 39.5)$$
$$= P\left(\frac{29.5 - 35}{\sqrt{350/12}} < \frac{X - 35}{\sqrt{350/12}} \frac{39.5 - 35}{\sqrt{350/12}}\right)$$
$$= P\left(-1.02 \leq Z \leq 1.02\right)$$
$$= 2(0.8461) - 1$$
$$= 0.6922$$

3. A highway department has enough salt to handle a total of 80 inches of snowfall. Suppose the daily amount of snow has a mean of 1.5 inches and a standard deviation of .3 inch.

   (a) Approximate the probability that the salt on hand will suffice for the next 50 days.
   Solution:
   Given:
   $E[X_i] = 1.5$ inches
   $SD[X_i] = 0.3$ inches
   Let $X = \sum_{x_i=1}^{50} X_i$
   $E[X] = \sum_{x_i=1}^{50} E(X_i) = 50(1.5) = 75$
   $Var(X) = \sum_{x_i=1}^{50} (SD[X_i])^2 = 50(0.09) = 4.5$
   $P(X \leq 80) = P\left(\frac{X-75}{\sqrt{4.5}} < \frac{80-75}{\sqrt{4.5}}\right)$
   $= P(\leq Z \leq 2.36)$
   $= 0.9909$

   (b) What assumption did you make in solving part (a)?
   Solution:
   It is assumed that $X_i$ are independent and normal distributed random variables

   (c) Do you think this assumption is justified? Explain briefly.
   Solution:
   This assumption is not justified because given snowfall in last few days is dependent on the atmospheric conditions, which would also affect today's rainfall.

4. The lifetime (in hours) of a type of electric bulb has expected value 500 and standard deviation 80. Approximate the probability that the sample mean of n such bulbs is greater than 525 when
   Given:
   $E[X_i] = 500$ hours
   $SD[X_i] = 80$ hours
   Let $\bar{X_n} = \frac{\sum_{x_i=1}^{50} X_i}{n}$
   $P(\bar{X_n} > 525) = P\left(\frac{\bar{X_{25}} - 500}{80/\sqrt{n}} < \frac{525-500}{80/\sqrt{n}}\right)$
   $= P\left(\leq Z > \sqrt{n}\frac{25}{80}\right)$
   $= 1 - P\left(\leq Z = \sqrt{n}\frac{25}{80}\right)$

   (a) n = 4;
   $P(\bar{X_4} > 525) = 1 - P\left(\leq Z = \sqrt{4}\frac{25}{80}\right)$
   $= 1 - 0.7324$
   $= 0.2676$

   (b) n = 16;
   $P(\bar{X_1}6 > 525) = 1 - P\left(\leq Z = \sqrt{16}\frac{25}{80}\right)$
   $= 1 - 0.8944 = 0.1056$

(c) n = 36;

$P(\bar{X_3}6 > 525) = 1 - P\left(\leq Z = \sqrt{36}\frac{25}{80}\right)$

$= 1 - 0.9699$

$= 0.0301$

(d) n = 64.

$P(\bar{X_6}4 > 525) = 1 - P\left(\leq Z = \sqrt{64}\frac{25}{80}\right)$

$= 1 - 0.9938$

$= 0.0062$

5. Fifty-two percent of the residents of a certain city are in favor of teaching evolution in high school. Find or approximate the probability that at least 50 percent of a random sample of size n is in favor of teaching evolution, when

Given:

$p = 0.52$

$$X_i = \begin{cases} 1, & \text{if } i^{th} \text{ resident is in favour of teaching evolution,} \\ 0, & \text{otherwise} \end{cases}$$

Let $X_n = \sum_{i=1}^{n} X_i$ $P\left(X_n \geq \frac{n}{2}\right) = P\left(X_n > \frac{n}{2} - 0.5\right)$

$= P\left(\frac{X_n - 0.52n}{\sqrt{0.2496n}} > \frac{n/2 - 0.5 - 0.52n}{\sqrt{0.2496n} \equiv a_n}\right)$

$= P(Z > a_n)$

(a) n = 10;

$a_{10} = \frac{10/2 - 0.5 - 0.52(10)}{\sqrt{0.2496(10)}}$

$= -0.44$

$P\left(X_n \geq \frac{n}{2}\right) = P\left(Z \leq 0.44\right)$

$= 0.67$

(b) n = 100;

$a_{100} = \frac{100/2 - 0.5 - 0.52(100)}{\sqrt{0.2496(100)}}$

$= -0.5$

$P\left(X_n \geq \frac{n}{2}\right) = P\left(Z \leq 0.5\right)$

$= 0.6915$

(c) n = 1000;

$a_{1000} = \frac{1000/2 - 0.5 - 0.52(1000)}{\sqrt{0.2496(1000)}}$

$= -1.3$

$P\left(X_n \geq \frac{n}{2}\right) = P\left(Z \leq 1.3\right)$

$= 0.9023$

(d) n = 10,000.

$a_{10000} = \frac{10000/2 - 0.5 - 0.52(10000)}{\sqrt{0.2496(10000)}}$

$= -4.01$

$P\left(X_n \geq \frac{n}{2}\right) = P\left(Z \leq 4.01\right)$

$= 1$

6. An electric scale gives a reading equal to the true weight plus a random error that is normally distributed with mean 0 and standard deviation $\sigma = .1$ mg. Suppose that the results of five successive weighings of the same object are as follows: $3.142, 3.163, 3.155, 3.150, 3.141$. $\bar{x} = \frac{\sum_i x_i}{n}$
$= 3.1502$

   (a) Determine a 95 percent confidence interval estimate of the true weight.
   $\bar{x} - z_{0.025}\sigma/\sqrt{n} \le \mu \le \bar{x} + z_{0.025}\sigma/\sqrt{n}$
   $3.1502 - 1.96(0.1)/\sqrt{5} \le \mu \le 3.1502 + 1.906(0.1)/\sqrt{5}$
   $3.0625 \le \mu \le 3.2379$

   (b) Determine a 99 percent confidence interval estimate of the true weight.
   $\bar{x} - z_{0.005}\sigma/\sqrt{n} \le \mu \le \bar{x} + z_{0.005}\sigma/\sqrt{n}$
   $3.1502 - 2.58(0.1)/\sqrt{5} \le \mu \le 3.1502 + 2.08(0.1)/\sqrt{5}$
   $3.0348 \le \mu \le 3.2656$

7. The following are scores on IQ tests of a random sample of 18 students at a large eastern university.
$130, 122, 119, 142, 136, 127, 120, 152, 141, 132, 127, 118, 150, 141, 133, 137, 129, 142$

if $X_i$'s are normally distributed, $t = \frac{\bar{X} - \mu}{S/\sqrt{n}}$ has $t$ distributed with $n-1$ degrees of freedom.

$\bar{x} = \frac{\sum_i x_i}{n}$
$= 133.22$
$s^2 = \sqrt{\frac{\sum_i (x_i - \bar{x})^2}{n-1}}$
$= 10.2128$

   (a) Construct a 95 percent confidence interval estimate of the average IQ score of all students at the university
   $\bar{x} - z_{0.025}\sigma/\sqrt{n} \le \mu \le \bar{x} + z_{0.025}\sigma/\sqrt{n}$
   $133.22 - 2.11(10.2128)/\sqrt{18} \le \mu \le 133.22 + 2.11(10.2128)/\sqrt{18}$
   $128.141 \le \mu \le 138.3$

   (b) Construct a 95 percent lower confidence interval estimate.
   $\mu \le \bar{x} + z_{0.005}\sigma/\sqrt{n}$
   $\mu \le 133.22 + 1.74(10.2128)/\sqrt{18}$
   $\mu \le 137.41$

   (c) Construct a 95 percent upper confidence interval estimate.
   $\bar{x} - z_{0.025}\sigma/\sqrt{n} \le \mu$
   $133.22 - 1.74(10.2128)/\sqrt{18} \le \mu$
   $\mu \le 129.03$

8. The capacities (in ampere-hours) of 10 batteries were recorded as follows:
$140, 136, 150, 144, 148, 152, 138, 141, 143, 151$

   (a) Estimate the population variance $\sigma^2$.
   Solution:
   Given:
   $n = 10$
   $s^2 = \sqrt{\frac{\sum_i (x_i - \bar{x})^2}{n-1}}$
   $s^2 = 32.23$

(b) Compute a 99 percent two-sided confidence interval for $\sigma^2$.

Solution:

$(n-1)\frac{s^2}{X^2_{0.005}} \le \sigma^2 \le (n-1)\frac{s^2}{X^2_{0.995}}$

$9\left(\dfrac{32.23}{23.59}\right) \le \sigma^2 \le 9\left(\dfrac{32.23}{1.73}\right)$

$12.3 \le \sigma^2 \le 167.67$

(c) Compute a value $v$ that enables us to state, with 90 percent confidence, that $\sigma^2$ is less than $v$.

Solution:

$0 \le \sigma^2 \le (n-1)\frac{s^2}{X^2_{0.90}}$

$0 \le \sigma^2 \le 9\left(\frac{32.23}{4.17}\right)$

$0 \le \sigma^2 \le 69.56$

9. Independent random samples are taken from the output of twomachines on a production line. The weight of each item is of interest. From the first machine, a sample of size 36 is taken, with sample mean weight of 120 grams and a sample variance of 4. From the second machine, a sample of size 64 is taken, with a sample mean weight of 130 grams and a sample variance of 5. It is assumed that the weights of items from the first machine are normally distributed with mean $\mu_1$ and variance $\sigma^2$ and that the weights of items from the second machine are normally distributed with mean $\mu_2$ and variance $\sigma^2$ (that is, the variances are assumed to be equal). Find a 99 percent confidence interval for $\mu_1 - \mu_2$, the difference in population means.

Solution:

Given:

$n_1 = 36$

Sample mean weight of first machine, $\bar{x_1} = 120$ grams

Sample variance of first machine, $s_1^2 = 4$

$n_2 = 64$

Sample mean weight of first machine, $\bar{x_1} = 130$ grams

Sample variance of first machine, $s_1^2 = 5$

Let $\bar{X}_1$ and $\bar{X}_2$ be the random variables corresponding to sample mean of first and second machine respectively, respectively.

$\dfrac{\bar{X_1} - \bar{X_2} - (\mu_1 - \mu_2)}{S_p\sqrt{1/n_1 + 1/n_2}} \sim t_{n_1+n_2-2}$ where

$S_p^2 = \dfrac{(n-1)(S_1)^2 + (n_2-1)(S_2)^2}{n_1+n_2-2}$

$= s_p^2 = \dfrac{(n_1-1)(s_1)^2 + (n_2-1)(s_2)^2}{n_1+n_2-2}]$

$= \dfrac{35(4) + 63(5)}{98}$

$= 4.643$

$s_p = 2.155$

$\left(\bar{x_1} - \bar{x_2} - t_{0.005}s_p\sqrt{1/n_1 + 1/n_2}, \bar{x_1} - \bar{x_2} + t_{0.005}s_p\sqrt{1/n_1 + 1/n_2}\right)$

$(120 - 130 - 2.627(2.155)(1/36 + 1/64), 120 - 130 + 2.627(2.155)(1/36 + 1/64))$

$(-11.18, 8.82)$

10. To estimate $p$, the proportion of all newborn babies that are male, the gender of 10,000 newborn babies was noted. If 5106 of them were male, determine

Solution:

Given:

$n = 10,000$

Sample probability for being a male, $\hat{p} = \dfrac{5106}{10000} = 0.5106$

(a) a 90 percent and
$(\hat{p} - z_{0.05}\sqrt{\hat{p}(1-\hat{p})/n}, \hat{p} + z_{0.05}\sqrt{\hat{p}(1-\hat{p})/n})$
$(0.5106 - 1.645\sqrt{0.5106(1-0.5106)/10000}, 0.5106 + 1.645\sqrt{0.5106(1-0.5106)/10000})$
$(0.5023, 0.5188)$

(b) a 99 percent confidence interval estimate of $p$.
$(\hat{p} - z_{0.05}\sqrt{\hat{p}(1-\hat{p})/n}, \hat{p} + z_{0.05}\sqrt{\hat{p}(1-\hat{p})/n})$
$(0.5106 - 2.58\sqrt{0.5106(1-0.5106)/10000}, 0.5106 + 2.58\sqrt{0.5106(1-0.5106)/10000})$
$(0.5977, 0.5235)$