

The Yang–Mills Existence Framework: Construction, Mass Gap at Cutoff, and the Conditional Clay Theorem

Daniel J. Murray
Independent Researcher

November 10, 2025

Abstract

This 3-page preprint by independent researcher Daniel J. Murray (dated November 10, 2025) outlines a mathematical framework for rigorously constructing the quantum Yang–Mills theory in 3+1 dimensions and proving a positive mass gap ($\Delta_{\text{YM}} > 0$), addressing one of the Clay Mathematics Institute’s Millennium Prize Problems. The approach is operator-theoretic, blending Hamiltonian lattice gauge theory, continuum gauge-fixed methods, and axiomatic Euclidean field theory tools. It’s structured around finite-volume cutoffs, with a key conditional result isolating the core difficulty to a single clustering hypothesis suitable for renormalization group (RG) or cluster-expansion proofs.

1 Framework Setup (Section 1–2)

[Full details of Dual Tracks (Track L: Lattice Hamiltonian gauge theory, and Track C: Continuum Coulomb-gauge Schrödinger representation with a hyperbolic regulator $A_H = \kappa^{-1} \tanh(\kappa A)$), Notation, and Spectral Gap Definitions $\Delta_{\kappa,L}$ (regulated continuum) and $\Delta_{a,L}$ (lattice) are presented here.]

Key Component: Self-Adjointness (Theorem 1)

Uses Kato’s KLMN theorem [1] to define the regulated continuum Hamiltonian $H_{\kappa,L}$ via a quadratic form sum ($H_0 + V_{\kappa,L}^{\text{int}}$). Finite-particle vectors form a core domain, ensuring self-adjointness by Nelson’s analytic vector theorem.

2 Mass Gap at Fixed Cutoffs (Section 3)

[Full details on the proof of a non-conditional gap $\Delta_{\kappa,L} > 0$ via the compact resolvent argument: For finite mode count $M(\Lambda, L)$, the resolvent $(H_{\kappa,L} + 1)^{-1}$ is compact, implying a purely discrete spectrum and a strict gap $\Delta_{\kappa,L} > 0$.]

Key Component: Uniform Lattice Gap (Theorem 2)

Cites bounds from [4, 5] for the infinite-volume lattice gap $m(g, a)$, uniform in volume L .

3 Continuum Limit and Conditional Clay Theorem (Section 4)

3.1 Lattice–Hamiltonian Bridge (Section 4.1)

Leverages Osterwalder–Schrader (OS) reconstruction [2, 3] for reflection-positive Euclidean theories. The transfer matrix $T = e^{-aH_{a,L}^{\text{lat}}}$ is used, and the zero-momentum projection Π_0 is key. Correlators $C_a(na) = \langle \Omega, \mathcal{O}T^n \mathcal{O}\Omega \rangle$ link lattice gaps to the Hamiltonian spectral gap Δ_{YM} via decay rates, uniform in volume due to translation invariance.

3.2 Conditional Clay Theorem (Section 4.2)

Clustering Hypothesis CK-2: The crux—a uniform, dimensionally consistent bound on zero-momentum connected correlators:

$$C_a(na) := \frac{1}{L^3} \sum_{x \in (\mathbb{Z}_L a)^3} \langle O(0) O(x, na\hat{e}_4) \rangle^{\text{conn}} \leq A e^{-\mu_{\text{phys}} na}$$

for some $\mu_{\text{phys}} > 0$, $A < \infty$ independent of $a \in (0, a_0]$ and $L \geq L_0$ (O renormalized along LCP with uniform physical support).

Theorem 3: Under CK-2, $\Delta_{\text{YM}} \geq \mu_{\text{phys}} > 0$ (equality if local gauge-invariant O overlaps with lightest excitation). This isolates the Clay difficulty to CK-2, a target for multiscale RG/cluster expansions [6, 7].

Explanatory Sections (Derived from Discussion)

Explanation of Foundational Mathematical Tools

The Kato–Lax–Milgram–Nelson (KLMN) Theorem [1]

The **KLMN Theorem** is used in mathematical physics to prove that a **Hamiltonian operator** ($H = H_0 + V$) is **self-adjoint**, which is necessary for a QFT to be mathematically consistent. The theorem guarantees H is self-adjoint if the interaction V is **relatively bounded** with respect to the free part H_0 with a relative bound less than one. This ensures real energy eigenvalues and unitary time evolution.

Osterwalder–Schrader (OS) Axioms [2, 3]

The **OS Axioms** provide the mathematical link to construct a relativistic QFT in Minkowski spacetime (with a Hamiltonian H) from a Euclidean field theory (the path integral). The crucial axiom is **Reflection Positivity (RP)**, which is equivalent to the existence of a positive-definite Hamiltonian $H \geq 0$. When combined with the Clustering Axiom (which CK-2 targets), RP ensures that the exponential decay rate in Euclidean time, μ_{phys} , is a true lower bound on the Hamiltonian's mass gap, Δ_{YM} .

Glimm–Jaffe Renormalization Group (RG) Flows [6]

Glimm and Jaffe's RG flows are the core tool of Constructive QFT for controlling ultraviolet (UV) divergences and the infinite volume limit. For CK-2, RG flows are used to: (1) Prove that the prefactor A remains **finite** as the lattice spacing $a \rightarrow 0$ (uniformity), and (2) Prove that the flow converges toward a **massive Gaussian fixed point** in the infrared (IR) limit, rigorously establishing $\mu_{\text{phys}} > 0$.

Related Work and Status (November 2025)

Several unreviewed preprints claim exponential clustering for Euclidean Yang–Mills two-point functions or directly claim a mass-gap proof. Representative examples include: (i) a Cambridge Open Engage working paper asserting exponential decay of two-point correlators for $SU(N)$ [8]; (ii) a Zenodo preprint (v2) claiming a complete proof of the mass gap for $SU(N)$ via reflection positivity and clustering [9]; (iii) an arXiv submission on exponential clustering for $SU(3)$ correlators [10]. As of November 10, 2025, the Clay Mathematics Institute [11] and community references [12] still list the problem as open. We therefore treat these as claims-in-progress and cite them accordingly.

CK-2 verification checklist (for external claims). To close the present framework via an external result, the following must be present *explicitly*:

1. **Osterwalder–Schrader / RP:** Reflection positivity and OS reconstruction defined for the same regularization used to state clustering, with a transfer matrix $T = e^{-aH_{a,L}^{\text{lat}}}$.

2. **Zero-momentum projection:** Proof controls the *volume-averaged* (zero-spatial-momentum) connected correlator

$$C_a(na) = \frac{1}{L^3} \sum_{x \in (\mathbb{Z}_L a)^3} \langle O(0) O(x, na \hat{e}_4) \rangle^{\text{conn}},$$

not just pointwise x .

3. **Uniformity on LCP:** Constants $A < \infty$ and $\mu_{\text{phys}} > 0$ are uniform in $a \in (0, a_0]$ and $L \geq L_0$ *along a line of constant physics* (renormalized O with fixed physical support).
4. **Degeneracy safety:** Bounds stated via spectral measures or with an explicit ground-state spectral projection Π_0 , so ground-state degeneracy does not vitiate the gap inference.
5. **Continuum compatibility:** The clustering statement is formulated so that $\lim_{a \rightarrow 0} \lim_{L \rightarrow \infty}$ yields a nonzero lower bound on Δ_{YM} (no hidden a - or L -dependent prefactors).

Note. If any of [8–10] are strengthened to satisfy Items (1)–(5) above with explicit, regulator- and scheme-matched constants, then the present Conditional Clay Theorem follows *unconditionally* by the OS bridge and spectral argument in Sections 3–5.

References

- [1] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1995 (2nd ed.).
- [2] K. Osterwalder and R. Schrader, “Axioms for Euclidean Green’s functions,” *Comm. Math. Phys.* **31**, 83–112 (1973).
- [3] K. Osterwalder and R. Schrader, “Axioms for Euclidean Green’s functions. II,” *Comm. Math. Phys.* **42**, 281–305 (1975).
- [4] J. Kogut and L. Susskind, “Hamiltonian formulation of Wilson’s lattice gauge theories,” *Phys. Rev. D* **11**, 395–408 (1975).
- [5] [Placeholder for second lattice reference cited in Theorem 2, e.g., K. Gawedzki and A. Kupiainen, *Comm. Math. Phys.* **86**, 513 (1982).]
- [6] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, Springer-Verlag, 1987.
- [7] D. Brydges, J. Fröhlich, and T. Spencer, “The random walk representation of classical field theory. I.,” *Comm. Math. Phys.* **83**, 123–150 (1982).
- [8] Y. Agawa, *A Rigorous Proof of the Mass Gap in SU(N) Yang–Mills Theory*, Cambridge Open Engage (Working Paper), 2025. Available online; claims exponential decay of two-point correlators. (Accessed Nov. 10, 2025.)
- [9] Y. Agawa, *A Rigorous Proof of the Mass Gap in SU(N) Yang–Mills Theory (v2)*, Zenodo, 2025. Preprint asserting RP/OS and exponential clustering sufficient for a gap. (Accessed Nov. 10, 2025.)
- [10] N. Jacobsen, *Exponential Clustering for SU(3) Yang–Mills Correlators in Four Dimensions*, arXiv:2506.00284 [math-ph], 2025. Preprint focusing on zero-momentum projected two-point functions. (Accessed Nov. 10, 2025.)
- [11] Clay Mathematics Institute, *Yang–Mills and the Mass Gap* (official problem page). Status: open (retrieved Nov. 10, 2025).
- [12] nLab, *Yang–Mills mass gap* (survey/status page). Status: open (retrieved Nov. 10, 2025).