

# The Yang–Mills Existence Framework: Construction, Mass Gap at Cutoff, and the Conditional Clay Theorem

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## Abstract

This 3-page preprint by independent researcher Daniel J. Murray (dated November 10, 2025) outlines a mathematical framework for rigorously constructing the quantum Yang–Mills theory in 3+1 dimensions and proving a positive mass gap ( $\Delta_{\text{YM}} > 0$ ), addressing one of the Clay Mathematics Institute’s Millennium Prize Problems. The approach is operator-theoretic, blending Hamiltonian lattice gauge theory, continuum gauge-fixed methods, and axiomatic Euclidean field theory tools. It’s structured around finite-volume cutoffs, with a key conditional result isolating the core difficulty to a single clustering hypothesis suitable for renormalization group (RG) or cluster-expansion proofs.

## 1 Framework Setup (Section 1–2)

[Full details of Dual Tracks (Track L: Lattice Hamiltonian gauge theory, and Track C: Continuum Coulomb-gauge Schrödinger representation with a hyperbolic regulator  $A_H = \kappa^{-1} \tanh(\kappa A)$ ), Notation, and Spectral Gap Definitions  $\Delta_{\kappa,L}$  (regulated continuum) and  $\Delta_{a,L}$  (lattice) are presented here.]

### Key Component: Self-Adjointness (Theorem 1)

Uses Kato’s KLMN theorem [1] to define the regulated continuum Hamiltonian  $H_{\kappa,L}$  via a quadratic form sum ( $H_0 + V_{\kappa,L}^{\text{int}}$ ). Finite-particle vectors form a core domain, ensuring self-adjointness by Nelson’s analytic vector theorem.

## 2 Mass Gap at Fixed Cutoffs (Section 3)

[Full details on the proof of a non-conditional gap  $\Delta_{\kappa,L} > 0$  via the compact resolvent argument: For finite mode count  $M(\Lambda, L)$ , the resolvent  $(H_{\kappa,L} + 1)^{-1}$  is compact, implying a purely discrete spectrum and a strict gap  $\Delta_{\kappa,L} > 0$ .]

### Key Component: Uniform Lattice Gap (Theorem 2)

Cites bounds from [4, 5] for the infinite-volume lattice gap  $m(g, a)$ , uniform in volume  $L$ .

## 3 Continuum Limit and Conditional Clay Theorem (Section 4)

### 3.1 Lattice–Hamiltonian Bridge (Section 4.1)

Leverages Osterwalder–Schrader (OS) reconstruction [2, 3] for reflection-positive Euclidean theories. The transfer matrix  $T = e^{-aH_{a,L}^{\text{lat}}}$  is used, and the zero-momentum projection  $\Pi_0$  is key. Correlators  $C_a(na) = \langle \Omega, \mathcal{O}T^n \mathcal{O} \Omega \rangle$  link lattice gaps to the Hamiltonian spectral gap  $\Delta_{\text{YM}}$  via decay rates, uniform in volume due to translation invariance.

### 3.2 Conditional Clay Theorem (Section 4.2)

**Clustering Hypothesis CK-2:** The crux—a uniform, dimensionally consistent bound on zero-momentum connected correlators:

$$C_a(na) := \frac{1}{L^3} \sum_{x \in (\mathbb{Z}_L a)^3} \langle O(0) O(x, na\hat{e}_4) \rangle^{\text{conn}} \leq A e^{-\mu_{\text{phys}} na}$$

for some  $\mu_{\text{phys}} > 0$ ,  $A < \infty$  independent of  $a \in (0, a_0]$  and  $L \geq L_0$  ( $O$  renormalized along LCP with uniform physical support).

**Theorem 3:** Under CK-2,  $\Delta_{\text{YM}} \geq \mu_{\text{phys}} > 0$  (equality if local gauge-invariant  $O$  overlaps with lightest excitation). This isolates the Clay difficulty to CK-2, a target for multiscale RG/cluster expansions [6, 7].

## Explanatory Sections (Derived from Discussion)

### Explanation of Foundational Mathematical Tools

#### The Kato–Lax–Milgram–Nelson (KLMN) Theorem [1]

The **KLMN Theorem** is used in mathematical physics to prove that a **Hamiltonian operator** ( $H = H_0 + V$ ) is **self-adjoint**, which is necessary for a QFT to be mathematically consistent. The theorem guarantees  $H$  is self-adjoint if the interaction  $V$  is **relatively bounded** with respect to the free part  $H_0$  with a relative bound less than one. This ensures real energy eigenvalues and unitary time evolution.

#### Osterwalder–Schrader (OS) Axioms [2, 3]

The **OS Axioms** provide the mathematical link to construct a relativistic QFT in Minkowski spacetime (with a Hamiltonian  $H$ ) from a Euclidean field theory (the path integral). The crucial axiom is **Reflection Positivity (RP)**, which is equivalent to the existence of a positive-definite Hamiltonian  $H \geq 0$ . When combined with the Clustering Axiom (which CK-2 targets), RP ensures that the exponential decay rate in Euclidean time,  $\mu_{\text{phys}}$ , is a true lower bound on the Hamiltonian’s mass gap,  $\Delta_{\text{YM}}$ .

#### Glimm–Jaffe Renormalization Group (RG) Flows [6]

**Glimm and Jaffe’s** RG flows are the core tool of Constructive QFT for controlling ultraviolet (UV) divergences and the infinite volume limit. For CK-2, RG flows are used to: (1) Prove that the prefactor  $A$  remains **finite** as the lattice spacing  $a \rightarrow 0$  (uniformity), and (2) Prove that the flow converges toward a **massive Gaussian fixed point** in the infrared (IR) limit, rigorously establishing  $\mu_{\text{phys}} > 0$ .

### Related Work and Status (November 2025)

Several unreviewed preprints claim exponential clustering for Euclidean Yang–Mills two-point functions or directly claim a mass-gap proof. Representative examples include: (i) a Cambridge Open Engage working paper asserting exponential decay of two-point correlators for  $\text{SU}(N)$  [8]; (ii) a Zenodo preprint (v2) claiming a complete proof of the mass gap for  $\text{SU}(N)$  via reflection positivity and clustering [9]; (iii) an arXiv submission on exponential clustering for  $\text{SU}(3)$  correlators [10]. As of November 10, 2025, the Clay Mathematics Institute [11] and community references [12] still list the problem as open. We therefore treat these as claims-in-progress and cite them accordingly.

**CK-2 verification checklist (for external claims).** To close the present framework via an external result, the following must be present *explicitly*:

1. **Osterwalder–Schrader / RP:** Reflection positivity and OS reconstruction defined for the same regularization used to state clustering, with a transfer matrix  $T = e^{-aH_{a,L}^{\text{lat}}}$ .

2. **Zero-momentum projection:** Proof controls the *volume-averaged* (zero-spatial-momentum) connected correlator

$$C_a(na) = \frac{1}{L^3} \sum_{x \in (\mathbb{Z}_L a)^3} \langle O(0) O(x, na \hat{e}_4) \rangle^{\text{conn}},$$

not just pointwise  $x$ .

3. **Uniformity on LCP:** Constants  $A < \infty$  and  $\mu_{\text{phys}} > 0$  are uniform in  $a \in (0, a_0]$  and  $L \geq L_0$  *along a line of constant physics* (renormalized  $O$  with fixed physical support).
4. **Degeneracy safety:** Bounds stated via spectral measures or with an explicit ground-state spectral projection  $\Pi_0$ , so ground-state degeneracy does not vitiate the gap inference.
5. **Continuum compatibility:** The clustering statement is formulated so that  $\lim_{a \rightarrow 0} \lim_{L \rightarrow \infty}$  yields a nonzero lower bound on  $\Delta_{\text{YM}}$  (no hidden  $a$ - or  $L$ -dependent prefactors).

**Note.** If any of [8–10] are strengthened to satisfy Items (1)–(5) above with explicit, regulator- and scheme-matched constants, then the present Conditional Clay Theorem follows *unconditionally* by the OS bridge and spectral argument in Sections 3–5.

## References

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