The Yang–Mills Existence Framework: Construction, Mass Gap at Cutoff, and the Conditional Clay Theorem

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Abstract

We give an operator-theoretic construction of the finite-volume Coulomb-gauge Yang–Mills Hamiltonian ($\mathbf{H}^{\kappa,\mathbf{L}}$) with a hyperbolic regulator via KLMN [4, 3], prove compact resolvent (trace-class semigroup) and thus a cutoff mass gap ($\Delta_{\kappa,L} > 0$). Using Osterwalder–Schrader (OS) reconstruction [1, 2] with a zero-momentum projection, we transfer a volume-uniform lattice gap to the Hamiltonian setting. Assuming a single uniform clustering hypothesis CK-2 along a line of constant physics (LCP), the continuum Hamiltonian gap satisfies $\Delta_{\mathrm{YM}} \geq \mu_{\mathrm{phys}} > 0$. This work isolates the Clay difficulty to a single, dimensionally correct clustering statement, providing a clean target for multiscale RG/cluster-expansion techniques [7, 8].

Keywords: Yang–Mills, mass gap, reflection positivity, OS reconstruction, KLMN, compact resolvent, clustering, lattice Hamiltonian bridge.

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1. Introduction and The Nature of the Problem

The rigorous construction of the quantum theory for a $3+1$D non-abelian gauge field and the proof of a spectral gap $\Delta_{\mathrm{YM}} > 0$ constitutes the Yang–Mills Existence and Mass Gap problem.

Notation. Boldface denotes operators ($\mathbf{H}, \mathbf{N}, \mathbf{T}$); scalars are upright. Let $M = M(\Lambda, L)$ be the *single–particle mode count* (finite for fixed cutoffs).

The framework proceeds via two tracks: Track L (Hamiltonian Lattice Gauge Theory [5]) and Track C (Continuum Gauge-Fixed) with a regulator $\mathbf{A^H} = \kappa^{-1} \tanh(\kappa\mathbf{A})$.

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2. Rigorous Construction and Spectral Definition

We work with the Coulomb-gauge Schrödinger Hamiltonian; denote the $\kappa \to 0$ limit by $\mathbf{H}_{\mathrm{YM}}^{\mathrm{Coul},\mathbf{L}}$ and the regulated operator by $\mathbf{H}^{\kappa,\mathbf{L}}$.

Gap Definitions.

$$\Delta_{\kappa,L} := \inf\big(\sigma(\mathbf{H}^{\kappa,\mathbf{L}}) \setminus \{E_0\}\big) - E_0, \quad \Delta_{a,L} := \inf\big(\sigma(\mathbf{H}^{\mathrm{lat}}_{\mathbf{a},\mathbf{L}}) \setminus \{E_0\}\big) - E_0.$$

Theorem 1 (**KLMN Self-Adjointness**, [4])**.** *The Hamiltonian $\mathbf{H}^{\kappa,\mathbf{L}}$ is defined via the form sum $\mathbf{H_0} \dot{+} \mathbf{V}^{\kappa,\mathbf{L}}_{\mathrm{int}}$. The form sum is uniquely defined and self-adjoint on the form domain $\mathcal{D}(q_K)$.*

Core. The finite-particle vectors are a common form core for all forms $q_{\kappa,L}$. They are also a core for $\mathbf{H}^{\kappa,\mathbf{L}}$ and $\mathbf{H}^{\mathrm{Coul},\mathbf{L}}_{\mathrm{YM}}$ by Nelson's analytic vector theorem.

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3. The Mass Gap at Fixed Cutoffs

Proposition 1 (**Trace-Class Semigroup $\Rightarrow$ Compact Resolvent**)**.** *Since $\mathbf{H}^{\kappa,\mathbf{L}} \geq \alpha\mathbf{N} - \beta$ ($\alpha > 0$), the semigroup satisfies*

$$e^{-t\mathbf{H}^{\kappa,\mathbf{L}}} \;\leq\; e^{t\beta}\, e^{-\alpha t\mathbf{N}}.$$

The trace is

$$\mathrm{Tr}\; e^{-\alpha t\mathbf{N}} = \prod_{j=1}^{M} \frac{1}{1 - e^{-\alpha t}} = (1 - e^{-\alpha t})^{-M} < \infty.$$

*Since M is finite, $e^{-t\mathbf{H}^{\kappa,\mathbf{L}}}$ is trace-class. The resolvent $(\mathbf{H}^{\kappa,\mathbf{L}} + 1)^{-1}$ is therefore **trace-class and hence compact**, guaranteeing a purely discrete spectrum and $\Delta_{\kappa,L} > 0$.*

Theorem 2 (**Uniform Gap at Strong Coupling**, [5, 6])**.** *The infinite-volume lattice gap $m(g,a)$ satisfies:*

$$m(g,a) \geq \frac{\kappa_E(N_c)}{a} g^2 - \frac{c_{N_c}}{a\, g^2}.$$

The bound holds uniformly in L.

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4. The Continuum Limit and the Conditional Clay Theorem

The limits must be taken along the line of constant physics (LCP): $\Delta_{\mathrm{YM}} = \lim_{a\to 0} \lim_{L\to\infty} \Delta_{a,L}$.

4.1. Lattice $\leftrightarrow$ Hamiltonian Bridge

Assume translation invariance and Reflection positivity (OS). Let $\mathbf{T} = e^{-a\mathbf{H}^{\mathrm{lat}}_{\mathbf{a},\mathbf{L}}}$ and let $\mathbf{\Pi_0}$ be its spectral projection onto the ground-state subspace. OS reconstruction yields

$$C_a(na) = \langle \Omega, \widetilde{O}\, \mathbf{T}^n\, \widetilde{O}\, \Omega \rangle \quad \text{with } \widetilde{O} = \sum_x O(x, 0).$$

The zero-momentum projection ensures the decay rate μ_{phys} bounds $\Delta_{a,L}$ from below irrespective of ground-state degeneracy.

4.2. Conditional Theorem

The entire framework is conditional on the following uniform bound.

CK-2 (Uniform Clustering at Zero Momentum). The zero-momentum connected correlator is:

$$C_a(na) := \frac{1}{L^3} \sum_{x \in (\mathbb{Z}_L a)^3} \langle O(0)\, O(x, na\, \hat{e}_4) \rangle^{\mathrm{conn}}.$$

The hypothesis is:

$$\boxed{\exists\, \mu_{\mathrm{phys}} > 0,\ A < \infty \text{ independent of } a \in (0, a_0],\ L \geq L_0 :\ C_a(na) \leq A\, e^{-\mu_{\mathrm{phys}}\, na}\ \forall n \in \mathbb{Z}_{\geq 0}}$$

(with O renormalized along the LCP and uniform physical support).

Theorem 3 (**Existence of a Positive Continuum Mass Gap**). *Assuming CK-2: The continuum mass gap is bounded below by the clustering rate:*

$\Delta_{\mathrm{YM}} \ \geq\ \mu_{\mathrm{phys}} > 0, \qquad \Delta_{\mathrm{YM}} = \mu_{\mathrm{phys}}$ *if some local gauge-invariant O has nonzero overlap with the l*

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