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LESSION 2.2:

Second Order

Ordinary Differential Equations (ODEs)

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OBJECTIVE:

- To solve Homogeneous linear equations for various cases
- To apply Homogeneous Solution for initial- and boundary-value problems

Outlines

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- **Second Order Homogeneous Linear Equations with Constant Coefficients**

Second-Order Homogeneous Equations with Constant Coefficients

- Standard Form of a Second-Order Homogeneous Equation with Constant Coefficients is

$$y'' + ay' + by = 0$$

whose coefficients a and b are

constants.

How to solve:

$$y'' + ay' + by = 0$$

$$\text{Try } y = e^{\lambda x}$$

$$\Rightarrow (e^{\lambda x})'' + a(e^{\lambda x})' + be^{\lambda x} = 0$$

$$\Rightarrow \lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0$$

$$\Rightarrow (\lambda^2 + a\lambda + b)e^{\lambda x} = 0$$

Characteristic Equation

$$\lambda^2 + a\lambda + b = 0$$

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

- Case 1:

$$a^2 - 4b > 0$$

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

- Case 2:

$$a^2 - 4b = 0$$

$$y = (c_1 + c_2 x) e^{\lambda x}$$

- Case 3:

$$a^2 - 4b < 0$$

$$y = e^{sx}(A\cos tx + B\sin tx)$$

Case I: Two distinct real roots λ_1 and $\lambda_2 (\lambda_1 \neq \lambda_2)$

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$$y_1 = e^{\lambda_1 x},$$
$$y_2 = e^{\lambda_2 x}$$

$$\therefore \ \frac{y_1}{y_2} = e^{(\lambda_1 - \lambda_2)x}$$

$$\neq constant$$

General solution :

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

Example 1

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Find the general solution of ,

Solution

$$y'' - 9y' + 20y = 0.$$

Diff. Equation : $y'' - 9y' + 20y = 0$

The characteristic equation :

$$\lambda^2 - 9\lambda + 20 = 0$$

$$(\lambda - 4)(\lambda - 5) = 0$$

Roots : $\lambda_1 = 4,\ \lambda_2 = 5$

Hence the general solution :

$$y = c_1 e^{4x} + c_2 e^{5x}$$

Case II : Real double root $\lambda\ (\lambda_1 = \lambda_2)$

$$a^2 - 4b = 0 \;\Rightarrow\; so\ \lambda = \lambda_1 = \lambda_2 = -{}^{a}/_{2}$$

$$y_1 = e^{\lambda x} = e^{-ax/2} \qquad\qquad y_2 = uy_1$$

$$y'' + ay' + by = 0$$

$$u = \int \frac{1}{{y_1}^2} e^{-\int a dx}\, dx = \int \frac{e^{-ax}}{e^{-ax}} dx = x,$$

$$\therefore\ y_2 = xy_1 = xe^{-ax/2}$$

General solution : $y = (c_1 + c_2 x)e^{\lambda x} \;\Rightarrow\; y = (c_1 + c_2 x)e^{-ax/2}$

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Example Initial Value Problem in the Case of a Double Root

$$y'' + y' + 0.25y = 0, y(0) = 3.0, y'(0) = -3.5$$

Solution

Diff. Equation : $\boldsymbol{y'' + y' + 0.25y = 0, y(0) = 3.0, y'(0) = -3.5}$

The characteristic equation :

$$\boxed{\boldsymbol{\lambda^2 + \lambda + 0.25 = 0}}$$

$$\boldsymbol{(\lambda + 0.5)(\lambda + 0.5) = 0}$$

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Roots : $\boldsymbol{\lambda = \lambda_1 = \lambda_2 = -0.5}$

The general solution is

$$y = (c_1 + c_2 x)e^{-0.5x}$$

$$y' = c_2 e^{-0.5x} - 0.5(c_1 + c_2 x)e^{-0.5x}$$

$$y(0) = 3.0 \Rightarrow 3.0 = (c_1 + c_2 \cdot 0)e^0 \Rightarrow c_1 = 3.0$$

$$y'(0) = -3.5 \Longrightarrow -3.5 = c_2 e^0 - 0.5(c_1 + c_2 . 0)e^0 \Longrightarrow c_2 = -2$$

The particular solution is

$$y = (3.0 - 2x)e^{-0.5x}$$

Case III : Complex roots

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When $a^2 - 4b < 0.$ The Roots :

$$\lambda = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4b}\,,$$

$$\lambda = s \pm i\, t$$

$$y_1 = e^{sx}\cos tx\,, \qquad y_2 = e^{sx}\ \sin tx$$

$$where, \qquad s = -\frac{a}{2}\ and\ t = \frac{1}{2}\sqrt{4b - a^2}$$

General solution : $y = c_1 e^{sx}\ \cos tx + c_2 e^{sx}\ \sin tx$

Example Initial Value Problem in the Case of a Double Root

$$y'' + 0.4y' + 9.04y = 0, y(0) = 0, y'(0) = 3$$

Solution

Diff. Equation : $\boldsymbol{y'' + 0.4y' + 9.04\,y = 0, y(0) = 0, y'(0) = 3}$

The characteristic equation :

$$\boxed{\boldsymbol{\lambda^2 + 0.4\lambda + 9.04 = 0}}$$

$$\boldsymbol{\lambda = \frac{-0.4 \pm \sqrt{0.16 - 36.16}}{2} = -0.2 \pm 3i}$$

The general solution is

$$y = c_1 e^{-0.2x} \cos 3x + c_2 e^{-0.2x} \sin 3x$$

$$y(0) = 0 \Rightarrow 0 = (c_1 \overset{1}{\cancel{\cos 0}} + c_2 \overset{0}{\cancel{\sin 0}})e^0 \Rightarrow c_1 = 0$$

$$y = c_2 e^{-0.2x} \sin 3x$$

$$y' = 3c_2 e^{-0.2x} \cos 3x - 0.2c_2 e^{-0.2x} \sin 3x$$

$$y'(0) = 3 \Longrightarrow 3 = 3c_2 e^0 \cos 0 - 0.2c_2 e^0 \sin 0 \Longrightarrow c_2 = 1$$

The particular solution is

$$y = e^{-0.2x} \sin 3x$$

Summary of Case I-III

Case	Roots	Basis: y_1, y_2	General Solution
I	Distinct real $\lambda_1 \neq \lambda_2$	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$\boldsymbol{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$
II	Real double root $\lambda = \lambda_1 = \lambda_2$ $= -a/2$	$e^{\lambda x}, xe^{\lambda x}$ $e^{-\frac{ax}{2}}, xe^{-\frac{ax}{2}}$	$\boldsymbol{y = (c_1 + c_2 x)e^{\lambda x}}$ $\boldsymbol{y = (c_1 + c_2 x)e^{-ax/2}}$
III	$\lambda_1 = s + it$ $\lambda_2 = s - it$	$e^{sx} \cos tx$ $e^{sx} \sin tx$	$\boldsymbol{y = c_1 e^{sx} \cos tx + c_2 e^{sx} \sin tx}$

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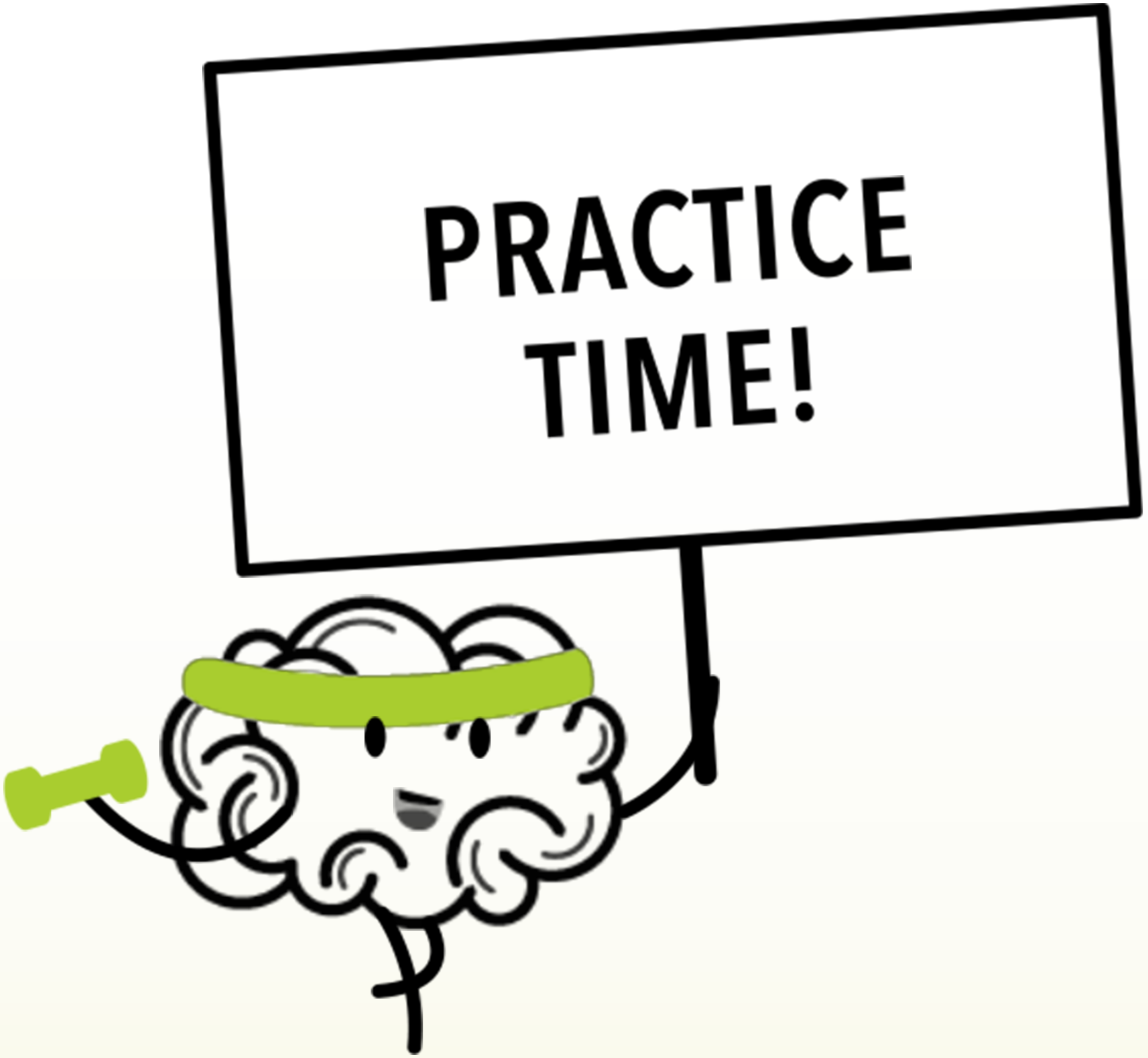

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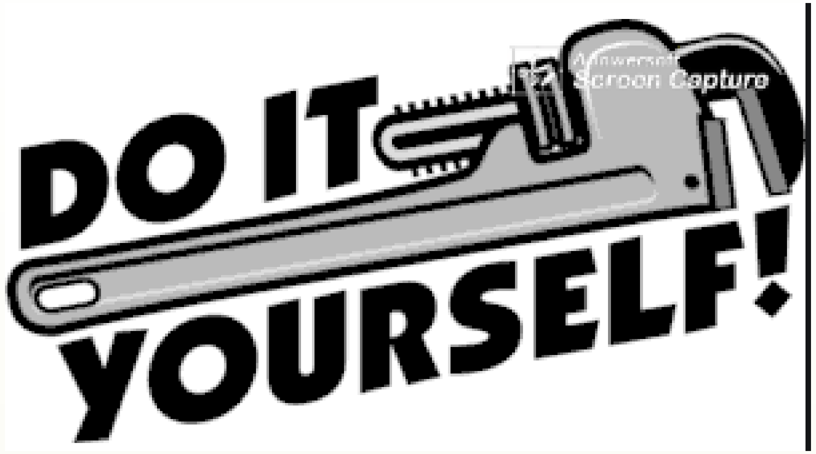
Solve the initial value problem,

$$y'' + y' - 6y = 0 \qquad y(0) = 10, \qquad y'(0) = 0$$



Solution

Diff. Equation : $y'' + y' - 6y = 0$

The characteristic equation : $\boldsymbol{\lambda^2 + \lambda - 6 = 0}$

Factorization : $(\lambda + 3)(\lambda - 2) = 0$

Roots : $\boldsymbol{\lambda_1 = -3,\ \lambda_2 = 2}$

The general solution : $\boldsymbol{y(x) = c_1 e^{-3x} + c_2 e^{2x}}$

$$y'(x) = -3c_1 e^{-3x} + 2c_2 e^{2x}$$

Initial conditions : $y(0) = 10,\ y'(0) = 0$

$$y(0) = c_1 + c_2 = 10, \qquad y'(0) = -3c_1 + 2c_2 = 0$$

$$\therefore\ c_1 = 4, c_2 = 6 \Rightarrow \textit{Particular solution}: \boldsymbol{y = 4e^{-3x} + 6e^{2x}}$$

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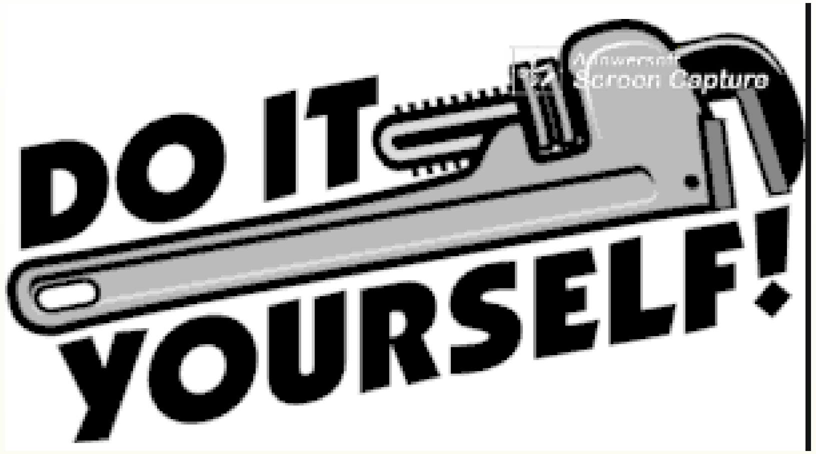
Give the general solution of differential equation

$$4y'' + 4y' + 10y = 0$$



Solution

Diff. Equation : $4y'' + 4y' + 10y = 0$

The characteristic equation : $4\lambda^2 + 4\lambda + 10 = 0$

Roots : $\lambda = \frac{-4 \pm\sqrt{16-160}}{8} = \frac{-4 \pm 12\, i}{8}$

$$\Rightarrow \lambda_1 = -\frac{1}{2} + \frac{3i}{2} \qquad \lambda_2 = -\frac{1}{2} - \frac{3i}{2}$$

The general solution :

$$y = e^{-x/2}(c_1 \cos(\tfrac{3}{2})x + c_2 \sin(\tfrac{3}{2})x)$$

Find an ODE $y'' + ay' + by = 0$ for the given basis.

$$e^{2.6x}, e^{-4.3x}$$

Solution

The given basis: $e^{2.6x}, e^{-4.3x}$

Compare the basis: $e^{\lambda_1 x}, e^{\lambda_2 x}$

$\therefore it's\ root: \lambda_1 = 2.6, \qquad \lambda_2 = -4.3$

$\therefore \lambda - 2.6 = 0, \qquad \lambda + 4.3 = 0$

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Continue

$$(\lambda - 2.6)(\lambda + 4.3) = 0$$

$$\lambda^2 + 1.7\lambda - 11.18 = 0$$

$\therefore$the ODE is $y'' + 1.7y' - 11.8y = 0$

Find an ODE $y'' + ay' + by = 0$ for the given basis.

$$cos2\pi x, sin2\pi x$$

Solution

The given basis: $cos2\pi x, sin2\pi x$

Compare the basis: $e^{sx}\ costx$
 $e^{sx} sintx$

$$\therefore it's\ root{:}\ \lambda_1 = 0 + i2\pi, \qquad \lambda_2 = 0 - i2\pi$$

$$\therefore \lambda\ -2\pi i = 0, \qquad \lambda\ + 2\pi i = 0$$

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Continue

$$(\lambda - 2\pi i)(\lambda + 2\pi i) = 0$$

$$\lambda^2 - 4\pi^2 i^2 = 0$$

$$\lambda^2 + 4\pi^2 = 0$$

∴the ODE is $y'' + 4\pi^2 = 0$

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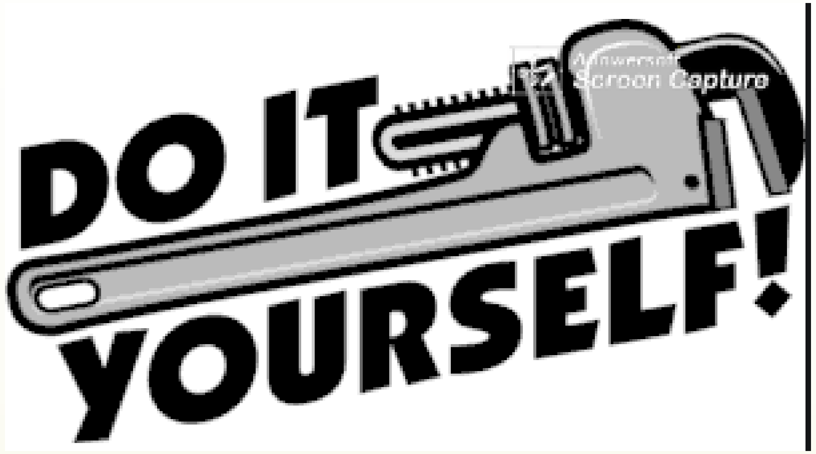
Find an ODE $y'' + ay' + by = 0$ for the given basis.

$$e^{-\sqrt{5}x}, xe^{-\sqrt{5}x}$$



Solution

The given basis: $\boxed{e^{-\sqrt{5}x}, xe^{-\sqrt{5}x}}$

Compare the basis: $\boxed{e^{\lambda x}, xe^{\lambda x}}$

$$\therefore\ it's\ root: \lambda_1 = -\sqrt{5}, \qquad \lambda_2 = -\sqrt{5}$$

$$\therefore\ (\lambda + \sqrt{5})^2 = 0,$$

$$\lambda^2 + 2\sqrt{5}\lambda + 5 = 0$$

∴the ODE is $\boxed{y'' + 2\sqrt{5}y' + 5y = 0}$

Home Work

Find a general solution. Check your answer by substitution.

$$4y'' - 25y = 0$$

$$y'' + 2\pi y' + \pi^2 y = 0$$

$$y'' + 4.5y' = 0$$

Home Work

Solve the IVP. Check that your answer satisfies the ODE as well as the initial conditions. Show the details of your work.

$$y'' + 25y = 0, \mathrm{y}(0) = 4.6, \mathrm{y}'(0) = -1.2$$

$$4y'' - 4y' - 3y = 0, y(-2) = e, y'(-2) = -e/2$$

Practice Makes Perfect

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