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Winning strategies for infinite games: from large cardinals to computer science extended abstract

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Denis Richard in his roaring sixtie

Abstract

(1) Set Theory's topic of Large Cardinals is the most **infinitary** part of Mathematics. At the other end, the study of Finite State machines is the very first chapter of **Computer Science**. Can we combine these two opposite extremes fruitfully and use ideas coming from large cardinals to produce results about finite state machines?

(2) Using the large cardinal axiom of **"sharps"** Martin proved **analytic determinacy:** the existence of a winning strategy for one of the players in every infinite game of perfect information between two players, provided the winning set of one of the players happens to be an analytic one. I modify and complement his proof so as to obtain a new proof of the Rabin, Buechi-Landweber, Gurevich-Harrington theorem of **finite state** determinacy: existence of a winning strategy computed by a finite state machine, when the player's winning sets are themselves finite state accepted. This 4th proof of finite state determinacy is again a totally new one—as must be the case since it still makes use of the large cardinal axiom, to prove such an effective result!

(3) Thus to our question (1) the new proof answers with a clear and surprising yes ... of modest bearing, since it only concerns an old result. But we shall explain why the new proof is more suggestive and useful than former ones, in order to address today's two main unsolved problems connecting effective determinacy with Computer Science: the P-time realization of finite state strategies, and the P-time decision of the winner of a parity game. Indeed: adding to our proof an effective elimination of the very restricted part of the axiom of sharps that it really uses, may lead to useful new results, ideas and methods around these two hard, crucial problems. (Of course our use of sharps is eliminated in advance in 3 ways: namely the 3 former proofs of Rabin, of Buechi-Landweber and of Gurevich-Harrington. But the new proof seems to be more suggestive than the old ones when **feasibility** questions are at stake....)

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1. Introduction

Let A be a set of binary "infinite words": $A \subset 2^\omega$; the associated **game** $G(A)$ lets player I and player II choose $z \in 2^\omega$. This **play** z is won by player I iff z belongs to A, and by player II otherwise. (Thus A is player I 's **winning set**; while the winning set of player II is the complement $2^\omega - A$). The choice of z is made in an infinite succession of turns: at turn n player I chooses $z(2n) \in 2$ and player II replies with $z(2n+1) \in 2$.

$G(A)$ is **determined** iff one of the player has a **winning strategy.** That is, a map $\sigma : 2^{<\omega} \longrightarrow 2$ such that the player is guaranteed to win every play z in which he played $z(n) = \sigma(z|n)$ whenever it was his turn to play.

Martin proved **"analytic determinacy":** determinacy of $G(A)$ in case the winning set of one of the players belongs to the class of all analytic sets. The latter is the most important class of well behaved, non pathological sets of infinite sequences—in Mathematics, in Physics and in Computer Science as well. Of course this analytic class includes all "effective" sets such as the **FS** (:= finite state) accepted sets; but it is immensely richer. To the effect that in order to deal with them, Martin's proof uses **"sharps":** a large cardinal axiom, much stronger and daring than ordinary Set Theory although the strength of the latter is already way beyond the current needs of mathematics (as of today). And Harrington proved a form of converse, to the effect that this axiom is in a sense **required** to prove analytic determinacy. This underlies the fact that the winning strategy proved to exist is in the general case of the most extremely **non** effective kind. In fact, the theory of large cardinals is the most infinitary part of mathematics; and analytic (more generally projective) determinacy is the most spectacular outcome of this infinitary riot.

See [2] for a self-contained exposition of these results.

Nevertheless we shall supplement Martin's proof so that it yields again **"finite state determinacy"—**which is one of the few mathematical theorems of recognized bearing for Computer Science: existence of some FS computable winning strategy in $G(A)$, whenever the winning set A itself is FS accepted.

The present paper is only the extended abstract of [3]; the latter gives our new proof in full details, eliminates the use of a large cardinal from it. Ressayre [3] also gives quite a few reasons why our new proof of FS determinacy should be of interest for many experts in TCS; even the original analytic determinacy proof of Martin should be valuable to them. Therefore Ressayre [3] in addition presents Martin's proof in a way we hope appropriate for TCS: in particular we use a new form of the large cardinal axiom "sharps" (since the original one is hopelessly far away from CS).

2. FS determinacy

Recall that A is a **finite state accepted** subset of 2^ω if there is an FS machine such that A consists of the words $z \in 2^\omega$ which are accepted by this machine. We call it the **acceptor** A, we denote Q its **set of states** (so Q is finite) and for every $s \in 2^{<\omega}$ we denote $q(s)$ the state which the acceptor A enters upon reading the word s. Thus when acceptor A reads an infinite word $z \in 2^{\omega}$, Q has a subset $\mathrm{Inf}\, z := \{q;\ q = q(z|n)$ for infinitely many $n < \omega\}$ which forms a loop in the transition graph of the acceptor A. And the acceptor has a fixed family $\mathscr{F}$ of **accepting** loops such that z is accepted iff $\mathrm{Inf}\, z$ belongs to $\mathscr{F}$.

Let us call **rejecting** all other loops in Q. In order for player I to win the game $G(A)$, he must make sure that no rejecting loop gets eventually repeated during the play z. Here is an idea that player I could apply to ensure this: $*$) player I plays ordinals in addition to his moves, in such a way that whenever a rejecting loop gets repeated by the acceptor A (reading the play z during its performance), the ordinals chosen by player I at successive occurrences of the repetition are strictly decreasing.

Remark 1. All If player I manages for the whole play z to ensure (*), then every sequence of repetitions of any rejecting loop is finite; hence $z \in A$ and this play of $G(A)$ is won by player I.

This idea can be made precise in different ways, which are not equivalent. Indeed, the whole sequel depends on a carefully selected form of (*)—given in the full version of this work. For any ordinal γ we denote $G(\gamma)$ the version of the game $G(A)$ in which in addition to his moves $z(2n)$, player I produces (and **shows** to player II) ordinals $< \gamma$ in the way (*); and he wins iff he manages to do so for the whole infinite play. If he cannot, player II **immediately** wins. Thus the only cases where player II wins z^* are the **finite** steps of the plays where player I is unable to provide an ordinal subject to (*). Hence $G(\gamma)$ is an **open** game for player II: she wins iff she already does so at a finite stage of the play. Openness of the game $G(\gamma)$ provides an easy way to define (but not to compute effectively!) a winning strategy σ_γ for one of the players: σ_γ is based on backtrack analysis from the set of all positions in $G(\gamma)$ that are already won by player II. (Nota Bene: this simply recalls in a particular case the **Gale–Stewart theorem**—determinacy of all games that are open in the above sense.)

Below we distinguish two cases: in Case 1, player I has a winning strategy for the original game $G(A)$ and in Case 2 it is player II.

Case 1: There is some γ such that the above strategy σ_γ is winning for player I. Then by the above Remark, σ_γ is winning for player I also in the original game $G(A)$ (just omitting to exhibit the ordinals provided by σ_γ).

Case 2: Not Case 1, hence σ_γ is winning for player II for any γ. In contrast with the first case, this strategy σ_γ for player II in $G(\gamma)$ is no longer a strategy at all in $G(A)$: for it depends on ordinals that player I provides in $G(\gamma)$ but not in $G(A)$. Here enter the axiom “sharps” and Martin’s analytic determinacy proof: as soon as A is analytic (and the more so when it is FS), sharps enable Martin to define a **mean value** $\sigma = \int_\gamma \sigma_\gamma$ of these strategies for all **countable** γ’s. This mean value no longer depends on the ordinals provided by player I during the play of $G(\gamma)$. In this way σ becomes a strategy for player II in the original game $G(A)$. And using the fact that the strategies σ_γ which σ comes from were winning ones, Martin’s proof shows that σ is winning for player II.

The disjunction between the two cases produced a winning strategy for $G(A)$ in any case, hence proves its determinacy. But it does so in a way that seems infinitely far away from FS determinacy. For the set of countable ordinals, and the strategies σ_γ—let alone their "mean value", are everything but effective—let alone FS!

Nevertheless there are many possible small variations in the definition of the open games $G(\gamma)$ and their winning strategy σ_γ. And we found one of these variations which in Case 2 yields an average strategy σ that **is** FS. (For full details, see our full version [3].) This shows FS determinacy in case player II has a winning strategy. And interchanging the roles of the two players in the proof allows to show the other case, thus ending the proof.

3. Conclusion

Let us recall that behind FS determinacy there is an issue of great practical and effective content. For present days industry raises a large number of problems of the form: design a (non-terminating and FS) processor P working in real time interaction with its environment, which satisfies for a certain specification. FS determinacy is the theoretical background of a successful modelization of this problem.

(a) One imagines an infinite game in which player player I is the environment: its possible moves are all stimuli which the environment might send at once to the processor. Player player II is the processor: its possible moves are all the reactions which the processor might have to make at once. The plays are all infinite sequences of alternate moves of player I and player II—coded so as to coincide with all elements of 2^ω.

(b) The specification is then represented by the set A of all plays $z \in 2^\omega$ such that the moves of player II are a satisfactory response to the moves of player I, according to the specification; thus our problem becomes: find a transducer σ which is a w.s. (:= **winning strategy** for player II in the game $G(A)$).

(c) If A happens to be FS accepted, then by FS determinacy one of the players has a w.s. σ which is an FS transducer. If this player is player II, then σ is the abstract form of the desired processor P; and if it is player I, then no processor of any kind will ever satisfy the specification. Moreover using Büchi's lemma saying that an FS acceptor accepts an infinite word iff it accepts an ultimately periodic one, we can **effectively** determine which player has the w.s.—and find out the transducer σ which realizes it.

The infinite length of $G(A)$ is an imaginary feature; and one expects that this turns the above "model of real world processors" to a particularly crude, falsely idealized one. But if a w.s. σ for $G(A)$ is performed by an FS processor P then σ does everything it has to do every time a loop is completed in the transition graph of P. (For otherwise player I could induce infinite repetition of the unsatisfactory loop, to win the play and defeat σ). Now suppose P is a real world processor: it has about 10^6 states, say. Today, its speed is counted in Gigaherz: P completes a loop every fraction of a second!

Thus although $G(A)$ seems to allow σ unlimited amounts of time to complete its tasks, in practice σ is very quickly effective. So that the above model of processor design has some (heuristic) value for a very large class of applications—for instance, in the design of processors used in modern planes. Indeed, the expanding mathematical theory of this model should give guide marks and ideas of algorithms to the engineers in charge of designing such processors. Which is an extraordinary fate for a theoretical research about **infinite** games...

But FS determinacy is only the beginning of this remarkable story. Today, behind the problem solved by the theorem of FS determinacy there are unsolved theoretical problems which are as beautiful, but much finer and of much stronger bearing on applied CS... only they are much harder. Namely (see [4] for their precise form and motivation):

(1) P-time realization of FS strategies.

(2) P-time decision of the winner of a parity game (= a special case of FS games). While FS determinacy contents itself with the mere existence of a FS winning strategy, these problems ask to quickly compute it and to quickly decide which of the two players has it. In fact if N is the number of states of the acceptor A, then these problems allocate you only **constant** time N^k (where $k < \omega$ is independent of A) to compute the next move of the winning strategy or to decide which player has it. The first, obvious step in the study of these two problems is to examine the now 4 existing proofs of FS determinacy and see whether we might extract from them the required additional information (on the strategy, or on the player). The answer looks negative for the proofs of Rabin, of Büchi-Landweber. In my opinion and for some precise reason the answer is also negative even in case of the optimal Gurevich-Harrington proof (as reshaped in [4]). Whereas our new proof offers at least the beginning of a track:

- The choices of $G(\gamma)$, of σ_γ and of the averaging procedure have many possible variants.
- Girard (unpublished) defined one very subtle variant of Martin's averaging procedure and proved that it also yields analytic determinacy.
- Berardi [1] has studied Girard's procedure in a case where it becomes effective (while Martin's procedure remains uneffective in that case).
- Applying this to the two above P-time problems has not been tried so far, but should lead to very novel algorithms.
- Other effective determinacy problems might also benefit from these ideas.

In conclusion, beyond the present work there is a series of ideas, methods and results, as a base of future research on the P-time version of FS determinacy and other challenges of effective determinacy.

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