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Theoretical Computer Science 374 (2007) 203–213
Theoretical Computer Science
www.elsevier.com/locate/tcs

Ranks of graphs: The size of acyclic orientation cover for deadlock-free packet routing☆

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Received 28 February 2004; received in revised form 28 September 2006; accepted 5 October 2006

Communicated by D. Peleg

Abstract

Given a graph G, the problem is to determine an acyclic orientation of G which minimizes the maximal number of changes of orientation along any shortest path in G. The corresponding value is called the *rank* of the graph G. The motivation for this graph theoretical problem comes from the design of deadlock-free packet routing protocols [G. Tel, Deadlock-free packet switching networks, in: Introduction to Distributed Algorithms, Cambridge University Press, Cambridge, UK, 1994 (Chapter 5)].

This acyclic orientation covering problem on the shortest path systems has been studied in [J.-C. Bermond, M. Di Ianni, M. Flammini, S. Perennes, Acyclic orientations for deadlock prevention in interconnection networks, in: 23rd International Workshop on Graph-Theoretic Concepts in Computer Science (WG), in: Lecture Notes in Computer Science, vol. 1335, Springer-Verlag, 1997, pp. 52–64] where it was shown that the general problem of determining the rank is NP-complete and some upper and lower bounds on the rank were presented for particular topologies, such as grids, tori and hypercubes. The main unresolved problem stated in [J.-C. Bermond, M. Di Ianni, M. Flammini, S. Perennes, Acyclic orientations for deadlock prevention in interconnection networks, in: 23rd International Workshop on Graph-Theoretic Concepts in Computer Science (WG), in: Lecture Notes in Computer Science, vol. 1335, Springer-Verlag, 1997, pp. 52–64] was to determine the rank values for other well-known interconnection networks and also for more general classes of graphs.

In this paper we give a general lower bound argument for the rank problem and apply it to the class of involution-generated Cayley graphs which among others include hypercubes, star graphs, pancake graphs and transposition-tree based graphs [S.B. Akers, B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, IEEE Transactions on Computers 38 (4) (1989) 555–565]. We also present a large class $\mathcal{LCP}_{(T,SP)}$ of graphs with constant rank. This class of graphs is defined as the layered cross product [S. Even, A. Litman, Layered cross product—A technique to construct interconnection networks, Networks 29 (1997) 219–223] of layered trees and series–parallel graphs and includes among others butterflies, Beneš networks, fat-trees and meshes of trees. For some special topologies, improved lower bounds on the rank are also presented. We consider some of the modified versions of the rank problem as well.

Keywords: Communication networks; Packet routing; Acyclic orientation cover

☆ This research was supported in part by VEGA 1/3106/06.

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doi:10.1016/j.tcs.2006.10.036

1. Introduction

In this paper we focus on the graph theoretical problem of determining the *rank* of a graph. The problem is informally stated as follows. Consider a graph G and the path system $\mathcal{P}$ consisting of all shortest paths between every pair of vertices. Given an acyclic orientation of G, we denote as a rank of a path P the number of changes of orientation caused by P. The goal is to find an acyclic orientation of G which minimizes the maximal rank of a path from $\mathcal{P}$. This value is called the rank of G.

The motivation for this problem follows from the design of deadlock-free packet routing protocols. Consider a point-to-point (store-and-forward) network, where each node has a number of buffers. Messages are only allowed to be sent to and received from buffers (i.e. in order to send a message, it must be stored into a free buffer on the sender's side first; then a free buffer on the recipient's side must be found; otherwise the communication is delayed). The aim of the protocol is to avoid deadlocks (i.e. the situations where messages are cyclically waiting for buffers) with minimal number of buffers per node.

One of the studied techniques [11,6,3,8,10,9] for constructing deadlock-free protocols is based on acyclic orientations (i.e. directed acyclic graphs obtained by orienting edges of the given undirected graph). A sequence of acyclic orientations is constructed and in each node there is one buffer reserved for each orientation. Messages are sent to the buffer i only in the direction of the i-th acyclic orientation. Moreover, each message has to use buffers in non-decreasing order which gives the desired deadlock avoidance property.

Recently [3], attention has been devoted to a special case of this deadlock prevention technique in which the constructed sequence $\mathcal{S}$ of acyclic orientations has to consist of two alternating orientations $\mathcal{S} = \langle AO_1, AO_2, AO_1, AO_2, \dots\rangle$. Note that in this case AO_2 has to be the dual orientation (i.e. obtained by reversing the orientation on each edge) to AO_1. Hence, the rank of a graph represents the number of buffers per node required for this method.

In [3] the NP-completeness of determining the rank for arbitrary graph was shown together with lower and upper bounds on some topologies.

In this paper we present a lower bound on the rank of involution-generated Cayley graphs which include hypercubes, star graphs, pancake graphs, transposition-tree based graphs and others. Next, we present a large class $\mathcal{LCP}_{(T,SP)}$ of graphs with constant rank (including butterflies, series–parallel graphs, fat-trees, meshes of trees and Beneš networks) and some bounds for particular topologies. To conclude, we consider some modifications of the problem.

The paper is organized as follows. In Section 2 we give the basic notions and definitions which are used throughout the paper. In Section 3 we present the general lower bound argument on the rank and we show the lower bound for involution-generated Cayley graphs. In Section 4 we present a large class of graphs having constant rank. In Section 5 we give new lower and upper bounds for specific network topologies and finally Section 6 discusses some modifications of the rank problem. In the Conclusions section we add some remarks and we address some open problems.

2. Definitions

In this section we present some basic definitions that will be extensively used throughout the rest of the paper. When talking about graphs we will consider symmetric digraphs obtained from undirected graphs by replacing every edge by a pair of opposite links. Unless stated otherwise, N denotes the number of vertices in the graph.

Definition 1 (*[3]*). An acyclic orientation of a symmetric digraph $G = (V, E)$ is an acyclic digraph $\vec{G} = (V, \vec{E})$ such that $\vec{E} \subseteq E$. We say that two consecutive arcs in E cause a change of orientation if and only if exactly one of them belongs to $\vec{E}$.

Definition 2 (*[3]*). Let $\vec{G} = (V, \vec{E})$ be an acyclic orientation of $G = (V, E)$. Given a dipath $P = [u_1, u_2, \dots, u_n]$ in G, let $c(P, \vec{G})$ be the number of changes of orientation caused by all pairs of consecutive arcs along P. We define the rank $rank(P, \vec{G})$ of P with respect to $\vec{G}$ as $rank(P, \vec{G}) = c(P, \vec{G}) + 1$ if $(u_1, u_2) \in \vec{E}$ and $rank(P, \vec{G}) = c(P, \vec{G}) + 2$ if $(u_1, u_2) \notin \vec{E}$.

Given a set $\mathcal{P}$ of dipaths in G, the rank of $\mathcal{P}$ is defined as

$$rank_G(\mathcal{P}) = \min_{\vec{G}} \max_{P \in \mathcal{P}} rank(P, \vec{G})$$

where the minimum is taken over all acyclic orientations of G.

Given the set $\mathcal{P}_{\mathrm{All}}$ of all shortest dipaths between all pairs of vertices in G, the rank of G is defined as $rank_G = rank_G(\mathcal{P}_{\mathrm{All}})$.

For the sake of efficiency we consider the shortest dipaths only. When the path system is not specified explicitly, we consider the system of all shortest dipaths between all pairs of vertices. Note that in this case the diameter of G plus 1 is a trivial upper bound on $rank_G$.

3. Lower bound results

We first present a general technique for obtaining lower bounds and then use it on a class of Cayley graphs generated by involutions.

3.1. General lower bound argument

Our lower bound technique is based on a counting argument. Consider any system $\mathcal{P}$ of shortest paths. In order to bound the maximal rank of any shortest path we restrict ourselves only to the paths from $\mathcal{P}$. Moreover, instead of bounding the maximum rank of a path we bound the average rank.

Let $\mathcal{C}$ be a set of cycles. Clearly, in each acyclic orientation there must be at least two pairs of incident edges with opposite orientation in each cycle. If we could, for each cycle from $\mathcal{C}$, guarantee that there are exactly q paths from $\mathcal{P}$ going through any pair of incident edges we could bound the average rank using the cardinality of $\mathcal{C}$.

These thoughts are formalized in the following lemma:

Lemma 1. *Consider a graph $G = (V, E)$ and a path system $\mathcal{P}$ of shortest dipaths in G. Given an acyclic orientation $\vec{G} = (V, \vec{E})$, define $w(\mathcal{P}, \vec{G}) = |\{[u_1, u_2, \ldots] \in \mathcal{P}; (u_1, u_2) \notin \vec{E}\}|$. Further consider sets of cycles $\mathcal{C}_1, \ldots, \mathcal{C}_k$ in G with the properties that:*

- *any two incident edges belong to at most one cycle,*
- *for each $\mathcal{C}_i$ there exists q_i such that every two incident edges belonging to a cycle $C \in \mathcal{C}_i$ belong to at least q_i dipaths from $\mathcal{P}$.*

Then

$$rank_G \geq 1 + \frac{1}{|\mathcal{P}|}\left(2\sum_{i=1}^{k} |\mathcal{C}_i| q_i + \min_{\vec{G}} w(\mathcal{P}, \vec{G})\right)$$

where the minimum is taken over all acyclic orientations of G.

Proof. From the definition of the rank it follows for any shortest path system $\mathcal{P}$ that

$$rank_G = \min_{\vec{G}} \max_{P \in \mathcal{P}_{\mathrm{All}}} rank(P, \vec{G}) \geq \frac{1}{|\mathcal{P}|} \min_{\vec{G}} \sum_{P \in \mathcal{P}} rank(P, \vec{G}).$$

For a given acyclic orientation $\vec{G}$ it holds that

$$\sum_{P \in \mathcal{P}} rank(P, \vec{G}) = \sum_{P \in \mathcal{P}} \left(c(P, \vec{G}) + 1 + ([u_1, u_2] \notin \vec{E})\right) = \sum_{P \in \mathcal{P}} c(P, \vec{G}) + |\mathcal{P}| + w(\mathcal{P}, \vec{G}).$$

In each acyclic orientation there must be at least two pairs of incident edges in each cycle with different orientation. Thus $\sum_{P \in \mathcal{P}} c(P, \vec{G}) \geq 2 \sum_{i=1}^{k} |\mathcal{C}_i| q_i$. $\square$

Note that $w(\mathcal{P}, \vec{G}) \leq |\mathcal{P}|$; hence the contribution of the second term in the lower bound of the rank is at most constant.

3.2. Lower bound for involution-generated Cayley graphs

Cayley graphs over finite groups form a broad family of graphs which plays an important role in many areas of graph theory. Among other properties, Cayley graphs are vertex symmetric which makes them a very convenient tool for the design of interconnection networks.

Let $S = \{g_1, \ldots, g_n\}$ be a generating set of a finite group such that $S = S^{-1}$, i.e. S is closed under inverses. The Cayley graph (CG) [2] is defined to have vertices corresponding to the elements of the group and edges corresponding to the actions of g_i's. That means there is an edge from an element u to an element v exactly if there is a g_i such that $u \circ g_i = v$ in the group. Let us call the elements of S generators of the corresponding graph.

Every CG is uniquely determined by the set of its generators. As every finite group is isomorphic to some group of permutations it is sufficient to consider the generators to be permutations.

In this section we consider a more restricted class of CGs, called involution-generated Cayley graphs, for which every generator g_i is an involution (self-inverse), i.e. a permutation consisting of a set of disjoint transpositions.

In order to obtain a lower bound on the rank, we apply Lemma 1 to a set of cycles that are formed by two alternating generators and a path system of N shortest paths of length $diam(G)$ all having the same sequence of generators.

First, we prove the following technical lemma:

Lemma 2. *Let a be an integer. Consider the problem of maximizing the product $r = \prod_{i=1}^{p}(l_i + 1)$ over all integers $p, l_1, \ldots, l_p$, subject to $1 \le l_1 < l_2 < \cdots < l_p$ and $\sum_{i=1}^{p} l_i = a$. Then r is maximized only if $\{l_1, \ldots, l_p\} = \{1, \ldots, p+1\} \setminus \{k\}$ for some $1 \le k \le p+1$.*

Proof. The lemma states that the integers l_i which maximize the product form a sequence with at most one "hole". To prove this fact we show that every sequence with at least two holes can be altered in such a way that the sum does not change and the product increases.

If $l_1 \ge 3$ (i.e. there are two holes at the beginning of the sequence) it is possible to replace l_1 by $(1, l_1 - 1)$. The contribution to the product increases from $l_1 + 1$ to $2l_1$.

If $l_1 = 2$ and $l_i < l_{i+1} - 1$ (i.e. there is one hole at the beginning and one between the ith and $i+1$st elements) it is possible to replace l_{i+1} by $(1, l_{i+1} - 1)$ increasing the contribution to the product.

If $l_i < l_{i+1} - 2$ (i.e. there are at least two consecutive holes) we can replace (l_i, l_{i+1}) by $(l_i + 1, l_{i+1} - 1)$. Finally, if $l_i < l_{i+1} - 1$ and $l_j < l_{j+1} - 1$ for some $i < j$ it is possible to replace (l_i, l_{j+1}) by $(l_i + 1, l_{j+1} - 1)$. □

Now we apply the general lower bound to the Cayley graphs. In order to use Lemma 1 we need to find suitable sets of cycles. We consider, for every pair of generators g_a, g_b, the set of cycles formed by alternating application of g_a and g_b. The previous lemma is used to bound the length of these cycles.

Theorem 1. *Let G be an involution-generated Cayley graph such that every generator consists of at most c disjoint transpositions. Then*

$$rank_G \ge \frac{3}{2} + (diam(G) - 1)\,\frac{(p+1)(p+2) - 4c + 2}{2(p+2)!}$$

where $p = \left\lfloor \sqrt{4c + \frac{1}{4}} - \frac{1}{2} \right\rfloor$.

Proof. Let G be an involution-generated CG with N vertices and n generators. By ϕ we denote the identity permutation. Let $k = diam(G)$ and $[v, v \circ g_{i_1}, v \circ g_{i_1} \circ g_{i_2}, \ldots, v \circ g_{i_1} \circ \cdots \circ g_{i_k}]$ be a shortest path of length $diam(G)$. Let

$$\mathcal{P} = \left\{[v, v \circ g_{i_1}, v \circ g_{i_1} \circ g_{i_2}, \ldots, v \circ g_{i_1} \circ \cdots \circ g_{i_k}] \mid \text{ for each } v\right\}$$

be a system of N shortest dipaths. Let $\mathcal{C}_{\{a,b\}}$ denote the set of cycles consisting of alternating edges g_a and g_b, i.e. cycles of the form

$$\{[v, v \circ g_a, v \circ g_a \circ g_b, v \circ g_a \circ g_b \circ g_a, \ldots] \mid \text{ for each } v\}\,.$$

From the symmetry it follows that all cycles from $\mathcal{C}_{\{a,b\}}$ have the same length; denote this length as $k_{\{a,b\}}$. Thus $|\mathcal{C}_{\{a,b\}}| = \frac{N}{k_{\{a,b\}}}$. Let $q_{\{a,b\}}$ be the number of occurrences of generators a, b in the sequence of generators from $\mathcal{P}$, i.e. $q_{\{a,b\}} = |\{i \mid \{g_i, g_{i+1}\} = \{a, b\}\}|$. For any two consecutive edges generated by a, b there are $q_{\{a,b\}}$ dipaths from $\mathcal{P}$ going through them. Thus, as $w(\mathcal{P}, \vec{G}) = \frac{1}{2}|\mathcal{P}|$, Lemma 1 leads to

$$rank_G \geq \frac{3}{2} + 2\sum_{\{a,b\}} \frac{q_{\{a,b\}}}{k_{\{a,b\}}}. \tag{1}$$

Clearly $k_{\{a,b\}} = 2\, ord(a \circ b)$, where $ord(\varphi)$ is the order of a permutation φ (i.e. the smallest i such that $\varphi^i = \phi$), which is the least common multiple of the lengths of its cycles. In the following we bound the order of any permutation φ which can be written as a composition of at most $2c$ transpositions. Let φ consist of p cycles of different lengths $l_1 + 1, \ldots, l_p + 1$ ($l_i > 0$). Then it holds that $\sum_{i=1}^{p} l_i \leq 2c$ and $ord(\varphi) \leq \prod_{i=1}^{p}(l_i + 1)$. Increasing any l_i also increases the product, so it is sufficient to maximize the product over p and all distinct l_i such that $\sum_{i=1}^{p} l_i = 2c$. Using Lemma 2 we get that $\{l_1, \ldots, l_p\} = \{1, \ldots, p+1\} \setminus \{k\}$ for some k. Hence, $ord(\varphi) \leq \frac{(p+2)!}{k+1}$. Moreover, $\sum_{i=1}^{p} l_i = 2c = \sum_{i=1}^{p+1} i - k = (p+1)(p+2)/2 - k$ leading to $ord(\varphi) \leq \frac{2(p+2)!}{(p+1)(p+2)-4c+2}$. Combining the last result with formula (1) it holds that

$$rank(G) \geq \frac{3}{2} + \frac{(p+1)(p+2) - 4c + 2}{2(p+2)!} \sum_{\{a,b\}} q_{\{a,b\}}.$$

This bound is decreasing in p so the minimum is obtained for maximal value of p. Since l_i's are distinct and $l_1 > 0$, we have $\sum_{i=1}^{p} l_i = 2c \geq p(p+1)/2$ thus getting $p \leq \sqrt{4c + \frac{1}{4}} - \frac{1}{2}$ which concludes the proof. □

Many well-known interconnection networks are transposition-generated CGs (i.e. $c = 1$), e.g. hypercubes (with d generators, where $g_i = (2i-1, 2i)$), star graphs (with $d-1$ generators, where $g_i = (1, i+1)$), or transposition-tree based graphs [2]. For all these classes of graphs Theorem 1 gives the bound $rank(G) \geq Diam(G)/3 + 7/6$. This bound is asymptotically optimal; however, the leading term is in general not tight. Consider e.g. the hypercube. The best known bound is $rank(H_d) \geq \lceil (d+1)/2 \rceil$ (see [4]).

However, in a hypercube, for every pair of generators g_a, g_b the order $ord(g_a \circ g_b) = 2$. Hence, for Eq. (1) in the proof of Theorem 1 we get $k_{\{a,b\}} = 4$, $q_{\{a,a+1\}} = 1$, and otherwise $q_{\{a,b\}} = 0$. Plugging this into Eq. (1) we get $rank(H_d) \geq \frac{3}{2} + \frac{d-1}{2}$.

The previous theorem gives the bound in terms of the diameter and the maximal number of transpositions used as generators. Indeed, it is not possible to obtain a non-trivial lower bound for the involution-generated CG in terms of the diameter only: note that a circle of length $4n$ is an involution-generated CG generated by two involutions $(1,2)(3,4)\cdots(2n-1,2n)$ and $(2,3)(4,5)\cdots(2n-3,2n-1)$ with diameter $2n$ and rank 3 (the bound obtained by Theorem 1 is 2).

Theorem 1 allows us to specify a large class of graphs having $rank = \Omega(D)$, where D is the diameter of the graph, while $D+1$ is a trivial upper bound. As an example consider involution-generated CGs with constant c.

The class of hierarchical CGs [2] is another large class of involution-generated CGs known from the literature. A Cayley graph is called strongly hierarchical if for any ordering $g_1, \ldots, g_n$ of generators it holds that g_i is outside the subgroup generated by $g_1, \ldots, g_{i-1}$. It is easy to see that the generators of strongly hierarchical CGs are involutions (self-inverses), i.e. permutations consisting of a set of disjoint transpositions. The class of strongly hierarchical CGs includes among others hypercubes, star graphs and transposition-tree based graphs, but not pancake graphs.

4. Graphs with small rank

A trivial upper bound on the rank of a graph G is $rank_G \leq diameter(G) + 1$. However, for some graphs much better upper bounds can be achieved. In this section we present a large class of graphs with constant rank. These graphs are layered graphs obtained by the operation LCP (layered cross product) from trees and series–parallel graphs.

Definition 3. A layered graph $G = (\{V_i\}_{i=1}^{n}, E)$ is a graph with n disjoint layers of vertices where each edge connects vertices from consecutive layers.

Let us adopt the convention that the vertices from V_i are in the i-th layer.

The operation of a layered cross product is defined as follows:

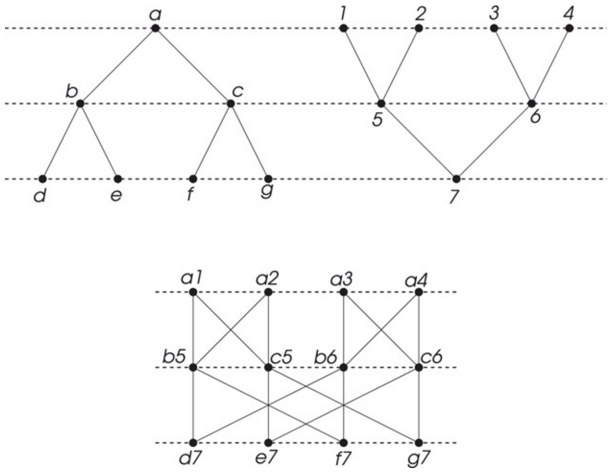

Fig. 1. A layered cross product of two layered trees.

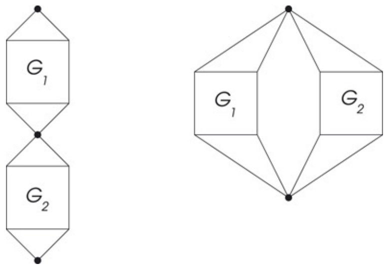

Fig. 2. Serial and parallel composition.

Definition 4 *([5])*. Let $G_1 = (\{V_i^{(1)}\}_{i=1}^{n+1}, E^{(1)})$, $G_2 = (\{V_i^{(2)}\}_{i=1}^{n+1}, E^{(2)})$ be layered graphs, each of $n+1$ layers. Then $LCP(G_1, G_2) = (\{V_i^{(1)} \times V_i^{(2)}\}_{i=1}^{n+1}, E)$ is a layered graph where $(\langle u, v\rangle, \langle u', v'\rangle) \in E$ if and only if $(u, u') \in E^{(1)}$ and $(v, v') \in E^{(2)}$.

Every tree is a layered graph; however for a given tree the assignment of layers is not unique. We will use only those assignments in which paths from root are monotone. Moreover, we require that all leaves are on the same layer (see Fig. 1).

Definition 5. A layered tree is a rooted tree for which each path from the root to a leaf passes through increasing or decreasing layers only, and all leaves are on the same layer.

We use a straightforward generalization of the notion of series–parallel graphs to take into account the layered structure (see Fig. 2).

Definition 6. If $G_1 = (\{V_i^{(1)}\}_{i=1}^{n}, E^{(1)})$ and $G_2 = (\{V_i^{(2)}\}_{i=1}^{n}, E^{(2)})$ are n-layered graphs, $|V_1^{(1)}| = |V_1^{(2)}| = |V_n^{(1)}| = |V_n^{(2)}| = 1$, $V_i^{(1)} \cap V_i^{(2)} = \emptyset$, then:

- the parallel composition of G_1 and G_2 is $(V_1^{(1)}, V_2^{(1)} \cup V_2^{(2)}, \ldots, V_{n-1}^{(1)} \cup V_{n-1}^{(2)}, V_n^{(1)}, E^{(1)} \cup E^{(2)})$, $V_1^{(1)} \equiv V_1^{(2)}$, $V_n^{(1)} \equiv V_n^{(2)}$, and is an n-layered series–parallel graph,
- the serial composition of G_1 and G_2 is $(V_1^{(1)}, \ldots, V_n^{(1)}, V_2^{(2)}, \ldots, V_n^{(2)}, E^{(1)} \cup E^{(2)})$, $V_n^{(1)} \equiv V_1^{(2)}$, and is a $(2n-1)$ layered series–parallel graph.

Layered series–parallel graphs are those layered graphs which can be obtained by a finite number of serial and parallel compositions of K_2.

Definition 7. $\mathcal{LCP}_{(T,SP)}$ is the smallest class of graphs which contains layered trees and layered series–parallel graphs and is closed under the LCP.

The class $\mathcal{L}CP_{(T,SP)}$ includes common topologies used in parallel computing, as can be seen from the following claims.

Claim 1 ([5]). *The butterfly, mesh of trees and fat-tree can be obtained by LCP from layered trees.*

Claim 2 ([5]). *The Beneš network can be obtained by LCP from a layered tree and a layered series–parallel graph.*

Theorem 2. *Let G be an $\mathcal{L}CP_{(T,SP)}$ graph. Then $rank_G \leq 4$.*

Proof. For a walk $W = (u_1, u_2, \ldots, u_r)$ in a layered graph, denote as the *characteristic sequence* of W a sequence $(s_1, s_2, \ldots, s_{r-1})$ of $+1, -1$ such that $s_i = 1$ if u_{i+1} is on higher layer than u_i, and $s_i = -1$ otherwise. If a graph $G = LCP(G_1, G_2)$ then every walk W in G induces in a natural way walks W_1 in G_1 and W_2 in G_2 such that W, W_1, and W_2 have the same characteristic sequence. In a characteristic sequence, call the maximal continuous subsequence of equal elements a *block*.

Let $G \in \mathcal{L}CP_{(T,SP)}$ and $G = LCP(G_1, \ldots, G_n)$. Consider two vertices $u = [u_1, \ldots, u_n]$, $v = [v_1, \ldots, v_n]$ in G. We prove that the characteristic sequence of any shortest path between u, v consists of at most three blocks.

First label the layers in the following manner: the layer containing the vertex u is the layer 0; the layers above are numbered with increasing positive numbers, the layers below with decreasing negative numbers. Denote as $l(v)$ the layer containing the vertex v, and let $l_{\min}$ and $l_{\max}$ be the minimum and maximum layer, respectively.

Note that since every G_i is a tree or a layered series–parallel graph, it has the following property:
There exist a non-negative integer a_i and a non-positive integer b_i such that a given sequence $S = (s_1, \ldots, s_m)$ of $+1, -1$ is a characteristic sequence of a walk in G_i from u_i to v_i if and only if $\sum_{k=1}^{m} s_k = l(v_i) = l(v)$, for each j it holds that $l_{\min} \leq \sum_{k=1}^{j} s_k \leq l_{\max}$, and there exists j such that either $\sum_{k=1}^{j} s_k = a_i$ or $\sum_{k=1}^{j} s_k = b_i$.
The "if" part of the property states that every $u - v$ walk either goes up to some layer a_i or down to some level b_i. This is easily seen: for trees, every $u - v$ walk must go through the nearest common ancestor; for the layered series–parallel graphs every $u - v$ walk must go through one of the nearest composition points.

The "only if" part is an immediate consequence of the fact that every vertex has neighbors on both neighboring layers.

Now consider a sequence $S = (s_1, \ldots, s_m)$ to be a characteristic sequence of a shortest path between u and v in G and suppose it consists of at least four blocks. W.l.o.g. consider the first block to be positive and denote the lengths of the first four blocks as x, y, z, q. We distinguish two cases. First let $z - q \geq 0$. Consider a sequence S' obtained from S by replacing the first four blocks by one block with the sum $x - y$ followed by a block with the sum $\max\{z, y\}$ followed by a block with sum $z - q - \max\{z, y\}$. S' is shorter than S, has fewer blocks and the sum of S' equals the sum of S. To show that S' is a characteristic sequence of a shortest path in G we show that it fulfills the above property for each G_i. Clearly S contains for each G_i an index j_i such that $\sum_{k=1}^{j_i} s_k \in \{a_i, b_i\}$. If this j_i is not in the first four blocks it is also in S'. So it is sufficient to note that for every number from $\langle 0, .., \max\{x, x + z - y\}\rangle$ and $\langle x - y, .., 0\rangle$ there is a prefix of S' with this sum.

For the case $z - q < 0$ we replace the first four blocks with blocks having sums $\max\{x, x - y + z\}$, $\min\{0, y - z\} - y$, $x + z - q - \max\{0, z - y\}$. The case for the first negative block is similar.

We have proven that if a sequence is a characteristic sequence of a shortest path in G then it consists of at most three blocks.

Now consider a *layered* orientation. Every path in G with a given characteristic sequence causes a change of orientation only between two blocks. Thus every path in the LCP causes at most two changes of orientation. □

Claim 3. *Consider a graph G' obtained from G by subdivision of some edges. Then $rank_{G'} = rank_G$.*

Corollary 1. *Let SP be a series–parallel graph. Then $rank_{SP} \leq 4$.*

5. Fixed topology results

In this section we present some upper and lower bound results on the rank problem for some well-known interconnection networks. These results complete previous known results (given in [3,7]) on the rank problem for some classes of graphs which are widely used in parallel and distributed systems. Namely, for $p \times q$ grid $G_{p,q}$ with $p \geq q$ it holds that $\lceil (2 - \sqrt{2})q \rceil - 1 \leq rank_{G_{p,q}} \leq \frac{3}{5}q + o(q)$. For $p \times q$ torus $T_{p,q}$ with $p \geq q$ it holds that $\lfloor \frac{q}{2} \rfloor + 2 \leq rank_{T_{p,q}} \leq \lceil \frac{q}{2} \rceil + 4$. And for hypercube Q_d of degree d it holds that $\lceil \frac{d+1}{2} \rceil \leq rank_{Q_d} \leq d + 1$.

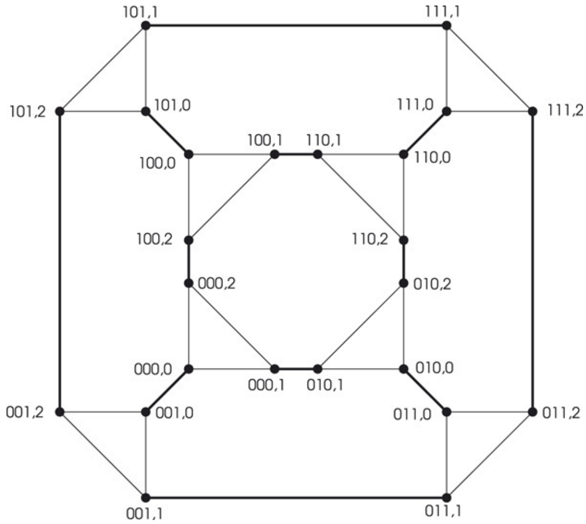

Fig. 3. CCC_3 with bold S-edges.

5.1. Cube connected cycles

Definition 8. A cube connected cycle graph of degree d (see Fig. 3), denoted as CCC_d, is a graph (V, E) where $V = \left\{\langle u, p\rangle \mid u \in \{0,1\}^d, p \in \{0,\ldots,d-1\}\right\}$ and $E = \{(\langle\alpha, p\rangle, \langle\alpha', p'\rangle) \mid p' = p \oplus 1, \alpha' = \alpha$ or $p' = p \ominus 1, \alpha' = \alpha$ or $p' = p, \alpha$ differs from α' only in the p-th position$\}$.

Since CCC_d is not an involution-generated Cayley graph, the lower bound in Theorem 1 does not apply for CCC_d. Following the argument used in Lemma 1, we are able to prove the next lower bound.

Theorem 3. $rank_{CCC_d} \geq \lfloor \frac{d}{2} \rfloor + 1$.

Proof. Call an arc $(\langle\alpha, p\rangle, \langle\alpha', p\rangle)$ an S-arc, an arc $(\langle\alpha, p\rangle, \langle\alpha, p \oplus 1\rangle)$ an L-arc and an arc $(\langle\alpha, p\rangle, \langle\alpha, p \ominus 1\rangle)$ an R-arc.

Consider $\mathcal{P}$ to be the set of all shortest paths between all pairs of vertices of the form $\langle\alpha, p\rangle \to \langle\overline{\alpha}, p \ominus 1\rangle$. These paths are of the form $(SL)^{d-1}S$ and $|\mathcal{P}| = d2^d$. For a given $\vec{G}$ it holds that $w(\mathcal{P}, \vec{G}) = \frac{|\mathcal{P}|}{2}$.

Let $\mathcal{C}$ be the set of cycles of length 8 of the form $LSRSLSRS$. Clearly, $|\mathcal{C}| = d2^{d-2}$ and each pair of consecutive arcs is in at most one cycle from $\mathcal{C}$. There are $d-1$ paths from $\mathcal{P}$ passing via each such pair of arcs. The reason is the following: each such pair can be uniquely denoted as a vertex v and a sequence of LS or SL arcs. Which form may have the starting vertex of a path from $\mathcal{P}$ going through this pair of arcs? If we fix the cursor position p, the bit-string α is determined uniquely. In each case, there are $d-1$ possible cursor positions. Each starting vertex represents exactly one path.

Thus we have

$$rank_{CCC_d} \geq \frac{3}{2} + \frac{1}{d2^d} 2(d-1)d2^{d-2} = \frac{d}{2} + 1. \quad \square$$

The following upper bound on the rank of CCC_d is an improvement of the result in [8].

Theorem 4. $rank_{CCC_d} \leq d + 8$.

Proof. We construct an acyclic orientation AO such that each shortest path causes at most $d+6$ changes of orientation by the consecutive pairs of links.

The orientation AO is defined as follows: for each binary string $\alpha = a_0 \ldots a_{d-1}$ the cycle $[\langle\alpha, 0\rangle, \langle\alpha, 1\rangle, \ldots, \langle\alpha, d-1\rangle]$ is oriented as $\langle\alpha, 0\rangle \to \cdots \to \langle\alpha, d-1\rangle$ and $\langle\alpha, 0\rangle \to \langle\alpha, d-1\rangle$ if the number of ones in α is odd; otherwise the orientation is the opposite. The remaining edges are oriented arbitrarily provided that the resulting orientation is acyclic.

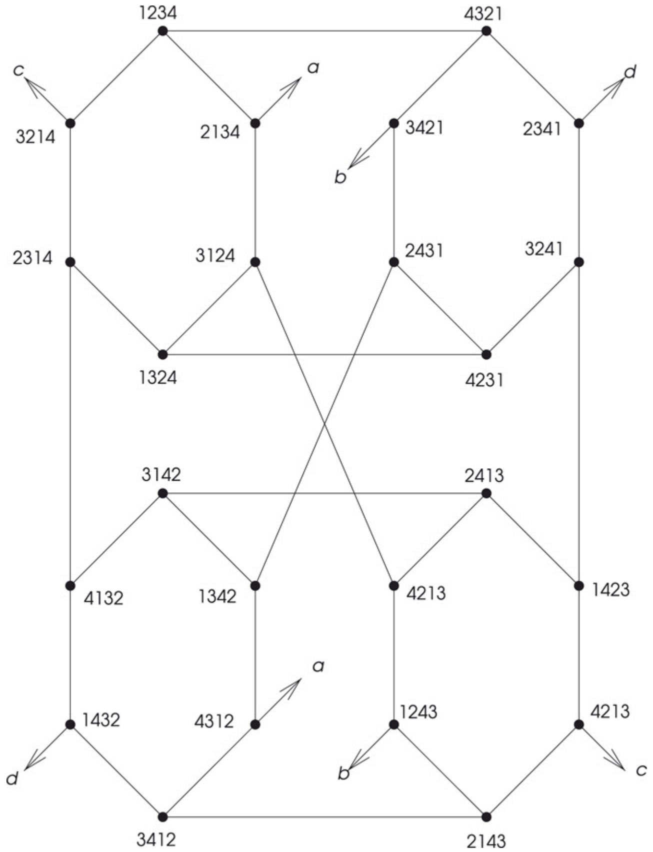

Fig. 4. Pancake graph P_4 with generators $g_1 = (1, 2)$, $g_2 = (1, 3)$ and $g_3 = (1, 4)(2, 3)$.

Consider a shortest path $P = [\langle a_0, p_0\rangle, \dots, \langle a_k, p_k\rangle]$. Call an arc for which $p_i = p_{i+1}$ an S-arc. By a cycle segment we mean a maximal sub-path of P that contains no S-arcs. Clearly P consists of non-empty cycle segments separated by S-arcs. A cycle segment that contains only arcs for which $p_{i+1} = p_i \oplus 1$ is called a left segment. A cycle segment that contains only arcs for which $p_{i+1} = p_i \ominus 1$ is called a right segment. Each segment is either right or left. The shortest path P can be divided into three sub-paths P_1, P_2, P_3, each of which contains only segments of one type (left/right). By a zero segment we mean a segment that contains an arc $(\alpha, 0), (\alpha, d-1)$. There are at most two zero segments in the whole path P. Consider a sub-path P_i and let us forget about the zero segments for a while. Each segment (non-zero) is contained either in AO or in its dual. Two consecutive segments are separated by a single S-arc, so they are not both contained in the same orientation. Thus regardless of the orientation of the S-arc, there is exactly one change of orientation between any two consecutive segments. Each zero segment contributes two changes and the borders of sub-paths add one change each. Since there are at most d S-arcs in P, the number of changes is $d + 6$. □

5.2. Pancake graph

Definition 9. A pancake graph of degree d (see Fig. 4) is a Cayley graph with $d - 1$ generators $g_1, \dots, g_{d-1}$ where

$$g_i = (1, i+1) \circ (2, i) \circ (3, i-1) \circ \cdots \circ (\lfloor (i+1)/2 \rfloor, i + 2 - \lfloor (i+1)/2 \rfloor).$$

The following theorem is an improvement over the bound from Theorem 1 which is constant.

Theorem 5. *$rank_{P_d} \geq \lfloor \ln d \rfloor + O(1)$.*

Proof. The generators of P_d are $g_1 = (1, 2)$, $g_2 = (1, 3)$, $g_3 = (1, 4)(2, 3), \dots, g_{d-1} = (1, d)(2, d-1) \dots$. Denote as $\varphi_{(i_1,\dots,i_k)}$ the permutation $\varphi \circ g_{i_1} \circ \cdots \circ g_{i_k}$. Denote by ϕ the identity permutation.

Choose a path system $\mathcal{P} = \{[\varphi, \varphi_{(1)}, \dots, \varphi_{(1,2,\dots,d-1)}] \mid \text{ for each } \varphi\}$. We show that it is a system of shortest paths. Because P_d is vertex symmetric, consider only the path $P = [\phi, \dots, \phi_{(1,\dots,d-1)}]$. The path is of length $d - 1$, starts in

$[1, 2, 3, \ldots, d]$ and ends in $[d, d-2, d-4, \ldots, 1, 2, 4, 6, \ldots, d-1]$ for odd d (for even d it is similar). Consider an arbitrary shortest path P' from ϕ to $\phi_{(1,..,d-1)}$. Check the number of consecutive elements in each intermediate vertex (in ϕ it is $d-1$; in $\phi_{(1,..,d-1)}$ it is 1). An application of each generator decreases this number by at most 1; thus the length of P' is at least $d-2$. However the element d is at the first position in the destination, so P' must use g_{d-1} at least once. Because it corresponds to reverting the sequence it does not decrease the number of consecutive elements, so the length of P' is at least $d-1$.

$|\mathcal{P}| = d!$ and $w(\mathcal{P}, \vec{G}) = \frac{d!}{2}$.

Let $\mathcal{C}_i$ for $i = 1, \ldots, d-2$ be the set of cycles of the form $[\varphi, \varphi_{(i)}, \varphi_{(i,i+1)}, \ldots]$ (with alternating application of g_i, g_{i+1}) for each φ. The length of each cycle in $\mathcal{C}_i$ is $2(i+2)$, because $\varphi_{(i,i+1)}$ represents the cyclic shift of the prefix of the length $i+2$. Thus $|\mathcal{C}_i| = \frac{d!}{2(i+2)}$ and $q_i = 1$.

Thus $rank_{P_d} \geq \frac{3}{2} + \sum_{i=1}^{d-2} \frac{1}{i+2} = \ln d + O(1)$. □

The best upper bound on the $rank_{P_d}$ so far is the diameter which is known to be of magnitude $O(d)$.

6. Modifications

In this section we present two variations of the rank problem. One possibility is to allow the cover to consist of up to k distinct acyclic orientations. For $k = 3$ a lower bound similar to the one from Lemma 1 holds:

Lemma 3. *Consider a graph $G = (V, E)$ and a path system $\mathcal{P}$ of shortest dipaths in G such that $\mathcal{P}$ is closed under reversal. Given an acyclic orientation $\vec{G} = (V, \vec{E})$, define $w(\mathcal{P}, \vec{G}) = |\{[u_1, u_2, \ldots] \in \mathcal{P}; (u_1, u_2) \notin \vec{E}\}|$. Further consider sets of cycles $\mathcal{C}_1, \ldots, \mathcal{C}_k$ in G with the properties that:*

- *the length of each cycle is at least 4,*
- *any two incident edges belong to at most one cycle,*
- *for each $\mathcal{C}_i$ there exists q_i such that every dipath of length 2 formed from edges from a cycle $C \in \mathcal{C}_i$ belongs to at least q_i dipaths from $\mathcal{P}$.*

Then for any acyclic orientation cover $\mathcal{S}$ consisting of at most three different orientations it holds that

$$|\mathcal{S}| \geq 1 + \frac{1}{|\mathcal{P}|}\left(2\sum_{i=1}^{k} |\mathcal{C}_i| q_i + \min_{\vec{G}} w(\mathcal{P}, \vec{G})\right)$$

where the minimum is taken over all acyclic orientations of G.

Proof. It is easy to prove by induction that in every cycle C of length at least 4 there are at least two dipaths of length 2 that are not entirely covered by any of the three orientations. The proof is then concluded similarly to the proof of Lemma 1. □

Using this lemma, results can be obtained for the considered topologies that are of a multiplicative factor 2 worse than the upper bounds for rank.

For some graphs, introduction of more than two different orientations decreases the length of the cover. Consider for example $q \times q$ grid. It is known (from [3]) that $rank_{G_{p,q}} \geq \lceil (2-\sqrt{2})q \rceil - 1$. Using Lemma 3, for three-orientation cover it holds that $|\mathcal{S}| \geq \lceil (2-\sqrt{2})q/2 \rceil$. However, using four acyclic orientations the size of the cover is 4.

On the other hand, in [10] it has been proven that every acyclic orientation cover for hypercube Q_d or cube connected cycles CCC_d has the size at least $d/\log d$.

Another variation of the rank problem is to consider other path systems. Of special interest are path systems with only one shortest path between every pair of vertices. This restriction refers to non-adaptive routing methods. For some graphs like grids or hypercubes, there exist such path systems allowing acyclic cover of size 2. On the other hand, from the proof of Theorem 3 it follows that for CCC_d every single shortest path system induces an acyclic cover of size at least $\lfloor \frac{d}{2} \rfloor + 1$. A similar result can be obtained for star graphs S_d (the paths of the length of the diameter are unique), where every single shortest path system induces an acyclic cover of size at least $\lfloor \frac{d}{2} + \frac{1}{3} \rfloor$. (Note that both lower bounds hold even in the case when in Definition 2 the all-to-all shortest path system $\mathcal{P}_{\text{All}}$ is replaced by a permutation (i.e. a 1-relation) path system, in which each node is the source and the destination of at most one routing path.)

Using the bounds on rank of grids and hypercubes from [3], one can get results for path systems consisting of k different shortest paths between all pairs of vertices in G (or all paths if the number is less than k). This restriction refers to k-adaptive routing methods.

Claim 4. *Choose a such that $\binom{2a-2}{a-1} \geq k$. Let $r = \min\{a, q\}$. Then $\lceil (2-\sqrt{2})r \rceil \leq |\mathcal{S}|_{G_{p,q}} \leq \min\{\frac{3}{5}q + o(q), 2r\}$.*

Note that as $\binom{2a-2}{a-1} \geq \frac{1}{\sqrt{\pi(a-1)}} 4^{a-1}$ it is sufficient that a be of the order $o(\log k)$.

Claim 5. *Choose a such that $a! \geq k$. Let $r = \min\{a, d\}$. Then $\lceil \frac{r+1}{2} \rceil \leq |\mathcal{S}|_{Q_d} \leq r + 1$.*

7. Conclusions

We have studied upper and lower bounds on the rank for certain classes of graphs. We identified a large class of graphs with constant rank. We presented the lower bound on the rank for involution-generated Cayley graphs and also some improved results for fixed topologies.

Open questions involve determining larger classes of graphs with high ranks (of the order of diameter) and improved lower bounds for usual topologies. There is also a lack of techniques for constructing upper bounds better than the diameter. Other questions involve the modified versions of the problem. As regards the latter issue it will be interesting to see whether there are differences between the size of the alternating acyclic orientation cover and the size of the general acyclic orientation cover for hypercubes and cube connected cycles.

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