



Enforcing and defying associativity, commutativity, totality, and strong noninvertibility for worst-case one-way functions[☆]

Lane A. Hemaspaandra^a, Jörg Rothe^{b,*}, Amitabh Saxena^c

^a Department of Computer Science, University of Rochester, Rochester, NY 14627, USA

^b Institut für Informatik, Heinrich-Heine-Universität Düsseldorf, 40225 Düsseldorf, Germany

^c Department of Computer Science and Computer Engineering, La Trobe University, Bundoora, VIC 3086, Australia

ARTICLE INFO

Article history:

Received 19 June 2007

Received in revised form 24 January 2008

Accepted 16 March 2008

Communicated by O. Watanabe

Keywords:

Computational complexity

Worst-case one-way functions

Associativity

Commutativity

Strong noninvertibility

ABSTRACT

Rabi and Sherman [M. Rabi, A. Sherman, An observation on associative one-way functions in complexity theory, *Information Processing Letters* 64 (5) (1997) 239–244; M. Rabi, A. Sherman, Associative one-way functions: A new paradigm for secret-key agreement and digital signatures, *Tech. Rep. CS-TR-3183/UMIACS-TR-93-124*, Department of Computer Science, University of Maryland, College Park, MD, 1993] proved that the hardness of factoring is a sufficient condition for there to exist one-way functions (i.e., p -time computable, honest, p -time noninvertible functions; this paper is in the worst-case model, not the average-case model) that are total, commutative, and associative but not strongly noninvertible. In this paper we improve the sufficient condition to $P \neq NP$.

More generally, in this paper we completely characterize which types of one-way functions stand or fall together with (plain) one-way functions—equivalently, stand or fall together with $P \neq NP$. We look at the four attributes used in Rabi and Sherman's seminal work on algebraic properties of one-way functions (see [M. Rabi, A. Sherman, An observation on associative one-way functions in complexity theory, *Information Processing Letters* 64 (5) (1997) 239–244; M. Rabi, A. Sherman, Associative one-way functions: A new paradigm for secret-key agreement and digital signatures, *Tech. Rep. CS-TR-3183/UMIACS-TR-93-124*, Department of Computer Science, University of Maryland, College Park, MD, 1993]) and subsequent papers – strongness (of noninvertibility), totality, commutativity, and associativity – and for each attribute, we allow it to be required to hold, required to fail, or “don't care”. In this categorization there are $3^4 = 81$ potential types of one-way functions. We prove that each of these 81 feature-laden types stands or falls together with the existence of (plain) one-way functions.

© 2008 Elsevier B.V. All rights reserved.

[☆] A preliminary version was presented at the 2005 ICTCS conference [L. Hemaspaandra, J. Rothe, A. Saxena, Enforcing and defying associativity, commutativity, totality, and strong noninvertibility for one-way functions in complexity theory, in: *Proceedings of the 9th Italian Conference on Theoretical Computer Science*, in: *Lecture Notes in Computer Science*, vol. 3701, Springer-Verlag, 2005].

* Corresponding author. Tel.: +49 211 81 12188; fax: +49 211 81 11667.

E-mail address: rothe@cs.uni-duesseldorf.de (J. Rothe).

URLs: <http://www.cs.rochester.edu/u/lane/> (L.A. Hemaspaandra), <http://ccc.cs.uni-duesseldorf.de/~rothe/> (J. Rothe), <http://homepage.cs.latrobe.edu.au/asaxena/> (A. Saxena).

1. Introduction

1.1. Motivation

In this paper, we study the properties of one-way functions, i.e., properties of functions that are easy to compute, but hard to invert. One-way functions are important cryptographic primitives and are the key building blocks in many cryptographic protocols. Various models to capture “noninvertibility” and, depending on the model used, various candidates for one-way functions have been proposed. The notion of noninvertibility is usually based on the average-case (where the “average-case” refers to the difficulty of inversion) complexity model (see, e.g., the book [9] and the references therein) in cryptographic applications, whereas noninvertibility for complexity-theoretic one-way functions is usually defined in the worst-case model (see the definitions of this paper). Though the average-case model is very important, we note that even the challenge of showing that any type of one-way function exists in the “less challenging” worst-case model remains an open issue after many years of research. It is thus natural to wonder, as a first step, what assumptions are needed to create various types of complexity-theoretic one-way functions. In this paper, we seek to characterize this existence issue in terms of class separations. (In addition, we mention that the seminal work on associativity, commutativity, and strong noninvertibility of one-way functions, which was done by Rabi and Sherman [25,26] who also proposed concrete protocols to be based on such one-way functions, is itself in the worst-case model.)

Complexity-theoretic one-way functions of various sorts, and related notions, were studied early on by, for example, Berman [3], Brassard, Fortune, and Hopcroft [7,6], Ko [24], and especially Grollmann and Selman [11], and have been much investigated ever since; see, e.g., [1,32,33,12,30,25,10,20,26,16,4,17,27,8,22,21,15]. The four properties of one-way functions to be investigated in this paper are strongness, totality, commutativity, and associativity. Intuitively, strong noninvertibility – a notion proposed by Rabi and Sherman [26,25] and more recently studied in [16,21,15] – means that for a two-ary function, given some function value and one of the corresponding arguments, it is hard to determine the other argument. It has been known for decades that one-way functions exist if and only if $P \neq NP$. But the Rabi–Sherman paper brought out the natural issue of trying to understand what complexity-theoretic assumptions characterized the existence of one-way functions with certain algebraic properties. Eventually, Hemaspaandra and Rothe [16] proved that strong, total, commutative, associative one-way functions exist if and only if $P \neq NP$. (As mentioned earlier, one-way functions with these properties are the key building blocks in Rabi, Rivest, and Sherman’s cryptographic protocols for secret-key agreement and for digital signatures (see [26,25]).) The surprising work of Homan [21] both strengthens the results of Rabi and Sherman on the ambiguity that must be present in total, associative functions and proves that if one-to-one one-way functions exist, then there exist strong, total, associative one-way functions having relatively low ambiguity.

This paper (a preliminary version of which appeared as [18]) provides a detailed study of the four properties of one-way functions mentioned above. For each possible combination of possessing, not possessing, and being oblivious to possession of the property, we study the question of whether such one-way functions can exist. Why should one be interested in knowing if a one-way function possesses “negative” properties, such as noncommutativity? On the one hand, negative properties can also have useful applications. For example, Saxena, Soh, and Zantidis [28,29] propose authentication protocols for mobile agents and digital cash with signature chaining that use as their key building blocks strong, associative one-way functions for which commutativity in fact is a disadvantage—though they need commutativity to not merely fail but to fail far more often than is achieved in the failure constructions of the present paper. More generally and more importantly, given that complexity-theoretic one-way functions have already been studied for decades (see the citations above, going as far back as the 1970s), it seems natural to try to understand and catalog which types of one-way functions are created by, for example, simply assuming $P \neq NP$. This paper does that completely with respect to strongness, totality, commutativity, and associativity.

1.2. Summary of our results

This paper is organized as follows. In Sections 2 and 3, we formally define the notions and notation used, and we provide some basic lemmas that allow us to drastically reduce the number of cases we have to consider. We will state the full definitions later, but stated merely intuitively, a function is said to be strongly noninvertible if given the output and one argument one cannot efficiently find a corresponding other argument; and a function is said to be strong if it is polynomial-time computable, strongly noninvertible, and satisfies the natural honesty condition related to strong noninvertibility (so-called *s-honesty*). In Section 4, we prove that the condition $P \neq NP$ characterizes all 27 cases induced by one-way functions that are strong. (The number is $27 = 3^3$ because we are dealing with the case of strong functions, and for each of the other three attributes – totality, commutativity, and associativity – there are three possible cases to handle, namely, that the property holds, that the property does not hold, and that we are oblivious to whether or not the property holds. And the two other blocks of 27 cases that we are about to mention each have 27 cases for the analogous reason.) As a corollary, we also obtain a $P \neq NP$ characterization of all 27 cases where one requires one-way-ness but is oblivious to whether or not the functions are strong. In Section 5, we consider functions that are required to be one-way but to not be strong. We show that $P \neq NP$ characterizes all of these 27 cases. Thus, $P \neq NP$ characterizes all 81 cases overall.

Table 1 summarizes the support for our results for the 16 key cases in which each of the four properties considered is either enforced or defied. (In light of the forthcoming Lemma 3.2, those cases in which one is oblivious to whether some

Table 1

Summary of support for the 16 key cases

Properties (s, t, c, a)	$P \neq$ NP characterization for this case is established by	Properties (s, t, c, a)	$P \neq$ NP characterization for this case is established by
(N, N, N, N)	Lemma 5.2 + Lemma 3.4	(Y, N, N, N)	Lemma 4.5 + Lemma 3.4
(N, N, N, Y)	Lemma 5.5 + Lemma 3.4	(Y, N, N, Y)	Lemma 4.4 + Lemma 3.4
(N, N, Y, N)	Lemma 5.1 + Lemma 3.4	(Y, N, Y, N)	Lemma 4.3 + Lemma 3.4
(N, N, Y, Y)	Lemma 5.5 + Lemma 3.4	(Y, N, Y, Y)	Lemma 4.2 + Lemma 3.4
(N, Y, N, N)	[15]; see Lemma 5.2	(Y, Y, N, N)	Lemma 4.5
(N, Y, N, Y)	Lemma 5.5	(Y, Y, N, Y)	Lemma 4.4
(N, Y, Y, N)	Lemma 5.1	(Y, Y, Y, N)	Lemma 4.3
(N, Y, Y, Y)	Lemma 5.5	(Y, Y, Y, Y)	[16], here restated as Lemma 4.2

property holds follow immediately from the cases stated in Table 1.) Definition 2.4 provides the classification scheme used in this table. The left column of Table 1 has 16 quadruples of the form (s, t, c, a) , where s regards “strong”, t means “total”, c means “commutative”, and a means “associative”. The variables s, t, c , and a take on a value from $\{Y, N\}$, where Y means presence (i.e., “yes”), and N means absence (i.e., “no”) of the given property. The right column of Table 1 gives the references to the proofs of the results stated.

1.3. General proof strategy

We do not attempt to brute-force all 81 cases. Rather, we seek to turn the cases’ structure and connectedness against themselves. So, in Section 3 we will reduce the 81 cases to their 16 key cases that do not contain “don’t care” conditions. Then, also in Section 3, we will show how to derive the nontotal cases from the total cases, thus further reducing our problem to 8 key cases.

As Corollary 4.6 and, especially, much of Section 5 will show, even among the 8 key cases we share attacks, and find and exploit implications.

Thus, the proof in general consists both of specific constructions – concrete realizations forcing given patterns of properties – and the framework that minimizes the number of such constructions needed.

2. Preliminaries and notations

Fix the alphabet $\Sigma = \{0, 1\}$. The set of strings over Σ is denoted by Σ^* . Let ε denote the empty string. Let $\Sigma^+ = \Sigma^* - \{\varepsilon\}$. For any string $x \in \Sigma^*$, let $|x|$ denote the length of x . Let $\langle \cdot, \cdot \rangle : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ be some standard pairing function, that is, some total, polynomial-time computable bijection that has polynomial-time computable inverses and is nondecreasing in each argument when the other argument is fixed. Let FP denote the class of polynomial-time computable functions (this includes both total and nontotal functions). This paper focuses completely on mappings from $\Sigma^* \times \Sigma^*$ to Σ^* (they are allowed to be many-to-one and they are allowed to be nontotal, i.e., they may map many distinct pairs of strings from $\Sigma^* \times \Sigma^*$ to one and the same string in Σ^* , and they need not be defined for all pairs in $\Sigma^* \times \Sigma^*$). (The study of 2-argument one-way functions of course is needed if associativity and commutativity are to be studied.) For each function f , let $\text{domain}(f)$ denote the set of input pairs on which f is defined, and denote the image of f by $\text{image}(f)$.

Definition 2.1 presents the standard notion of a (complexity-theoretic, many-one) one-way function, suitably tailored to the case of two-ary functions in the standard way; see [25,26,16,21,15]. (For general introductions to or surveys on one-way functions, see [30], [4], and [13, Chapter 2]. For general background on complexity see, e.g., [13,5].) Our one-way functions are based on noninvertibility in the worst-case model, as opposed to noninvertibility in the average-case model that is more appealing for cryptographic applications. The notion of honesty in Definition 2.1 below is needed in order to preclude functions from being noninvertible simply due to the trivial reason that some family of images lacks polynomially short preimages.

Definition 2.1 (*One-Way Function*). Let σ be a function (it may be either total or nontotal) mapping from $\Sigma^* \times \Sigma^*$ to Σ^* .

- (i) We say σ is *honest* if and only if there exists a polynomial p such that for each $z \in \text{image}(\sigma)$, there exists a pair $(x, y) \in \text{domain}(\sigma)$ such that $\sigma(x, y) = z$ and $|x| + |y| \leq p(|z|)$.
- (ii) We say σ is (*polynomial-time*) *noninvertible* if and only if there exists no function f in FP such that for all $z \in \text{image}(\sigma)$, we have $\sigma(f(z)) = z$.
- (iii) We say σ is a *one-way function* if and only if σ is polynomial-time computable, honest, and noninvertible.

The four properties of one-way functions that we will study in this paper are strongness, totality, commutativity, and associativity. A function σ mapping from $\Sigma^* \times \Sigma^*$ to Σ^* is said to be *total* if and only if σ is defined for each pair in $\Sigma^* \times \Sigma^*$, and is said to be *nontotal* if it is not total. We say that a function is *partial* if it is either total or nontotal; this says nothing, but makes it clear that we are not demanding that the function be total.

We now define the remaining three properties. Rabi, Rivest, and Sherman (see [26,25]) introduced the notion of strongly noninvertible associative one-way functions (strong AOWFs, for short). Rivest and Sherman (as attributed in [26, 25]) designed cryptographic protocols for two-party secret-key agreement and Rabi and Sherman designed cryptographic

protocols for digital signatures, both of which need strong, total AOWFs as their key building blocks. They also sketched protocols for multiparty secret-key agreement that require strong, total, commutative AOWFs. Strong (and sometimes total and commutative) AOWFs have been intensely studied in [16,4,21,15].

Though Rabi and Sherman's [26] notion of associativity is meaningful for total functions, it is not meaningful for nontotal two-ary functions, as has been noted and discussed in [16]. Thus, we here follow Hemaspaandra and Rothe's [16] notion of associativity, which is appropriate for both total and nontotal two-ary functions, and is designed as an analog to Kleene's 1952 [23] notion of complete equality of partial functions. (We mention in passing that if in bullets (i) and (ii) of the following definition the Σ^* is replaced by $\Sigma^* \cup \{\perp\}$, the notions defined remain unchanged.)

Definition 2.2 (*Associativity and Commutativity*). Let σ be any partial function mapping from $\Sigma^* \times \Sigma^*$ to Σ^* . Extend Σ^* by $\Gamma = \Sigma^* \cup \{\perp\}$, where \perp is a special symbol indicating, in the usage " $\sigma(x, y) = \perp$ ", that σ is not defined for the pair (x, y) . Define an extension $\hat{\sigma}$ of σ , which maps from $\Gamma \times \Gamma$ to Γ , as follows:

$$\hat{\sigma}(x, y) = \begin{cases} \sigma(x, y) & \text{if } x \neq \perp \text{ and } y \neq \perp \text{ and } (x, y) \in \text{domain}(\sigma) \\ \perp & \text{otherwise.} \end{cases} \quad (2.1)$$

- (i) We say σ is *associative* if and only if for each $x, y, z \in \Sigma^*$, $\hat{\sigma}(\hat{\sigma}(x, y), z) = \hat{\sigma}(x, \hat{\sigma}(y, z))$.
- (ii) We say σ is *commutative* if and only if for each $x, y \in \Sigma^*$, $\hat{\sigma}(x, y) = \hat{\sigma}(y, x)$.

Informally speaking, strong noninvertibility (see [26,25]) means that even if a function value and one of the corresponding two arguments are given, it is hard to compute the other argument. It is known that, unless $P = NP$, some noninvertible functions are not strongly noninvertible [15]. And, perhaps counterintuitively, it is known that, unless $P = NP$, some strongly noninvertible functions are not noninvertible [15]. That is, unless $P = NP$, strong noninvertibility does not imply noninvertibility. Strong noninvertibility requires a variation of honesty that is dubbed s-honesty in [15]. The notion defined now, as "strong (function)" in Definition 2.3, is in the literature typically called a "strong one-way function". This is quite natural. However, to avoid any possibility of confusion as to when we refer to that and when we refer to the notion of a "one-way function" (see Definition 2.1; as will be mentioned later, neither of these notions necessarily implies the other), we will throughout this paper simply call the notion below "strong" or "a strong function", rather than "strong one-way function".

Definition 2.3 (*Strong Function*). Let σ be any partial function mapping from $\Sigma^* \times \Sigma^*$ to Σ^* .

- (i) We say σ is *s-honest* if and only if there exists a polynomial p such that the following two conditions are true:
 - (a) For each $x, z \in \Sigma^*$ with $\sigma(x, y) = z$ for some $y \in \Sigma^*$, there exists some string $\hat{y} \in \Sigma^*$ such that $\sigma(x, \hat{y}) = z$ and $|\hat{y}| \leq p(|x| + |z|)$.
 - (b) For each $y, z \in \Sigma^*$ with $\sigma(x, y) = z$ for some $x \in \Sigma^*$, there exists some string $\hat{x} \in \Sigma^*$ such that $\sigma(\hat{x}, y) = z$ and $|\hat{x}| \leq p(|y| + |z|)$.
- (ii) We say σ is (*polynomial-time*) *invertible with respect to the first argument* if and only if there exists an inverter $g_1 \in \text{FP}$ such that for every string $z \in \text{image}(\sigma)$ and for all $x, y \in \Sigma^*$ with $(x, y) \in \text{domain}(\sigma)$ and $\sigma(x, y) = z$, $\sigma(x, g_1((x, z))) = z$.
- (iii) We say σ is (*polynomial-time*) *invertible with respect to the second argument* if and only if there exists an inverter $g_2 \in \text{FP}$ such that for every string $z \in \text{image}(\sigma)$ and for all $x, y \in \Sigma^*$ with $(x, y) \in \text{domain}(\sigma)$ and $\sigma(x, y) = z$, $\sigma(g_2((y, z)), y) = z$.
- (iv) We say σ is *strongly noninvertible* if and only if σ is neither invertible with respect to the first argument nor invertible with respect to the second argument.
- (v) We say σ is *strong* if and only if σ is polynomial-time computable, s-honest, and strongly noninvertible.

In this paper, we will look at the $3^4 = 81$ categories of one-way functions that one can get by requiring the properties strong/total/commutative/associative to either: hold, fail, or "don't care". For each, we will try to characterize whether such one-way functions exist.

We now define a classification scheme suitable to capture all possible combinations of these four properties of one-way functions.

Definition 2.4 (*Classification Scheme for One-Way Functions*). For each $s, t, c, a \in \{Y, N, *\}$, we say that a partial function $\sigma : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ is an (s, t, c, a) *one-way function* (an (s, t, c, a) -OWF, for short) if and only if all the following hold: σ is a one-way function, if $s = Y$ then σ is strong, if $s = N$ then σ is not strong, if $t = Y$ then σ is a total function, if $t = N$ then σ is a nontotal function, if $c = Y$ then σ is a commutative function, if $c = N$ then σ is a noncommutative function, if $a = Y$ then σ is an associative function, and if $a = N$ then σ is a nonassociative function.

For example, a function is a (Y, Y, Y, Y) -OWF exactly if it is a strong, total, commutative, associative one-way function. And note that, under this definition, whenever a setting is $*$, we do not place any restriction as to whether the corresponding property holds or fails to hold—that is, $*$ is a "don't care" designator. For example, a function is a $(*, Y, *, *)$ -OWF exactly if it is a total one-way function. Of course, all (Y, Y, Y, Y) -OWFs are $(*, Y, *, *)$ -OWFs. That is, our 81 classes do not seek to partition, but rather to allow all possible simultaneous settings and "don't care"s for these four properties. However, the 16 such classes with no stars are certainly pairwise disjoint.

3. Groundwork: Reducing the cases

In this section, we show how to tackle our ultimate goal, stated as [Goal 3.1](#), by drastically reducing the number of cases that are relevant among the 81 possible cases.

Goal 3.1. For each $s, t, c, a \in \{Y, N, *\}$, characterize the existence of (s, t, c, a) -OWFs in terms of some suitable complexity-theoretic condition.

Since $*$ is a “don’t care”, for a given $*$ position the characterization that holds with that $*$ is simply the “or” of the characterizations that hold with each of Y and N substituted for the $*$. For example, clearly there exist $(Y, Y, Y, *)$ -OWFs if and only if either there exist (Y, Y, Y, Y) -OWFs or there exist (Y, Y, Y, N) -OWFs. And cases with more than one $*$ can be “unwound” by repeating this. So, to characterize all 81 cases, it suffices to characterize the 16 cases stated in [Table 1](#).

- Lemma 3.2.** (i) For each $t, c, a \in \{Y, N, *\}$, there exist $(*, t, c, a)$ -OWFs if and only if either there exist (Y, t, c, a) -OWFs or there exist (N, t, c, a) -OWFs.
(ii) For each $s, c, a \in \{Y, N, *\}$, there exist $(s, *, c, a)$ -OWFs if and only if either there exist (s, Y, c, a) -OWFs or there exist (s, N, c, a) -OWFs.
(iii) For each $s, t, a \in \{Y, N, *\}$, there exist $(s, t, *, a)$ -OWFs if and only if either there exist (s, t, Y, a) -OWFs or there exist (s, t, N, a) -OWFs.
(iv) For each $s, t, c \in \{Y, N, *\}$, there exist $(s, t, c, *)$ -OWFs if and only if either there exist (s, t, c, Y) -OWFs or there exist (s, t, c, N) -OWFs.

It is well known (see [2] and Proposition 1 of [30]) that $P \neq NP$ if and only if $(*, *, *, *)$ -OWFs exist, i.e., $P \neq NP$ if and only if there exist one-way functions, regardless of whether or not they possess any of the four properties. So, in the upcoming proofs, we will often focus on just showing that $P \neq NP$ implies the given type of OWF exists.

Lemma 3.3. For each $s, t, c, a \in \{Y, N, *\}$, if there are (s, t, c, a) -OWFs then $P \neq NP$.

Next, we show that all cases involving nontotal one-way functions can be easily reduced to the corresponding cases involving total one-way functions. Thus, we have eliminated the eight “nontotal” of the remaining 16 cases, provided we can solve the eight “total” cases.

Lemma 3.4. For each $s, c, a \in \{Y, N\}$, if there exists an (s, Y, c, a) -OWF, then there exists an (s, N, c, a) -OWF.

Proof. Fix any $s, c, a \in \{Y, N\}$, and let σ be any given (s, Y, c, a) -OWF. For each string $w \in \Sigma^*$, let w^+ denote the successor of w in the standard lexicographic ordering of Σ^* , and for each string $w \in \Sigma^+$, let w^- denote the predecessor of w in the standard lexicographic ordering of Σ^* .

Define a function $\rho : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ by

$$\rho(x, y) = \begin{cases} (\sigma(x^-, y^-))^+ & \text{if } x \neq \varepsilon \neq y \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Note that ρ is nontotal, since it is not defined on the pair $(\varepsilon, \varepsilon)$. It is a matter of routine to check that ρ is a one-way function, i.e., polynomial-time computable, honest, and noninvertible. It remains to show that ρ inherits all the other properties from σ as well. To this end, we show the following claim.

- Claim 1.** (i) σ is commutative if and only if ρ is commutative.
(ii) σ is associative if and only if ρ is associative.
(iii) σ is strong if and only if ρ is strong.

Proof of Claim 1. We check these properties separately.

- (i) *Commutativity:* Suppose that σ is commutative. Given any strings $x, y \in \Sigma^*$, if $x = \varepsilon$ or $y = \varepsilon$, then both $\rho(x, y)$ and $\rho(y, x)$ are undefined. If $x \neq \varepsilon \neq y$, then the commutativity of σ implies that $\rho(x, y) = (\sigma(x^-, y^-))^+ = (\sigma(y^-, x^-))^+ = \rho(y, x)$. So $\widehat{\rho}(x, y) = \widehat{\rho}(y, x)$. By [Definition 2.2](#), ρ is commutative.

Conversely, suppose that σ is noncommutative. Since σ is total, we do not have to worry about holes in the domain of σ . Let a and b be fixed strings in Σ^* such that $\sigma(a, b) \neq \sigma(b, a)$. It follows that $\rho(a^+, b^+) \neq \rho(b^+, a^+)$. Thus, ρ is noncommutative.

- (ii) *Associativity:* Suppose that σ is associative. Let x, y , and z be any strings in Σ^* . If $x = \varepsilon$ or $y = \varepsilon$ or $z = \varepsilon$, then both $\rho(x, \rho(y, z))$ and $\rho(\rho(x, y), z)$ are undefined. If none of x, y , and z equals the empty string, then the associativity of σ implies $\rho(x, \rho(y, z)) = (\sigma(x^-, \sigma(y^-, z^-)))^+ = (\sigma(\sigma(x^-, y^-), z^-))^+ = \rho(\rho(x, y), z)$. So $\widehat{\rho}(x, \widehat{\rho}(y, z)) = \widehat{\rho}(\widehat{\rho}(x, y), z)$. By [Definition 2.2](#), ρ is associative.

Conversely, suppose that σ is nonassociative. Let a, b , and c be fixed strings in Σ^* such that $\sigma(a, \sigma(b, c)) \neq \sigma(\sigma(a, b), c)$. Since σ is total, each of $\sigma(a, b)$, $\sigma(b, c)$, $\sigma(a, \sigma(b, c))$, and $\sigma(\sigma(a, b), c)$ is defined. So $(\rho(a^+, \rho(b^+, c^+)))^- = \sigma(a, \sigma(b, c)) \neq \sigma(\sigma(a, b), c) = (\rho(\rho(a^+, b^+), c^+))^-$, which implies $\rho(a^+, \rho(b^+, c^+)) \neq \rho(\rho(a^+, b^+), c^+)$. Thus, $\widehat{\rho}(a^+, \widehat{\rho}(b^+, c^+)) \neq \widehat{\rho}(\widehat{\rho}(a^+, b^+), c^+)$. By [Definition 2.2](#), ρ is nonassociative.

(iii) **Strongness:** First, we note that σ is s-honest if and only if ρ is s-honest. Let p be some polynomial witnessing the s-honesty of σ as per Definition 2.3:

- (a) For each $x, z \in \Sigma^*$ with $\sigma(x, y) = z$ for some $y \in \Sigma^*$, there exists some string $\hat{y} \in \Sigma^*$ such that $\sigma(x, \hat{y}) = z$ and $|\hat{y}| \leq p(|x| + |z|)$.
- (b) For each $y, z \in \Sigma^*$ with $\sigma(x, y) = z$ for some $x \in \Sigma^*$, there exists some string $\hat{x} \in \Sigma^*$ such that $\sigma(\hat{x}, y) = z$ and $|\hat{x}| \leq p(|y| + |z|)$.

Since ρ shifts the arguments and the function value of σ just by one position in the lexicographic ordering on Σ^* , the polynomial $q(n) = p(n) + 1$ witnesses the s-honesty of ρ . The converse is proven analogously.

Now, we show that σ is strongly noninvertible if and only if ρ is strongly noninvertible. Suppose that σ is invertible with respect to the first argument via some inverter g_1 in FP. That is, for each string $z \in \text{image}(\sigma)$ and for all $x, y \in \Sigma^*$ with $(x, y) \in \text{domain}(\sigma)$ and $\sigma(x, y) = z$, we have $\sigma(x, g_1(\langle x, z \rangle)) = z$. From g_1 we construct an inverter $f_1 \in \text{FP}$ that inverts ρ with respect to the first argument as follows. Let z be any string in $\text{image}(\rho)$, and let $x, y \in \Sigma^*$ be any strings such that $(x, y) \in \text{domain}(\rho)$ and $\rho(x, y) = z$. Given $\langle x, z \rangle$, f_1 computes $(g_1(\langle x^-, z^- \rangle))^+$. Note that ρ never maps to the empty string, so $z \neq \varepsilon$ and z^- is well-defined. Similarly, $x \neq \varepsilon$ because $(x, y) \in \text{domain}(\rho)$, so x^- is well-defined. Thus, $\rho(x, f_1(\langle x, z \rangle)) = \rho(x, (g_1(\langle x^-, z^- \rangle))^+) = z$. Similarly, an inverter with respect to the second argument can be built for ρ given one for σ .

Conversely, given an inverter for ρ with respect to the first (respectively, second) argument, an inverter for σ with respect to the first (respectively, second) argument can be constructed by reverting the shifting above. Thus, if ρ is not strongly noninvertible, neither is σ .

This completes the proof of Claim 1 and Lemma 3.4. ■

Lemmas 3.2–3.4 imply that it suffices to deal with only the “total” cases. That is, to achieve Goal 3.1, it would be enough to show that if $P \neq \text{NP}$ then each of the following eight types of one-way functions exist: (Y, Y, Y, Y)-OWFs, (Y, Y, Y, N)-OWFs, (Y, Y, N, Y)-OWFs, (Y, Y, N, N)-OWFs, (N, Y, Y, Y)-OWFs, (N, Y, Y, N)-OWFs, (N, Y, N, Y)-OWFs, and (N, Y, N, N)-OWFs. In the following sections, we will study each of these cases. Table 1 summarizes where each of these is established.

4. Strongness and being oblivious to strongness

In this section, we consider the “strong”-is-required cases (the (Y, t, c, a)-OWF cases) and those cases where the property of strongness is a “don’t care” issue (the (*, t, c, a)-OWF cases). We start with the 27 “strong” cases. Theorem 4.1 below characterizes each of these cases by the condition $P \neq \text{NP}$. The proof of Theorem 4.1 follows from the upcoming Lemma 4.2–4.5, via Lemmas 3.2–3.4. (That is, as mentioned earlier, the number $27 = 3^3$ comes from the fact that “strong” is set to “Y”, and for each of “total”, “commutative”, and “associative”, the cases “Y”, “N”, and “*” must be handled. However, the lemmas from Section 3 will allow us to establish these results via simply establishing the first four of the eight conditions mentioned in the final paragraph of Section 3.)

Theorem 4.1. For each $t, c, a \in \{Y, N, *\}$, there exist (Y, t, c, a)-OWFs if and only if $P \neq \text{NP}$.

Lemma 4.2 is already known from Hemaspaandra and Rothe’s work [16].

Lemma 4.2. If $P \neq \text{NP}$ then there exist (Y, Y, Y, Y)-OWFs.

The equivalence (due to [16], and following immediately from Lemma 4.2 in light of Lemma 3.3) of $P \neq \text{NP}$ and the existence of (Y, Y, Y, Y)-OWFs will be exploited in the upcoming proofs of Lemmas 4.3–4.5. That is, in these proofs, we start from a strong, total, commutative, associative one-way function.

Lemma 4.3. If $P \neq \text{NP}$ then there exist (Y, Y, Y, N)-OWFs.

The proof is omitted here, but is available in detail in the technical report version of this paper [19].

Lemma 4.4. If $P \neq \text{NP}$ then there exist (Y, Y, N, Y)-OWFs.

Proof. Assume $P \neq \text{NP}$. By Lemma 4.2, let σ be a (Y, Y, Y, Y)-OWF. Define a function $\rho : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ by

$$\rho(x, y) = \begin{cases} y & \text{if } x, y \in \{0, 1\} \\ (\sigma(x^{3-}, y^{3-}))^{3+} & \text{if } x \notin \{\varepsilon, 0, 1\} \wedge y \notin \{\varepsilon, 0, 1\} \\ \varepsilon & \text{otherwise,} \end{cases}$$

where we use the following shorthand: Recall from the proof of Lemma 3.4 that, in the standard lexicographic ordering of Σ^* , w^+ denotes the successor of $w \in \Sigma^*$ and w^- denotes the predecessor of $w \in \Sigma^+$. For $w \in \Sigma^*$, let $w^{3+} = ((w^+)^+)^+$, and for $w \in \Sigma^*$ with $w \notin \{\varepsilon, 0, 1\}$, let $w^{3-} = ((w^-)^-)^-$.

It is easy to see, given the fact that σ is a (Y, Y, Y, Y)-OWF, that ρ is a strongly noninvertible, s-honest, total one-way function. However, unlike σ , ρ is noncommutative, since $\rho(0, 1) = 1 \neq 0 = \rho(1, 0)$. To see that ρ , just like σ , is associative, let three arbitrary strings be given, say a, b , and c . Distinguish the following cases:

Case 1: Each of a, b , and c is a member of $\{0, 1\}$. Then, associativity follows from the definition of ρ : $\rho(a, \rho(b, c)) = \rho(a, c) = c = \rho(b, c) = \rho(\rho(a, b), c)$.

Case 2: None of a, b , and c is a member of $\{\varepsilon, 0, 1\}$. Then the associativity of ρ follows immediately from the associativity of σ . That is, $\rho(a, \rho(b, c)) = \rho(a, (\sigma(b^{3-}, c^{3-}))^{3+}) = (\sigma(a^{3-}, \sigma(b^{3-}, c^{3-})))^{3+} = (\sigma(\sigma(a^{3-}, b^{3-}), c^{3-}))^{3+} = \rho((\sigma(a^{3-}, b^{3-}))^{3+}, c) = \rho(\rho(a, b), c)$. Note here that both $(\sigma(a^{3-}, b^{3-}))^{3+}$ and $(\sigma(b^{3-}, c^{3-}))^{3+}$ are strings that are not members of $\{\varepsilon, 0, 1\}$.

Case 3: At least one of a, b , and c is not a member of $\{0, 1\}$, and at least one of a, b , and c is a member of $\{\varepsilon, 0, 1\}$. In this case, it follows from the definition of ρ that $\rho(a, \rho(b, c)) = \varepsilon = \rho(\rho(a, b), c)$.

Thus, ρ is a (Y, Y, N, Y) -OWF. ■

Lemma 4.5. If $P \neq NP$ then there are (Y, Y, N, N) -OWFs.

The proof is omitted here, but is available in detail in the technical report version of this paper [19].

Next, we note Corollary 4.6, which follows immediately from Theorem 4.1 via Lemmas 3.2 and 3.3. That is, in light of Lemmas 3.2 and 3.3, Theorem 4.1 provides also a $P \neq NP$ characterization of all 27 cases where one requires one-way-ness but is oblivious to whether or not the functions are guaranteed to be strong.

Corollary 4.6. For each $t, c, a \in \{Y, N, *\}$, there are $(*, t, c, a)$ -OWFs if and only if $P \neq NP$.

5. Nonstrongness

It remains to prove the 27 “nonstrong” cases (i.e., the (N, t, c, a) -OWF cases). All 27 have $P \neq NP$ as a necessary condition. For each of them, we also completely characterize the existence of such OWFs by $P \neq NP$. (The number $27 = 3^3$ comes from the fact that “strong” is set to “N”, and for each of “total”, “commutative”, and “associative”, the cases “Y”, “N”, and “*” must be handled. However, the lemmas from Section 3 will allow us to establish these results via simply establishing the final four of the eight conditions mentioned in the final paragraph of Section 3.)

First, we consider two “total” and “nonstrong” cases in Lemmas 5.1 and 5.2 below. Note that Hemaspaandra, Pasanen, and Rothe [15] constructed one-way functions that in fact are not strongly noninvertible. Unlike Lemmas 5.1 and 5.2, however, they did not consider associativity and commutativity. Note that, in the proofs of Lemmas 5.1 and 5.2, we achieve “nonstrongness” while ensuring that the functions constructed are s -honest. That is, they are not “nonstrong” because they are not s -honest, but rather they are “nonstrong” because they are not strongly noninvertible.

Lemma 5.1. If $P \neq NP$ then there exist (N, Y, Y, N) -OWFs.

Proof. Assuming $P \neq NP$, we define an (N, Y, Y, N) -OWF that is akin to a function constructed in Theorem 3.4 of [15] (which is also available as Theorem 3 of [14]).

Define a function $\sigma : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ by

$$\sigma(x, y) = \begin{cases} 1\rho(x) & \text{if } x = y \\ 0 \min(x, y) \max(x, y) & \text{if } x \neq y, \end{cases}$$

where $\min(x, y)$ denotes the lexicographically smaller of x and y , $\max(x, y)$ denotes the lexicographically greater of x and y , and $\rho : \Sigma^* \rightarrow \Sigma^*$ is a total one-ary one-way function, which exists assuming $P \neq NP$. Note that σ is polynomial-time computable, total, honest, and s -honest. Clearly, if σ could be inverted in polynomial time then ρ could be too. Thus, σ is a one-way function. However, although σ is s -honest, it is not strong. To prove that σ is not strongly noninvertible, we show that it is invertible with respect to each of its arguments. Define a function $f_1 : \Sigma^* \rightarrow \Sigma^*$ by

$$f_1(a) = \begin{cases} y & \text{if } (\exists x, y, z \in \Sigma^*) [a = \langle x, 0z \rangle \wedge z = xy \wedge x <_{\text{lex}} y] \\ y & \text{if } (\exists x, y, z \in \Sigma^*) [a = \langle x, 0z \rangle \wedge z = yx \wedge y <_{\text{lex}} x] \\ x & \text{if } (\exists x, z \in \Sigma^*) [a = \langle x, 1z \rangle] \\ \varepsilon & \text{otherwise,} \end{cases}$$

where $x <_{\text{lex}} y$ indicates that x is strictly smaller than y in the lexicographic ordering of Σ^* . Note that f_1 is in FP and that f_1 inverts σ with respect to the first argument. Although this is already enough to defy strong noninvertibility of σ , we note that one can analogously show that σ also is invertible with respect to the second argument.

To see that σ is commutative, note that if $x \neq y$ then $\sigma(x, y) = 0 \min(x, y) \max(x, y) = \sigma(y, x)$. (Although the $x = y$ case does not need to be discussed to establish commutativity, for completeness we mention that if $x = y$ then $\sigma(x, y) = 1\rho(x) = \sigma(y, x)$.) To see that σ is nonassociative, note that $\sigma(\sigma(1, 0), 001) = \sigma(001, 001) = 1\rho(001) \neq 0100001 = \sigma(1, 00001) = \sigma(1, \sigma(0, 001))$.

Thus, σ is an (N, Y, Y, N) -OWF, which completes the proof. ■

Lemma 5.2. If $P \neq NP$ then there exist (N, Y, N, N) -OWFs.

Proof. Assume that $P \neq NP$. So there exists a total one-argument one-way function $\rho : \Sigma^* \rightarrow \Sigma^*$. In Theorem 3.4 of [15], a function $\sigma : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ is constructed as follows:

$$\sigma(x, y) = \begin{cases} 1\rho(x) & \text{if } x = y \\ 0_{xy} & \text{if } x \neq y. \end{cases}$$

It is shown in [15] that σ is a total, s-honest one-way function that is not strongly noninvertible.

To see that σ is noncommutative, note that $\sigma(0, 1) = 001 \neq 010 = \sigma(1, 0)$. To see that σ is nonassociative, note that $\sigma(\sigma(0, 1), 001) = 1\rho(001) \neq 0001001 = \sigma(0, \sigma(1, 001))$. Thus, σ is an (N, Y, N, N)-OWF, which completes the proof. ■

Next, we observe that the two remaining “total” and “nonstrong” cases are connected: Lemma 5.3 shows that, given an (N, Y, Y, Y)-OWF, one can construct an (N, Y, N, Y)-OWF. Thus, by Lemma 3.4, characterizing via $P \neq NP$ just the case of (N, Y, Y, Y)-OWFs will suffice to solve all the four remaining cases (namely, NYYY, NYNY, NNYY, and NNNY) at once.

Lemma 5.3. *If there exist (N, Y, Y, Y)-OWFs, then there exist (N, Y, N, Y)-OWFs.*

The proof of Lemma 5.3 is closely related to that of Lemma 4.4, and can be found in the technical report version of this paper [19].

We now turn to completely characterizing the existence of (N, Y, Y, Y)-OWFs. A transformation from the literature that might seem to come close to establishing “if $P \neq NP$, then (N, Y, Y, Y)-OWFs exist” has been shown to be flawed unless an unlikely complexity class collapse occurs.¹ However, the following result of Rabi and Sherman does provide evidence that (N, Y, Y, Y)-OWFs indeed exist.

Theorem 5.4 ([26,25]). *If factoring is not in polynomial time, then there exist (N, Y, Y, Y)-OWFs.*

We now improve that sufficient condition to $P \neq NP$.

Lemma 5.5. *If $P \neq NP$ then there exist (N, Y, Y, Y)-OWFs and (N, Y, N, Y)-OWFs.*

Proof. By Lemma 5.3, it suffices to handle the case of (N, Y, Y, Y)-OWFs. So, assume $P \neq NP$. This implies that there exists a total, one-way function $f : \Sigma^* \rightarrow \Sigma^*$. Define the function $g : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ by

$$g(x, y) = \begin{cases} 0f(a) & \text{if } x = 1a \text{ and } y = 1a \\ \varepsilon & \text{otherwise.} \end{cases}$$

g is clearly a one-way function. g also is clearly total and commutative. g is associative since it is not hard to see that $(\forall a, b, c)[g(a, g(b, c)) = g(g(a, b), c) = \varepsilon]$. Though g is easily seen to be s-honest, g fails to be strongly noninvertible, and so is not strong. In particular, given the output and a purported first argument, here is how to find a second argument consistent with the first argument when one exists. If the output is ε and the purported first argument is z , then output ε as a second argument. If the output is $0y$ and the purported first argument is $1x$, then if $f(x) = y$ a good second argument is $1x$. In every other case, the output and purported first argument cannot have any second argument that is consistent with them, so we safely (though irrelevantly, except for achieving totality of our inverter if one desires that) in this case have our inverter output ε . ■

Theorem 5.6. *For each $t, c, a \in \{Y, N, *\}$, there exist (N, t, c, a)-OWFs if and only if $P \neq NP$.*

The proof of Theorem 5.6 follows immediately from Lemmas 5.1, 5.2 and 5.5, via Lemmas 3.2–3.4.

In conclusion, this paper studied the question of whether one-way functions can exist, where one imposes either possession, nonpossession, or being oblivious to possession of the properties of strongness, totality, commutativity, and associativity. We have shown that $P \neq NP$ is a necessary and sufficient condition in each of the possible 81 cases.

¹ In more detail: Rabi and Sherman [25,26], assuming $P \neq NP$, constructed a *nontotal*, commutative, associative (in a slightly weaker model of associativity for partial functions that completely coincides with our model when speaking of total functions) one-way function that appears to fail to possess strong noninvertibility. They also proposed a construction that they claim can be used to transform every nontotal AOWF whose domain is in P to a total AOWF. However, their claim does not provide an (N, Y, Y, Y)-OWF, due to some subtle technical points. First, Rabi and Sherman’s construction – even if their claim were valid – is not applicable to the nonstrong, nontotal, commutative AOWF they construct, since this function seems to not have a domain in P . Second, it is not at all clear that their above-mentioned “construction to add totality” has the properties they assert for it. In particular, let UP as usual denote Valiant’s [31] class representing “unambiguous polynomial time”. Hemaspaandra and Rothe showed in [16] that any proof that the Rabi–Sherman claim about their transformation’s action is in general valid would immediately prove that $UP = NP$, which is considered unlikely.

Acknowledgments

We are grateful to the referee and the editor for helpful comments and suggestions. The first author was supported in part by NSF grant CCF-0426761, an Alexander von Humboldt Foundation TransCoop grant, and a Friedrich Wilhelm Bessel Research Award. His work was done in part while visiting Julius-Maximilians-Universität Würzburg and Heinrich-Heine-Universität Düsseldorf. The second author was supported in part by an Alexander von Humboldt Foundation TransCoop grant and DFG grants RO 1202/9-1, RO 1202/9-3, and RO 1202/11-1. His work was done in part while visiting the University of Rochester and Julius-Maximilians-Universität Würzburg.

References

- [1] E. Allender, R. Rubinfeld, P-printable sets, *SIAM Journal on Computing* 17 (6) (1988) 1193–1202.
- [2] J. Balcázar, J. Díaz, J. Gabarró, Structural Complexity I, in: *EATCS Monographs on Theoretical Computer Science*, 2nd ed., Springer-Verlag, 1995.
- [3] L. Berman, Polynomial reducibilities and complete sets, Ph.D. Thesis, Cornell University, Ithaca, NY, 1977.
- [4] A. Beygelzimer, L. Hemaspaandra, C. Homan, J. Rothe, One-way functions in worst-case cryptography: Algebraic and security properties are on the house, *SIGACT News* 30 (4) (1999) 25–40.
- [5] D. Bovet, P. Crescenzi, *Introduction to the Theory of Complexity*, Prentice Hall, 1993.
- [6] G. Brassard, A note on the complexity of cryptography, *IEEE Transactions on Information Theory* 25 (2) (1979) 232–233.
- [7] G. Brassard, S. Fortune, J. Hopcroft, A note on cryptography and $NP \cap coNP = P$, Tech. Rep. TR-338, Department of Computer Science, Cornell University, Ithaca, NY, Apr. 1978.
- [8] S. Fenner, L. Fortnow, A. Naik, J. Rogers, Inverting onto functions, *Information and Computation* 186 (1) (2003) 90–103.
- [9] O. Goldreich, *Foundations of Cryptography*, Cambridge University Press, 2001.
- [10] E. Grädel, Definability on finite structures and the existence of one-way functions, *Methods of Logic in Computer Science* 1 (1994) 299–314.
- [11] J. Grollmann, A. Selman, Complexity measures for public-key cryptosystems, *SIAM Journal on Computing* 17 (2) (1988) 309–335.
- [12] J. Hartmanis, L. Hemachandra, One-way functions and the nonisomorphism of NP-complete sets, *Theoretical Computer Science* 81 (1) (1991) 155–163.
- [13] L. Hemaspaandra, M. Ogihara, *The Complexity Theory Companion*, in: *EATCS Texts in Theoretical Computer Science*, Springer-Verlag, Berlin, Heidelberg, New York, 2002.
- [14] L. Hemaspaandra, K. Pasanen, J. Rothe, If $P \neq NP$ then some strongly noninvertible functions are invertible, in: *Proceedings of the 13th International Symposium on Fundamentals of Computation Theory*, in: *Lecture Notes in Computer Science*, vol. 2138, Springer-Verlag, 2001.
- [15] L. Hemaspaandra, K. Pasanen, J. Rothe, If $P \neq NP$ then some strongly noninvertible functions are invertible, *Theoretical Computer Science* 362 (1–3) (2006) 54–62.
- [16] L. Hemaspaandra, J. Rothe, Creating strong, total, commutative, associative one-way functions from any one-way function in complexity theory, *Journal of Computer and System Sciences* 58 (3) (1999) 648–659.
- [17] L. Hemaspaandra, J. Rothe, Characterizing the existence of one-way permutations, *Theoretical Computer Science* 244 (1–2) (2000) 257–261.
- [18] L. Hemaspaandra, J. Rothe, A. Saxena, Enforcing and defying associativity, commutativity, totality, and strong noninvertibility for one-way functions in complexity theory, in: *Proceedings of the 9th Italian Conference on Theoretical Computer Science*, in: *Lecture Notes in Computer Science*, vol. 3701, Springer-Verlag, 2005.
- [19] L. Hemaspaandra, J. Rothe, A. Saxena, Enforcing and defying associativity, commutativity, totality, and strong noninvertibility for one-way functions in complexity theory, Technical Report TR-854, Department of Computer Science, University of Rochester, Rochester, NY, Dec. 2004. Revised Nov. 2007. Also appears as ACM Computing Research Repository (CoRR) Technical Report cs.CC/0503049 (March 2005, revised Nov. 2007).
- [20] L. Hemaspaandra, J. Rothe, G. Wechsung, On sets with easy certificates and the existence of one-way permutations, in: *Proceedings of the 3rd Italian Conference on Algorithms and Complexity*, in: *Lecture Notes in Computer Science*, vol. 1203, Springer-Verlag, 1997.
- [21] C. Homan, Tight lower bounds on the ambiguity of strong, total, associative, one-way functions, *Journal of Computer and System Sciences* 68 (3) (2004) 657–674.
- [22] C. Homan, M. Thakur, One-way permutations and self-witnessing languages, *Journal of Computer and System Sciences* 67 (3) (2003) 608–622.
- [23] S. Kleene, *Introduction to Metamathematics*, D. van Nostrand Company, Inc., New York, Toronto, 1952.
- [24] K. Ko, On some natural complete operators, *Theoretical Computer Science* 37 (1) (1985) 1–30.
- [25] M. Rabi, A. Sherman, Associative one-way functions: A new paradigm for secret-key agreement and digital signatures, Tech. Rep. CS-TR-3183/UMIACS-TR-93-124, Department of Computer Science, University of Maryland, College Park, MD, 1993.
- [26] M. Rabi, A. Sherman, An observation on associative one-way functions in complexity theory, *Information Processing Letters* 64 (5) (1997) 239–244.
- [27] J. Rothe, L. Hemaspaandra, On characterizing the existence of partial one-way permutations, *Information Processing Letters* 82 (3) (2002) 165–171.
- [28] A. Saxena, B. Soh, A novel method for authenticating mobile agents with one-way signature chaining, in: *Proceedings of the 7th International Symposium on Autonomous Decentralized Systems*, IEEE Computer Society Press, 2005.
- [29] A. Saxena, B. Soh, D. Zantidis, A digital cash protocol based on additive zero knowledge, in: *Proceedings of the 3rd International Workshop on Internet Communications Security*, in: *Lecture Notes in Computer Science*, vol. 3482, Springer Verlag, 2005.
- [30] A. Selman, A survey of one-way functions in complexity theory, *Mathematical Systems Theory* 25 (3) (1992) 203–221.
- [31] L. Valiant, The relative complexity of checking and evaluating, *Information Processing Letters* 5 (1) (1976) 20–23.
- [32] O. Watanabe, On hardness of one-way functions, *Information Processing Letters* 27 (3) (1988) 151–157.
- [33] O. Watanabe, On one-way functions, in: D. Du, H. Guodong (Eds.), *Combinatorics, Computing and Complexity*, Kluwer Academic Publishers, 1989, pp. 98–131.