Theoretical Computer Science 406 (2008) 187–206

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

A new model for selfish routing☆

Thomas Lücking[a], Marios Mavronicolas[b,∗], Burkhard Monien[a], Manuel Rode[a]

[a] Faculty of Computer Science, Electrical Engineering and Mathematics, University of Paderborn, Fürstenallee 11, 33102 Paderborn, Germany

[b] Department of Computer Science, University of Cyprus, P.O. Box 20537, Nicosia CY-1678, Cyprus

ARTICLE INFO

Keywords:
Game theory
Nash equilibrium
Coordination ratio

ABSTRACT

In this work, we introduce and study a new, potentially rich model for *selfish routing* over non-cooperative networks, as an interesting hybridization of the two prevailing such models, namely the KP *model* [E. Koutsoupias, C.H. Papadimitriou, Worst-case equilibria, in: G. Meinel, S. Tison (Eds.), Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science, in: Lecture Notes in Computer Science, vol. 1563, Springer-Verlag, 1999, pp. 404–413] and the W *model* [J.G. Wardrop, Some theoretical aspects of road traffic research, Proceedings of the of the Institute of Civil Engineers 1 (Pt. II) (1952) 325–378].

In the hybrid model, each of *n users* is using a *mixed strategy* to ship its unsplittable *traffic* over a *network* consisting of *m* parallel *links*. In a *Nash equilibrium*, no *user* can unilaterally improve its *Expected Individual Cost*. To evaluate Nash equilibria, we introduce *Quadratic Social Cost* as the *sum* of the expectations of the *latencies*, incurred by the squares of the accumulated traffic. This modeling is unlike the KP model, where *Social Cost* [E. Koutsoupias, C.H. Papadimitriou, Worst-case equilibria, in: G. Meinel, S. Tison (Eds.), Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science, in: Lecture Notes in Computer Science, vol. 1563, Springer-Verlag, 1999, pp. 404–413] is the expectation of the *maximum* latency incurred by the accumulated traffic; but it is like the W model since the Quadratic Social Cost can be expressed as a weighted sum of Expected Individual Costs. We use the Quadratic Social Cost to define *Quadratic Coordination Ratio*. Here are our main findings:

- Quadratic Social Cost can be computed in polynomial time. This is unlike the $\#\mathcal{P}$-completeness [D. Fotakis, S. Kontogiannis, E. Koutsoupias, M. Mavronicolas, P. Spirakis, The structure and complexity of Nash equilibria for a selfish routing game, in: P. Widmayer, F. Triguero, R. Morales, M. Hennessy, S. Eidenbenz, R. Conejo (Eds.), Proceedings of the 29th International Colloquium on Automata, Languages and Programming, in: Lecture Notes in Computer Science, vol. 2380, Springer-Verlag, 2002, pp. 123–134] of computing Social Cost for the KP model.
- For the case of identical users and identical links, the *fully mixed Nash equilibrium* [M. Mavronicolas, P. Spirakis, The price of selfish routing, Algorithmica 48 (1) (2007) 91–126], where each user assigns positive probability to every link, maximizes Quadratic Social Cost.

☆ A preliminary version of this work appeared in the *Proceedings of the 21st International Symposium on Theoretical Aspects of Computer Science*, V. Diekert and M. Habib eds., pp. 547–558, Vol. 2996, Lecture Notes in Computer Science, Springer-Verlag, Montpellier, France, March 2004. This work has been partially supported by the IST Program of the European Union under projects ALCOM-FT (contract number IST-1999-14186). FLAGS (contract number IST-2001-33116) and DELIS (contract number 001907).

∗ Corresponding author.
E-mail addresses: luck@uni-paderborn.de (T. Lücking), mavronic@ucy.ac.cy (M. Mavronicolas), bm@uni-paderborn.de (B. Monien), rode@uni-paderborn.de (M. Rode).

doi:10.1016/j.tcs.2008.06.045

- As our main result, we present a comprehensive collection of *tight,* constant (that is, independent of m and n), strictly less than 2, lower and upper bounds on the Quadratic Coordination Ratio for several, interesting special cases. Some of the bounds stand in contrast to corresponding super-constant bounds on the Coordination Ratio previously shown in [A. Czumaj, B. Vöcking, Tight bounds for worst-case equilibria, ACM Transactions on Algorithms 3 (1) (2007); E. Koutsoupias, M. Mavronicolas, P. Spirakis, Approximate equilibria and ball fusion, Theory of Computing Systems 36 (6) (2003) 683–693; E. Koutsoupias, C.H. Papadimitriou, Worst-case equilibria, in: G. Meinel, S. Tison (Eds.), Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science, in: Lecture Notes in Computer Science, vol. 1563, Springer-Verlag, 1999, pp. 404–413; M. Mavronicolas, P. Spirakis, The price of selfish routing, Algorithmica 48 (1) (2007) 91–126] for the KP model.

1. Introduction

1.1. Motivation and framework

1.1.1. Outline

We propose a new model for *selfish routing* over *non-cooperative networks*, as a hybridization of the two prevailing such models, namely the KP *model*, due to Koutsoupias and Papadimitriou [26], and the W *model* due to Wardrop [36]. Although proposed only recently in the context of studying selfish traffic over the Internet, the KP model has yet received a lot of interest — see, e.g., [3,12,13,15,16,18,20,23,25,28,29] or [17,24] for surveys. The W model dates back to studies of transportation networks in the 1950s; however, much recent work on selfish routing (see [34] and references therein), has witnessed a revival of interest into the W model.

Within the new model, we study an interesting *strategic game,* originally proposed in [26]; we are especially interested in the associated *Nash equilibria* [31,32]. At a Nash equilibrium, no player (here, *user*) can unilaterally improve their objective by switching to a different *strategy* (here, *link*). In a *pure* Nash equilibrium, each player chooses exactly one strategy (with probability one); in a *mixed* Nash equilibrium, each player uses a probability distribution over strategies, and in the *fully mixed* Nash equilibrium [29] all probabilities are (strictly) positive.

1.1.2. The KP *model and the* W *model*

The two models differ with respect to their assumptions about the following parameters:

- The structure and topology of the underlying *network* they consider; in both cases, two distinguished nodes, the *source* and the *destination*, are considered.
- The *splittability* or *unsplittability* of the selfish traffic; *unsplittable* traffic is routed all together along a single path, while *splittable* traffic may split into infinitesimal pieces.
- The type of equilibria (*pure* or *mixed*) they consider.
- The definition of *(Expected) Individual Cost* they adopt; these are used for defining Nash equilibria and *Wardrop equilibria*, respectively.
- The definitions of *Social Cost* (a performance measure for equilibria) and *Optimum* (an optimality measure for general assignments). In turn, these are used for defining *Coordination Ratio* [26], as the *worst-case* ratio of Social Cost over Optimum, over all equilibria. A *worst-case* equilibrium maximizes the particular Social Cost.

We continue to describe separately the KP model and the W model.

- The KP model has considered a simple network consisting of *m parallel links*, from source to destination; each link bears a *capacity* — its traffic processing rate. Selfish traffic is modeled as a finite collection of *users,* each bearing an unsplittable traffic, and shipping it using a *mixed strategy* — a probability distribution over links. The *Expected Individual Cost* of a user is its expected (conditional) latency on a link it chooses; in a Nash equilibrium, no Expected Individual Cost can be unilaterally decreased. The Social Cost is the expectation of the maximum link latency; the Optimum is the least possible maximum link latency. The KP model may be viewed as a *weighted congestion game* [30] equipped with Social Cost and Coordination Ratio.
- The W model has considered arbitrary *multicommodity* networks (that is, networks with multiple sources and destinations), with a *latency function* for each link. Selfish traffic is modeled as a *splittable flow*. The *Individual Cost* for a path (from source to destination), is the sum of the latencies incurred on its links. In a *Wardrop equilibrium,* all (used) paths have the same Individual Cost. In the W model, users may be thought of as (infinitely many) *non-atomic* entities, each carrying infinitesimal traffic; then, the definitions of Individual Cost for the W and the KP models are the same, while pure and mixed equilibria coincide in the W model. The Social Cost is the sum, over all paths, of Individual Costs.[1] The Optimum is the least possible, over all flows, Social Cost.

[1] Social Cost as the expectation of maximum latency has also been considered for the W model [11,35].

1.1.3. The hybrid model

We follow the KP model to consider the parallel links network, unsplittable traffic, and mixed Nash equilibria. The Expected Individual Cost we adopt is the expected latency incurred to a user. This generalized definition applies to arbitrary latency functions, while it matches the original definition of Expected Individual Cost for the KP model [26], in the case of linear latency functions. Hence, the Nash equilibria for our new model are exactly those for the KP model. However, we follow the W model to model Social Cost as some kind of total latency, which turns out to be a certain sum of Expected Individual Costs. So, our hybrid model represents the *first* step towards accommodating unsplittable traffic and mixed strategies within the W model, but also the *first* step towards accommodating total latency within the KP model.

We continue to describe the hybrid model in some more detail. For any link, consider the latency incurred by the square of the total traffic on the link; add up the expectations of these latencies over all links. This results in *Quadratic Social Cost*. (Similar modelings that have employed quadratic cost functions can be found in both the scheduling literature [1, 8,10,27], and the networking literature [2].) *Quadratic Optimum* is the best possible Quadratic Social Cost, over all pure assignments. *Quadratic Coordination Ratio* is the worst-case ratio, over all Nash equilibria, of Quadratic Social Cost over Quadratic Optimum.

Note that expectation and maximum used in the definition of Social Cost for the KP model do not commute. However, expectation and sum used, in the definition of Quadratic Social Cost for our model do commute. This commutativity allows some hope for a more tractable analysis of Nash equilibria in the hybrid model.

1.2. Contribution and significance

Some of our results deal specifically with the cases of *identical users* and *identical links*, with equal (unit) user traffics and link capacities, respectively; the most general cases are those of *arbitrary users* and *related links*, respectively.

1.2.1. Quadratic social cost

We observe that Quadratic Social Cost can be expressed as a weighted sum of Expected Individual Costs, where the weight is the user traffic. This observation allows for the computation of Quadratic Social Cost for any mixed assignment in polynomial time $O(mn)$ (Theorem 4.1).

For the fully mixed Nash equilibria, we obtain two very simple combinatorial expressions for Quadratic Social Cost, in the two cases of identical links and identical users (Theorems 4.2 and 4.4, respectively). For these two cases, the two expressions imply corresponding polynomial time algorithms (Corollaries 4.3 and 4.5). For the case of identical users, the link capacities enter the expression for Quadratic Social Cost, via their sum appearing in the denominator. This dependence excludes *Braess-like paradoxes* [7], while it implies that the Quadratic Social Cost of the fully mixed Nash equilibrium is insensitive to reallocating capacity among the links.

We use a combinatorial analysis to prove that, for the case of identical users and identical links, the fully mixed Nash equilibrium is the worst-case Nash equilibrium with respect to Quadratic Social Cost (Theorem 4.8). We formulate the *Quadratic Fully Mixed Nash Equilibrium Conjecture* (Conjecture 4.1) to speculate that this happens in the general case.

1.2.2. Quadratic coordination ratio

As our main result, we obtain a collection of *tight* bounds on the Quadratic Coordination Ratio. All bounds we prove are either equal to or bounded by a constant strictly less than 2.

- We first consider pure Nash equilibria.[2] To establish upper bounds, we prove some new structural properties of optimal assignments and pure Nash equilibria, which may be of independent interest. For example, Proposition 3.2 provides an efficient characterization of optimal assignments in the hybrid model, which implies that those can be decided in polynomial time (Corollary 3.3). We obtain:
  - For the case of identical users, the Quadratic Coordination Ratio is $\frac{4}{3}$ (Theorem 5.1).
  - For the case of identical links, the Quadratic Coordination Ratio is $\frac{9}{8}$ (Theorem 5.2).

  These different tight bounds imply a Quadratic Coordination Ratio separation between the cases of identical links and of identical users, restricted to pure Nash equilibria.
- We continue to consider mixed Nash equilibria.
  - For the case of identical links, we prove a *tight* bound of $1 + \frac{\min\{n,m\}-1}{m}$ on Quadratic Coordination Ratio for the fully mixed Nash equilibrium.
  - For the case of identical users and identical links, we prove a (sometimes better) upper bound of $1 + \min\{\frac{m-1}{n}, \frac{n-1}{m}\}$, which holds for all Nash equilibria.

[2] These always exist as they coincide with those for the KP model, which are known to exist [18, Theorem 1].

1.3. Directly related work and comparison

Computing the Social Cost of a Nash equilibrium in the KP model is known to be $\#\mathcal{P}$-complete [18, Theorem 8]; this applies even if links are identical. This stands in contrast to the obtained polynomial computation of Quadratic Social Cost, in the hybrid model.

Fully mixed Nash equilibria were introduced and analyzed in [29]. The (yet unproven) *Fully Mixed Nash Equilibrium Conjecture* asserts that the worst-case Nash equilibrium for the KP model, is the fully mixed Nash equilibrium. This conjecture has been motivated by some results in [18]; it was explicitly formulated in [23] and further studied in [21,22,28].

Bounds on Coordination Ratio for the KP model were proved in [3,13,16,20,25,29]. These include (*tight*) bounds of $\Theta(\frac{\lg m}{\lg\lg m})$ for the case of identical links [13,25,26,29], and of $\Theta(\frac{\lg m}{\lg\lg\lg m})$ for the case of related links [13]. These bounds are contrasted by the corresponding constant bounds on the Quadratic Coordination Ratio, proved here for the hybrid model. (For the W model, there have been shown constant bounds on Coordination Ratio — see [34] and references therein.)

1.4. Organization

Section 2 presents the hybrid model. Some preliminary properties of Nash equilibria, and optimal assignments are articulated in Section 3. The Quadratic Social Cost is studied in Section 4. The bounds on the Quadratic Coordination Ratio are presented in Section 5. We conclude, in Section 6, with a discussion of our results and some open problems.

2. The model

Our definitions for the new model are built on top of those for the KP model [26]; those definitions are extended to accommodate features from the W model [36]. The definitions for the KP model are patterned after those in [29, Section 2], [18, Section 2], [16, Section 2], [23, Section 2] and [28, Section 2], which, in turn, were based on those in [26, Sections 1 & 2].

Throughout, denote for any integer $m \geq 1$, $[m] = \{1, \dots, m\}$; take $[0] = \emptyset$. For a random variable X, with associated probability distribution $\mathbf{P}$; denote $\mathbb{E}_{\mathbf{P}}(X)$ the *expectation* of X. For a probability p, denote $\overline{p} = 1 - p$.

2.1. General

We consider a simple *network* consisting of a set of m parallel *links* $1, 2, \dots, m$, from a *source* node to a *destination* node. Each of n *users* $1, 2, \dots, n$ wishes to route a particular amount of traffic along a (non-fixed) link, from source to destination. Assume throughout that $m \geq 2$ and $n \geq 2$. (Throughout, we will be using subscripts for users and superscripts for links.)

Denote w_i the *traffic* of user $i \in [n]$. Define the $n \times 1$ *traffic vector* $\mathbf{w}$ in the natural way. Without loss of generality, assume that $w_1 \geq w_2 \geq \cdots \geq w_n$. Denote $W = \sum_{i\in[n]} w_i$, $W_1 = \sum_{i\in[n]} w_i^2$ and $W_2 = \sum_{i,j\in[n],i<j} w_i w_j$. (These quantities will be used in our later proofs.) Note that $W^2 = W_1 + 2W_2$. It is a well known simple fact that $W_1 \geq \frac{W^2}{n}$.

Denote $c^\ell > 0$ the *capacity* of link $\ell \in [m]$, representing the rate at which the link processes traffic. So, the *latency* for traffic w through link ℓ equals w/c^ℓ. Assume throughout, without loss of generality, that $c^1 \geq c^2 \geq \cdots \geq c^m$. Denote $C = \sum_{j\in[m]} c^j$. An *instance* is a pair $\langle \mathbf{w}, \mathbf{c} \rangle$.

In the case of *identical users*, all user traffic is 1; in the case of *identical links*, all link capacities are 1. In the general case, we talk about *arbitrary users* and *related links*.

2.2. Strategies and assignments

A *pure strategy* for user $i \in [n]$ is some specific link. A *mixed strategy* for user $i \in [n]$ is a probability distribution over pure strategies; so, it is a probability distribution over links.

A *pure assignment* is an n-tuple $\mathbf{L} = \langle \ell_1, \ell_2, \dots, \ell_n \rangle \in [m]^n$; a *mixed assignment* is an $n \times m$ *probability matrix* $\mathbf{P}$ of nm probabilities p_i^j, $i \in [n]$ and $j \in [m]$, where p_i^j is the probability that user i chooses link j. Throughout, we will cast a pure assignment as a special case of a mixed assignment, in which all (mixed) strategies are pure. Moreover, the mixed assignment $\mathbf{P}$ can be cast as a collection of pure assignments $\mathbf{L}_\mathbf{P} = \langle \ell_1, \ell_2, \dots, \ell_n \rangle$, such that for all users $i \in [n]$, $p_i^{\ell_i} > 0$; any such pure assignment $\mathbf{L}_\mathbf{P}$ will be called *consistent* with $\mathbf{P}$. So, a mixed assignment $\mathbf{P}$ induces a probability distribution $\mathsf{P}_\mathbf{P}$ on the space of all pure assignments consistent with $\mathbf{P}$. A mixed assignment $\mathbf{P}$ is *fully mixed* [29, Section 2.2], if for all users $i \in [n]$ and links $j \in [m]$, $p_i^j > 0$. A user $i \in [n]$ is *solo* in the pure assignment $\mathbf{L}$ if no other user is assigned to link ℓ_i; in such case, say that link ℓ_i is *solo* as well.

Fix now a mixed assignment $\mathbf{P}$. The *load* $\delta^\ell(\mathbf{P})$ on link $\ell \in [m]$, induced by $\mathbf{P}$ is the total traffic assigned to the link according to $\mathbf{P}$; so, $\delta^\ell(\mathbf{P})$ is a random variable. For each link $\ell \in [m]$, denote $\theta^\ell(\mathbf{P})$ the *expected load* on link $\ell \in [m]$; thus, $\theta^\ell(\mathbf{P}) = \mathbb{E}_\mathbf{P}(\delta^\ell(\mathbf{P})) = \sum_{i\in[n]} p_i^\ell w_i$. The *latency* $\Delta^\ell(\mathbf{P})$ on link $\ell \in [m]$, induced by $\mathbf{P}$ is the latency due to the load assigned to the link according to $\mathbf{P}$; so, $\Delta^\ell(\mathbf{P})$ is a random variable and $\Delta^\ell(\mathbf{P}) = \frac{\delta^\ell(\mathbf{P})}{c^\ell}$.

2.3. Costs

2.3.1. Individual cost and expected individual cost

For a pure assignment $\mathbf{L}$, the *Individual Cost* for user $i \in [n]$, denoted $\lambda_i(\mathbf{L})$, is $\lambda_i(\mathbf{L}) = \Delta^{\ell_i}(\mathbf{L})$; so, the Individual Cost for a user is the latency of the link it chooses. For a mixed assignment $\mathbf{P}$, the *Expected Individual Cost* for user $i \in [n]$, denoted $\lambda_i(\mathbf{P})$, is the expectation according to $\mathbf{P}$ of the Individual Cost for the user in any pure assignment $\mathbf{L_P}$ consistent with $\mathbf{P}$; so, $\lambda_i(\mathbf{P}) = \mathbb{E}_{\mathbf{P}}\left(\lambda_i(\mathbf{L_P})\right)$.

2.3.2. Quadratic social cost

Associated with an instance $\langle \mathbf{w}, \mathbf{c}\rangle$, and a mixed assignment $\mathbf{P}$ is the *Quadratic Social Cost*, denoted as $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$, which is the expectation according to $\mathbf{P}$, of the sum of the link latencies due to the squares of the incurred loads, in a pure assignment $\mathbf{L}$, consistent with $\mathbf{P}$; so,

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) = \mathbb{E}_{\mathbf{P}}\left(\sum_{\ell\in[m]} \frac{(\delta^{\ell}(\mathbf{L_P}))^2}{c^{\ell}}\right).$$

For a pure assignment $\mathbf{L_P} = \langle \ell_1, \ldots, \ell_n\rangle$, changing the summation order, and using the definition of Individual Cost, yields that $\sum_{\ell\in[m]} \frac{(\delta^{\ell}(\mathbf{L_P}))^2}{c^{\ell}} = \sum_{i\in[n]} w_i \cdot \frac{\delta^{\ell_i}(\mathbf{L_P})}{c^{\ell_i}} = \sum_{i\in[n]} w_i\lambda_i(\mathbf{L_P})$. So, linearity of expectation and the definition of Expected Individual Cost imply that

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) = \sum_{i\in[n]} w_i\, \lambda_i(\mathbf{P}).$$

Thus, the Quadratic Social Cost is a weighted sum of Expected Individual Costs, where the weight is the user traffic. There is a counterpart of Quadratic Social Cost in the KP model. Associated with an instance $\langle \mathbf{w}, \mathbf{c}\rangle$, and a mixed assignment $\mathbf{P}$ is the *Maximum Social Cost* [26], denoted $\mathsf{MSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$, which is the expectation according to $\mathbf{P}$, of the maximum of the incurred link latencies in a pure assignment $\mathbf{L_P}$; so,

$$\mathsf{MSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) = \mathbb{E}_{\mathbf{P}}\left(\max_{\ell\in[m]} \Delta^{\ell}(\mathbf{L_P})\right).$$

2.3.3. Quadratic optimum

Associated with an instance $\langle \mathbf{w}, \mathbf{c}\rangle$ is the *Quadratic Optimum,* denoted $\mathsf{QOPT}(\mathbf{w}, \mathbf{c})$, which is the least possible, over all pure assignments, sum of link latencies incurred by the squares of the total traffic on the links; so,

$$\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) = \min_{\mathbf{L}\in[m]^n} \sum_{\ell\in[m]} \frac{(\delta^{\ell}(\mathbf{L}))^2}{c^{\ell}}.$$

Note that $\mathsf{QOPT}(\mathbf{w}, \mathbf{c})$ refers to the *optimal* pure assignment. Formally, a pure assignment $\mathbf{L}$ is *optimal* for the instance $\langle \mathbf{w}, \mathbf{c}\rangle$, if $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{L}) = \mathsf{QOPT}(\mathrm{w}, \mathbf{c})$. Clearly, for the case of identical users and identical links, $\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) \geq \max\{n, \frac{n^2}{m}\} = n \max\{1, \frac{n}{m}\}$. There is a counterpart of Quadratic Optimum in the KP model. This is the *Maximum Optimum* [26], denoted $\mathsf{MOPT}(\mathbf{w}, \mathbf{c})$, and defined as the least possible, over all pure assignments, maximum incurred link latency; so,

$$\mathsf{MOPT}(\mathbf{w}, \mathbf{c}) = \min_{\mathbf{L}\in[m]^n} \max_{\ell\in[m]} \Delta^{\ell}(\mathbf{L}).$$

2.4. Nash equilibria

Given an instance $\langle \mathbf{w}, \mathbf{c}\rangle$, the mixed assignment $\mathbf{P}$, is a *Nash equilibrium* [26, Section 2] if for each user $i \in [n]$, it minimizes the Expected Individual Cost $\lambda_i(\mathbf{P})$, over all mixed assignments $\mathbf{Q}$ that differ from $\mathbf{P}$, only with respect to the mixed strategy of user i; that is, for all such mixed assignments $\mathbf{Q}$, $\lambda_i(\mathbf{P}) \leq \lambda_i(\mathbf{Q})$. Thus, in a Nash equilibrium, there is no incentive for a user to unilaterally deviate from its mixed strategy.

Denote a fully mixed Nash equilibrium as $\mathbf{F}$. In the case of identical links, a fully mixed Nash equilibrium exists always and uniquely [29, Lemma 15]; there, all links are equiprobable (each chosen with probability $\frac{1}{m}$) for each user. The fully mixed Nash equilibrium does not necessarily exist in the case of identical users [29, Lemma 22]; when it exists, it is unique, but for each user, each link $\ell \in [m]$ is now chosen with probability $\frac{m+n-1}{(n-1)C}\, c^{\ell} - \frac{1}{n-1}$.

2.5. Quadratic coordination ratio

The *Quadratic Coordination Ratio,* denoted QCR, is the worst-case ratio $\frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})}$, over all instances $\langle \mathbf{w}, \mathbf{c}\rangle$, and associated Nash equilibria $\mathbf{P}$. This is similar to *Maximum Coordination Ratio,* denoted MCR, which was defined in [26] as the worst-case ratio $\frac{\mathsf{MSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{MOPT}(\mathbf{w},\mathbf{c})}$, over all instances $\langle \mathbf{w}, \mathbf{c}\rangle$ and associated Nash equilibria $\mathbf{P}$.

3. Preliminaries

We present some properties of Nash equilibria, and optimal assignments in the new model. We first prove that the Expected Individual Cost takes a special form for any (mixed) assignment.

Lemma 3.1. *Fix an instance $\langle \mathbf{w}, \mathbf{c}\rangle$, and a mixed assignment $\mathbf{P}$. Then, for each user $i \in [n]$,*

$$\lambda_i(\mathbf{P}) = \sum_{\ell \in [m]} p_i^\ell \, \frac{w_i + \sum\limits_{k \in [n]: k \neq i} p_k^\ell w_k}{c^\ell}.$$

Proof. By the definition of Individual Cost and Expected Individual Cost, we obtain that

$$\begin{aligned}
\lambda_i(\mathbf{P}) &= \sum_{\mathbf{L_P} \in [m]^{[n]}} \mathsf{P}_\mathbf{P}(\mathbf{L_P}) \cdot \lambda_i(\mathbf{L_P}) \\
&= \sum_{j \in [m]} \left(\sum_{\mathbf{L_P} \in [m]^{[n]}:\, l_i = j} \mathsf{P}_\mathbf{P}(\mathbf{L_P}) \cdot \lambda_i(\mathbf{L_P}) \right) \\
&= \sum_{j \in [m]} \left(\sum_{\mathbf{L_P} \in [m]^{[n]\setminus\{i\}}:\, l_i = j} p_i^j \, \mathsf{P}_\mathbf{P}(\mathbf{L_P}) \cdot \lambda_i(\mathbf{L_P}) \right) \\
&= \sum_{j \in [m]} p_i^j \left(\sum_{\mathbf{L_P} \in [m]^{[n]\setminus\{i\}}:\, l_i = j} \mathsf{P}_\mathbf{P}(\mathbf{L_P}) \cdot \lambda_i(\mathbf{L_P}) \right) \\
&= \sum_{j \in [m]} p_i^j \left(\sum_{\mathbf{L_P} \in [m]^{[n]\setminus\{i\}}} \mathsf{P}_\mathbf{P}(\mathbf{L_P}) \cdot \lambda_i(\mathbf{L_P}) \,\Big|_{l_i = j} \right) \\
&= \sum_{j \in [m]} p_i^j \left(\sum_{\mathbf{L_P} \in [m]^{[n]\setminus\{i\}}} \mathsf{P}_\mathbf{P}(\mathbf{L_P}) \cdot \left(\frac{\sum\limits_{k \in [n]: l_k = j} w_k}{c^j} \right) \right) \\
&= \sum_{j \in [m]} p_i^j \left(\sum_{\mathbf{L_P} \in [m]^{[n]\setminus\{i\}}} \mathsf{P}_\mathbf{P}(\mathbf{L_P}) \cdot \left(\frac{w_i + \sum\limits_{k \in [n]\setminus\{i\}: l_k = j} w_k}{c^j} \right) \right) \\
&= \sum_{j \in [m]} p_i^j \left(\frac{w_i}{c^j} + \sum_{\mathbf{L_P} \in [m]^{[n]\setminus\{i\}}} \mathsf{P}_\mathbf{P}(\mathbf{L_P}) \cdot \left(\frac{\sum\limits_{k \in [n]\setminus\{i\}: l_k = j} w_k}{c^j} \right) \right).
\end{aligned}$$

Reorder users so that $i = n$. We prove by backward induction on r, $0 \leq r \leq n-1$, that

$$\lambda_i(\mathbf{P}) = \sum_{j \in [m]} p_i^j \left(\frac{w_i}{c^j} + \sum_{k \in [n-1]\setminus[r]} \frac{p_k^j w_k}{c^j} + \sum_{\mathbf{L_P} = \langle l_1, \ldots, l_r\rangle \in [m]^{[r]}} \mathsf{P}_\mathbf{P}(\mathbf{L_P}) \cdot \left(\frac{\sum\limits_{k \in [r]: \ell_k = j} w_k}{c^j} \right) \right).$$

For the basis case, where $r = n-1$, $\sum_{k\in[n-1]\setminus[r]} \frac{p_k^j w_k}{c^j} = 0$; so, the claim reduces to

$$\lambda_i(\mathbf{P}) = \sum_{j\in[m]} p_i^j \left(\frac{w_i}{c^j} + \sum_{\mathbf{L_P}\in[m]^{[n-1]}} \mathsf{P}_{\mathbf{P}}(\mathbf{L_P}) \cdot \left(\frac{\sum_{k\in[n-1]:\ell_k=j} w_k}{c^j} \right) \right),$$

which has been shown. Assume the claim for r, $0 < r \leq n-1$. and prove it for $r-1$. So,

$$\begin{aligned}
&\sum_{\mathbf{L_P}\in[m]^{[r]}} \mathsf{P}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k\in[r+1]:\ell_k=j} w_k}{c^j} \right) \\
&= \sum_{\mathbf{L_P}\in[m]^{[r]}:\ell_r=j} \mathsf{P}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k\in[r]:\ell_k=j} w_k}{c^j} \right) + \sum_{\mathbf{L_P}\in[m]^{[r]}:\ell_r\neq j} \mathsf{P}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k\in[r]:\ell_k=j} w_k}{c^j} \right) \\
&= \sum_{\mathbf{L_P}\in[m]^{[r]}:\ell_r=j} \mathsf{P}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{w_r + \sum_{k\in[r-1]:\ell_k=j} w_k}{c^j} \right) + \sum_{\mathbf{L_P}\in[m]^{[r]}:\ell_r\neq j} \mathsf{P}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k\in[r-1]:\ell_k=j} w_k}{c^j} \right) \\
&= \sum_{\mathbf{L_P}\in[m]^{[r-1]}} \mathsf{P}_{\mathbf{P}}(\mathbf{L_P})\, p_r^j \left(\frac{w_r + \sum_{k\in[r-1]:\ell_k=j} w_k}{c^j} \right) + \sum_{\mathbf{L_P}\in[m]^{[r-1]}} \mathsf{P}_{\mathbf{P}}(\mathbf{L_P})\, \overline{p_r^j} \left(\frac{\sum_{k\in[r-1]:\ell_k=j} w_k}{c^j} \right) \\
&= \sum_{\mathbf{L_P}\in[m]^{[r-1]}} \mathsf{P}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{p_r^j w_r}{c^j} + \frac{\sum_{k\in[r-1]:\ell_k=j} w_k}{c^j} \right) \\
&= \frac{p_r^j w_r}{c^j} + \sum_{\mathbf{L_P}\in[m]^{[r-1]}} \mathsf{P}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k\in[r-1]:\ell_k=j} w_k}{c^j} \right).
\end{aligned}$$

Hence, by induction hypothesis, it follows that

$$\begin{aligned}
\lambda_i(\mathbf{P}) &= \sum_{j\in[m]} p_i^j \left(\frac{w_i}{c^j} + \sum_{k\in[n-1]\setminus[r]} \frac{p_k^j w_k}{c^j} + \frac{p_r^j w_r}{c^j} + \sum_{\mathbf{L_P}\in[m]^{[r-1]}} \mathsf{P}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k\in[r-1]:\ell_k=j} w_k}{c^j} \right) \right) \\
&= \sum_{j\in[m]} p_i^j \left(\frac{w_i}{c^j} + \sum_{k\in[n-1]\setminus[r-1]} \frac{p_k^j w_k}{c^j} + \sum_{\mathbf{L_P}\in[m]^{[r-1]}} \mathsf{P}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k\in[r-1]:\ell_k=j} w_k}{c^j} \right) \right),
\end{aligned}$$

as needed to prove the claim for r. Setting now $r = 0$ yields that

$$\lambda_i(\mathbf{P}) = \sum_{j\in[m]} p_i^j \frac{w_i + \sum_{k\in[n]:k\neq i} p_k^j w_k}{c^j}.$$

as needed. ∎

For each user $i \in [n]$ and link $j \in [m]$, define the *Conditional Expected Individual Cost* [26] of user i on link j as

$$\lambda_i^j(\mathbf{P}) = \frac{w_i + \sum_{k\in[n]:k\neq i} p_k^j w_k}{c^j}.$$

By Lemma 3.1, $\lambda_i(\mathbf{P}) = \sum_{j\in[m]} p_i^j \lambda_i^j(\mathbf{P})$; thus, the definition of Nash equilibrium implies that for each link $j \in [m]$, such that $p_i^j > 0$, for each link $j' \in [m]$, either $\lambda_i^j = \lambda_i^{j'}$, if $p_i^{j'} > 0$ or $\lambda_i^j \le \lambda_i^{j'}$ if $p_i^{j'} = 0$. (This property was used for defining Nash equilibria in the original work of Koutsoupias and Papadimitriou [26].)

We now consider the case of identical users, where we characterize optimal assignments.

Proposition 3.2 (*Global Optimality = Local Optimality*). *Consider a pure assignment* $\mathbf{Q}$*, for an instance* $\langle \mathbf{w}, \mathbf{c}\rangle$ *in the case of identical users. Then,* $\mathbf{Q}$ *is optimal, if and only if for every pair of distinct links* $j, j' \in [m]$*,*

$$\frac{2\delta^j(\mathbf{Q})+1}{c^j} \ge \frac{2\delta^{j'}(\mathbf{Q})-1}{c^{j'}}.$$

Proof. Assume first that $\mathbf{Q}$ is optimal, so that $\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{Q}) = \mathsf{QOPT}(\mathbf{w},\mathbf{c})$. Consider any pair of distinct links $j, j' \in [m]$, and use $\mathbf{Q}$ to construct a new pure assignment $\mathbf{Q}'$, by switching any single user from link j' to link j. Then, clearly,

$$\begin{aligned}\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{Q}') - \mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{Q}) &= \frac{(\delta^j(\mathbf{Q})+1)^2}{c^j} + \frac{(\delta^{j'}(\mathbf{Q})-1)^2}{c^{j'}} - \frac{(\delta^j(\mathbf{Q}))^2}{c^j} - \frac{(\delta^{j'}(\mathbf{Q}))^2}{c^{j'}}\\ &= \frac{2\delta^j(\mathbf{Q})+1}{c^j} - \frac{2\delta^{j'}(\mathbf{Q})-1}{c^{j'}}.\end{aligned}$$

Since $\mathbf{Q}$ is optimal, $\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{Q}') \ge \mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{Q})$, so that

$$\frac{2\delta^j(\mathbf{Q})+1}{c^j} \ge \frac{2\delta^{j'}(\mathbf{Q})-1}{c^{j'}},$$

as needed.

Assume now that for every pair of links $j, j' \in [m]$,

$$\frac{2\delta^j(\mathbf{Q})+1}{c^j} \ge \frac{2\delta^{j'}(\mathbf{Q})-1}{c^{j'}}.$$

We will prove that $\mathbf{Q}$, is an optimal assignment for the instance $\langle \mathbf{w}, \mathbf{c}\rangle$. To do so, we will consider an optimal assignment $\mathbf{R}$ for the same instance, and we will prove that $\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{Q}) = \mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{R})$. If $\delta^\ell(\mathbf{Q}) = \delta^\ell(\mathbf{R})$ for all links $\ell \in [m]$, then $\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{Q}) = \mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{R})$, and the optimality of $\mathbf{Q}$ follows. So assume otherwise. Since $\sum_{\ell\in[m]}\delta^\ell(\mathbf{Q}) = \sum_{\ell\in[m]}\delta^\ell(\mathbf{R}) = n$, it follows that there exist distinct links $j, j' \in [m]$ such that $\delta^j(\mathbf{R}) > \delta^j(\mathbf{Q})$ and $\delta^{j'}(\mathbf{Q}) > \delta^{j'}(\mathbf{R})$; that is, link j is assigned more traffic in $\mathbf{R}$ than in $\mathbf{Q}$, while link j' is assigned more traffic in $\mathbf{Q}$ than in $\mathbf{R}$. Since loads are integral in the case of identical users, it follows that $\delta^j(\mathbf{R}) - \delta^j(\mathbf{Q}) \ge 1$ and $\delta^{j'}(\mathbf{Q}) - \delta^{j'}(\mathbf{R}) \ge 1$.

Since $\mathbf{R}$ is optimal, it holds for links $j', j \in [m]$ that

$$\frac{2\delta^{j'}(\mathbf{R})+1}{c^{j'}} \ge \frac{2\delta^j(\mathbf{R})-1}{c^j}.$$

On the other hand, our assumption on $\mathbf{Q}$, implies that

$$\frac{2\delta^j(\mathbf{Q})+1}{c^j} \ge \frac{2\delta^{j'}(\mathbf{Q})-1}{c^{j'}}.$$

It follows that

$$\begin{aligned}\frac{2\delta^{j'}(\mathbf{R})+1}{c^{j'}} &\ge \frac{2\delta^j(\mathbf{R})-1}{c^j}\\ &\ge \frac{2\delta^j(\mathbf{Q})+1}{c^j} \text{ (since } \delta^j(\mathbf{R}) \ge \delta^j(\mathbf{Q})+1)\\ &\ge \frac{2\delta^{j'}(\mathbf{Q})-1}{c^{j'}}\\ &\ge \frac{2\delta^{j'}(\mathbf{R})+1}{c^{j'}} \text{ (since } \delta^{j'}(\mathbf{Q}) \ge \delta^{j'}(\mathbf{R})+1).\end{aligned}$$

It follows that all terms in this sequence of inequalities are equal. In particular,

$$\frac{2\delta^j(\mathbf{R})-1}{c^j} = \frac{2\delta^j(\mathbf{Q})+1}{c^j},$$

or $\delta^j(\mathbf{R}) = \delta^j(\mathbf{Q})+1$; also,

$$\frac{2\delta^{j'}(\mathbf{Q})-1}{c^{j'}} = \frac{2\delta^{j'}(\mathbf{R})+1}{c^{j'}},$$

or $\delta^{j'}(\mathbf{Q}) = \delta^{j'}(\mathbf{R}) + 1$. Thus, the difference of the loads on each of the links $j, j' \in [m]$ is exactly 1 (in absolute value) in both $\mathbf{Q}$ or $\mathbf{R}$. Since $\sum_{l\in[m]} \delta^l(\mathbf{Q}) = \sum_{l\in[m]} \delta^l(\mathbf{R})$, this implies that the number of links $j \in [m]$, such that $\delta^j(\mathbf{R}) > \delta^j(\mathbf{Q})$ and the number of links $j' \in [m]$ such that $\delta^{j'}(\mathbf{Q}) > \delta^{j'}(\mathbf{R})$ are equal. It also follows from the preceding sequence of inequalities that

$$\frac{2\delta^{j'}(\mathbf{R}) + 1}{c^{j'}} = \frac{2\delta^j(\mathbf{Q}) + 1}{c^j},$$

which implies that

$$\frac{(\delta^j(\mathbf{Q}))^2}{c^j} + \frac{(\delta^{j'}(\mathbf{R}) + 1)^2}{c^{j'}} = \frac{(\delta^{j'}(\mathbf{R}))^2}{c^{j'}} + \frac{(\delta^j(\mathbf{Q}) + 1)^2}{c^j}.$$

Since $\delta^j(\mathbf{R}) = \delta^j(\mathbf{Q}) + 1$ and $\delta^{j'}(\mathbf{Q}) = \delta^{j'}(\mathbf{R}) + 1$, it follows that

$$\frac{(\delta^j(\mathbf{Q}))^2}{c^j} + \frac{(\delta^{j'}(\mathbf{Q}))^2}{c^{j'}} = \frac{(\delta^{j'}(\mathbf{R}))^2}{c^{j'}} + \frac{(\delta^j(\mathbf{R}))^2}{c^j}.$$

Thus, links $j, j' \in [m]$ have the same contribution to the Quadratic Social Cost in both $\mathbf{Q}$ and $\mathbf{R}$. Since the number of links $j \in [m]$, such that $\delta^j(\mathbf{R}) > \delta^j(\mathbf{Q})$ and the number of links $j' \in [m]$, such that $\delta^{j'}(\mathbf{Q}) > \delta^{j'}(\mathbf{R})$ are equal, the total contribution to the Quadratic Social Cost of links $j \in [m]$ such that $\delta^j(\mathbf{R}) > \delta^j(\mathbf{Q})$, is equal to the total contribution to the Quadratic Social Cost of links $j' \in [m]$ such that $\delta^{j'}(\mathbf{Q}) > \delta^{j'}(\mathbf{R})$. It follows that $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{R})$, so that $\mathbf{Q}$ is optimal, as needed. ■

Proposition 3.2 establishes that the (global) optimality of a pure assignment is equivalent to a collection of $\Theta(m^2)$ *local* conditions – one for each pair of distinct links, and each checkable in time $\Theta(1)$, given the link loads. Hence, an immediate consequence of Proposition 3.2 follows.

Corollary 3.3. *Consider the case of identical users. Then, the optimality of a pure assignment can be decided in time $O(n + m^2)$.*

We continue to prove a relation between pure Nash equilibria, and optimal (pure) assignments, in the case of identical users.

Lemma 3.4 (*Optimal Assignment versus Nash Equilibrium*). *Consider an optimal assignment $\mathbf{Q}$, and a pure Nash equilibrium $\mathbf{P}$ for an instance $\langle \mathbf{w}, \mathbf{c}\rangle$ in the case of identical users. Then, for each link $j \in [m]$, $\delta^j(\mathbf{Q}) - \delta^j(\mathbf{P}) \le 1$.*

Proof. Assume, by way of contradiction, that there is some link $j \in [m]$ such that $\delta^j(\mathbf{Q}) - \delta^j(\mathbf{P}) \ge 2$. Since $\sum_{l\in[m]} \delta^l(\mathbf{Q}) = \sum_{l\in[m]} \delta^l(\mathbf{P})$, there also exists some link $j' \in [m]$, such that $\delta^{j'}(\mathbf{Q}) - \delta^{j'}(\mathbf{P}) \le -1$. Since $\mathbf{Q}$ is optimal, Proposition 3.2 implies that

$$\begin{aligned}
\frac{2\delta^j(\mathbf{Q}) - 1}{c^j} &\le \frac{2\delta^{j'}(\mathbf{Q}) + 1}{c^{j'}} \\
&\le \frac{2(\delta^{j'}(\mathbf{P}) - 1) + 1}{c^{j'}} \text{ (since } \delta^{j'}(\mathbf{Q}) \le \delta^{j'}(\mathbf{P}) - 1) \\
&= \frac{2\delta^{j'}(\mathbf{P}) - 1}{c^{j'}} \\
&\le 2\,\frac{\delta^j(\mathbf{P}) + 1}{c^j} - \frac{1}{c^{j'}} \text{ (since } \mathbf{P} \text{ is a Nash equilibrium)} \\
&\le 2\,\frac{\delta^j(\mathbf{Q}) - 2 + 1}{c^j} - \frac{1}{c^{j'}} \text{ (since } \delta^j(\mathbf{P}) \le \delta^j(\mathbf{Q}) - 2) \\
&= \frac{2\delta^j(\mathbf{Q}) - 1}{c^j} - \frac{1}{c^j} - \frac{1}{c^{j'}}.
\end{aligned}$$

It follows that $\frac{1}{c^j} + \frac{1}{c^{j'}} \le 0$. A contradiction. ■

We now consider an instance $\langle \mathbf{w}, \mathbf{c}\rangle$, for the case of identical links. Call user $i \in [n]$ *bursty* if $w_i > \frac{W}{m}$. Intuitively, the traffic of a bursty user exceeds the fair share of traffic for a link. We prove a simple property of bursty users:

Lemma 3.5 (*Bursty Users are Solo*). *Consider any instance in the case of identical links. Then, a bursty user is solo in either an optimal assignment, or a pure Nash equilibrium.*

Proof. Fix an instance $\langle \mathbf{w}, \mathbf{c}\rangle$ and a bursty user $i \in [n]$.

Consider first an optimal assignment $\mathbf{Q} = \langle q_1, \ldots, q_n \rangle$. Note that $\delta^{q_i}(\mathbf{Q}) \geq w_i$. Since i is a bursty user, it follows that $\delta^{q_i}(\mathbf{Q}) > \frac{W}{m}$; Since $\sum_{l \in [m]} \delta^l(\mathbf{Q}) = W$, there is some other link $j \in [m]$ with $j \neq q_i$, such that $\delta^j(\mathbf{Q}) < \frac{W}{m}$. Assume, by way of contradiction, that some user $k \neq i$ is assigned to link q_i. Modify $\mathbf{Q}$ to obtain $\mathbf{Q}'$ by switching user k to link j. Then,

$$\begin{aligned}\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}') - \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) &= \left(\delta^{q_i}(\mathbf{Q}')\right)^2 + \left(\delta^j(\mathbf{Q}')\right)^2 - \left(\delta^{q_i}(\mathbf{Q})\right)^2 - \left(\delta^j(\mathbf{Q})\right)^2 \\ &= w_i^2 + \left(\delta^j(\mathbf{Q}) + w_k\right)^2 - (w_i + w_k)^2 - \left(\delta^j(\mathbf{Q})\right)^2 \\ &= 2w_k\left(\delta^j(\mathbf{Q}) - w_i\right).\end{aligned}$$

Since $\delta^j(\mathbf{Q}) < \frac{W}{m}$ and $w_i > \frac{W}{m}$, it follows that $\delta^j(\mathbf{Q}) - w_i < 0$, so that $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}') < \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q})$. Since $\mathbf{Q}$ is optimal, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}') \geq \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q})$. A contradiction.

Consider now a pure Nash equilibrium $\mathbf{P} = \langle \ell_1, \ldots, \ell_n \rangle$. Note that $\delta^{\ell_i}(\mathbf{P}) \geq w_i > \frac{W}{m}$. Since $\sum_{l \in [m]} \delta^l(\mathbf{P}) = W$, there is some other link $j \in [m]$, with $j \neq \ell_i$, such that $\delta^j(\mathbf{P}) < \frac{W}{m}$. Assume, by way of contradiction, that some user $k \neq i$ is assigned to link ℓ_i. Then, $\lambda_k(\mathbf{P}) \geq w_i + w_k > \frac{W}{m} + w_k$. However, if user k switches to link j, its Individual Cost becomes $\delta^j(\mathbf{P}) + w_k < \frac{W}{m} + w_k$. Since $\mathbf{P}$ is a Nash equilibrium, $\delta^j(\mathbf{P}) + w_k \geq \lambda_k(\mathbf{P}) > \frac{W}{m} + w_k$. A contradiction. ■

Say that an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ is *bursty*, if some user $i \in [n]$ is bursty; otherwise, the instance is *non-bursty*. We start with a very simple observation about pure Nash equilibria associated with non-bursty instances, in the case of identical links.

Lemma 3.6 (*Non-Zero Loads in Nash Equilibrium for NonBursty Instance*). *Consider a pure Nash equilibrium* $\mathbf{P}$, *for a non-bursty instance* $\langle \mathbf{w}, \mathbf{c} \rangle$ *in the case of identical links. Then, for each link* $j \in [m]$, $\delta^j(\mathbf{P}) > 0$.

Proof. Assume, by way of contradiction, that there is a link $j \in [m]$ with $\delta^j(\mathbf{P}) = 0$. This implies that there is also some link $j' \in [m]$ with $\delta^{j'}(\mathbf{P}) \geq \frac{W}{m-1} > \frac{W}{m}$. Consider any user $i \in [n]$, assigned to link j'. Then, $\lambda_i(\mathbf{P}) = \delta^{j'}(\mathbf{P}) > \frac{W}{m}$, while $\lambda_i^j(\mathbf{P}) = w_i \leq \frac{W}{m}$ (since no user is bursty). This contradicts the assumption that $\mathbf{P}$ is a Nash equilibrium. ■

We continue with another preliminary property of pure Nash equilibria, associated with non-bursty instances, in the case of identical links; roughly speaking, we prove that link loads are balanced in a pure Nash equilibrium, for a non-bursty instance.

Lemma 3.7 (*Balanced Loads in Nash Equilibrium for NonBursty Instance*). *Consider a pure Nash equilibrium* $\mathbf{P}$ *for a non-bursty instance* $\langle \mathbf{w}, \mathbf{c} \rangle$ *in the case of identical links. Then, for each link* $j \in [m]$, $\delta^j(\mathbf{P}) \leq 2\min_{l \in [m]} \delta^l(\mathbf{P})$.

Proof. Assume, by way of contradiction, that there is some link $j \in [m]$ such that $\delta^j(\mathbf{P}) > 2\min_{l \in [m]} \delta^l(\mathbf{P})$. Choose link j so that it maximizes $\delta^l(\mathbf{P})$, over all links $l \in [m]$.

Clearly, $\delta^j(\mathbf{P}) \geq \frac{W}{m}$. Moreover, if $\delta^j(\mathbf{P}) = \frac{W}{m}$, then $\delta^l(\mathbf{P}) = \frac{W}{m}$ for all links $l \in [m]$ and the claim follows. So, assume that $\delta^j(\mathbf{P}) > \frac{W}{m}$. We proceed by case analysis.

- Assume first that there is a single solo user $i \in [n]$ on link j, so that $\delta^j(\mathbf{P}) = w_i$. Since link j maximizes latency, $w_i \geq \delta^l(\mathbf{P})$ for all links $l \in [m]$. Moreover, our assumption implies that $w_i > \min_{l \in [m]} \delta^l(\mathbf{P})$. It follows that $mw_i > \sum_{l \in [m]} \delta^l(\mathbf{P}) = W$, or $w_i > \frac{W}{m}$. Since $\langle \mathbf{w}, \mathbf{c} \rangle$ is a non-bursty instance, $w_i \leq \frac{W}{m}$. A contradiction.
- Assume now that at least two users are assigned to link j. Consider the smallest traffic w_i of some user $i \in [n]$ among all users assigned to link j. Then, clearly, $w_i \leq \frac{\delta^j(\mathbf{P})}{2}$. Hence, $\delta^j(\mathbf{P}) - w_i \geq \frac{\delta^j(\mathbf{P})}{2} > \min_{l \in [m]} \delta^l(\mathbf{P})$ (by assumption). So, $\min_{l \in [m]} \delta^l(\mathbf{P}) + w_i < \delta^j(\mathbf{P})$. Since $\mathbf{P}$ is a Nash equilibrium, $\min_{l \in [m]} \delta^l(\mathbf{P}) + w_i \geq \delta^j(\mathbf{P})$. A contradiction.

Since we obtained a contradiction in all possible cases, the proof is now complete. ■

4. Quadratic social cost

Some combinatorial expressions for the Quadratic Social Cost, and corresponding efficient algorithms, are presented in Section 4.1. Section 4.2, determines the worst-case Nash equilibrium with respect to the Quadratic Social Cost.

4.1. Combinatorial expressions

Recall that the Quadratic Social Cost can be expressed as the weighted sum of Expected Individual Costs. Hence, Lemma 3.1 implies that the Quadratic Social Cost of any mixed assignment $\mathbf{P}$, for the instance $\langle \mathbf{w}, \mathbf{c} \rangle$ can be written as

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) = \sum_{i \in [n]} w_i \left(\sum_{j \in [m]} p_i^j \frac{\overline{p_i^j}\, w_i + \theta^j(\mathbf{P})}{c^j} \right),$$

which immediately implies:

Proposition 4.1 (*Quadratic Social Cost of Mixed Assignment*). *The Quadratic Social Cost of any mixed assignment can be computed in time $O(nm)$.*

Since a probability matrix $\mathbf{P}$ has size $O(nm)$, Proposition 4.1 implies that the Quadratic Social Cost of any mixed assignment can be computed in linear time. We remark that this achieved efficiency has not needed the assumption that the mixed assignment is a Nash equilibrium. We next establish that the Quadratic Social Cost takes a particularly nice form, for the case of the fully mixed Nash equilibrium. We prove:

Theorem 4.2 (*Quadratic Social Cost of Fully Mixed Nash Equilibrium*). *Consider an instance $\langle \mathbf{w}, \mathbf{c}\rangle$ in the case of identical links. Then,*

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) = W_1 + \frac{2}{m} W_2 = \frac{W^2}{m} + \left(1 - \frac{1}{m}\right) W_2.$$

Proof. Since $W^2 = W_1 + 2W_2$, it suffices to prove the first equality. Recall that $f_i^j = \frac{1}{m}$ for all users $i \in [n]$ and links $j \in [m]$ in the fully mixed Nash equilibrium $\mathbf{F}$. Thus, by Lemma 3.1, $\lambda_i(\mathbf{F}) = \sum_{\ell \in [m]} \frac{1}{m} \left(w_i + \sum_{k \in [n]: k \neq i} \frac{1}{m} w_k \right) = w_i + \sum_{k \in [n]: k \neq i} \frac{1}{m} w_k$. Since $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) = \sum_{i \in [n]} w_i \lambda_i(\mathbf{F})$, the claim follows from the definitions of W_1 and W_2. ■

Theorem 4.2 immediately implies:

Corollary 4.3. *Consider the case of identical links. Then, the Quadratic Social Cost of the fully mixed Nash Equilibrium can be computed in time $O(n^2)$.*

We continue to prove:

Theorem 4.4. *Consider an instance $\langle \mathbf{w}, \mathbf{c}\rangle$, in the case of identical users, for which the fully mixed Nash equilibrium $\mathbf{F}$ exists. Then,*

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) = \frac{n\,(n + m - 1)}{C}.$$

Proof. Recall that in the case of identical users, for all users $i \in [n]$ and each link $j \in [m]$, $f_i^j = \frac{m+n-1}{(n-1)C} c^j - \frac{1}{n-1}$. Hence, by Lemma 3.1, $\lambda_i(\mathbf{F}) = \sum_{j \in [m]} \left(f_i^j \, \frac{1 + \sum_{k \in [n]: k \neq i} f_k^j}{c^j} \right) = \sum_{j \in [m]} \left(f_i^j \, \frac{1 + (n-1) f_i^j}{c^j} \right) = \sum_{j \in [m]} \left(f_i^j \, \frac{m+n-1}{C} \right) = \frac{m+n-1}{C}$. Since $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) = \sum_{i \in [n]} \lambda_i(\mathbf{F})$, the claim follows. ■

Theorem 4.4 immediately implies:

Corollary 4.5. *Consider the case of identical users. Then, the Quadratic Social Cost of the fully mixed Nash Equilibrium can be computed in time $O(m)$.*

For the special case of identical users and identical links, Theorem 4.4 immediately implies:

Corollary 4.6. *Consider the case of identical users and identical links. Then,*

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) = \frac{n\,(n + m - 1)}{m}.$$

4.2. The worst-case Nash equilibrium

We restrict our attention to the case of identical users, and identical links, for which the fully mixed Nash equilibrium always exists. We start by proving:

Lemma 4.7. *Consider the case of identical users, and identical links, and let $\mathbf{P}$ be any (mixed) Nash equilibrium. Then, for all users $i \in [n]$, $\lambda_i(\mathbf{P}) \leq \lambda_i(\mathbf{F})$.*

Proof. Assume, by way of contradiction, that $\lambda_i(\mathbf{P}) > \lambda_i(\mathbf{F})$, for some user $i \in [n]$.

For the Nash equilibrium $\mathbf{P}$, recall that all Conditional Expected Individual Costs λ_i^j for those links such that $p_i^j > 0$ are equal; hence, Lemma 3.1 implies that $\lambda_i(\mathbf{P}) = \lambda_i^j(\mathbf{P}) = 1 + \sum_{k \in [n]: k \neq i} p_k^j$. Moreover, all Conditional Expected Individual Costs for those links $j \in [m]$, such that $p_i^j = 0$ are no less than $\lambda_i(\mathbf{P})$. For the fully mixed Nash equilibrium $\mathbf{F}$, the same argument implies that $\lambda_i(\mathbf{F}) = \lambda_i^j(\mathbf{F})$ for any link $j \in [m]$.

By Lemma 3.1, $\lambda_i^j(\mathbf{P}) = 1 + \sum_{k \in [n]: k \neq i} p_k^j$ and $\lambda_i^j(\mathbf{F}) = 1 + \sum_{k \in [n]: k \neq i} f_k^j$. Since $\lambda_i(\mathbf{P}) > \lambda_i(\mathbf{F})$, $\sum_{k \in [n]: k \neq i} p_k^j > \sum_{k \in [n]: k \neq i} f_k^j$. But $\sum_{\ell \in [m]} \left(\sum_{k \in [n]: k \neq i} p_k^\ell \right) = \sum_{\ell \in [m]} \left(\sum_{k \in [n]: k \neq i} f_k^\ell \right) = n - 1$. It follows that there exists some link $j' \in [m]$ such that $\sum_{k \in [n]: k \neq i} p_k^{j'} < \sum_{k \in [n]: k \neq i} p_k^{j'}$. By definition of Conditional Expected Individual Cost, this implies that $\lambda_i^{j'}(\mathbf{P}) < \lambda_i^{j'}(\mathbf{F})$. However, $\lambda_i(\mathbf{P}) \leq \lambda_i^{j'}(\mathbf{P})$, while $\lambda_i(\mathbf{F}) = \lambda_i^{j'}(\mathbf{F})$. It follows that $\lambda_i(\mathbf{P}) < \lambda_i(\mathbf{F})$. A contradiction. ■

Since the Quadratic Social Cost is a weighted sum of Expected Individual Costs, Lemma 4.7 immediately implies:

Theorem 4.8. *Consider the case of identical users and identical links. Then, for any arbitrary Nash equilibrium* $\mathbf{P}$, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \leq \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F})$.

Theorem 4.8 proves that the fully mixed Nash equilibrium maximizes Quadratic Social Cost in the case of identical users and identical links. It is natural to ask if the same holds in all cases. In analogy to the yet unproven *Fully Mixed Nash Equilibrium Conjecture* [23], we conjecture:

Conjecture 4.1 (*Quadratic Fully Mixed Nash Equilibrium Conjecture*). *When it exists, the fully mixed Nash equilibrium maximizes Quadratic Social Cost.*

5. Quadratic coordination ratio

Some of our proofs in this section will make use of the following notation. Consider an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ with an associated pure assignment $\mathbf{P}$. Fix a set of links $\mathcal{L}$, inducing a set of users $\mathcal{U}$ that are assigned by the assignment $\mathbf{P}$ to links in $\mathcal{L}$. Then, $\mathbf{w} \setminus \mathcal{U}$ and $\mathbf{c} \setminus \mathcal{L}$, denote the vectors resulting from $\mathbf{w}$ and $\mathbf{c}$, respectively, by eliminating the entries corresponding to users in $\mathcal{U}$, and links in $\mathcal{L}$, respectively; $\mathbf{P} \setminus (\mathcal{U}, \mathcal{L})$ denotes the assignment induced by these eliminations. Pure and mixed Nash equilibria are considered in Sections 5.1 and 5.2.

5.1. Pure Nash equilibria

Identical users and identical links are considered in Sections 5.1.1 and 5.1.2, respectively.

5.1.1. Identical users

As our main result, we prove:

Theorem 5.1. *Consider the case of identical users, restricted to pure Nash equilibria. Then,* $\mathsf{QCR} = \frac{4}{3}$.

Proof. We first prove the upper bound. We start with an informal outline of our proof. We shall partition the set of links into a number of groups, so that in each group, the total loads on links of the group incurred by a Nash equilibrium, and an optimal assignment match each other. We then separately sum up the loads on links that are loaded more (resp., less) in the Nash equilibrium, than in the optimal assignment. We use simple properties of Nash equilibria to compare the corresponding partial sums of the Nash equilibria, and the optimal assignment. Adding together these two partial sums provides the required relation between the Quadratic Social Costs of the Nash equilibrium and the optimal assignment. We now continue with the details of the formal proof.

Consider any arbitrary instance $\langle \mathbf{w}, \mathbf{c} \rangle$, and an associated pure Nash equilibrium $\mathbf{P}$. Let $\mathbf{Q}$ be an optimal (pure) assignment for the instance $\langle \mathbf{w}, \mathbf{c} \rangle$; so, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = \mathsf{QOPT}(\mathbf{w}, \mathbf{c})$.

Call a link $j \in [m]$ *overloaded* if $\delta^j(\mathbf{P}) > \delta^j(\mathbf{Q})$, *underloaded* if $\delta^j(\mathbf{P}) < \delta^j(\mathbf{Q})$ and *indifferent* otherwise. Lemma 3.4 implies that $\delta^j(\mathbf{P}) = \delta^j(\mathbf{Q}) - 1$, is the only possibility for an underloaded link j. Note also that $\delta^j(\mathbf{Q}) \geq 1$, for each underloaded link j. Let $j_1, \ldots, j_k$ be the overloaded links. Partition the set of links $[m]$ into $k+1$ groups $\mathcal{I}, \mathcal{L}_1, \ldots, \mathcal{L}_k$ as follows:

- $\mathcal{I} = \{j \in [m] \mid \delta^j(\mathbf{P}) = \delta^j(\mathbf{Q})\}$ is the set of indifferent links. So, clearly, $\sum_{j \in \mathcal{I}} \delta^j(\mathbf{P}) = \sum_{j \in \mathcal{I}} \delta^j(\mathbf{Q})$.
- For each $l \in [k]$, the set $\mathcal{L}_l$ contains the overloaded link j_l and $\delta^{j_l}(\mathbf{P}) - \delta^{j_l}(\mathbf{Q})$ underloaded links. So, $|\mathcal{L}_l| = \delta^{j_l}(\mathbf{P}) - \delta^{j_l}(\mathbf{Q}) + 1$.

Such a partition is possible due to Lemma 3.4. Note that for each $l \in [k]$,

$$\begin{aligned}
\sum_{j \in \mathcal{L}_l} \delta^j(\mathbf{P}) &= \delta^{j_l}(\mathbf{P}) + \sum_{j \in \mathcal{L}_l \setminus \{j_l\}} \delta^j(\mathbf{P}) \\
&= \delta^{j_l}(\mathbf{P}) + \sum_{j \in \mathcal{L}_l \setminus \{j_l\}} (\delta^j(\mathbf{Q}) - 1) \\
&= \delta^{j_l}(\mathbf{P}) + \sum_{j \in \mathcal{L}_l \setminus \{j_l\}} \delta^j(\mathbf{Q}) - (|\mathcal{L}_l| - 1) \\
&= \delta^{j_l}(\mathbf{P}) + \sum_{j \in \mathcal{L}_l \setminus \{j_l\}} \delta^j(\mathbf{Q}) - (\delta^{j_l}(\mathbf{P}) - \delta^{j_l}(\mathbf{Q})) \\
&= \sum_{j \in \mathcal{L}_l} \delta^j(\mathbf{Q}).
\end{aligned}$$

We proceed to analyze the Quadratic Social Costs of $\mathbf{P}$ and $\mathbf{Q}$.

- By definition of $\mathcal{I}$, it follows that

$$\sum_{j\in\mathcal{I}} \frac{(\delta^j(\mathbf{P}))^2}{c^j} = \sum_{j\in\mathcal{I}} \frac{(\delta^j(\mathbf{Q}))^2}{c^j}.$$

- For each overloaded link j_l, $l \in [k]$, our construction implies that $\delta^{j_l}(\mathbf{P}) = \delta^{j_l}(\mathbf{Q}) + |\mathcal{L}_l| - 1$. So,

$$\sum_{l\in[k]} \frac{(\delta^{j_l}(\mathbf{P}))^2}{c^{j_l}} = \sum_{l\in[k]} \frac{(\delta^{j_l}(\mathbf{Q}))^2 + 2\,(|\mathcal{L}_l| - 1)\delta^{j_l}(\mathbf{Q}) + (|\mathcal{L}_l| - 1)^2}{c^{j_l}}.$$

- For each underloaded link $j \in \bigcup_{l\in[k]} (\mathcal{L}_l \setminus \{j_l\})$, $\delta^j(\mathbf{P}) = \delta^j(\mathbf{Q}) - 1$. So,

$$\begin{aligned}\sum_{l\in[k]} \sum_{j\in\mathcal{L}_l\setminus\{j_l\}} \frac{(\delta^j(\mathbf{P}))^2}{c^j} &= \sum_{l\in[k]} \sum_{j\in\mathcal{L}_l\setminus\{j_l\}} \frac{(\delta^j(\mathbf{Q}))^2 - 2\delta^j(\mathbf{Q}) + 1}{c^j} \\ &\le \sum_{l\in[k]} \sum_{j\in\mathcal{L}_l\setminus\{j_l\}} \frac{(\delta^j(\mathbf{Q}))^2 - \delta^j(\mathbf{Q})}{c^j}.\end{aligned}$$

Summing up we obtain that

$$\begin{aligned}&\sum_{j\in\mathcal{I}} \frac{(\delta^j(\mathbf{P}))^2}{c^j} + \sum_{l\in[k]} \frac{(\delta^{j_l}(\mathbf{P}))^2}{c^{j_l}} + \sum_{l\in[k]} \sum_{j\in\mathcal{L}_l\setminus\{j_l\}} \frac{(\delta^j(\mathbf{P}))^2}{c^j} \\ &\quad\le \sum_{j\in\mathcal{I}} \frac{(\delta^j(\mathbf{Q}))^2}{c^j} + \sum_{l\in[k]} \frac{(\delta^{j_l}(\mathbf{Q}))^2 + 2\,(|\mathcal{L}_l| - 1)\delta^{j_l}(\mathbf{Q}) + (|\mathcal{L}_l| - 1)^2}{c^{j_l}} + \sum_{l\in[k]} \sum_{j\in\mathcal{L}_l\setminus\{j_l\}} \frac{(\delta^j(\mathbf{Q}))^2 - \delta^j(\mathbf{Q})}{c^j} \\ &\quad= \sum_{j\in\mathcal{I}} \frac{(\delta^j(\mathbf{Q}))^2}{c^j} + \sum_{l\in[k]} \frac{(\delta^{j_l}(\mathbf{Q}))^2}{c^{j_l}} + \sum_{l\in[k]} \sum_{j\in\mathcal{L}_l\setminus\{j_l\}} \frac{(\delta^j(\mathbf{Q}))^2}{c^j} \\ &\qquad+ \sum_{l\in[k]} \frac{2\,(|\mathcal{L}_l| - 1)\delta^{j_l}(\mathbf{Q}) + (|\mathcal{L}_l| - 1)^2}{c^{j_l}} - \sum_{l\in[k]} \sum_{j\in\mathcal{L}_l\setminus\{j_l\}} \frac{\delta^j(\mathbf{Q})}{c^j}\end{aligned}$$

or

$$\begin{aligned}\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) &\le \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) + \sum_{l\in[k]} \frac{2\,(|\mathcal{L}_l| - 1)\delta^{j_l}(\mathbf{Q}) + (|\mathcal{L}_l| - 1)^2}{c^{j_l}} - \sum_{l\in[k]} \sum_{j\in\mathcal{L}_l\setminus\{j_l\}} \frac{\delta^j(\mathbf{Q})}{c^j} \\ &= \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) + \sum_{l\in[k]} \frac{(|\mathcal{L}_l| - 1)\delta^{j_l}(\mathbf{Q})}{c^{j_l}} + \sum_{l\in[k]} \frac{(|\mathcal{L}_l| - 1)(\delta^{j_l}(\mathbf{Q}) + |\mathcal{L}_l| - 1)}{c^{j_l}} - \sum_{l\in[k]} \sum_{j\in\mathcal{L}_l\setminus\{j_l\}} \frac{\delta^j(\mathbf{Q})}{c^j} \\ &= \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) + \sum_{l\in[k]} \frac{(|\mathcal{L}_l| - 1)\delta^{j_l}(\mathbf{Q})}{c^{j_l}} + \sum_{l\in[k]} \sum_{j\in\mathcal{L}_l\setminus\{j_l\}} \frac{\delta^{j_l}(\mathbf{Q}) + |\mathcal{L}_l| - 1}{c^{j_l}} - \sum_{l\in[k]} \sum_{j\in\mathcal{L}_l\setminus\{j_l\}} \frac{\delta^j(\mathbf{Q})}{c^j} \\ &= \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) + \sum_{l\in[k]} \frac{(|\mathcal{L}_l| - 1)\delta^{j_l}(\mathbf{Q})}{c^{j_l}} + \sum_{l\in[k]} \sum_{j\in\mathcal{L}_l\setminus\{j_l\}} \left(\frac{\delta^{j_l}(\mathbf{Q}) + |\mathcal{L}_l| - 1}{c^{j_l}} - \frac{\delta^j(\mathbf{Q})}{c^j} \right) \\ &= \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) + \sum_{l\in[k]} \frac{(|\mathcal{L}_l| - 1)\delta^{j_l}(\mathbf{Q})}{c^{j_l}} + \sum_{l\in[k]} \sum_{j\in\mathcal{L}_l\setminus\{j_l\}} \left(\frac{\delta^{j_l}(\mathbf{P})}{c^{j_l}} - \frac{\delta^j(\mathbf{Q})}{c^j} \right).\end{aligned}$$

We will analyze separately each sum in the last right expression.

- Recall that for every $x, y \in \mathbb{R}$, $xy \le \frac{1}{4}\,(x+y)^2$. So, for every overloaded link j_l, $(|\mathcal{L}_l| - 1)\delta^{j_l}(\mathbf{Q}) \le \frac{1}{4}\,(\delta^{j_l}(\mathbf{Q}) + |\mathcal{L}_l| - 1)^2 = (\delta^{j_l}(\mathbf{P}))^2$. Summing up over all overloaded links yields that

$$\begin{aligned}\sum_{l\in[k]} \frac{(|\mathcal{L}_l| - 1)\delta^{j_l}(\mathbf{Q})}{c^{j_l}} &\le \frac{1}{4} \sum_{l\in[k]} \frac{(\delta^{j_l}(\mathbf{P}))^2}{c^{j_l}} \\ &\le \frac{1}{4}\,\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}).\end{aligned}$$

- Consider any pair of an overloaded link j_l, and an undeloaded link $j \in \mathcal{L}_l \setminus \{j_l\}$. Recall that $\delta^j(\mathbf{Q}) = \delta^j(\mathbf{P}) + 1$. Hence, since $\mathbf{P}$ is a Nash equilibrium, $\frac{\delta^{j_l}(\mathbf{P})}{c^{j_l}} \le \frac{\delta^j(\mathbf{P})+1}{c^j} = \frac{\delta^j(\mathbf{Q})}{c^j}$. This implies that the second sum is non-positive.

Hence, it follows that $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \leq \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) + \frac{1}{4}\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$, or $\frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} \leq \frac{4}{3}$. Since the instance $\langle \mathbf{w}, \mathbf{c}\rangle$, and the associated pure Nash equilibrium $\mathbf{P}$ were chosen arbitrarily, this implies that $\mathsf{QCR} \leq \frac{4}{3}$, and the proof of the upper bound is complete.

We continue to prove the lower bound. Fix an instance $\langle \mathbf{w}, \mathbf{c}\rangle$, with $n = 2(m-1)$; for each link $l \in [m]$, set $c_l = 2(m-1)$ if $l = 1$ and 1 otherwise. Consider pure assignments $\mathbf{P}$ and $\mathbf{Q}$, such that:

- In $\mathbf{P}$, all users are assigned to link 1. Note that for each user $i \in [2(m-1)]$ $\lambda_i^1 = \frac{2(m-1)}{2(m-1)} = 1$ and for each link $l \in [m]$, $l \neq 1$, $\lambda_i^l = 1$; so, $\mathbf{P}$ is a Nash equilibrium. Clearly, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) = \frac{(2(m-1))^2}{2(m-1)} = 2(m-1)$.
- In $\mathbf{Q}$, $m-1$ users are assigned to link 1 and each of the rest is assigned to each link $2, \ldots, m$, respectively. Clearly, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = \frac{(m-1)^2}{2(m-1)} + (m-1)\frac{1^2}{1} = \frac{3}{2}(m-1)$.

So, $\mathsf{QCR} \geq \frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} \geq \frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{Q})} = \frac{4}{3}$, as needed. ■

5.1.2. Identical links

We prove:

Theorem 5.2. *Consider the case of identical links, restricted to pure Nash equilibria. Then,* $\mathsf{QCR} = \frac{9}{8}$.

Proof. We first prove the upper bound. Consider any arbitrary instance $\langle \mathbf{w}, \mathbf{c}\rangle$, with associated pure Nash equilibrium $\mathbf{P}$ and optimal assignment $\mathbf{Q}$. We consider two cases:

(1) The instance $\langle \mathbf{w}, \mathbf{c}\rangle$ is non-bursty. Recall that in this case, by Lemma 3.7, for each link $j \in [m]$, $\delta^j(\mathbf{P}) \leq 2\min_{l\in[m]} \delta^l(\mathbf{P})$. So transform the set of loads $\left\{\delta^l(\mathbf{P}) \mid l \in [m]\right\}$, into a new set of loads $\left\{\widehat{\delta^l}(\mathbf{P}) \mid l \in [m]\right\}$ as the output of the following repetitive procedure:

for each link $l \in [m]$, **do**
$\quad\widehat{\delta^l}(\mathbf{P}) := \delta^l(\mathbf{P})$;
while there are distinct $j, j' \in [m]$ with $\min\limits_{l\in[m]} \delta^l(\mathbf{P}) < \widehat{\delta^j}(\mathbf{P}) \leq \widehat{\delta^{j'}}(\mathbf{P}) < 2\min\limits_{l\in[m]} \delta^l(\mathbf{P})$ **do**
$\quad\widehat{\delta^j}(\mathbf{P}) := \widehat{\delta^j}(\mathbf{P}) - \min\left\{\widehat{\delta^j}(\mathbf{P}) - \min\limits_{l\in[m]} \delta^l(\mathbf{P}),\ 2\min\limits_{l\in[m]} \delta^l(\mathbf{P}) - \widehat{\delta^{j'}}(\mathbf{P})\right\}$;
$\quad\widehat{\delta^{j'}}(\mathbf{P}) := \widehat{\delta^{j'}}(\mathbf{P}) + \min\left\{\widehat{\delta^j}(\mathbf{P}) - \min\limits_{l\in[m]} \delta^l(\mathbf{P}),\ 2\min\limits_{l\in[m]} \delta^l(\mathbf{P}) - \widehat{\delta^{j'}}(\mathbf{P})\right\}$
end while

Intuitively, our transformation procedure chooses at each step, two intermediate loads $\delta^j(\mathbf{P})$ and $\delta^{j'}(\mathbf{P})$ (that is, two loads that are not yet pushed either to the upper end or to the lower end of the interval of link loads); it transfers the (strictly) positive quantity $\min\left\{\widehat{\delta^j}(\mathbf{P}) - \min_{l\in[m]} \delta^l(\mathbf{P}),\ 2\min_{l\in[m]} \delta^l(\mathbf{P}) - \widehat{\delta^{j'}}(\mathbf{P})\right\}$, from the small load $\delta^j(\mathbf{P})$, to the large load $\delta^{j'}(\mathbf{P})$. Clearly, each step of the procedure either pushes the small load $\delta^j(\mathbf{P})$, to the lower end $\min_{l\in[m]} \delta^l(\mathbf{P})$ of the interval of link loads, or pushes the large load $\delta^{j'}(\mathbf{P})$, to the upper end $2\min_{l\in[m]} \delta^l(\mathbf{P})$ of the interval of link loads (or both). So, when the procedure terminates, there is at most one intermediate load. Hence, by reordering links, we obtain that there exists an integer $\widehat{j}$, $0 \leq \widehat{j} \leq m-1$, such that for each link $j \in [m]$,

$$\widehat{\delta^j}(\mathbf{P}) = \begin{cases} 2\min\limits_{l\in[m]} \delta^l(\mathbf{P}), & j \in [\widehat{j}] \\ (1+x)\min\limits_{l\in[m]} \delta^l(\mathbf{P}), & j = \widehat{j}+1 \\ \min\limits_{l\in[m]} \delta^l(\mathbf{P}), & j \in [m] \setminus [\widehat{j}+1] \end{cases}$$

where $0 \leq x \leq 1$. Intuitively, $\widehat{j}$ is the number of overloaded links.

Note that this procedure maps a set of loads to a new set of loads, without explicitly mapping an instance to a new instance. However, for the sake of analysis, we will also consider that the procedure maps an instance $\langle \mathbf{w}, \mathbf{c}\rangle$, and a Nash equilibrium $\mathbf{P}$ to a new instance $\langle \widehat{\mathbf{w}}, \widehat{\mathbf{c}}\rangle$ and a new Nash equilibrium $\widehat{\mathbf{P}}$. Note also that the procedure preserves (at each step) the sum of loads. So, it also preserves the total traffic, so that $W = \widehat{W}$. For any single step transforming the load set $\left\{\delta^l(\mathbf{P}) \mid l \in [m]\right\}$ into $\left\{\widehat{\delta^l}(\mathbf{P}) \mid l \in [m]\right\}$,

$$\begin{aligned}&\mathsf{QSC}(\widehat{\mathbf{w}}, \widehat{\mathbf{c}}, \widehat{\mathbf{P}}) - \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \\ &\quad= \left(\widehat{\delta^j}(\mathbf{P}) - \min\left\{\widehat{\delta^j}(\mathbf{P}) - \min\limits_{l\in[m]} \delta^l(\mathbf{P}), 2\min\limits_{l\in[m]} \delta^l(\mathbf{P}) - \widehat{\delta^{j'}}(\mathbf{P})\right\}\right)^2\end{aligned}$$

$$
\begin{aligned}
&+\left(\widehat{\delta}^{j'}(\mathbf{P})+\min\left\{\widehat{\delta}^{j}(\mathbf{P})-\min_{l\in[m]}\delta^{l}(\mathbf{P}),2\min_{l\in[m]}\delta^{l}(\mathbf{P})-\widehat{\delta}^{j'}(\mathbf{P})\right\}\right)^2-\left(\widehat{\delta}^{j}(\mathbf{P})\right)^2-\left(\widehat{\delta}^{j'}(\mathbf{P})\right)^2\\
&=2\,\min\left\{\widehat{\delta}^{j}(\mathbf{P})-\min_{l\in[m]}\delta^{l}(\mathbf{P}),2\min_{l\in[m]}\delta^{l}(\mathbf{P})-\widehat{\delta}^{j'}(\mathbf{P})\right\}\cdot\left(\widehat{\delta}^{j'}(\mathbf{P})-\widehat{\delta}^{j}(\mathbf{P})\right)\\
&+2\,\left(\min\left\{\widehat{\delta}^{j}(\mathbf{P})-\min_{l\in[m]}\delta^{l}(\mathbf{P}),2\min_{l\in[m]}\delta^{l}(\mathbf{P})-\widehat{\delta}^{j'}(\mathbf{P})\right\}\right)^2\\
&>0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})\ &\leq\ \mathsf{QSC}(\widehat{\mathbf{w}},\widehat{\mathbf{c}},\widehat{\mathbf{P}})\\
&=\left(4\widehat{j}+(1+x)^2+(m-\widehat{j}-1)\right)\cdot\left(\min\left\{\widehat{\delta}^{j}(\mathbf{P})-\min_{l\in[m]}\delta^{l}(\mathbf{P}),2\min_{l\in[m]}\delta^{l}(\mathbf{P})-\widehat{\delta}^{j'}(\mathbf{P})\right\}\right)^2\\
&=\left(x^2+2x+m+3\widehat{j}\right)\cdot\left(\min\left\{\widehat{\delta}^{j}(\mathbf{P})-\min_{l\in[m]}\delta^{l}(\mathbf{P}),2\min_{l\in[m]}\delta^{l}(\mathbf{P})-\widehat{\delta}^{j'}(\mathbf{P})\right\}\right)^2.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\mathsf{QOPT}(\mathbf{w},\mathbf{c})\ &\geq\ \frac{W^2}{m}\\
&=\frac{\widehat{W}^2}{m}\\
&=\frac{\left(\sum_{j\in[m]}\widehat{\delta}^{j}(\mathbf{P})\right)^2}{m}\\
&=\frac{\left(2\widehat{j}+(1+x)+(m-\widehat{j}-1)\right)^2}{m}\cdot\left(\min\left\{\widehat{\delta}^{j}(\mathbf{P})-\min_{l\in[m]}\delta^{l}(\mathbf{P}),2\min_{l\in[m]}\delta^{l}(\mathbf{P})-\widehat{\delta}^{j'}(\mathbf{P})\right\}\right)^2\\
&=\frac{\left(x+m+\widehat{j}\right)^2}{m}\cdot\left(\min\left\{\widehat{\delta}^{j}(\mathbf{P})-\min_{l\in[m]}\delta^{l}(\mathbf{P}),2\min_{l\in[m]}\delta^{l}(\mathbf{P})-\widehat{\delta}^{j'}(\mathbf{P})\right\}\right)^2.
\end{aligned}
$$

It follows that

$$
\mathsf{QCR}\ \leq\ \frac{(x^2+2x+m+3\widehat{j})\,m}{(x+m+\widehat{j})^2}.
$$

Define the real function

$$
f(k)\ =\ \frac{(x^2+2x+m+3\widehat{j})m}{(x+m+\widehat{j})^2}
$$

of a real variable $\widehat{j}$. (The quantity x is taken as a parameter, while m is a fixed constant.) Clearly, $\mathsf{QCR}\leq\sup_{\widehat{j}}f(\widehat{j})$. So, we will determine $\sup_{\widehat{j}}f(\widehat{j})$.

To gain some intuition, write

$$
\begin{aligned}
f(\widehat{j})\ &=\ \frac{\left(\frac{m}{\widehat{j}}\right)^2+3\frac{m}{\widehat{j}}+\frac{m(x^2+2x)}{\widehat{j}^2}}{\left(\frac{m}{\widehat{j}}+1+\frac{x}{\widehat{j}}\right)^2}\\
&=\frac{y(y+3)+\frac{m(x^2+2x)}{\widehat{j}^2}}{\left(y+1+\frac{x}{\widehat{j}}\right)^2},
\end{aligned}
$$

where $y=\frac{m}{\widehat{j}}$ is the ratio of the total number of links to the number of overlaodad links $\widehat{j}$. Since we are interested in the ratio $\frac{m}{\widehat{j}}$ and $x\in[0,1]$, we can assume that $\widehat{j}$ is so large that $\frac{x}{\widehat{j}}$ is negligibe. Then, $f(\widehat{j})$ essentially behaves as the function $g(y)=\frac{y(y+3)}{(y+1)^2}$, which is maximized for $y=3$, achieving the value $\frac{9}{8}$. We now return to the formal proof.

To maximize the function $f(\widehat{j})$, observe that the first and second derivatives of $f(\widehat{j})$ are

$$\frac{\mathrm{d}f(\widehat{j})}{\mathrm{d}\widehat{j}} = \frac{m}{(x+m+\widehat{j})^3} \cdot \left(-2x^2 - x + m - 3\widehat{j}\right)$$

and

$$\frac{\mathrm{d}^2 f(\widehat{j})}{\mathrm{d}\widehat{j}^2} = m \cdot \left(\frac{-3(x+m+\widehat{j})^3 - (-2x^2 - x + m - 3\widehat{j}) \cdot 3(x+m+\widehat{j})^2}{(x+m+\widehat{j})^6} \right),$$

respectively. The only root of $\frac{\mathrm{d}f(\widehat{j})}{\mathrm{d}\widehat{j}}$ is $\widehat{j}_0 = \frac{m-x-2x^2}{3}$. For $\widehat{j} = \widehat{j}_0$, $\frac{\mathrm{d}^2 f(\widehat{j})}{\mathrm{d}\widehat{j}^2}$ evaluates to $\frac{-81m}{8(x+2m-x^2)^3} < 0$. Thus, $\widehat{j}_0$ is a local maximum of the function $f(\widehat{j})$. Since $f(\widehat{j})$ is a continuous function with a single extremum point that is a local maximum, it follows that $f(\widehat{j}) \leq f(\widehat{j}_0) = \frac{9m}{4(2m+x-x^2)} \leq \frac{9}{8}$ (since $x \in [0, 1]$), as needed.

(2) The instance $\langle \mathbf{w}, \mathbf{c} \rangle$ is bursty. Denote $\mathcal{U}$ the (non-empty) set of bursty users. Recall that, by Lemma 3.5, $\mathcal{U}$ induces sets of solo links $\mathcal{L}_{\mathbf{P}}$, and $\mathcal{L}_{\mathbf{Q}}$ for the Nash equilibrium $\mathbf{P}$ and the optimal assignment $\mathbf{Q}$, respectively, so that $|\mathcal{L}_{\mathbf{P}}| = |\mathcal{U}|$ and $|\mathcal{L}_{\mathbf{Q}}| = |\mathcal{U}|$. Since links are identical, we assume that $\mathcal{L}_{\mathbf{P}} = \mathcal{L}_{\mathbf{Q}} = \mathcal{L}$, with $|\mathcal{L}| \geq 1$. So,

$$\begin{aligned} \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) &= \sum_{l \in \mathcal{L}} \left(\delta^l(\mathbf{P})\right)^2 + \mathsf{QSC}(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L}, \mathbf{P} \setminus (\mathcal{U}, \mathcal{L})) \\ &= \sum_{i \in \mathcal{U}} w_i^2 + \mathsf{QSC}(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L}, \mathbf{P} \setminus (\mathcal{U}, \mathcal{L})) \end{aligned}$$

and

$$\begin{aligned} \mathsf{QOPT}(\mathbf{w}, \mathbf{c}) &= \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) \\ &= \sum_{l \in \mathcal{L}} \left(\delta^l(\mathbf{P})\right)^2 + \mathsf{QSC}(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L}, \mathbf{Q} \setminus (\mathcal{U}, \mathcal{L})) \\ &= \sum_{i \in \mathcal{U}} w_i^2 + \mathsf{QSC}(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L}, \mathbf{Q} \setminus (\mathcal{U}, \mathcal{L})). \end{aligned}$$

Note first that the assignment $\mathbf{P} \setminus (\mathcal{U}, \mathcal{L})$, is a Nash equilibrium for the instance $\langle \mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L} \rangle$. Moreover, since $\mathbf{Q}$ is an optimal assignment for the instance $\langle \mathbf{w}, \mathbf{c} \rangle$, it follows that $\mathbf{Q} \setminus (\mathcal{U}, \mathcal{L})$ is an optimal assignment for the instance $\langle \mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L} \rangle$, so that $\mathsf{QSC}(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L}, \mathbf{Q} \setminus (\mathcal{U}, \mathcal{L})) = \mathsf{QOPT}\,(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L})$. Thus,

$$\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) = \sum_{i \in \mathcal{U}} w_i^2 + \mathsf{QOPT}\,(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L})\,.$$

It follows that

$$\begin{aligned} \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} &= \frac{\sum_{i \in \mathcal{U}} w_i^2 + \mathsf{QSC}(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L}, \mathbf{P} \setminus (\mathcal{U}, \mathcal{L}))}{\sum_{i \in \mathcal{U}} w_i^2 + \mathsf{QOPT}\,(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L})} \\ &\leq \frac{\mathsf{QSC}(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L}, \mathbf{P} \setminus (\mathcal{U}, \mathcal{L}))}{\mathsf{QOPT}\,(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L})}. \end{aligned}$$

So consider the instance $\langle \mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L} \rangle$, and the associated pure Nash equilibrium $\mathbf{P} \setminus (\mathcal{U}, \mathcal{L})$. There are two possibilities according to the burstiness of the instance $\langle \mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L} \rangle$.

- Assume first that the smaller instance $\langle \mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L} \rangle$ is non-bursty. Then, we are reduced to the previous case of non-bursty instances, and the upper bound follows inductively.
- Assume now that the smaller instance $\langle \mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L} \rangle$ is bursty. We repeatedly identify the set of bursty users for the smaller instance, and we reduce this smaller instance to an even smaller instance that may be bursty or non-bursty. This procedure eventually yields a non-bursty instance (even the trivial one with one user), and the claim for the original bursty instance follows inductively.

The proof of the upper bound is now complete. We continue to prove the lower bound. Fix $n = 8$ and $m = 3$. Set $w_i = 1$ for $1 \leq i \leq 2$ and $w_i = \frac{1}{3}$ otherwise. Observe that $\mathbf{P} = \langle 1, 1, 2, 2, 2, 3, 3, 3 \rangle$, is a pure Nash equilibrium with $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) = 2^2 + 1^2 + 1^2 = 6$. Moreover, consider the pure assignment $\mathbf{Q} = \langle 1, 2, 1, 2, 3, 3, 3, 3 \rangle$ that achieves $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = 3 \cdot \left(\frac{4}{3}\right)^2 = \frac{16}{3}$. Thus, $\mathsf{QCR} \geq \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q})} = \frac{6}{\frac{16}{3}} = \frac{9}{8}$, as needed. ■

5.2. Mixed Nash equilibria

Throughout this section, we focus on the case of identical links. The fully mixed Nash equilibrium is treated in Section 5.2.1. The case of identical users is considered in Section 5.2.2.

5.2.1. Identical links and fully mixed Nash equilibrium

We prove:

Theorem 5.3. *Consider the case of identical links. Then, restricted to the fully mixed Nash equilibrium,*

$$\mathsf{QCR} = 1 + \frac{\min\{n, m\} - 1}{m}.$$

Proof. We first prove the upper bound. Consider any arbitrary instance $\langle \mathbf{w}, \mathbf{c} \rangle$, with an associated fully mixed Nash equilibrium $\mathbf{F}$, and optimal (pure) assignment $\mathbf{Q}$. Assume, without loss of generality, that $\delta^1(\mathbf{Q}) \geq \ldots \geq \delta^m(\mathbf{Q})$; that is, the links are indexed in non-increasing order of their loads in $\mathbf{Q}$. Clearly, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = \sum_{l\in[m]} (\delta^l(\mathbf{Q}))^2$, while by Theorem 4.2, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) = \frac{W^2}{m} + \left(1 - \frac{1}{m}\right) W_1$.

Transform the instance $\langle \mathbf{w}, \mathbf{c} \rangle$ to an instance $\langle \widehat{\mathbf{w}}, \widehat{\mathbf{c}} \rangle$ with m identical links and $\min\{n, m\}$ users with weights $\widehat{w}_i = \delta^i(\mathbf{Q})$ for $i \in [\min\{n, m\}]$. Clearly, $\widehat{W} = W$ and $\widehat{W}_1 \geq W_1$, while $\widehat{W}_1 \geq \frac{\widehat{W}^2}{\min\{n,m\}}$. Notice that if $n \leq m$, then $\delta^l(\mathbf{Q}) = 0$ for all links $l > n = \min\{n, m\}$, and the two instances are identical.

Denote $\widehat{\mathbf{Q}}$ the pure assignment for the new instance $\langle \widehat{\mathbf{w}}, \widehat{\mathbf{c}} \rangle$, that assigns user $i \in [\min\{n, m\}]$ to the link $i \in [m]$. Clearly, $\mathsf{QSC}(\widehat{\mathbf{w}}, \widehat{\mathbf{c}}, \widehat{\mathbf{Q}}) = \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q})$. Moreover, by construction $\mathsf{QSC}(\widehat{\mathbf{w}}, \widehat{\mathbf{c}}, \widehat{\mathbf{Q}}) = \widehat{W}_1$.

Denote also $\widehat{\mathbf{F}}$, the fully mixed Nash equilibrium for the new instance $\langle \widehat{\mathbf{w}}, \widehat{\mathbf{c}} \rangle$. By Theorem 4.2, $\mathsf{QSC}(\widehat{\mathbf{w}}, \widehat{\mathbf{c}}, \widehat{\mathbf{F}}) = \frac{\widehat{W}^2}{m} + \left(1 - \frac{1}{m}\right) \widehat{W}_1 \geq \frac{W^2}{m} + \left(1 - \frac{1}{m}\right) W_1 = \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F})$. Hence,

$$\begin{aligned}
\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} &= \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F})}{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q})} \\
&\leq \frac{\mathsf{QSC}(\widehat{\mathbf{w}}, \widehat{\mathbf{c}}, \widehat{\mathbf{F}})}{\mathsf{QSC}(\widehat{\mathbf{w}}, \widehat{\mathbf{c}}, \widehat{\mathbf{Q}})} \\
&= \frac{\left(1 - \frac{1}{m}\right) \widehat{W}_1 + \frac{\widehat{W}^2}{m}}{\widehat{W}_1} \\
&= 1 - \frac{1}{m} + \frac{1}{m} \frac{\widehat{W}^2}{\widehat{W}_1} \\
&\leq 1 - \frac{1}{m} + \frac{1}{m} \min\{n, m\} \\
&= 1 + \frac{\min\{n, m\} - 1}{m}.
\end{aligned}$$

Since the instance $\langle \mathbf{w}, \mathbf{c} \rangle$ was chosen arbitrarily, it follows that $\mathsf{QCR} \leq 1 + \frac{\min\{n,m\}-1}{m}$, and the proof of the upper bound is complete.

We continue to prove the lower bound. Fix $n = m$. Set $w_i = 1$ for all users $i \in [n]$. Then, clearly, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) = W_1 + \frac{2}{m} W_2 = m \cdot 1^2 + \frac{2}{m} \frac{m(m-1)}{2} \cdot 1 = 2m - 1$. Moreover, $\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) = m \cdot 1^2 = m$. Thus, $\mathsf{QCR} \geq \frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{F})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} = 2 - \frac{1}{m}$, as needed. ■

Note that since $n \geq 2$ and $m \geq 2$, the tight bound of $1 + \frac{\min\{n,m\}-1}{m}$ is at least $\frac{3}{2}$ and strictly less than 2.

5.2.2. Identical users and identical links

We prove:

Theorem 5.4. *Consider the case of identical users and identical links. Then,*

$$\mathsf{QCR} \leq 1 + \min\left\{\frac{m - 1}{n}, \frac{n - 1}{m}\right\}.$$

Proof. Consider any arbitrary instance $\langle \mathbf{w}, \mathbf{c} \rangle$, with an associated Nash equilibrium $\mathbf{P}$. Theorem 4.8, and Corollary 4.6 imply that $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \leq \frac{n(n+m-1)}{m}$. Since $\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) \geq n \max\left\{1, \frac{n}{m}\right\}$, it follows that

$$\begin{aligned}
\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} &\leq \frac{n + m - 1}{m} \min\left\{1, \frac{m}{n}\right\} \\
&= (n + m - 1) \min\left\{\frac{1}{n}, \frac{1}{m}\right\} \\
&= \min\left\{1 + \frac{m - 1}{n}, 1 + \frac{n - 1}{m}\right\}
\end{aligned}$$

$$= 1 + \min\left\{\frac{m-1}{n}, \frac{n-1}{m}\right\},$$

as needed. ■

A simple case analysis on the relation between n and m reveals that for all $m \geq 2$, and $n \geq 2$, $\min\left\{\frac{m-1}{n}, \frac{n-1}{m}\right\} + \max\left\{\frac{1}{n}, \frac{1}{m}\right\} \leq \frac{\min\{n,m\}-1}{m}$. This implies that the upper bound in Theorem 5.4, for the case of identical users and identical links, not only applies to all (mixed) Nash equilibria, but it is no worse than the tight bound in Theorem 5.3, for the case of identical links, which applies only to the fully mixed Nash equilibrium. So, intuitively, Theorems 5.3 and 5.4 together, suggest that for the case of identical links, considering arbitrary users, but restricting to the fully mixed Nash equilibrium, may have a more severe influence on the Quadratic Coordination Ratio, than considering all mixed Nash equilibria, but restricting to identical users.

6. Epilogue

6.1. Summary

We have presented a new, potentially rich model for selfish routing over non-cooperative networks, as an interesting hybridization of the two prevailing models for selfish routing, namely the KP model [26] and the W model [36]. Within this model, we focused on certain algorithmic and combinatorial properties of Nash equilibria; we also introduced and studied the Quadratic Social Cost and Quadratic Coordination Ratio, as interesting variants of the well studied Social Cost and Coordination Ratio, respectively, from the KP model [26].

Most interestingly, we presented a collection of *tight* bounds on the Quadratic Coordination Ratio for our model; these are the *first* known constant (or bounded by a constant) bounds (independent of the number of users and the number of links) for models with unsplittable traffic. Some of our proof techniques highlight several interesting transformations of instances (user traffics and link capacities), which could be useful for other applications.

6.2. Subsequent work

Some subsequent work has touched issues similar to those addressed in this work.

- Gairing et al. [22] introduce and study a discrete routing game as yet another hybridization of the KP model and the W model. In their model, the latency on a link is determined by an arbitrary, non-decreasing, convex function; in turn, the latency defines the Expected Individual Cost for a user (in the same way as in the KP model). However, the Social Cost is taken to be the sum of the Expected Individual Costs (as in the W model). When users are identical, and latency functions are linear, this discrete routing game reduces to the special case of identical users for the model studied in this paper. We note that the set of Nash equilibria for the model of Gairing et al. [22] (defined through arbitrary, convex latency functions) is (in general) different from the set of Nash equilibria for the model in this paper (which coincides with the set of Nash equilibria for the KP model).

  The following results in the work of Gairing et al. [22], are related to the model studied in this paper; some of them, although more general, become less tight than corresponding results in this paper for the special cases we considered.
  - Gairing et al. [22, Theorem 1] prove that the fully mixed Nash equilibrium (when it exists) is the worst-case Nash equilibrium for the case of identical users in their model. This extends our Theorem 4.8 to the case of links with arbitrary, convex latency functions.
  - For the case of identical users, and identical links with latency functions $f(x) = x^d$, where $d \geq 1$, Gairing et al. [22, Theorem 5] prove a tight bound of B_{d+1}, the $(d+1)$-th *Bell number,* on Coordination Ratio (for their model). For $d = 1$, this implies an upper bound of $B_2 = 2$ on the Quadratic Coordination Ratio for the case of identical users, and identical links in our model, which applies to all (mixed) Nash equilibria. Theorem 5.4 in this work provides a slightly better upper bound.
  - For the case of identical users, and arbitrary links with polynomial latency functions of maximum degree d, Gairing et al. [22, Corollary 1], prove an upper bound of $d + 1$ on the Coordination Ratio, restricted to pure Nash equilibria. For $d = 1$, this implies an upper bound of 2 on the Quadratic Coordination Ratio, for the case of identical users and identical links in our model, restricted to pure Nash equilibria. Theorem 5.1 provides a significantly better bound ($\frac{4}{3}$).
- Finally, Gairing et al. [20] introduce and study yet another hybridization of the KP model, and the W model. In their model, the latency on a link is a linear function of load on the link (as in both the KP model and the model studied in this paper). However, a so called *Polynomial Social Cost* is adopted that extends Quadratic Social Cost to arbitrary degrees. In turn, this yields Polynomial Coordination Ratio. Gairing et al. [20] only consider the case of identical users and identical links. For this case, they present a collection of upper bounds on Polynomial Coordination Ratio.
- Two recent papers [4,9] make significant progress towards determining the Coordination Ratio in the more general, related contexts of *congestion games* [33] and *weighted congestion games* [30].

- Awerbuch et al. [4] consider multicommodity *weighted network congestion games* [19] in general networks with linear latency functions; they study the Coordination Ratio with respect to total latency, which coincides with our Quadratic Coordination Ratio. Thus, our model can be cast as the special case of the model in [4] with a single commodity, and a parallel links network. Awerbuch et al. [4] prove that the Coordination Ratio is precisely $\frac{3+\sqrt{5}}{2} \approx 2.618$ in the general case, but, restricted to pure Nash equilibria and identical users, it reduces to $\frac{5}{2}$. Both bounds clearly apply as upper bounds to our model as well, but not as lower bounds since the constructions employed in [4] for the lower bounds use a multicommodity network, different from the parallel links network. Moreover, Theorem 5.1 implies that the $\frac{5}{2}$ bound is *not* tight for our model; the tightness of the $\frac{3+\sqrt{5}}{2}$ bound for our model is posed as an open problem.
- Christodoulou and Koutsoupias [9] consider general unweighted congestion games, both single-commodity and multicommodity, with linear cost functions; they study the Coordination Ratio, with respect to both the sum of Individual Costs and the maximum Individual Cost, the latter coinciding with our Quadratic Coordination Ratio (in the case of identical users). For single-commodity congestion games and pure Nash equilibria, Christodoulou and Koutsoupias [9] prove a tight bound of $\frac{5n-2}{2n+1}$ on the Coordination Ratio, which is strictly less than the corresponding (tight) bound of $\frac{5}{2}$ shown by Awerbuch et al. [4] for multicommodity networks, pure Nash equilibria and identical users. This bound clearly applies as an upper bound to our model as well, but not as a lower bound, since the construction employed in [9] for the lower bound uses a congestion game other than the parallel links network. Again, Theorem 5.1 implies that this bound is *not* tight for our model. Christodoulou and Koutsoupias [9] prove independently the same tight bound of $\frac{5}{2}$ on the Coordination Ratio for multicommodity congestion games with identical users and for pure Nash equilibria, and similar bounds on the Coordination Ratio as the maximum Individual Cost (for both single-commodity and multicommodity congestion games).

6.3. Open problems

We conclude this article with a collection of questions that naturally pose themselves within the new model for selfish routing, introduced in this work.

(1) What is the time complexity of computing the Quadratic Social Cost of an arbitrary Nash equilibrium? Proposition 4.1 provides a polynomial algorithm with time complexity $O(mn)$ for any arbitrary assignment. Is this optimal? (Lower bounds are totally missing.) Can this be improved for Nash equilibria?

(2) What is the time complexity of deciding optimality (with respect to Quadratic Social Cost) for a given pure assignment? Proposition 3.2 implies an $O(n + m^2)$ upper bound for the case of identical users. Is there a polynomial upper bound for the general case?

(3) Prove the Quadratic Fully Mixed Nash Equilibrium Conjecture (Conjecture 4.1).

(4) What is the value of the Quadratic Coordination Ratio, when restricted to pure Nash equilibria, for the new model? Theorems 5.1 and 5.2 consider separately the two cases of identical links and identical users. It is challenging to merge the two separate proofs for the two special cases into one, for the general case (restricted to pure Nash equilibria). (An upper bound of $\frac{3+\sqrt{5}}{2}$ follows from the recent works [4,9].)

(5) What is the value of the Quadratic Coordination Ratio when all Nash equilibria are considered? (Again, the upper bound of $\frac{3+\sqrt{5}}{2}$ from [4,9] applies here as well.)

Acknowledgments

We are grateful to the two anonymous referees for the numerous helpful comments and suggestions they provided. We especially thank one of the two referees for providing vast simplifications to our own original proofs of Theorems 5.1 and 5.3. We also thank Rainer Feldmann and Martin Gairing for several helpful discussions.

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