A note on the largest bipartite subgraph in point-hyperplane incidence graphs in four and five dimensions

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Abstract

Given m points and n hyperplanes in $\mathbb{R}^d$, if there are many incidences, we expect to find a big cluster $K_{r,s}$ in their incidence graph. In [1], Apfelbaum and Sharir found lower and upper bounds for the largest size of rs, which only match in three dimensions. In this paper we close the gap in four and five dimensions, up to some logarithmic factors.

1 Introduction

Given a set P of m points and a set Q of n hyperplanes in $\mathbb{R}^d$, their *incidence graph* $G(P, Q)$ is a bipartite graph with vertex set $P \cup Q$ and $(p, q) \in P \times Q$ forms an edge iff $p \in q$. It is proved in [1] that if this graph does not contain $K_{r,s}$ as a subgraph for some fixed r, s, then it can have at most $O_d((mn)^{d/(d+1)} + m + n)$ edges. Here the notations $f = O_d(g)$ means there exists some constant C that depends on d such that $f \leq Cg$.

Conversely, when the graph has many edges, we expect to find a big $K_{r,s}$ subgraph. How big can rs be in term of m, n and the number of edges? To make it precise, we use the following definition.

Definition 1.1. Given a set P of points and Q of hyperplanes in $\mathbb{R}^d$, let $rs(P, Q)$ be the maximum size of a complete bipartite subgraph of its incidence graph, and $rs_d(m, n, I)$ be the minimum of this quantity over all choices of m points and n hyperplanes in $\mathbb{R}^d$ with I incidences. To be precise:

$$rs(P, Q) := \max\{rs : K_{r,s} \subset G(P, Q)\}$$

$$rs_d(m, n, I) := \min_{|P|=m, |Q|=n, |G(P,Q)|=I} rs(P, Q).$$

Apfelbaum and Sharir in [1] proved that if $I = \Omega_d(mn^{1-\frac{1}{d-1}} + nm^{1-\frac{1}{d-1}})$ then

$$rs_d(m, n, I) = \Omega_d\left(\left(\frac{I}{mn}\right)^{d-1} mn\right). \tag{1}$$

Moreover, they show the following upper bound: if $I = \Omega_d((mn)^{1-\frac{1}{d-1}})$ then

$$rs_d(m, n, I) = O_d\left(\left(\frac{I}{mn}\right)^{\frac{d+1}{2}} mn\right). \tag{2}$$

These lower and upper bounds only match when $d = 3$. In this paper we close the gap in four and five dimensions. In particular, we improve the lower bound to match with the upper bound up to some logarithmic factors.

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Theorem 1.2. *When $d = 4$, there exist constants C_4 and C_4' such that if $I \geq C_4(mn^{2/3} + nm^{3/5})$ then*

$$rs_4(m, n, I) \geq C_4' \left(\frac{I}{mn}\right)^{5/2} mn(\log mn)^{-4}.$$

Theorem 1.3. *When $d = 5$, there exist constants C_5 and C_5' such that if $I \geq C_5(mn^{3/4} + nm^{3/4})$ then*

$$rs_5(m, n, I) \geq C_5' \left(\frac{I}{mn}\right)^{3} mn(\log mn)^{-10}.$$

The main tool used to prove Theorem 1.2 and Theorem 1.3 is an incidence bound between points and *nondegenerate* hyperplanes, which is reviewed in the next section. We then present the proof of Theorem 1.2 and sketch the proof of Theorem 1.3 in the subsequent sections. At the end we explain why our method does not work in six dimensions.

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2 Incidence with nondegenerate hyperplanes

We use the following notations. Let A and B be two sets of geometric objects in $\mathbb{R}^d$. Their *incidence graph* $G(A, B)$ is a bipartite graph on $A \times B$ where (a, b) forms an edge iff $a \subset b$. The number of incidences between A and B is the number of edges in this graph, and denoted by $I(A, B)$. In this paper, A are either a set of points or a set of lines, and B is a set of higher dimensional flats.

Given a set $\mathcal{S}$ of m points in $\mathbb{R}^d$ and some $\beta \in (0, 1)$, a hyperplane H is β-*nondegenerate* with respect to (w.r.t.) $\mathcal{S}$ if there does not exist a proper subflat $F \subset H$ that contains more than β fraction of the number of points of P in H, i.e. $|F \cap \mathcal{S}| > \beta |H \cap \mathcal{S}|$. Otherwise, H is β-*degenerate*. Elekes and Tóth proved the following incidence bound.

Theorem 2.1 (Elekes-Tóth [3])**.** *If $\mathcal{S}$ is a set of m points and $\mathcal{H}$ is a set of n β-non-degenerate hyperplanes (for any $0 < \beta < 1$)*[1] *in $\mathbb{R}^d$ then there exists a constant $C_{\beta,d}$ such that*

$$I(\mathcal{S}, \mathcal{H}) \leq C_{\beta,d} \left((mn)^{\frac{d}{d+1}} + mn^{1-\frac{1}{d-1}}\right). \tag{3}$$

This implies the maximum number of β-nondegenerate, k-rich (i.e. containing at least k points of $\mathcal{S}$) hyperplanes is $O_{\beta,d}\left(\frac{m^{d+1}}{k^{d+2}} + \frac{m^{d-1}}{k^{d-1}}\right)$. Actually this is what Elekes-Tóth proved. As shown in [1], this is equivalent to (3).

Since lines and hyperplanes are dual to each other, we also have a dual version of the above result. Given a set $\mathcal{H}$ of n hyperplanes in $\mathbb{R}^d$, a point p is β-nondengenerate with respect to $\mathcal{H}$ if there does not exist a line l such that $\#\{H \in \mathcal{H} : l \subset H\} \geq \beta \#\{H \in \mathcal{H} : p \in H\}$.

Corollary 2.2. *If $\mathcal{H}$ is a set of n hyperplanes in $\mathbb{R}^d$ and P is a set of m β-nondegenerate points w.r.t. $\mathcal{H}$ then there exists a constant $C'_{\beta,d}$*

$$I(\mathcal{S}, \mathcal{H}) \leq C'_{\beta,d} \left((mn)^{\frac{d}{d+1}} + nm^{1-\frac{1}{d-1}}\right). \tag{4}$$

Equivalently, given n hyperplanes in $\mathbb{R}^d$, the number of k-rich, β-nondegenerate points is $O_{\beta,d}\left(\frac{n^{d+1}}{k^{d+2}} + \frac{n^{d-1}}{k^{d-1}}\right)$.

[1]Elekes-Tóth actually proved this only for $\beta < \beta_d$ for some small β_d. It is later shown in [2] that we can take $\beta_d = \frac{1}{d-1}$ and in [4] that we can take $\beta_d = 1$.

3 Proof in four dimensions

We first outline our strategy. Let $\mathcal{S}$ be a set of m points, $\mathcal{H}$ be a set of n hyperplanes in $\mathbb{R}^4$. There are two ways to form a big $K_{r,s}$ in the incidence graph $G(\mathcal{H}, \mathcal{S})$: either a plane contains many points of $\mathcal{S}$ and belongs to many hyperplanes of $\mathcal{H}$, or a line does. By an averaging argument, we can assume each hyperplane is $\frac{I}{m}$-rich (i.e. contains at least $\frac{I}{m}$ points of $\mathcal{S}$). By Theorem 2.1, the contribution from β-nondegenerate hyperplanes is negligible, so we can assume each hyperplane is β-degenerate, i.e. it contains some plane with at least β of the total number of points, hence the plane is $\beta\frac{I}{m}$-rich. In this case, we say each hyperplane *degenerates* to a rich plane. Either one of those planes belongs to many hyperplanes, which would form a big $K_{r,s}$, or we can find a subset $\mathcal{P}_i$ of planes such that $I(\mathcal{S}, \mathcal{P}_i)$ is large. We repeat our argument: using the averaging argument and Corollary 2.2, we can assume each point in $\mathcal{S}$ belongs to many planes in $\mathcal{P}_i$ and degenerates to a line. Either one of those lines contains many points, which then form a big $K_{r,s}$, or we can find a subset $\mathcal{L}_j$ of lines such that $I(\mathcal{L}_j, \mathcal{P}_i)$ is large. But after some transformation, this number is the same with the number of incidences between points and lines in $\mathbb{R}^2$ and hence cannot be too large by Theorem 2.1 for $d = 2$, or equivalently, Szemerédi-Trotter's theorem in [5].

We now give the detailed proof.

Proof of Theorem 1.2. Assume $I \geq C_4(mn^{2/3} + nm^{3/5})$ for some big constant C_4 chosen later, but the incidence graph $G(\mathcal{S}, \mathcal{H})$ with I edges contains no $K_{r,s}$ of size $rs \gtrsim \left(\frac{I}{mn}\right)^{5/2} mn(\log mn)^{-4}$. We follow several steps to derive a contradiction.

Step 1: We can assume each hyperplane is $\frac{I}{4n}$-rich and β-degenerate with respect to $\mathcal{S}$ for some $\beta > 0$.

Indeed, remove all the hyperplanes that contain fewer than $\frac{I}{4n}$ points and the hyperplanes that is β-non-degenerate. The number of incidences from the non-rich hyperplanes is at most $n\frac{I}{4n} = \frac{I}{4}$. By Theorem 2.1, the number of incidences from the β-non-degenerate hyperplanes is at most $C_{\beta,4}((mn)^{4/5}+mn^{2/3}) < \frac{C_4}{4}(mn^{2/3} + nm^{3/5})$ for big enough C_4. Indeed, this only fails if $(mn)^{4/5} \gtrsim mn^{2/3}$ and $(mn)^{4/5} \gtrsim nm^{3/5}$, which is equivalent to $m \lesssim n^{2/3}$ and $n \lesssim m$, but they cannot happen at the same time for appropriate choices of constants. Therefore, after the removal, there remains at least $\frac{I}{2}$ incidences left. Assume there are n_1 hyperplanes left, where $n_1 \leq n$. In fact, throughout the proof, we always use n to upper bound n_1, so we can simply assume $n_1 = n$.

Step 2: For each $\frac{I}{4n}$-rich β-degerenate hyperplane H, we can find a plane $P \subset H$ so that $|P| \geq \beta|H| \geq \frac{\beta I}{4n}$. Let $\mathcal{P}$ denote the set of these planes. We claim that no plane in $\mathcal{P}$ belongs to more than s_0 hyperplanes in $\mathcal{H}$ where

$$s_0 := \frac{c_1 I^{3/2}}{m^{3/2}n^{1/2}(\log mn)^4} \tag{5}$$

Indeed, assume there are $\frac{c_1 I^{3/2}}{m^{3/2}n^{1/2}(\log mn)^4}$ hyperplanes that degenerate to a same plane for some constant c_1, then we have a configuration of $K_{r,s}$ where

$$rs \geq \frac{\beta I}{4n} \cdot \frac{c_1 I^{3/2}}{m^{3/2}n^{1/2}(\log mn)^4} \geq C'\left(\frac{I}{mn}\right)^{5/2} mn(\log mn)^{-4}$$

if we choose $C' < \frac{\beta c_1}{4}$. Contradiction.

Step 3: We use a dyadic decomposition to find a subset of planes with lots of incidences with $\mathcal{S}$. Let $\mathcal{P}_j$ denote the set of all planes that is assigned to at least 2^j and at most 2^{j+1} hyperplanes where $j < \log s_0 < \log n$ (here the logarithm is in base 2). We claim that there exists some i such that

$$I' := I(\mathcal{S}, \mathcal{P}_i) > 4C_{\beta,3}(|\mathcal{P}_i||\mathcal{S}|)^{3/4} + |\mathcal{P}_i||\mathcal{S}|^{1/2} \tag{6}$$

where β is the same as before, and $C_{\beta,3}$ is defined in Theorem 2.1.

Indeed, first notice that the contribution to incidences from the planes must be at least β-fraction the number of incidences from the β-degenerate hyperplanes, which implies $\sum_{j=0}^{\log s_0} 2^{j+1} I(\mathcal{S}, \mathcal{P}_j) \geq \frac{\beta}{4} I$. Hence there must exist some i such that

$$2^{i_0+1} I(\mathcal{S}, \mathcal{P}_i) \geq \frac{\beta I}{4 \log s_0}.$$

On the other hand, since each hyperplane is assigned to exactly one plane, we have $\sum_{j=0}^{\log s_0} 2^j |\mathcal{P}_j| \leq n_2 \leq n$. As a consequence, $|P_i| \leq \frac{n}{2^i}$. From (5), we have $2^i \leq s_0 \leq \frac{c_1 I^{3/2}}{m^{3/2} n^{1/2} (\log mn)^4}$. Assume (6) fails, then

$$\begin{aligned}
\frac{\beta I}{4 \log s_0} &\leq 2^{i+1} I' \\
&\leq 2^{i+1} 4 C_{\beta,3} \left((|\mathcal{P}_i||\mathcal{S}|)^{3/4} + |\mathcal{P}_i||\mathcal{S}|^{1/2} \right) \\
&\leq 8 C_{\beta,3} 2^i \left(\left(\frac{nm}{2^i} \right)^{3/4} + \frac{n}{2^i} m^{1/2} \right) \\
&\leq 8 C_{\beta,3} \left((mn)^{3/4} \left(\frac{c_1 I^{3/2}}{m^{3/2} n^{1/2} (\log mn)^4} \right)^{1/4} + nm^{1/2} \right)
\end{aligned}$$

Since $I \geq Cnm^{3/5} \gg nm^{1/2} \log s_0$, the first term in the right hand side must be at least $\frac{\beta I}{8 \log s_0}$. Rearranging we get

$$I^{5/8} \leq c_3 m^{3/8} n^{5/8} \frac{\log s_0}{\log mn}$$

where c_3 depends on $\beta, C_{\beta,3}$ and c_1. However, we can choose β and c_1 small enough and C_4 big enough so that $c_3^{8/5} < C_4$ and hence this contradicts with $I \geq C_4 nm^{3/5}$. So (6) must hold.

Step 4: Since the bound in (6) is the same with that in Corollary 2.2, we can use a similar argument with Step 1 to assume each point in S is $\frac{I'}{4m}$-rich (i.e. belongs to at least $\frac{I'}{4m}$ planes in $\mathcal{P}_i$), and is β-degenerate w.r.t. $\mathcal{P}_i$ (in the sense defined before Corollary 2.2. Each such point degenerates to a line that is $\beta \frac{I'}{4m}$-rich. Let $\mathcal{L}$ denote the set of all these lines. We claim that no line in $\mathcal{L}$ contains more than r_0 points where

$$r_0 := \frac{c_5 I^{3/2}}{m^{1/2} n^{3/2} (\log mn)^3} \tag{7}$$

Indeed, each line in $\mathcal{L}$ belongs to at least $\frac{\beta I'}{4m}$ planes in $\mathcal{P}_i$, and thus belongs to at least $\frac{\beta I' 2^i}{4m} \geq \frac{\beta I}{4m \log s_0}$ hyperplanes in $\mathcal{H}$ because each plane in $\mathcal{P}_i$ belongs to at least 2^i hyperplanes. If there are $\frac{c_5 I^{3/2}}{m^{1/2} n^{3/2} (\log mn)^3}$ points that degenerates (or belongs) to a same line for some constant c_5, then we have a configuration of $K_{r,s}$ where

$$rs \geq \frac{\beta I}{4m \log s_0} \cdot \frac{c_5 I^{3/2}}{m^{1/2} n^{3/2} (\log mn)^3} \geq C_4' \left(\frac{I}{mn} \right)^{5/2} mn (\log mn)^{-4}$$

for small enough C_4'. Contradiction.

Step 5: Similar to Step 3, we use a dyadic decomposition to find a subset of lines in $\mathcal{L}$ that has many incidences with $\mathcal{P}$. Here we say a line ℓ is incident to a plane P if $\ell \subset P$. Let $\mathcal{L}_k$ denote the set of all lines that contain at least 2^k and at most 2^{k+1} points where $k < \log r_0 < \log m$. Note that here for a line we only consider the points that degenerate to that line. We claim there must exist some j such that

$$I'' := I(\mathcal{L}_j, \mathcal{P}_i) \geq C_{\beta,2} \left(|\mathcal{P}_i|^{2/3} |\mathcal{L}_j|^{2/3} + |\mathcal{P}_i| + |\mathcal{L}_j| \right) \tag{8}$$

Indeed, first notice that the contribution to incidences from the lines must be at least β-fraction, which implies $\sum_{k=0}^{\log r_0} 2^{k+1} I(\mathcal{L}_k, \mathcal{P}_i) \geq \frac{\beta}{4} I'$. Hence there must exist some j such that

$$2^{j+1} I(\mathcal{L}_j, \mathcal{P}_i) \geq \frac{\beta I'}{4 \log r_0} \geq \frac{\beta^2 I}{16 \log s_0 \log r_0 2^i}.$$

On the other hand, since each point is assigned to exactly one line we have $\sum_{k=0}^{\log r_0} 2^k|\mathcal{L}_k| \leq m$. As a consequence, $|\mathcal{L}_j| \leq \frac{m}{2^j}$. From (7), $2^j \leq r_0 \leq \frac{c_5 I^{3/2}}{m^{1/2}n^{3/2}(\log mn)^3}$. Recall $|\mathcal{P}_i| \leq \frac{n}{2^i}$ and $2^i \leq s_0$. Assume (8) fails, then there exists some constant c_6 such that

$$
\begin{aligned}
I &\leq 32\beta^{-1}(\log r_0 \log s_0)2^{i+j} I(P_i, S_j) \\
&\leq c_6(\log m \log n)\left((s_0 r_0)^{1/3}(mn)^{2/3} + nr_0 + ms_0\right) \\
&\leq c_6(\log m \log n)\left((mn)^{2/3}\left(\frac{c_1 I^{3/2}}{m^{3/2}n^{1/2}(\log mn)^4}\frac{c_5 I^{3/2}}{m^{1/2}n^{3/2}(\log mn)^3}\right)^{1/3} + m\frac{c_1 I^{3/2}}{m^{3/2}n^{1/2}(\log mn)^4} + n\frac{c_5 I^{3/2}}{m^{1/2}n^{3/2}(\log mn)^3}\right) \\
&\leq I\left[\frac{c_1 c_5 c_6 \log m \log n}{(\log mn)^7} + \left(\frac{I}{mn}\right)^{1/2}\left(\frac{c_1 \log m \log n}{(\log mn)^4} + \frac{c_5 \log m \log n}{(\log mn)^3}\right)\right] \\
&\ll I.
\end{aligned}
$$

This contradiction implies (8) must hold.

Step 6: We claim that (8) violates Theorem 2.1 in two dimensions, or Szemerédi-Trotter's theorem.

Indeed, project the set of planes $\mathcal{P}_i$ and the set of lines $\mathcal{L}_j$ to a generic three dimensional subspace, then intersect them with generic plane Π within this subspace. After this transformation, $\mathcal{P}_i$ becomes a set of lines P^* and $\mathcal{L}_j$ becomes a set of points L^* in Π. We have $I(P^*, L^*) = I(\mathcal{L}_j, \mathcal{P}_i) \gtrsim |P^*|^{2/3}|L^*|^{2/3} + |P^*| + |L^*|$. This violation finishes our proof. □

4 Sketch of proof in five dimensions

The proof method is the same with that in four dimensions, but the exponents are different and the method is repeated one more time. In particular, we unwrap in three layers: hyperplanes degenerate to 3-flats, points degenerate to lines, and 3-flats degenerate to planes. At each layer, either we can find a big $K_{r,s}$, or the number of incidences remain larger than the nondegenerate bound in Theorem 2.1, and we can keep unwrapping. The detailed proof is quite similar to that in the four dimensions case, so we only give an outline here. For simplicity, we ignore all the constants and logarithmic factors.

Proof's sketch of Theorem 1.3. Prove by contradiction. Let $\mathcal{S}$ denote the set of m points and $\mathcal{H}$ denote the set of n hyperplanes in $\mathbb{R}^5$. Assume $I(\mathcal{S}, \mathcal{H}) \geq C_5(mn^{3/4} + ...)$ but their incidence graph does not contain any K_{rs} where $rs \gtrsim \left(\frac{I}{mn}\right)^3 mn(\log mn)^{-10}$.

Step 1 We can assume every hyperplane is $\frac{I}{n}$-rich, and β-degenerate with respect to $\mathcal{S}$ for some $\beta > 0$.

Step 2 For each such hyperplane H, we can find a 3-dimensional flat (or a 3-flat) F such that $F \subset H$ and $|F \cap \mathcal{S}| \geq \beta|H \cap \mathcal{S}| \geq \frac{\beta I}{n}$. Let $\mathcal{F}$ denote the set of these 3-flats. We show that no flat in $\mathcal{F}$ belong to more than s_0 hyperplanes where $s_0 \lesssim \frac{I^2}{m^2 n}$.

Step 3 Let $\mathcal{F}_j$ denote the set of all 3-flats in $\mathcal{F}$ that is assigned to at least 2^j and at most 2^{j+1} hyperplanes where $j \leq \log s_0 < \log n$. We show that there exists an i such that

$$I' := I(\mathcal{F}_i, \mathcal{S}) \gtrsim (|\mathcal{F}_i||\mathcal{S}|)^{4/5} + |\mathcal{F}_i||\mathcal{S}|^{2/3}.$$

Indeed, assume otherwise. Using $I' \gtrsim 2^i I$, $|\mathcal{F}_i| \leq \frac{n}{2^i}$ and $2^i \leq s_0 \lesssim \frac{I^2}{m^2 n}$, we have

$$I \lesssim 2^i I' \lesssim 2^i \left[\left(\frac{nm}{2^i}\right)^{4/5} + \frac{n}{2^i}m^{2/3}\right] \lesssim (mn)^{4/5}\left(\frac{I^2}{m^2 n}\right)^{1/5} + nm^{2/3}$$

which cannot happen given our condition $I \gtrsim mn^{4/5} + nm^{2/3}$.

Step 4 Since I' is large, using Corollary 2.2, we can assume each point in $\mathcal{S}$ is $\frac{I'}{m}$-rich (i.e. belongs to at least $\frac{I'}{m}$ flats in $\mathcal{F}_i$, and is β-degenerate w.r.t. $\mathcal{F}_i$. Each such point degenerates to a $\frac{\beta I'}{m}$-rich line. Let $\mathcal{L}$ denote that set of these lines. Then no line in $\mathcal{L}$ can contain more than r_0 points where $r_0 \lesssim \frac{I^2}{mn^2}$.

Step 5 We use a dyadic decomposition to find a subset of lines with many incidences with $\mathcal{F}_i$. Let $\mathcal{L}_k$ denote the set of all lines in $\mathcal{L}$ that contain more than 2^k and fewer than 2^{k+1} points. Then there exists a j such that

$$I'' := I(\mathcal{F}_i, \mathcal{L}_j) \gtrsim |\mathcal{F}_i|^{3/4}|\mathcal{L}_j|^{3/4} + |\mathcal{F}_i||\mathcal{L}_j|^{1/2}.$$

Indeed, assume otherwise. Using $I'' \gtrsim I'/2^j \gtrsim I/2^{i+j}$ $|\mathcal{F}_i| \le \frac{n}{2^i}$, $|\mathcal{L}_j| \le \frac{m}{2^j}$, $2^i \le s_0 \lesssim \frac{I^2}{m^2 n}$ and $2^j \le r_0 \lesssim \frac{I^2}{mn^2}$, we have

$$I \lesssim 2^{i+j} I'' \lesssim 2^{i+j}\left[\left(\frac{mn}{2^{i+j}}\right)^{3/4} + \frac{n}{2^i}\left(\frac{m}{2^j}\right)^{1/2}\right] \lesssim (mn)^{3/4}\left(\frac{I^2}{m^2 n}\frac{I^2}{mn^2}\right)^{1/4} + nm^{1/2}\left(\frac{I^2}{mn^2}\right)^{1/2} = 2I$$

This cannot happen with an appropriate choice of constants and logarithmic factors.

Step 6 Turn $I(\mathcal{F}_i, \mathcal{L}_j)$ into point-plane incidences in $\mathbb{R}^3$ by some transformation. This means we can assume each 3-flats in $\mathcal{F}_i$ degenerate to a plane. Let $\mathcal{P}$ denote the set of all such planes. Then no plane belongs to more than t_0 flats in $\mathcal{F}_i$ where $t_0 \lesssim \frac{I^2}{m^2 n}$

Step 7 Using a dyadic decomposition, there exists some subset $\mathcal{P}_k$ of planes, each belongs to at least 2^k and at most 2^{k+1} planes in $\mathcal{F}_i$ such that $I''' := I(\mathcal{L}_j, \mathcal{P}_k) \gtrsim |\mathcal{P}_k|^{2/3}|L_j|^{2/3} + |\mathcal{P}_k| + |\mathcal{L}_j|$.

Step 8 Turn I''' into point-line incidences, which leads to a violation with Szemerédi-Trotter's theorem. This finishes our proof.

$\square$

Remark 4.1. Our argument does not work in six dimensions and higher because when we write down the details of step 5 in the above outline, the exponents no longer match and we do not get a contradiction.

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