HÀM SỐ VÀ GIỚI HẠN

Các cặp vô cùng bé tương đương thông dụng

Dạng cơ bản: $x \to 0$	Dạng mở rộng: $u \to 0$ khi $x \to a$
$\sin x \sim x$	$\sin u \sim u$
$\arcsin x \sim x$	$\arcsin u \sim u$
$\tan x \sim x$	$\tan u \sim u$
$\arctan x \sim x$	$\arctan u \sim u$
$(1-\cos x) \sim \frac{x^2}{2}$	$(1-\cos u) \sim \frac{u^2}{2}$
$\ln(1+x) \sim x$	$\ln(1+u) \sim u$
$(e^x - 1) \sim x$	$(e^u - 1) \sim u$

Bài tập về nhà

$L_1=\lim\limits_{x\to 0}\dfrac{x\ln(\cos 3x)}{\arctan(\sin^3 x)}$	$L_2=\lim\limits_{x\to 0}\dfrac{\sin x-x}{x\ln(\cos x)}$	$L_3=\lim\limits_{x\to 0^+}(\arctan x)^{2x-x^2}$
$L_4=\lim\limits_{x\to 0}\dfrac{2(\tan x-\sin x)-x^3}{x^5}$	$L_5=\lim\limits_{x\to 0}\dfrac{1}{x}\left(\dfrac{1}{5x}-\dfrac{1}{\tan 5x}\right)$	$L_6=\lim\limits_{x\to +\infty}\left(x^2+3^x\right)^{\frac{4}{x}}$
$L_7=\lim\limits_{x\to 0}\left(3^x-x\ln 3\right)^{\frac{1}{x^2}}$	$L_8=\lim\limits_{x\to -\infty}\left(e^{-5x}+\sin 7x\right)^{\frac{2}{x}}$	$L_9=\lim\limits_{x\to \frac{\pi}{6}}\left(\dfrac{6x}{\pi}+\cos 3x\right)^{\tan 3x}$

Kết quả: $L_1=-\dfrac{9}{2}$; $L_2=1$; $L_3=0$; $L_4=\dfrac{1}{4}$; $L_5=\dfrac{5}{3}$; $L_6=e^{4\ln 3}=3^4=81$; $L_7=e^{\ln^2 3}$; $L_8=e^{-10}$; $L_9=e^{\frac{\pi-2}{\pi}}$

Kết quả	Hướng dẫn
$L_1=-\dfrac{9}{2}$	Sử dụng các cặp VCB tương đương sau, khi $u\to 0$ $\ln(1+u)\sim u$; $\arctan u\sim u$; $\sin u\sim u$; $(1-\cos u)\sim\dfrac{u^2}{2}$
$L_2=\dfrac{1}{3}$	Sử dụng các cặp VCB tương đương sau, khi $u\to 0:\ln(1+u)\sim u,\ (1-\cos u)\sim\dfrac{u^2}{2}$ Sau đó sử dụng quy tắc Lopitan
$L_3=1$	Xử lý bằng phương pháp Logarit hóa Sử dụng cặp vô cùng bé $\left(2x-x^3\right)\sim 2x$ trước khi biến đổi để Lopitan Sau đó dùng tiếp cặp $\arctan x\sim x$
$L_4=\dfrac{1}{4}$	Lopitan 2 lần liên tiếp rồi dừng lại (biểu thức lúc này quá cồng kềnh), tách biểu thức thành tổng của 2 giới hạn đơn giản hơn!
$L_5=\dfrac{5}{3}$	Quy đồng mẫu, rồi dùng VCB tương đương trước khi Lopitan
$L_6=81$	Logarit hóa, sau đó sử dụng Lopitan
$L_7=e^{\frac{1}{2}\ln^2 3}$	Logarit hóa, Lopitan kết hợp với việc tách thành tích 2 của giới hạn đơn giản
$L_8=e^{-10}$	Logarit hóa, Lopitan, sử dụng quy tắc Kẹp
$L_9=e^{\frac{\pi-2}{\pi}}$	Logarit hóa, dùng cặp $\ln(1+u)\sim u$, Lopitan

Đáp án

$$L_1=\lim_{x\to 0}\frac{x\ln(\cos 3x)}{\arctan(\sin^3 x)}=\lim_{x\to 0}\frac{x\ln[1+(\cos 3x-1)]}{\arctan(\sin^3 x)}\overset{(1)}{=}\lim_{x\to 0}\frac{x(\cos 3x-1)}{\sin^3 x}\overset{(2)}{=}\lim_{x\to 0}\frac{x\left[-\frac{(3x)^2}{2}\right]}{x^3}=\lim_{x\to 0}\left(-\frac{9}{2}\right)=-\frac{9}{2}$$

$$(1)\begin{cases}\ln(1+u)\sim u\\ \arctan u\sim u\end{cases};u\to 0;\quad (2)\begin{cases}(1-\cos u)\sim\frac{u^2}{2}\\ \sin u\sim u\end{cases};u\to 0$$

$$L_2=\lim_{x\to 0}\frac{\sin x-x}{x\ln(\cos x)}=\lim_{x\to 0}\frac{\sin x-x}{x\ln[1+(\cos x-1)]}\overset{(1)}{=}\lim_{x\to 0}\frac{\sin x-x}{x(\cos x-1)}\overset{(2)}{=}\lim_{x\to 0}\frac{\sin x-x}{x\left(-\frac{x^2}{2}\right)}=-2\lim_{x\to 0}\frac{\sin x-x}{x^3}\overset{(L)}{=}-2\lim_{x\to 0}\frac{\cos x-1}{3x^2}\overset{(L)}{=}$$

$$=-2\lim_{x\to 0}\frac{-\sin x}{6x}=\frac{1}{3}\qquad (1):\ln(1+u)\sim u,u\to 0\qquad (2):(1-\cos u)\sim\frac{u^2}{2};u\to 0$$

$$L_3=\lim_{x\to 0^+}(\arctan x)^{2x-x^2}$$

Đặt $y=(\arctan x)^{2x-x^2}\Leftrightarrow \ln y=(2x-x^2)\ln(\arctan x)$

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}(2x-x^2)\ln(\arctan x)\overset{(1)}{=}\lim_{x\to 0^+}2x.\ln(\arctan x)=\lim_{x\to 0^+}\frac{2\ln(\arctan x)}{\frac{1}{x}}\overset{(2)}{=}\lim_{x\to 0^+}\frac{2\frac{1}{(1+x^2)\arctan x}}{-\frac{1}{x^2}}=$$

$$=\lim_{x\to 0^+}\frac{-2x^2}{(1+x^2)\arctan x}\overset{(2)}{=}\lim_{x\to 0^+}\frac{-2x^2}{(1+x^2)x}=\lim_{x\to 0^+}\frac{-2x}{1+x^2}=0;\quad (1):(2x-x^2)\sim 2x;x\to 0\ (2):\arctan x\sim x;\ x\to 0$$

$$L_3=e^{\lim_{x\to 0^+}\ln y}=e^0=1$$

$$L_4=\lim_{x\to 0}\frac{2(\tan x-\sin x)-x^3}{x^5}\overset{(L)}{=}\lim_{x\to 0}\frac{2(1+\tan^2 x)-2\cos x-3x^2}{5x^4}\overset{(L)}{=}\lim_{x\to 0}\frac{4\tan(1+\tan^2 x)+2\sin x-6x}{20x^3}=$$

$$=\lim_{x\to 0}\frac{2\tan^3 x+2\tan x+\sin x-3x}{10x^3}\overset{(L)}{=}\lim_{x\to 0}\frac{6\tan^2 x(1+\tan^2 x)+2(1+\tan^2 x)+\cos x-3}{30x^2}=$$

$$=\lim_{x\to 0}\frac{6\tan^4 x+8\tan^2 x+\cos x-1}{30x^2}\overset{(L)}{=}\lim_{x\to 0}\frac{24\tan^3 x(1+\tan^2 x)+16\tan x(1+\tan^2 x)-\sin x}{60x}=$$

$$=\lim_{x\to 0}\left[\frac{1}{15}\frac{\tan x}{x}.\tan^2 x.(1+\tan^2 x)+\frac{16}{60}\frac{\tan x}{x}(1+\tan^2 x)-\frac{1}{60}\frac{\sin x}{x}\right]=0+\frac{16}{60}-\frac{1}{60}=\frac{1}{4}$$

$$L_5=\lim_{x\to 0}\frac{1}{x}\left(\frac{1}{5x}-\frac{1}{\tan 5x}\right)=\lim_{x\to 0}\frac{\tan 5x-5x}{5x^2\tan 5x}\overset{(*)}{=}\lim_{x\to 0}\frac{\tan 5x-5x}{5x^2.5x}=\lim_{x\to 0}\frac{\tan 5x-5x}{25x^3}\overset{(L)}{=}\lim_{x\to 0}\frac{5(1+\tan^2 5x)-5}{75x^2}=\lim_{x\to 0}\frac{\tan^2 5x}{15x^2}=$$

$$=\lim_{x\to 0}\frac{5}{3}\left(\frac{\tan 5x}{5x}\right)^2=\frac{5}{3};\qquad (*):\tan u\sim u,\ u\to 0$$

$L_6=\lim\limits_{x\to+\infty}\left(x^2+3^x\right)^{\frac{4}{x}}$:Đặt $y=\left(x^2+3^x\right)^{\frac{4}{x}}\Leftrightarrow \ln y=\dfrac{4\ln\left(x^2+3^x\right)}{x}$

$$\lim_{x\to+\infty}\ln y=\lim_{x\to+\infty}\frac{4\ln\left(x^2+3^x\right)}{x}\overset{(L)}{=}\lim_{x\to+\infty}\frac{4\left(2x+3^x\ln 3\right)}{x^2+3^x}\overset{(L)}{=}\lim_{x\to+\infty}\frac{4\left(2+3^x\ln^2 3\right)}{2x+3^x\ln 3}\overset{(L)}{=}\lim_{x\to+\infty}\frac{4.3^x\ln^3 3}{2+3^x\ln 3}\overset{(L)}{=}\lim_{x\to+\infty}\frac{4.3^x\ln^4 3}{3^x\ln^3 3}=4\ln 3$$

$$L_6=\lim_{x\to+\infty}y=e^{4\ln 3}=81$$

$L_7=\lim\limits_{x\to 0}\left(3^x-x\ln 3\right)^{\frac{1}{x^2}}$: Đặt $y=\left(3^x-x\ln 3\right)^{\frac{1}{x^2}}\Leftrightarrow \ln y=\dfrac{\ln\left(3^x-x\ln 3\right)}{x^2}$

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln\left(3^x-x\ln 3\right)}{x^2}=\lim_{x\to 0}\frac{\ln\left[1+\left(3^x-x\ln 3-1\right)\right]}{x^2}\overset{(*)}{=}\lim_{x\to 0}\frac{3^x-x\ln 3-1}{x^2}\overset{(L)}{=}\lim_{x\to 0}\frac{3^x\ln 3-\ln 3}{2x}\overset{(L)}{=}\lim_{x\to 0}\frac{3^x\ln^2 3}{2}=\frac{\ln^2 3}{2}$$

$L_7=\lim\limits_{x\to 0}y=e^{\frac{\ln^2 3}{2}}$; $(*):\ln(1+u)\sim u,\ u\to 0$

$L_8=\lim\limits_{x\to-\infty}\left(e^{-5x}+\sin 7x\right)^{\frac{2}{x}}$: Đặt $y=\left(e^{-5x}+\sin 7x\right)^{\frac{2}{x}}\Leftrightarrow \ln y=\dfrac{2\ln\left(e^{-5x}+\sin 7x\right)}{x}$

$$\lim_{x\to-\infty}\ln y=\lim_{x\to-\infty}\frac{2\ln\left(e^{-5x}+\sin 7x\right)}{x}\overset{(L)}{=}\lim_{x\to-\infty}\frac{2\left(-5e^{-5x}+7\cos 7x\right)}{e^{-5x}+\sin 7x}=\lim_{x\to-\infty}\frac{-10+14e^{5x}\cos 7x}{1+e^{5x}\sin 7x}$$

Ta thấy: $\lim\limits_{x\to-\infty}e^{5x}=0;\ |\sin 7x|\le 1;\ |\cos 7x|\le 1\Rightarrow \lim\limits_{x\to-\infty}e^{5x}\sin 7x=\lim\limits_{x\to-\infty}e^{5x}\cos 7x=0\Rightarrow \lim\limits_{x\to-\infty}\ln y=\dfrac{-10+0}{1+0}=-10$

$$L_8=\lim_{x\to-\infty}y=e^{-10}$$

$L_9=\lim\limits_{x\to\frac{\pi}{6}}\left(\dfrac{6x}{\pi}+\cos 3x\right)^{\tan 3x}$: Đặt $y=\left(\dfrac{6x}{\pi}+\cos 3x\right)^{\tan 3x}\Leftrightarrow \ln y=\tan 3x.\ln\left(\dfrac{6x}{\pi}+\cos 3x\right)$

$$\lim_{x\to\frac{\pi}{6}}\ln y=\lim_{x\to\frac{\pi}{6}}\tan 3x.\ln\left(\frac{6x}{\pi}+\cos 3x\right)=\lim_{x\to\frac{\pi}{6}}\frac{\ln\left[1+\left(\frac{6x}{\pi}+\cos 3x-1\right)\right]}{\cot 3x}\overset{(*)}{=}\lim_{x\to\frac{\pi}{6}}\frac{\frac{6x}{\pi}+\cos 3x-1}{\cot 3x}\overset{(L)}{=}\lim_{x\to\frac{\pi}{6}}\frac{\frac{6}{\pi}-3\sin 3x}{-3\left(1+\cot^2 3x\right)}=1-\frac{2}{\pi}$$

$L_9=\lim\limits_{x\to\frac{\pi}{6}}y=e^{1-\frac{2}{\pi}}$; $(*)\ln(1+u)\sim u,\ u\to 0$