FPT SCHOOL OF BUSINESS & TECHNOLOGY

Digital Signal Processing

Chuyển đổi Fourier số

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Phần 2:
Lý thuyết: Chuyển đổi Fourier 2-D

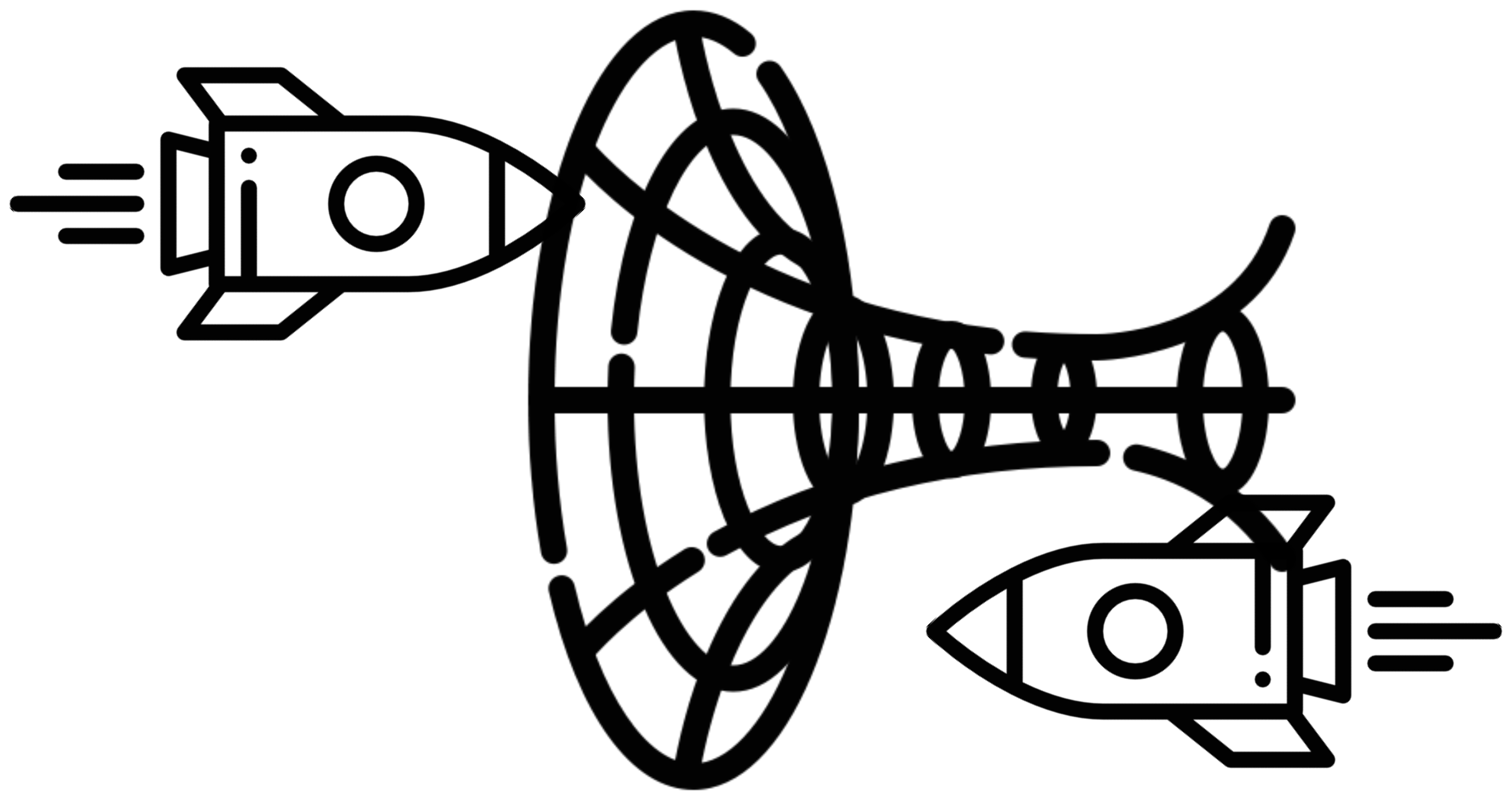
Recap

Tóm tắt chuyển đổi Fourier 1 chiều

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-i2\pi k \frac{n}{N}}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot e^{i2\pi k \frac{n}{N}}$$

- *time t*
- *Real number* $\mathbb{R}$
- $\sum_{n=0}^{\infty} x(nT_s)\delta(t - nT_s)$
- Rời rạc + không tuần hoàn
- Rời rạc + tuần hoàn



- *frequency* ν
- *Complex number* $\mathbb{C}$
- chồng chập về biên độ + pha của phổ xung Dirac
- Liên tục không tuần hoàn
- Hội tụ về rời rạc

Continuous Fourier transform

Fourier Transforms for periodic signals

Property	Equation
Linearity	$\mathcal{F}\{a \cdot f(t) + b \cdot g(t)\} = a \cdot \mathcal{F}\{f(t)\} + b \cdot \mathcal{F}\{g(t)\}$
Time Shift	$\mathcal{F}\{f(t - t_0)\} = e^{-j\omega t_0} \cdot \mathcal{F}\{f(t)\}$
Frequency Shift	$\mathcal{F}\{e^{j2\pi f_0 t} \cdot f(t)\} = F(\omega - 2\pi f_0)$
Scaling	$\mathcal{F}\{f(at)\} = \frac{1}{\lvert a \rvert} F\left(\frac{\omega}{a}\right)$
Conjugation	$\mathcal{F}\{\overline{f(t)}\} = \overline{F(-\omega)}$
Differentiation	$\mathcal{F}\{\frac{d}{dt} f(t)\} = j\omega F(\omega)$
Convolution	$\mathcal{F}\{f(t) * g(t)\} = F(\omega) \cdot G(\omega)$
Parseval's Theorem	$\int_{-\infty}^{\infty} \lvert f(t) \rvert^2 \, dt = \int_{-\infty}^{\infty} \lvert F(\omega) \rvert^2 \, d\omega$

Part 2:
2-D Fourier Transformation

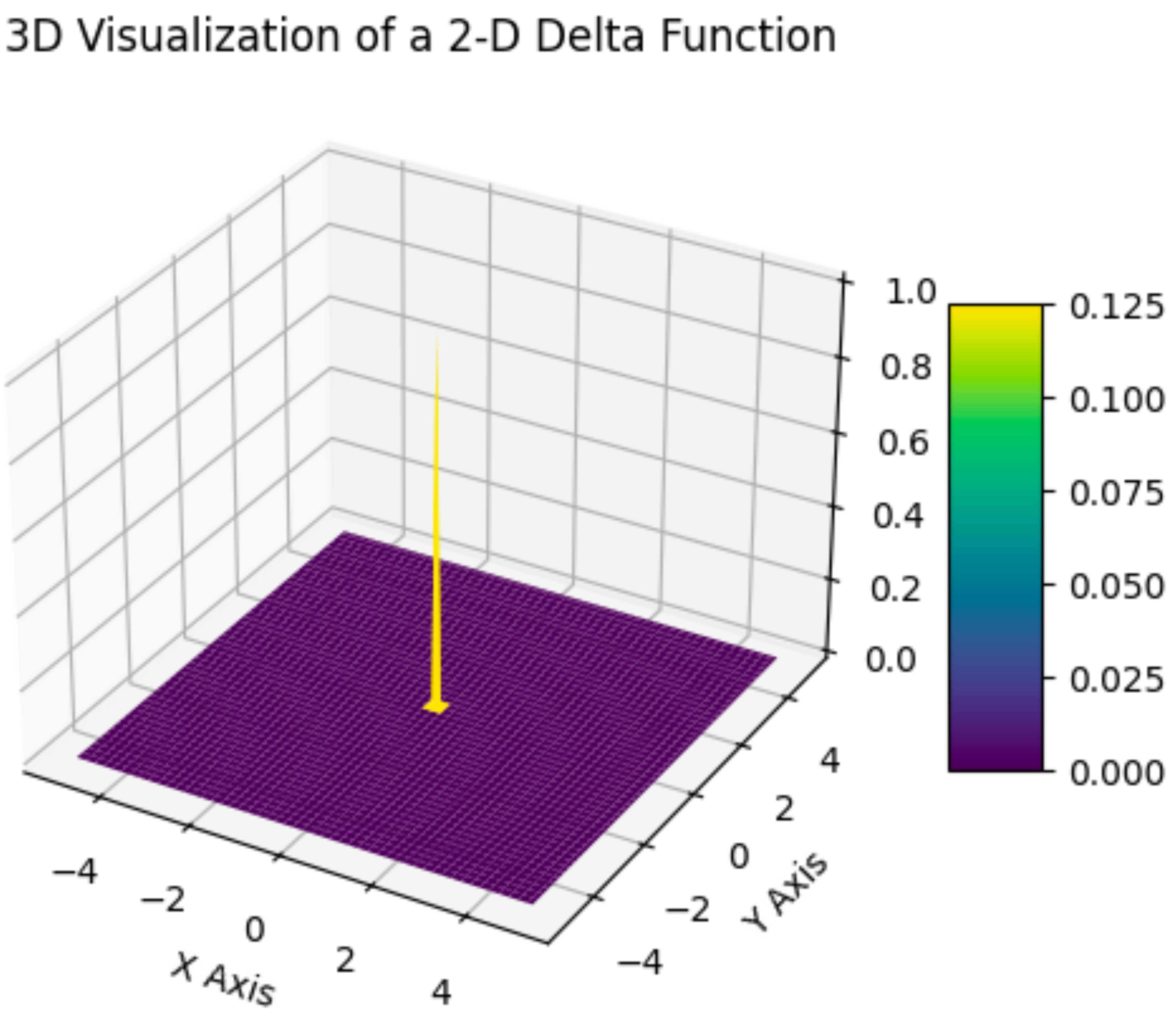
Chuyển đổi Fourier 2-D

Xung lý tưởng 2D-Impulse

$$\delta(t, z) = \begin{cases} 1, \text{ if } t = z = 0 \\ 0, \text{ otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t, z)\, dt\, dz = 1$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z)\delta(t - t_0, z - z_0)\, dt\, dz = f(t_0, z_0)$$



2D- impulse

Chuyển đổi Fourier 2-D

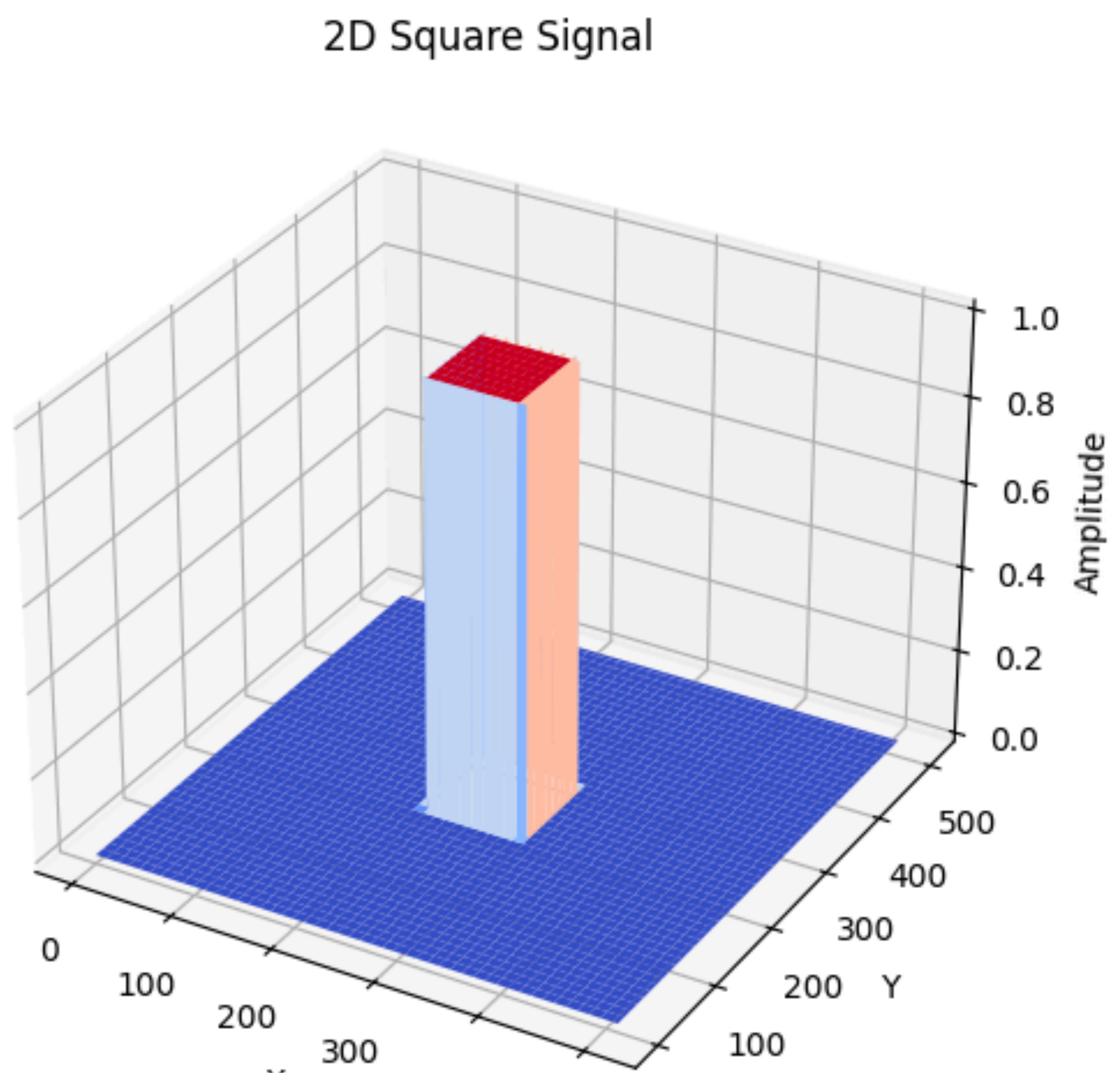
Phép chuyển đổi

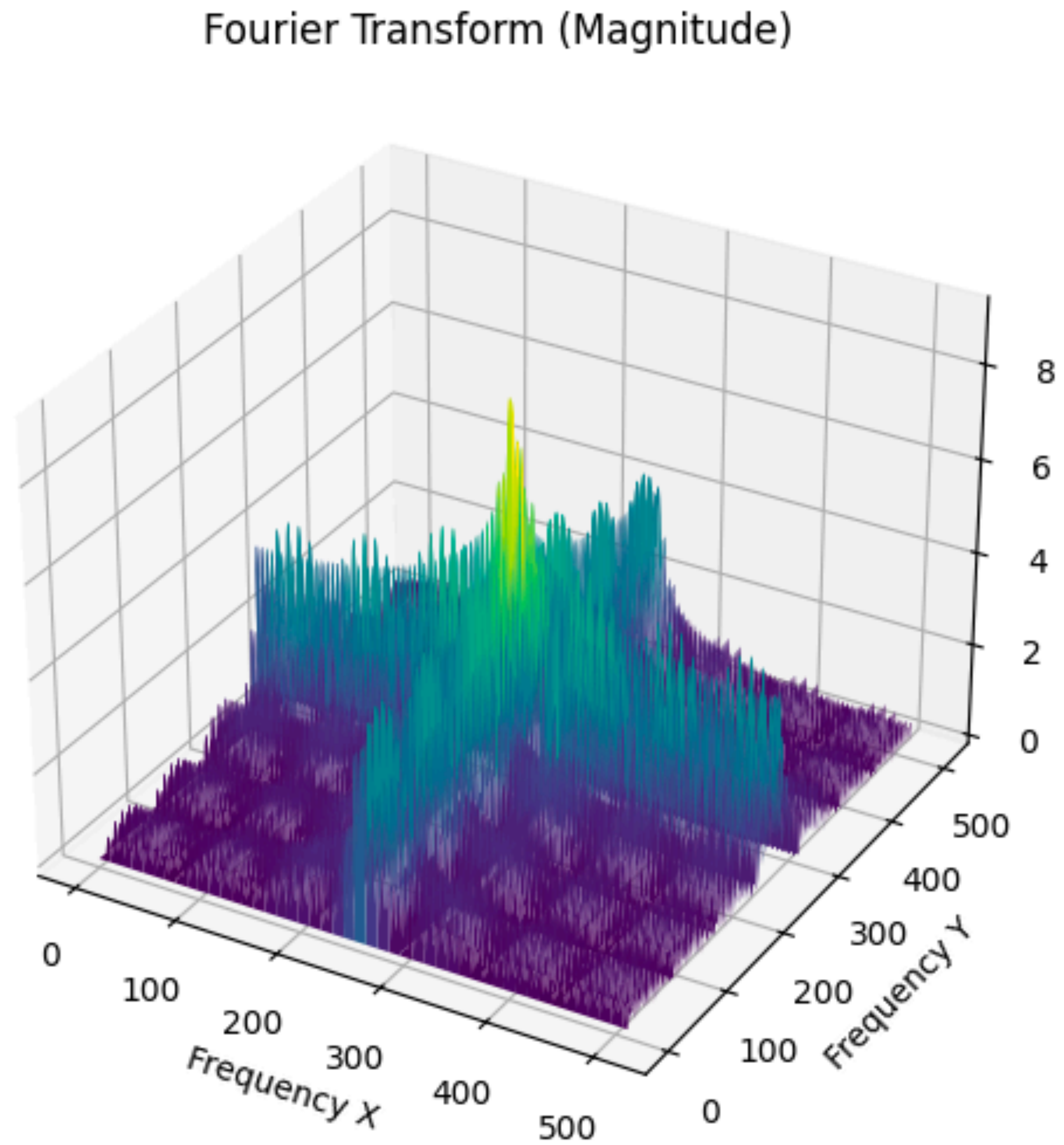
- **2D Fourier Transform**

$$F(\mu,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-i(\mu t+\nu z)}\,dt\,dz$$

- **Inverse 2D Fourier Transform**

$$f(t,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu,\nu)e^{i(\mu t+\nu z)}\,d\mu\,d\nu$$





Chuyển đổi Fourier 2-D

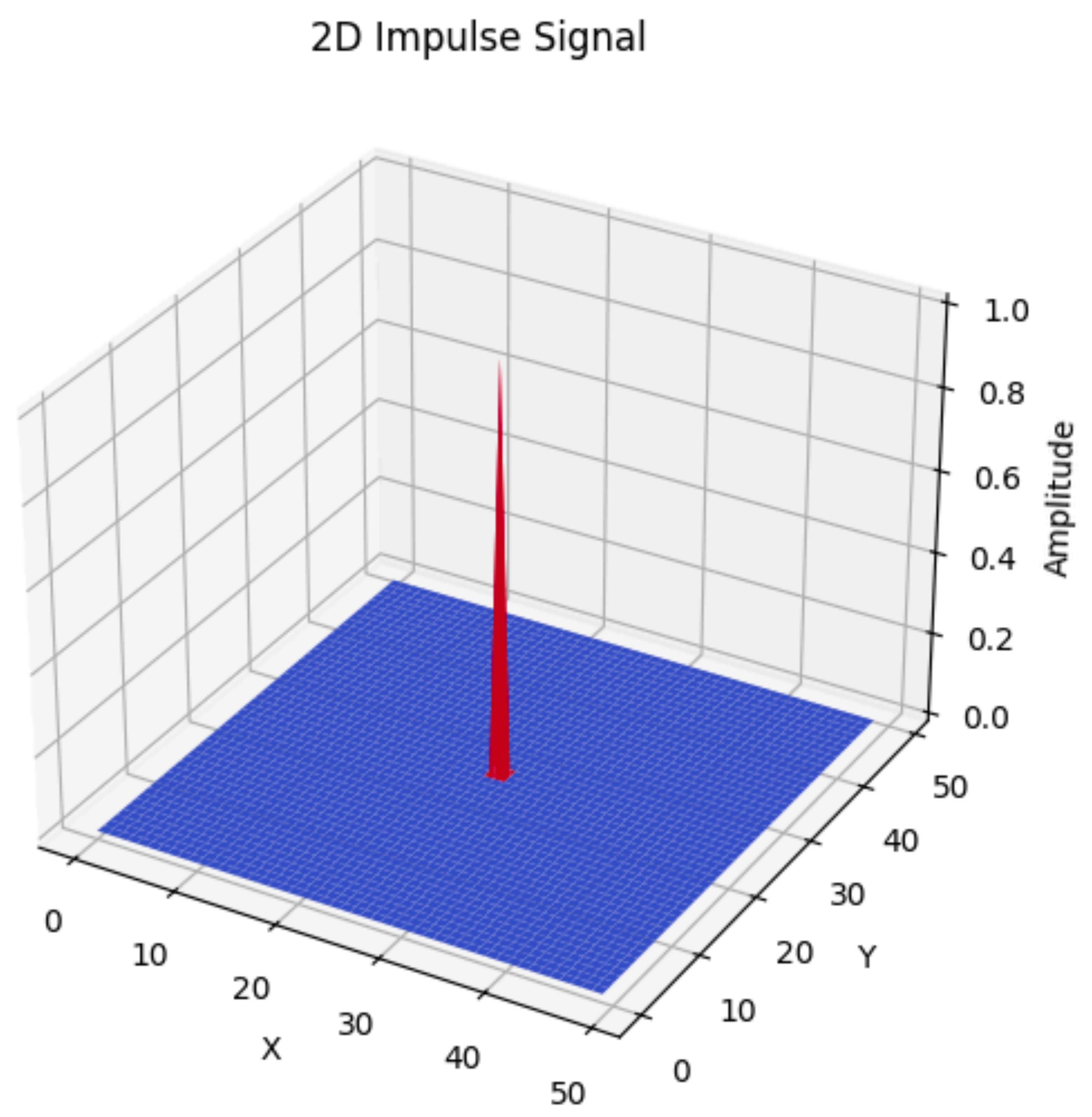
Ví dụ miền tần số của xung 2D impulse

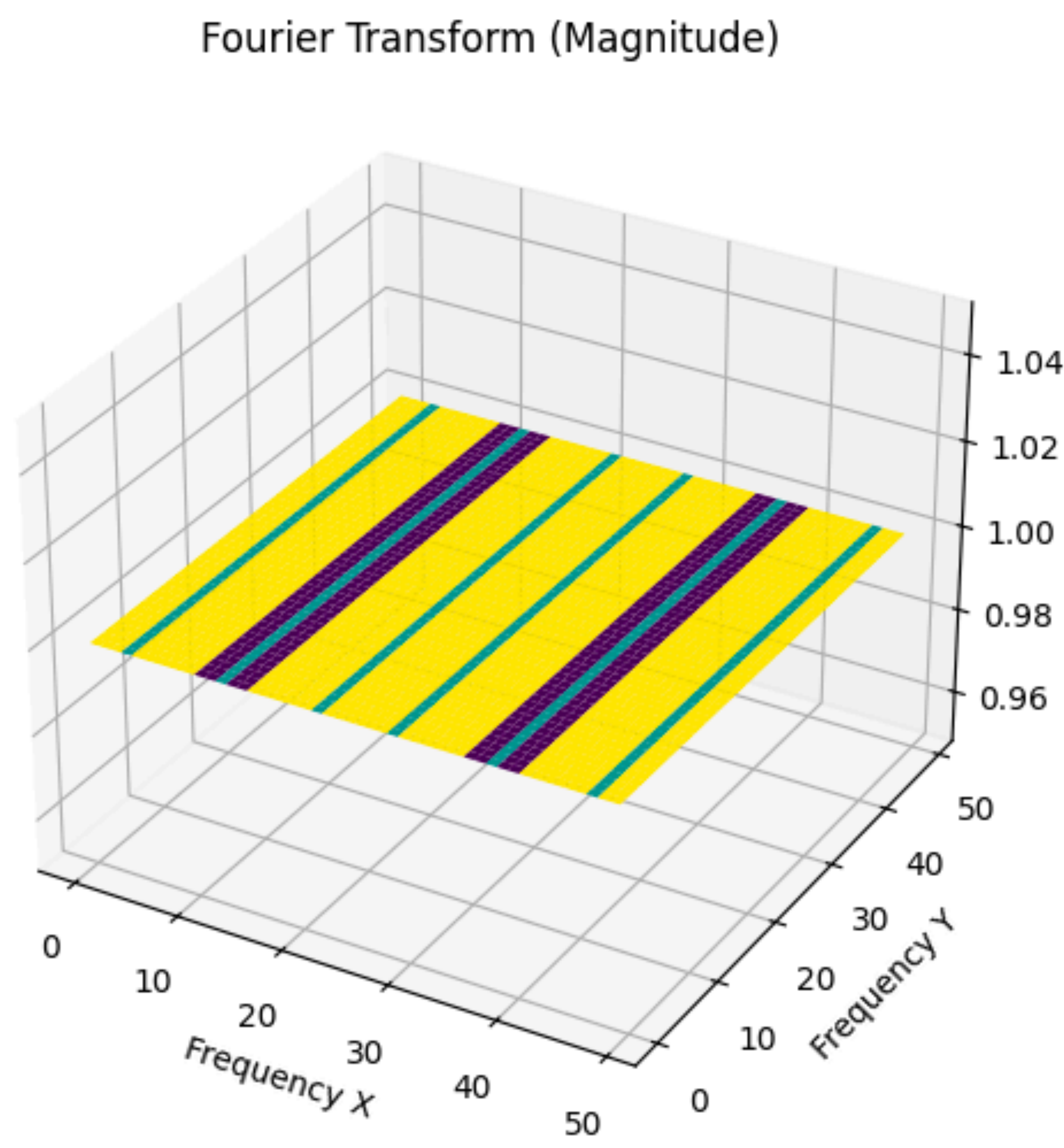
- Direct Transform:

$$F(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t, z) e^{-j2\pi(\mu t + \nu z)}\, dt\, dz = 1$$

- Inverse Transform:

$$f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 1 \cdot e^{j2\pi(\mu t + \nu z)}\, d\mu\, d\nu = \delta(t, z)$$

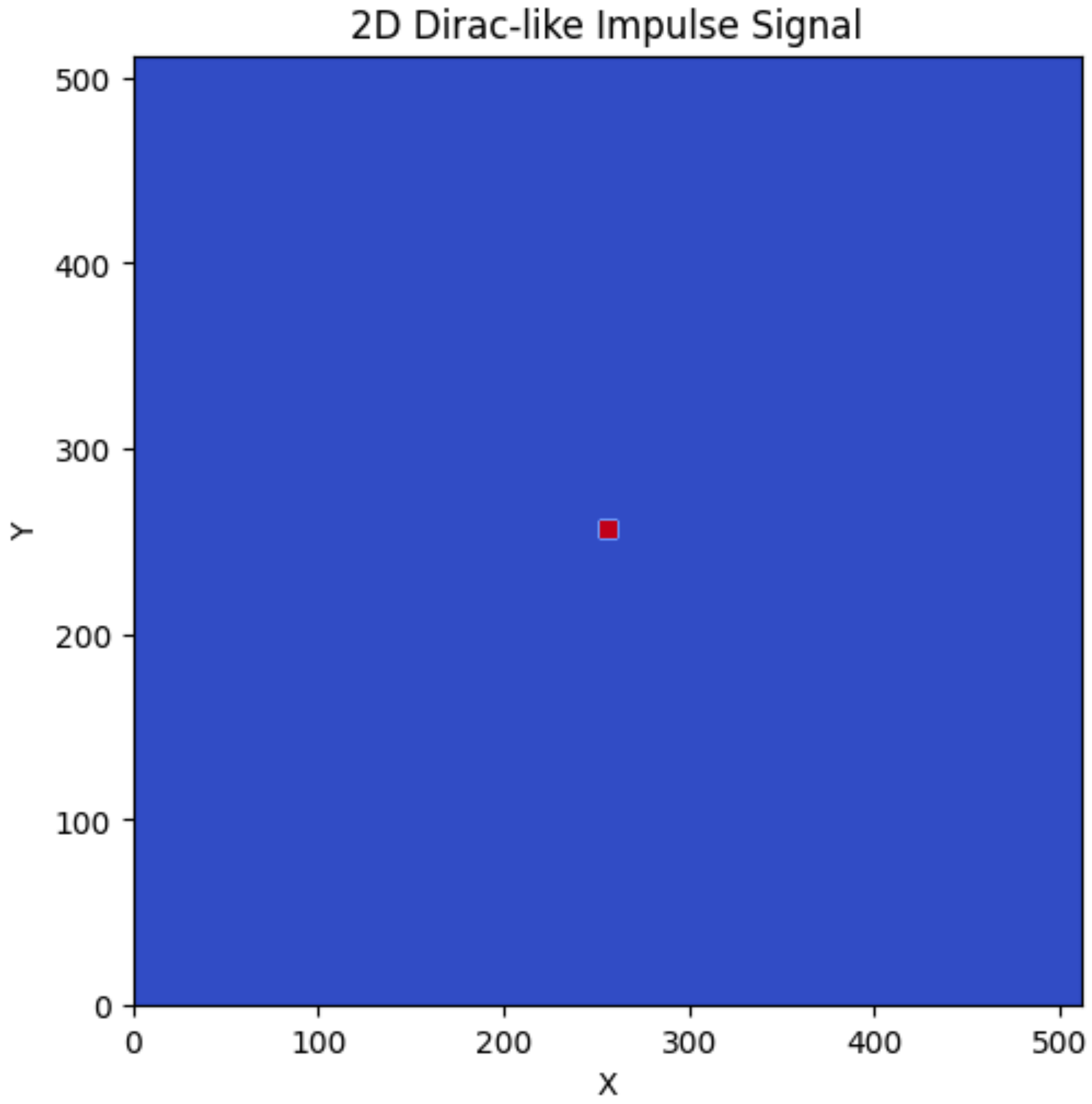


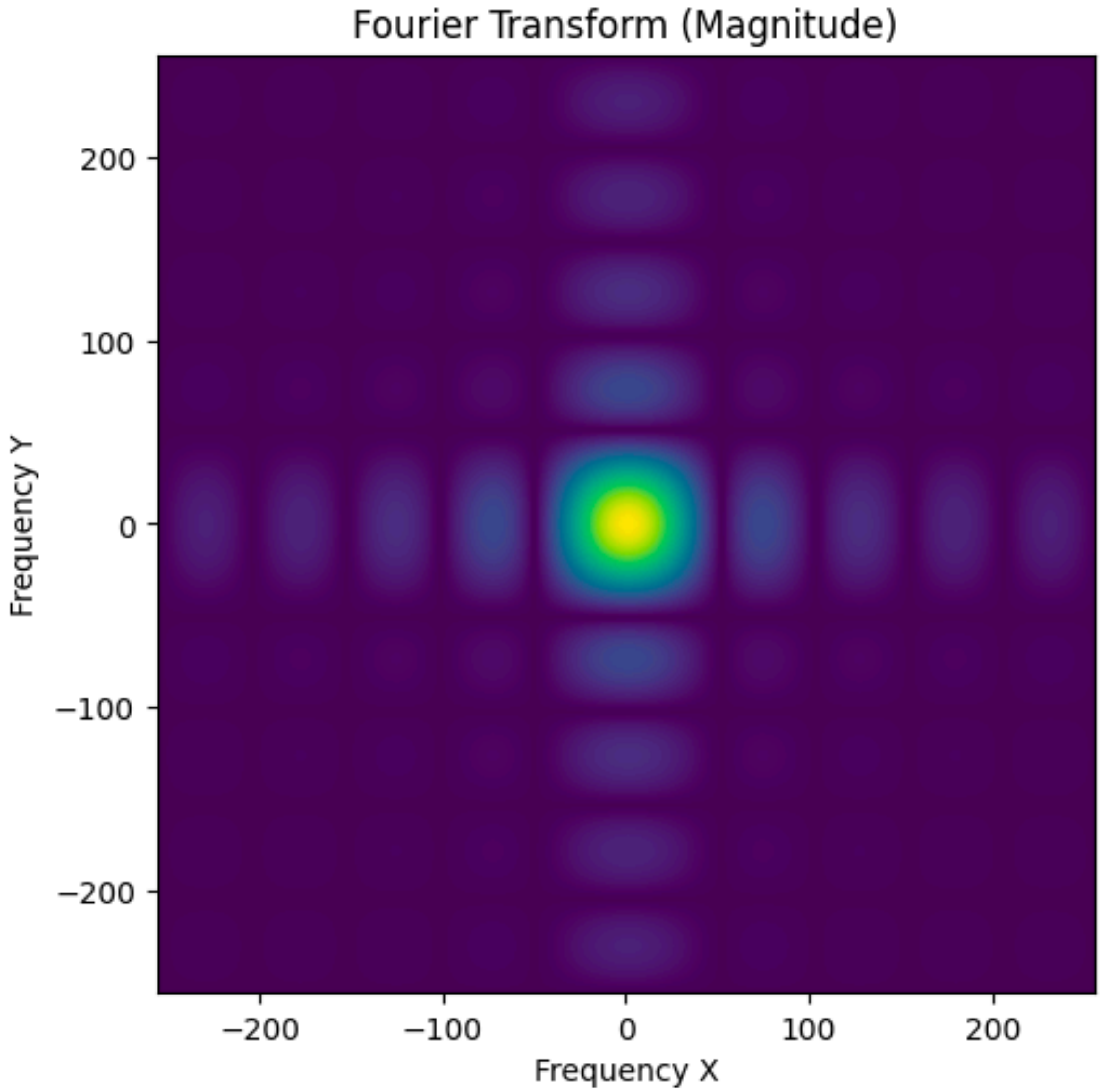


Chuyển đổi Fourier 2-D

Ví dụ miền tần số của xung gần giống 2D impulse

A narrow 2D square signal works as Dirac-like signal





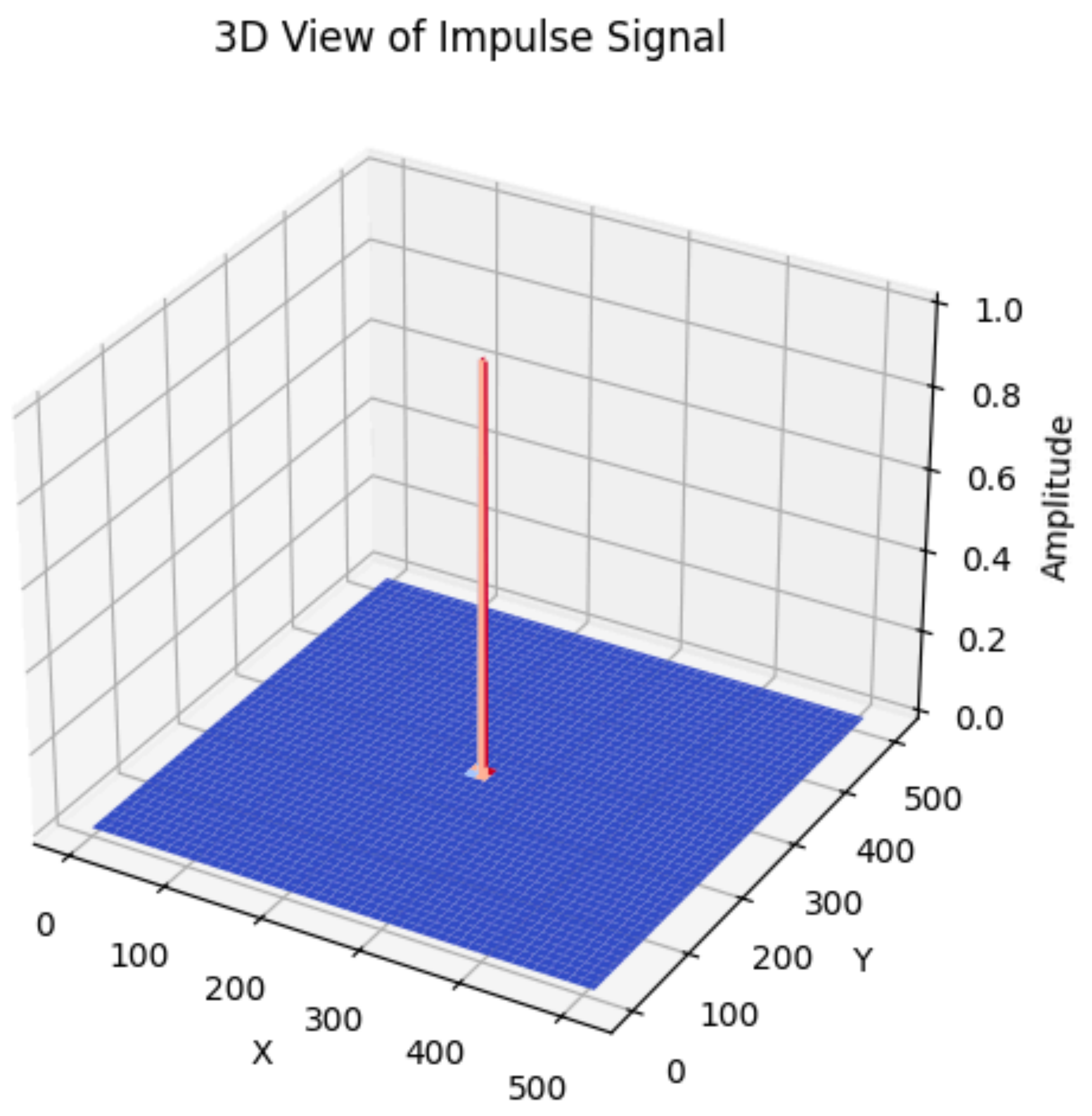
3D View of Impulse Signal

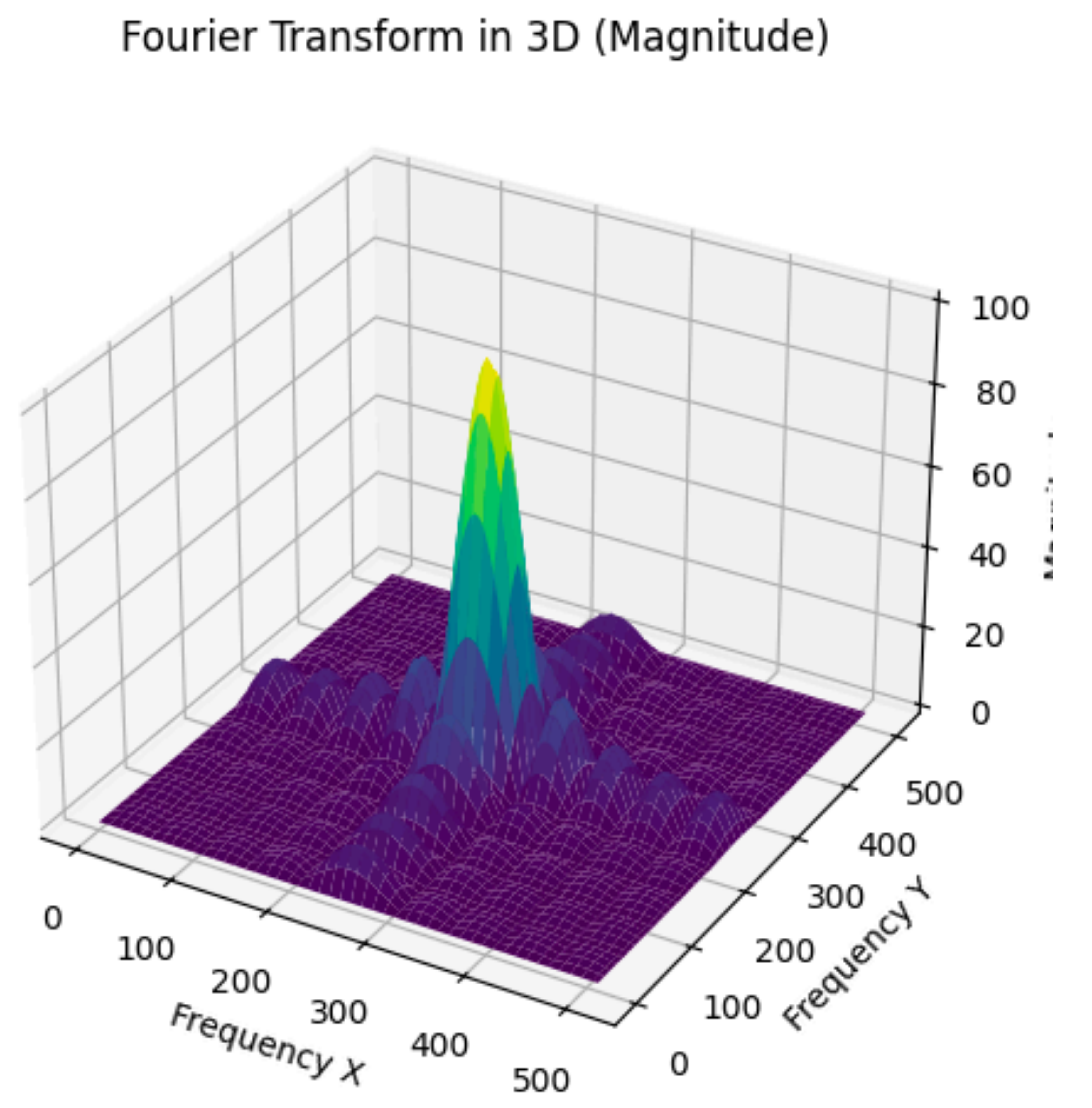
Fourier Transform in 3D (Magnitude)

Chuyển đổi Fourier 2-D

Ví dụ miền tần số của xung gần giống 2D impulse

A narrow 2D square signal works as Dirac-like signal





Chuyển đổi Fourier 2-D

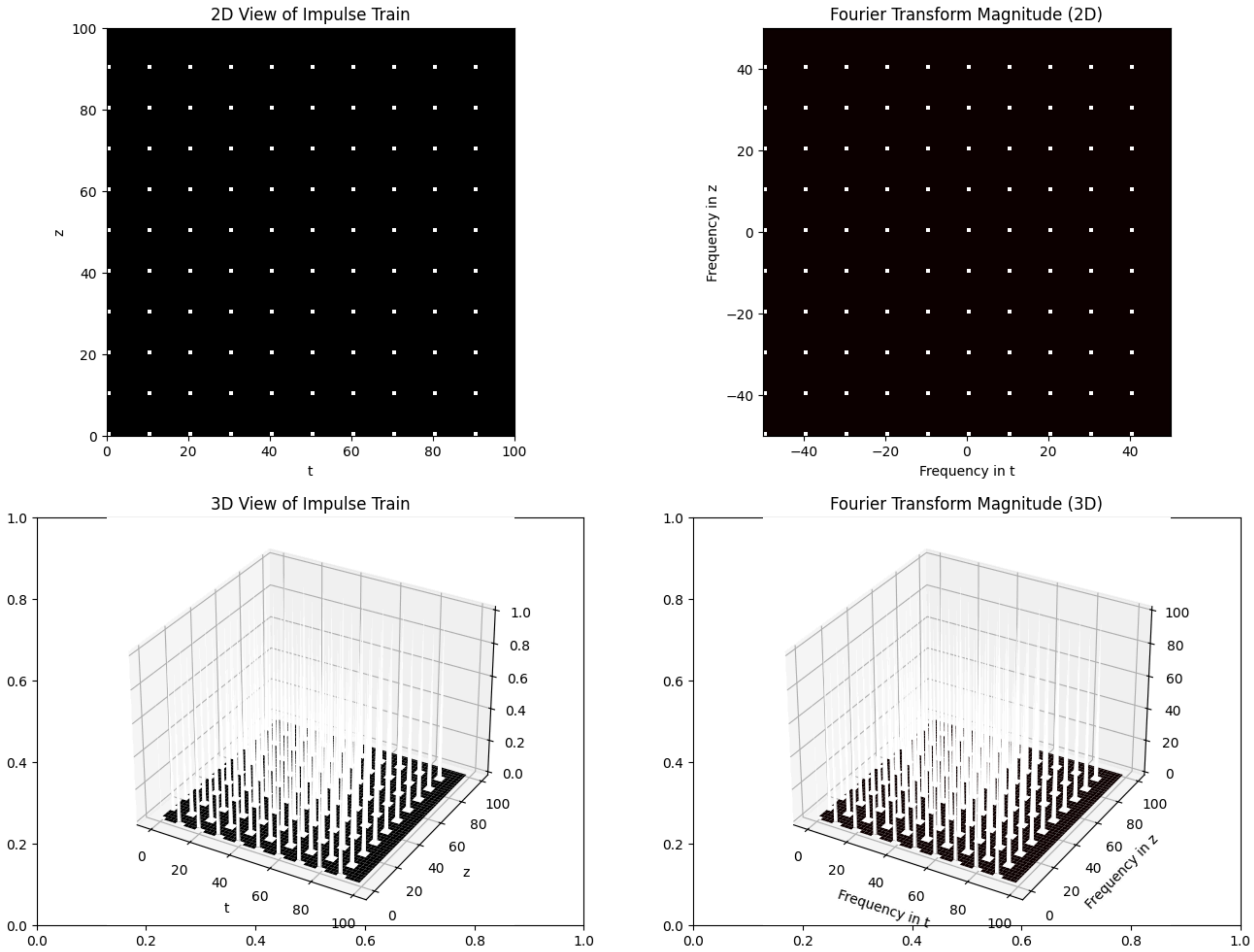
Chuỗi xung lý tưởng

- Impulse Train

$$s(t,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - mT, z - nT)$$

- Direct Transform:

$$S(\mu,\nu) = \frac{1}{T^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta\left(\mu - \frac{k}{T}, \nu - \frac{l}{T}\right)$$



Chuyển đổi Fourier 2-D

2D - Sampling

- Chuỗi xung lý tưởng 02 chiều:

$$s_{\Delta t,\Delta z}(t,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta t)\delta(z - n\Delta z)$$

- Chuyển đổi Fourier của chuỗi 2D Impulse:

$$\mathscr{F}\{s_{\Delta t,\Delta z}(t,z)\} = \frac{1}{\Delta t \Delta z} S_{\frac{1}{\Delta t},\frac{1}{\Delta z}}(\mu,\nu)$$

- Sampling of 2D Functions

$$\tilde{f}(t,z) = f(t,z) \cdot s_{\Delta t,\Delta z}(t,z)$$

- Chuyển đổi Fourier:

$$\mathscr{F}\{\tilde{f}(t,z)\} = \frac{1}{\Delta t \Delta z} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \tilde{F}(\mu - k/\Delta t, \nu - l/\Delta z)$$

Chuyển đổi Fourier 2-D

Ví dụ về chuyển Fourier 2D

- 2D Impulse Train

$$s_{\Delta t,\Delta z}(t,z) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(t-m\Delta t)\delta(z-n\Delta z)$$

- Fourier Transform of 2D Impulse Train

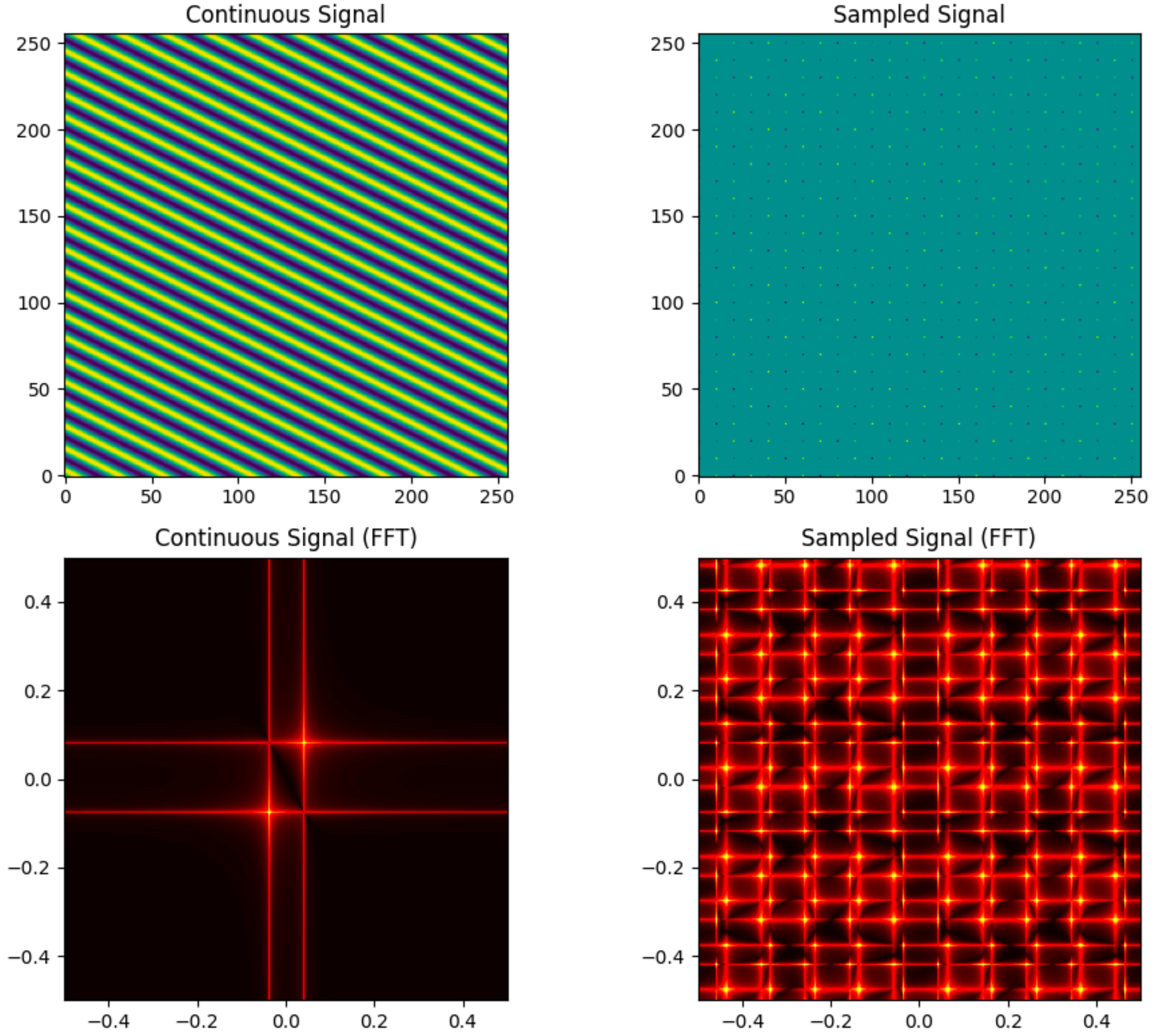
$$\mathcal{F}\{s_{\Delta t,\Delta z}(t,z)\} = \frac{1}{\Delta t\Delta z}S_{\frac{1}{\Delta t},\frac{1}{\Delta z}}(\mu,\nu)$$

- Sampling of 2D Functions

$$\tilde{f}(t,z) = f(t,z)\cdot s_{\Delta t,\Delta z}(t,z)$$

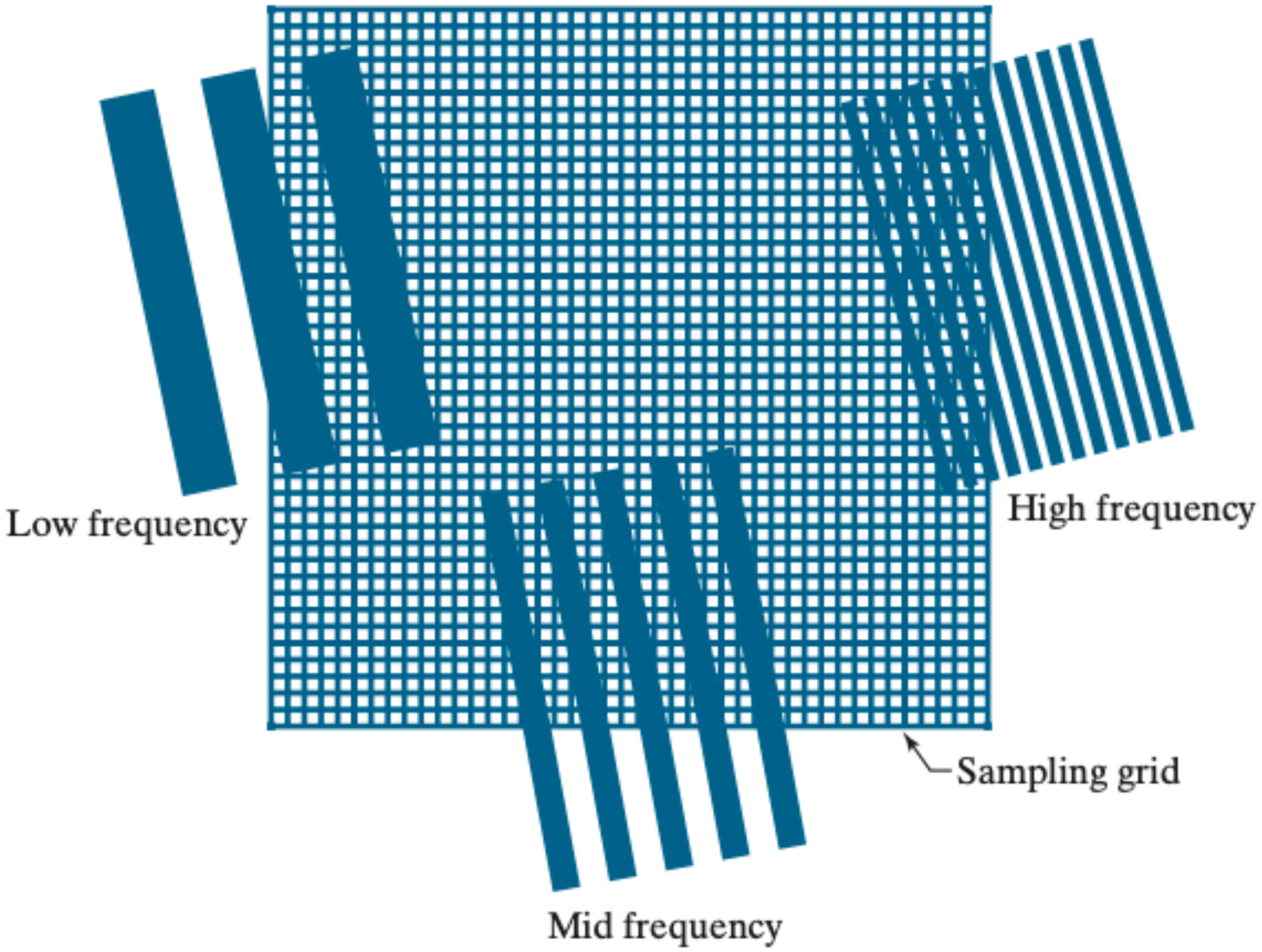
- The Fourier transform of the sampled function is:

$$\mathcal{F}\{\tilde{f}(t,z)\} = \frac{1}{\Delta t\Delta z}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\tilde{F}(\mu-k/\Delta t,\nu-l/\Delta z)$$



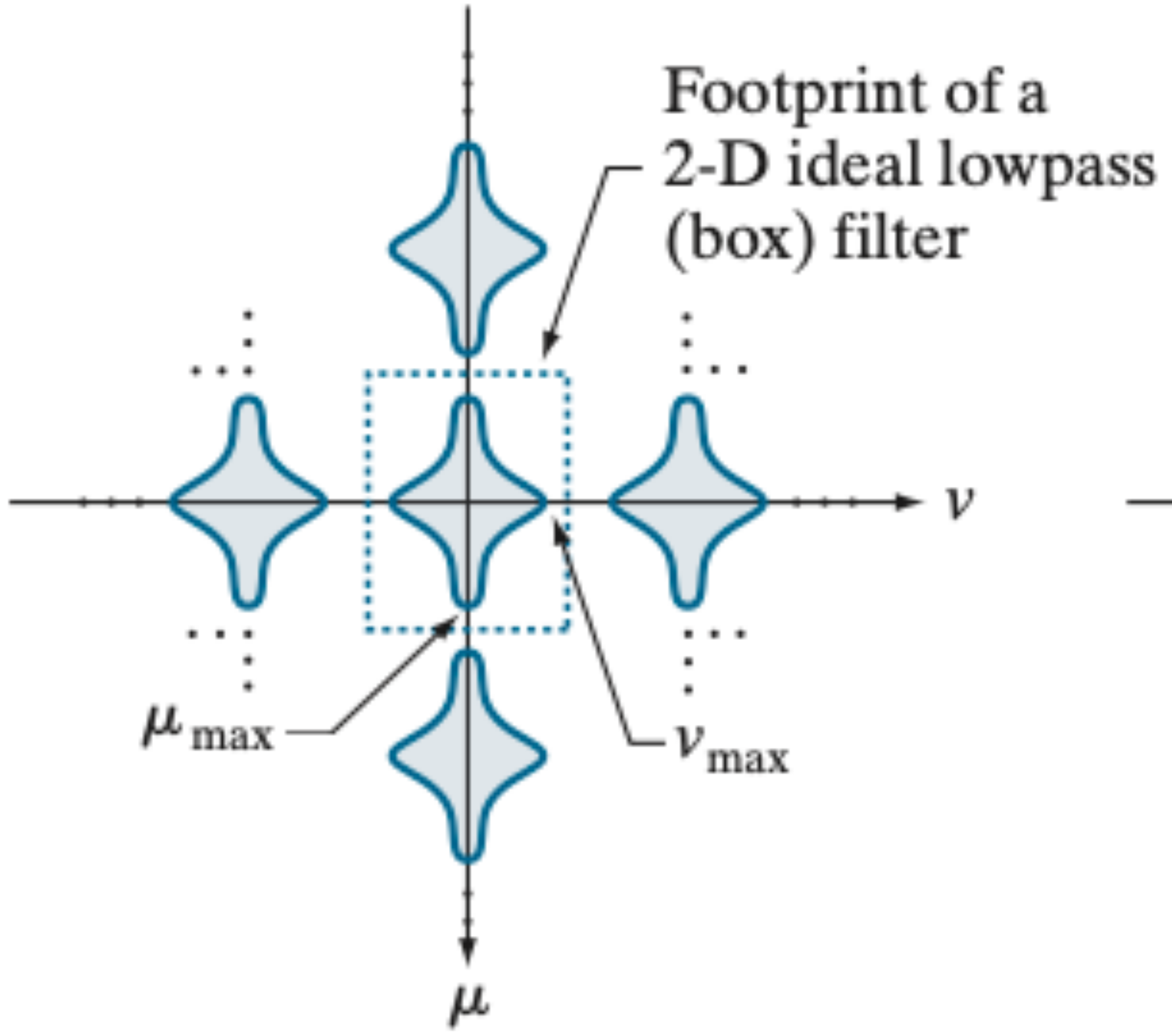
Chuyển đổi Fourier 2-D

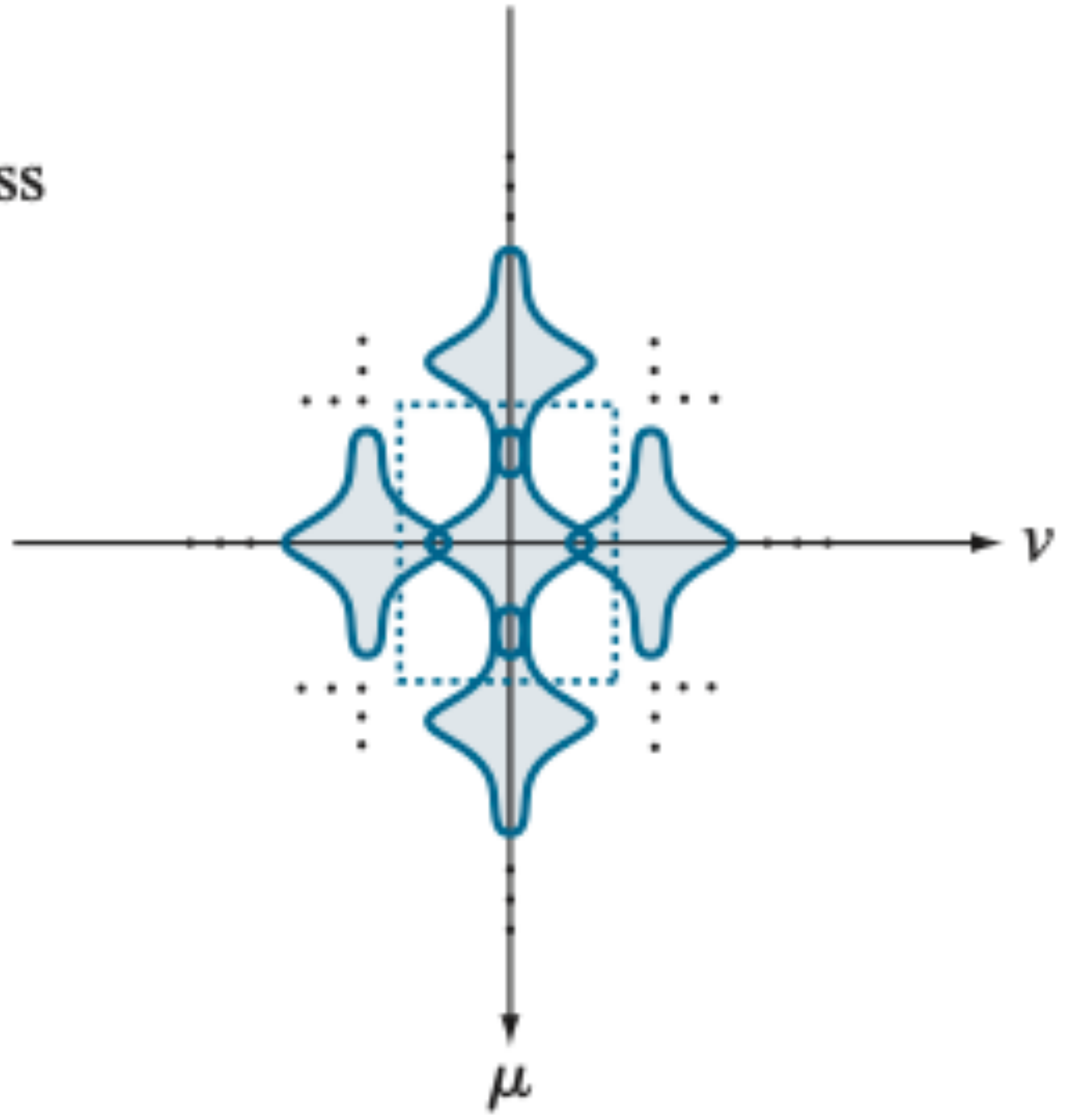
Hiện tượng Alias



$$\frac{1}{\Delta T} > \mu_{\max}, \frac{1}{\Delta Z} > \nu_{\max}$$

$$\frac{1}{\Delta T} < \mu_{\max}, \frac{1}{\Delta Z} < \nu_{\max}$$





Chuyển đổi Fourier 2-D

Ví dụ về Low pass Filter: Rectangular vs Circular Lowpass filter

- Rectangular Lowpass Filter

$$H(u,v) = \begin{cases} 1 & \text{if } |u - u_c| \le D_u \text{ and } |v - v_c| \le D_v \\ 0 & \text{otherwise} \end{cases}$$

- Circular Lowpass Filter

$$H(u,v) = \begin{cases} 1 & \text{if } \sqrt{(u - u_0)^2 + (v - v_0)^2} \le D_0 \\ 0 & \text{otherwise} \end{cases}$$

Chuyển đổi Fourier 2-D

Ví dụ về Low pass Filter: Rectangular vs Circular Lowpass filter

- Rectangular Lowpass Filter

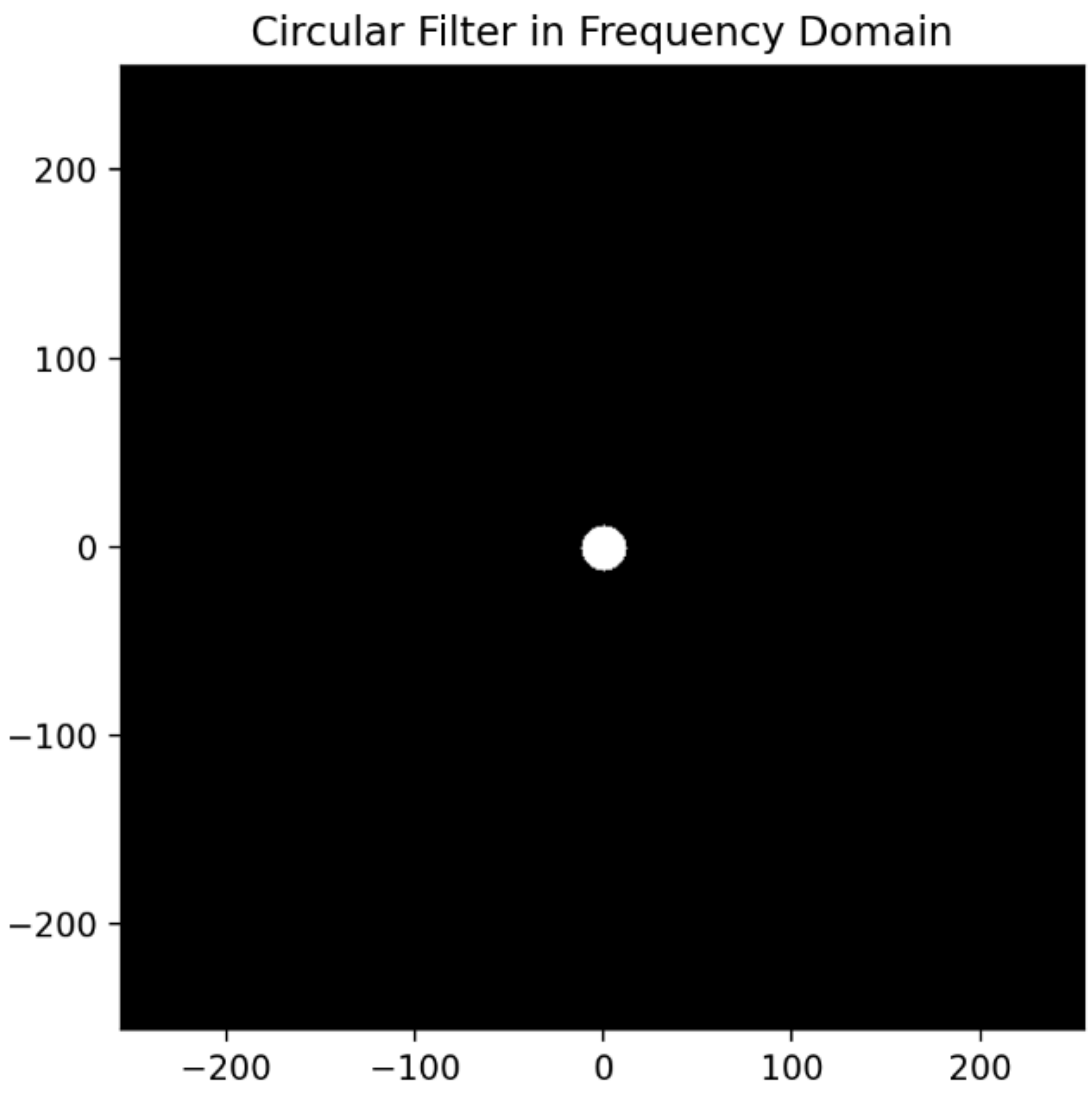
$$H(u,v) = \begin{cases} 1 & \text{if } \sqrt{\left(\dfrac{u - P/2}{P/2}\right)^2 + \left(\dfrac{v - Q/2}{Q/2}\right)^2} \leq \dfrac{D_0}{\min(P,Q)} \\ 0 & \text{otherwise} \end{cases}$$

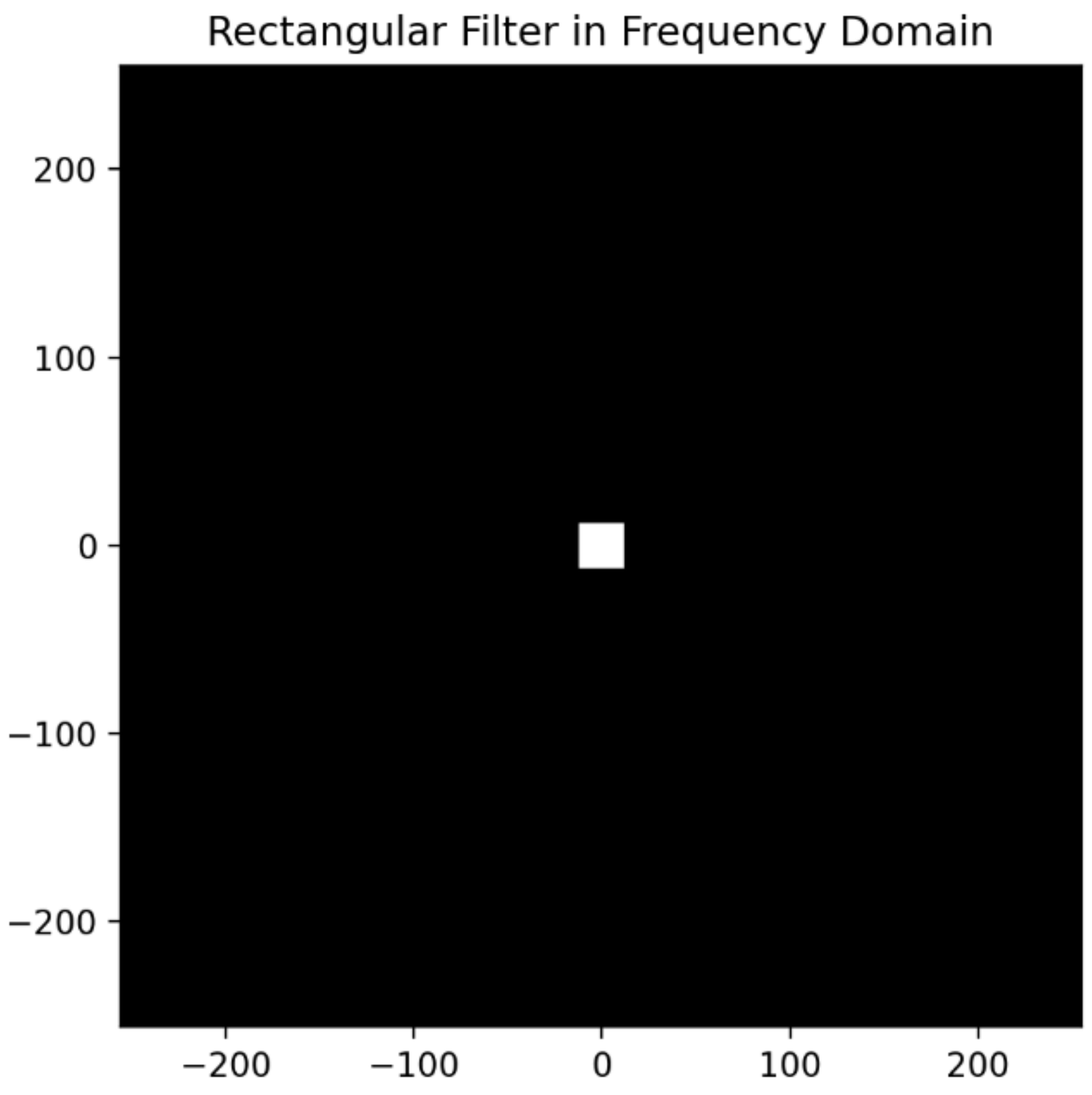
- Circular Lowpass Filter

$$H(u,v) = \begin{cases} 1 & \text{if } |u - \frac{P}{2}| \leq \frac{D_0}{2} \text{ and } |v - \frac{Q}{2}| \leq \frac{D_0}{2} \\ 0 & \text{otherwise} \end{cases}$$

Chuyển đổi Fourier 2-D

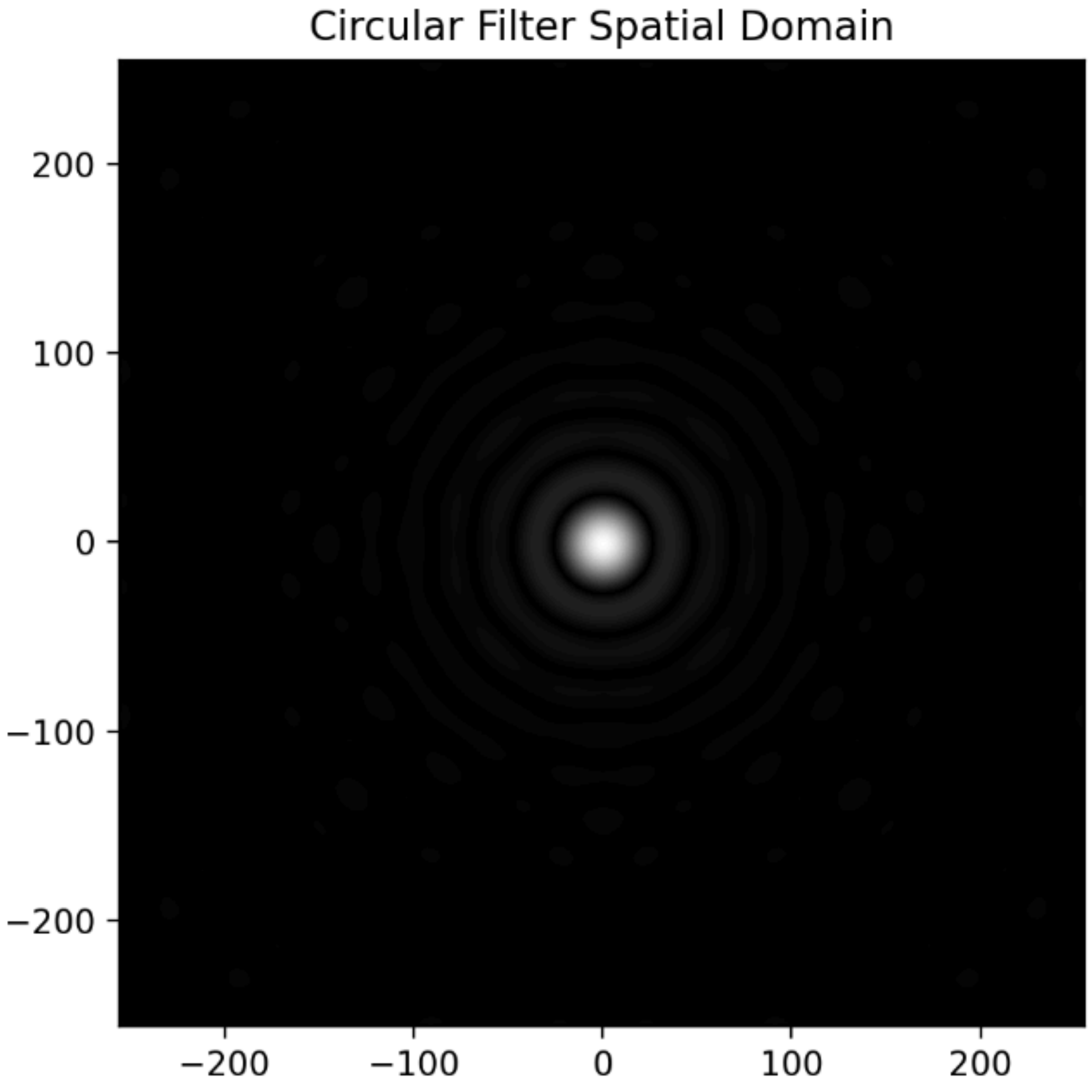
Visualization - Frequency Domain

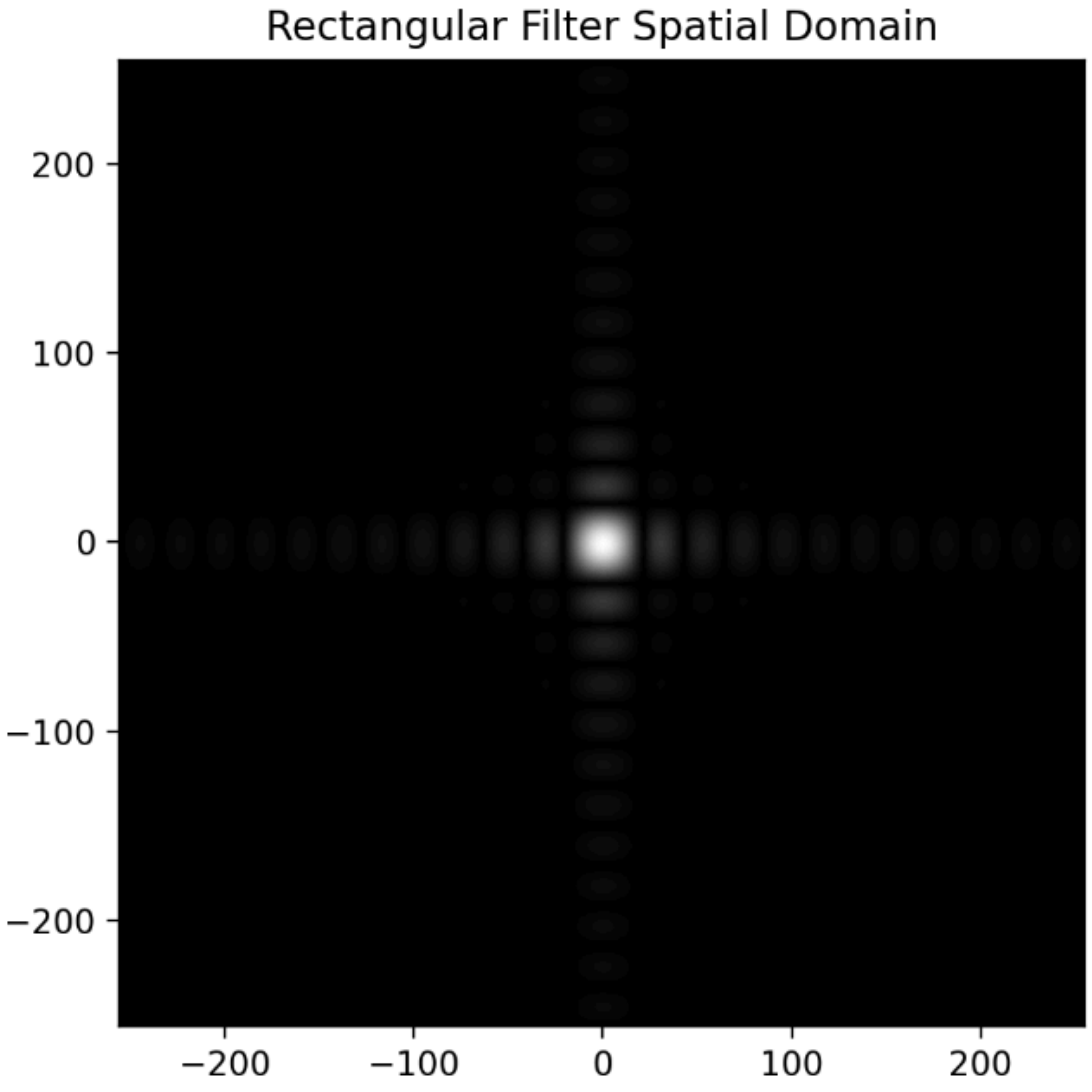




Ideal Lowpass Filter

Visualization - Time Domain

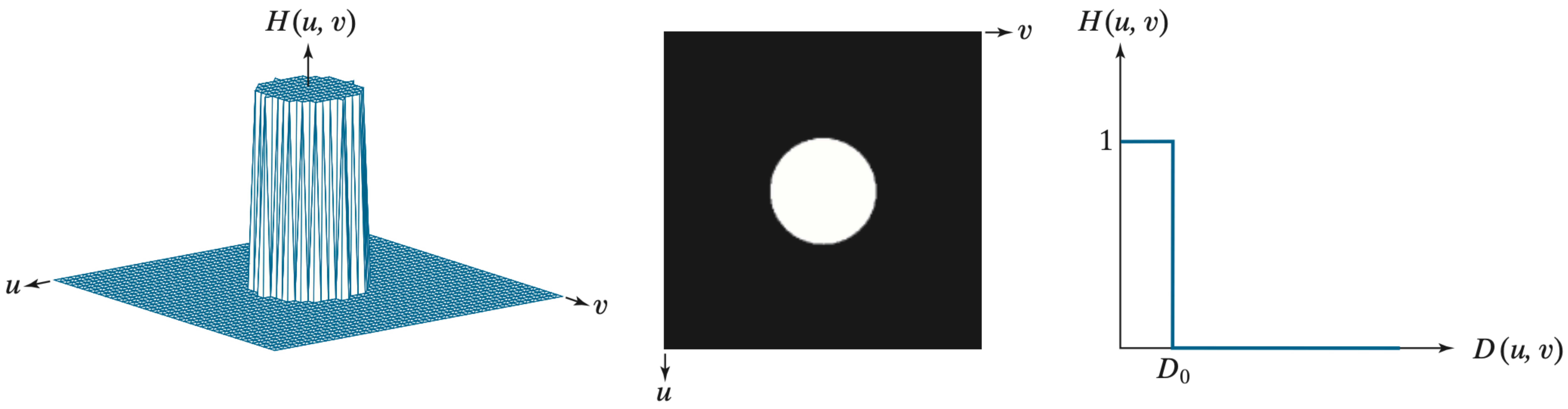




Chuyển đổi Fourier 2-D

Minh hoạ Low pass filter

Visualization of Ideal Lowpass Filter

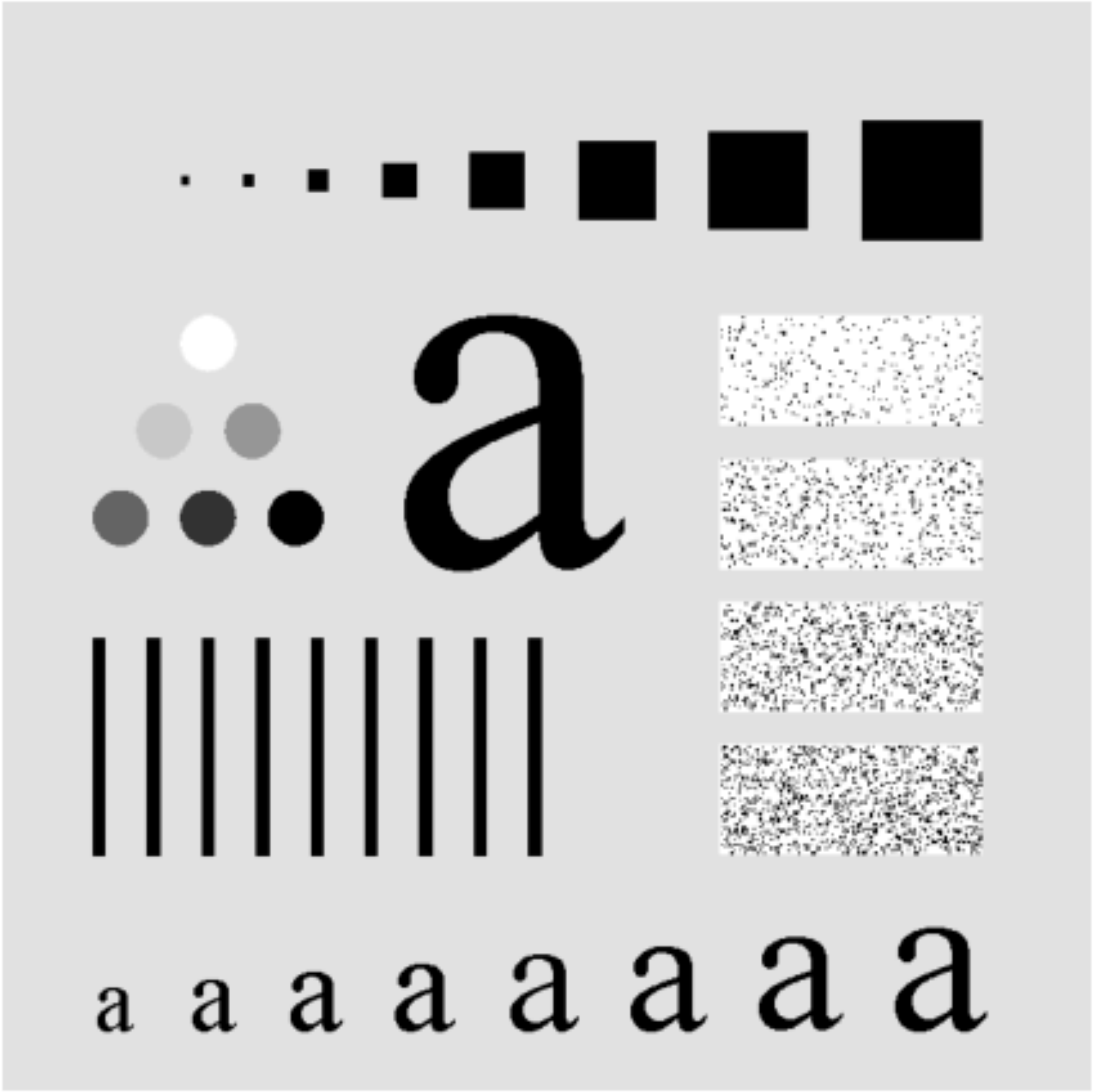


Ideal Lowpass Filter

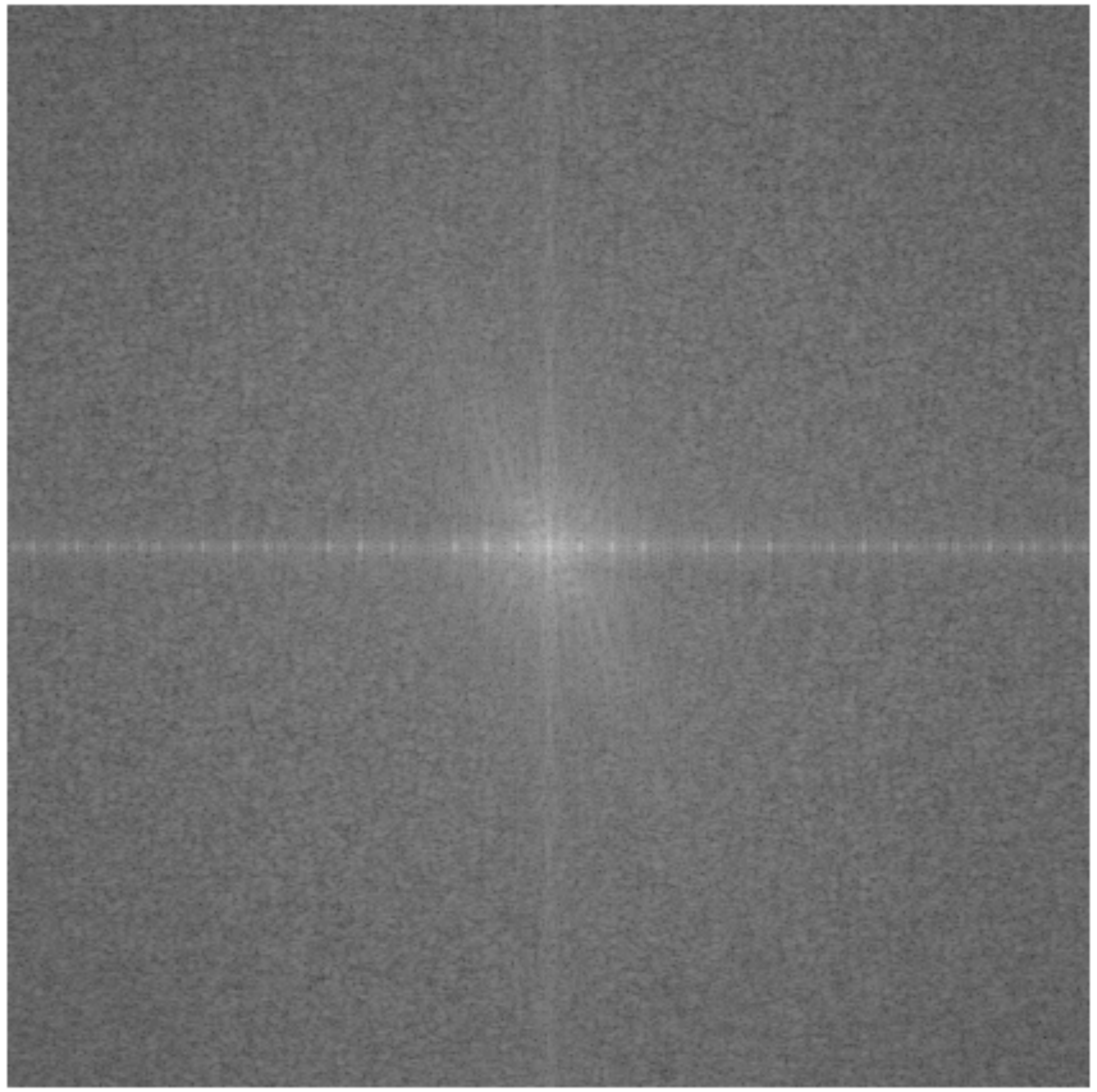
Setting the Cutoff Frequency

1 - Which kind of filter should be selected in this case?
2 - Which cutoff frequency should be selected in this case?

Original Image

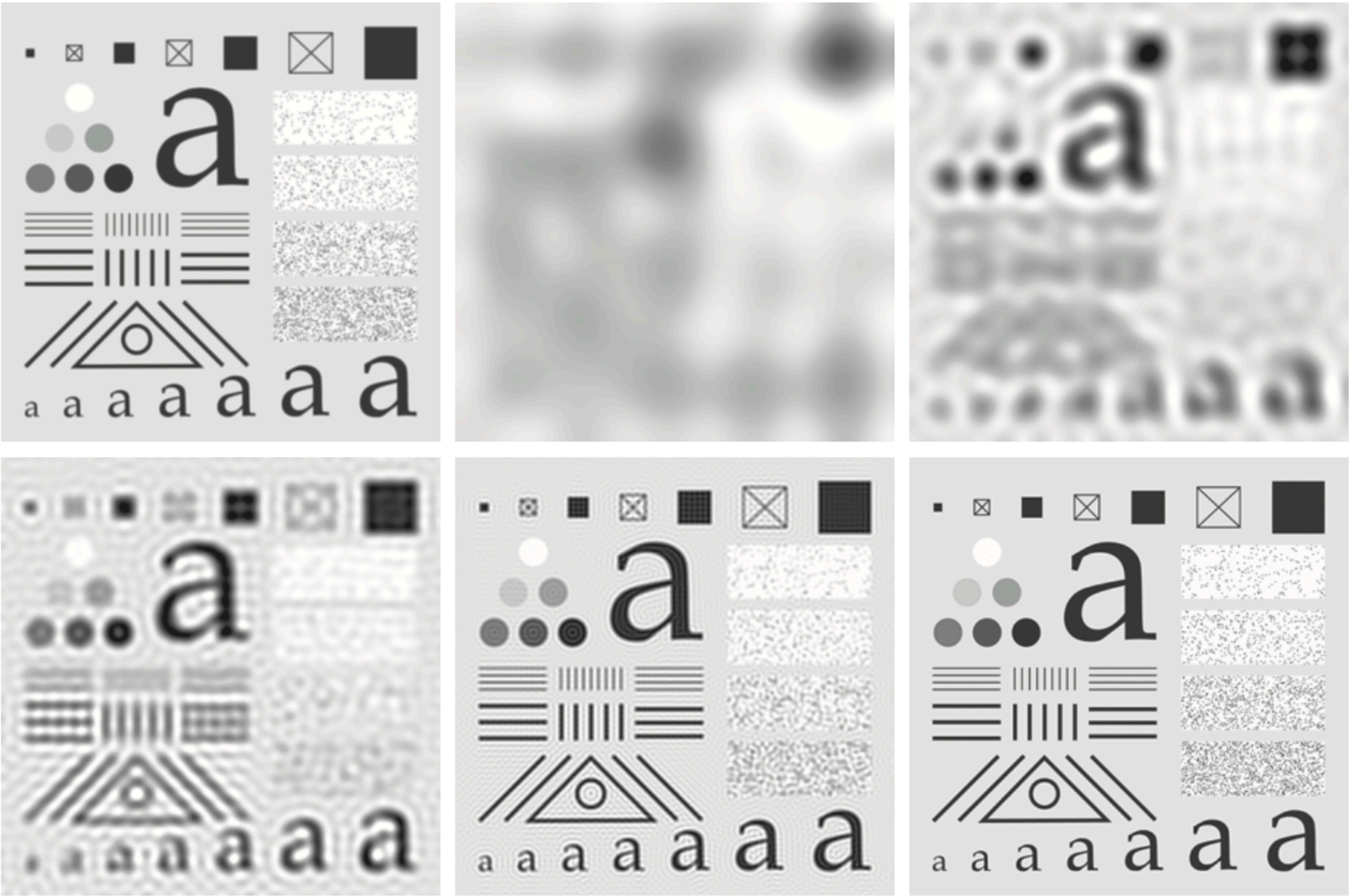


Magnitude Spectrum



Chuyển đổi Fourier 2-D

Setting the Cutoff Frequency



<u>Lab practice:</u>

Use various image and apply cutoff frequency setting to see the variants.

Chuyển đổi Fourier 2-D

Gaussian Lowpass Filter

- Continuous version

$$H(u, v) = e^{-\frac{D^2(u, v)}{2D_0^2}} \text{ where } D(u, v) = \sqrt{u^2 + v^2}$$

- Discrete version

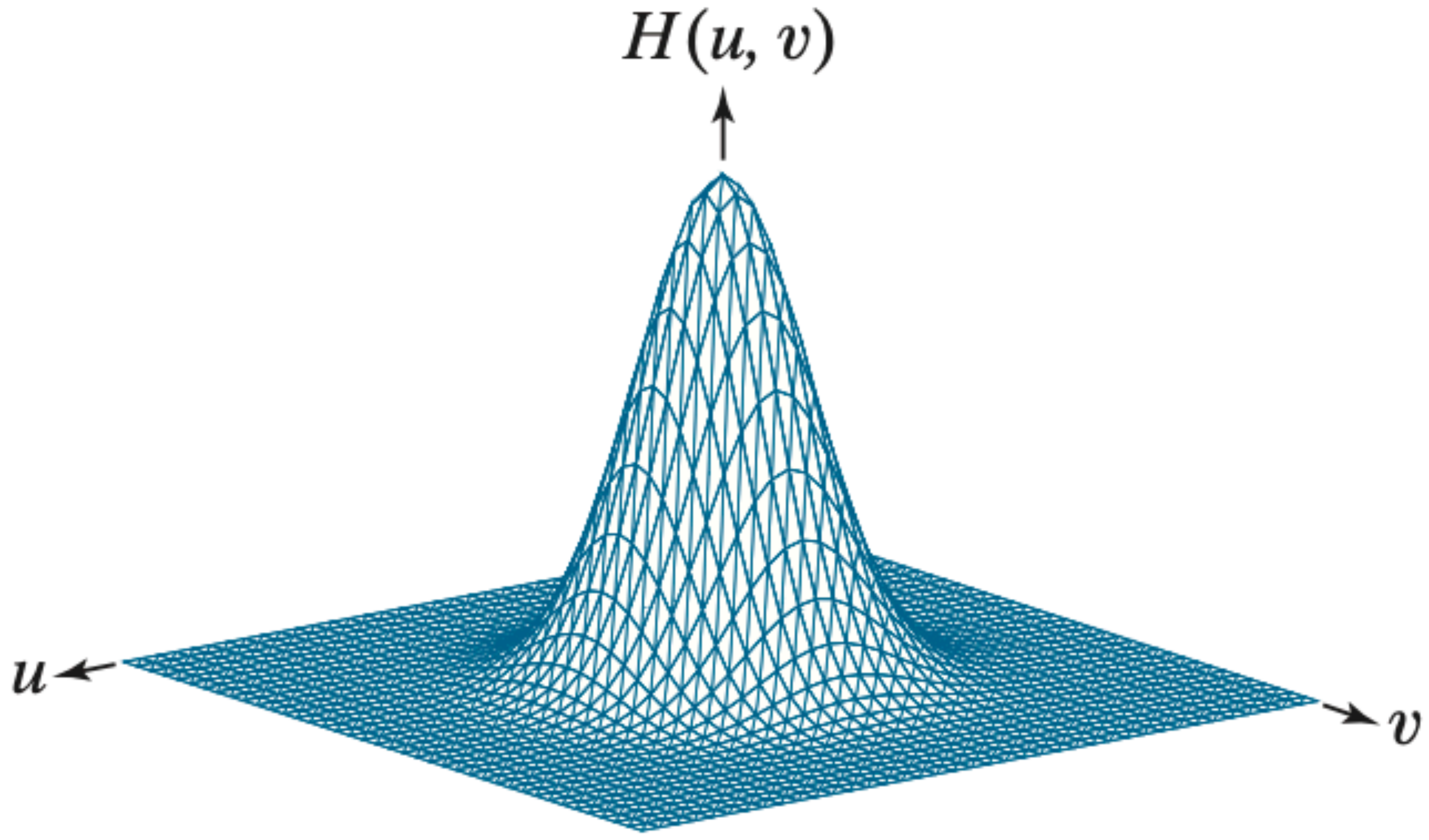
$$D(u, v) = \sqrt{(u - P/2)^2 + (v - Q/2)^2}$$

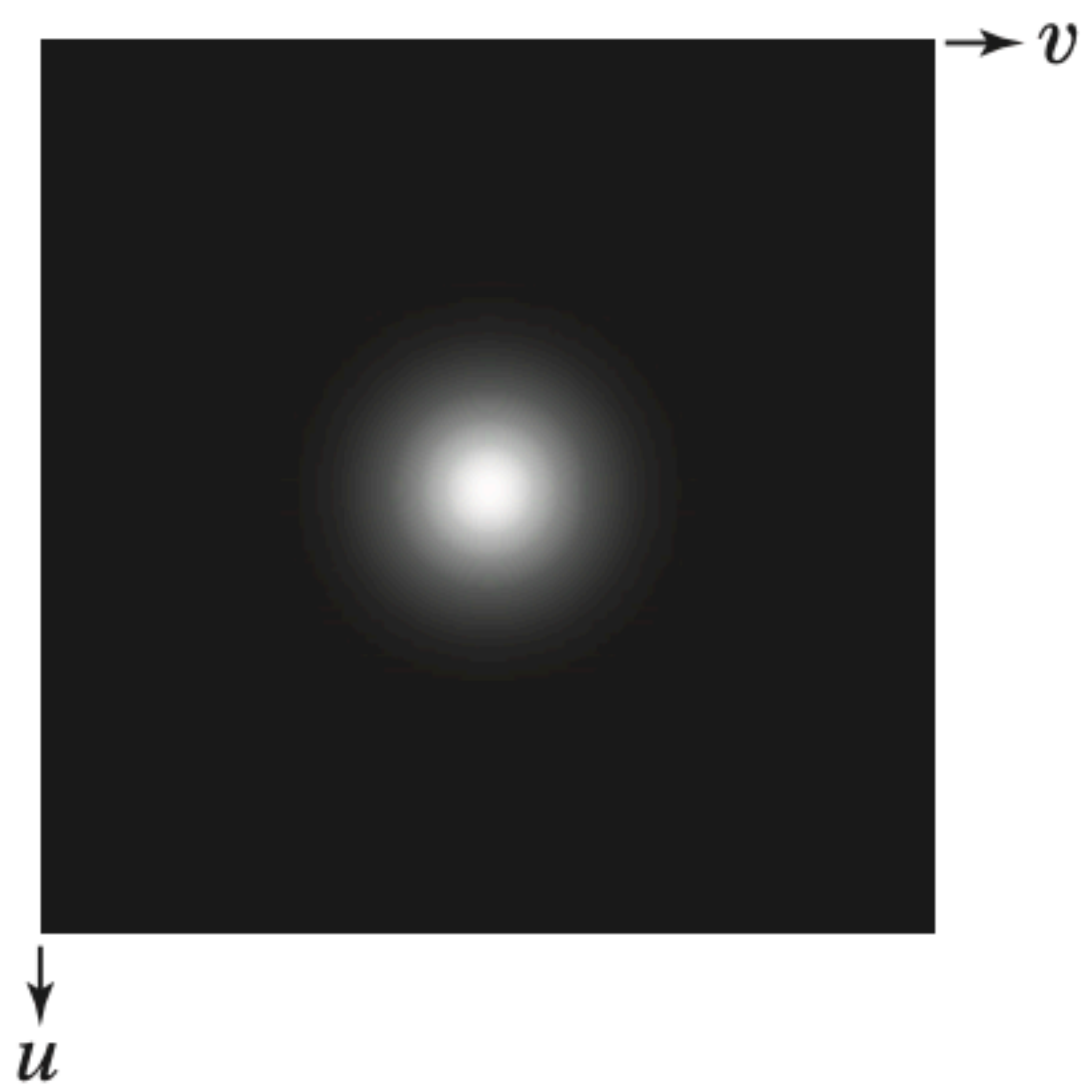
Chuyển đổi Fourier 2-D

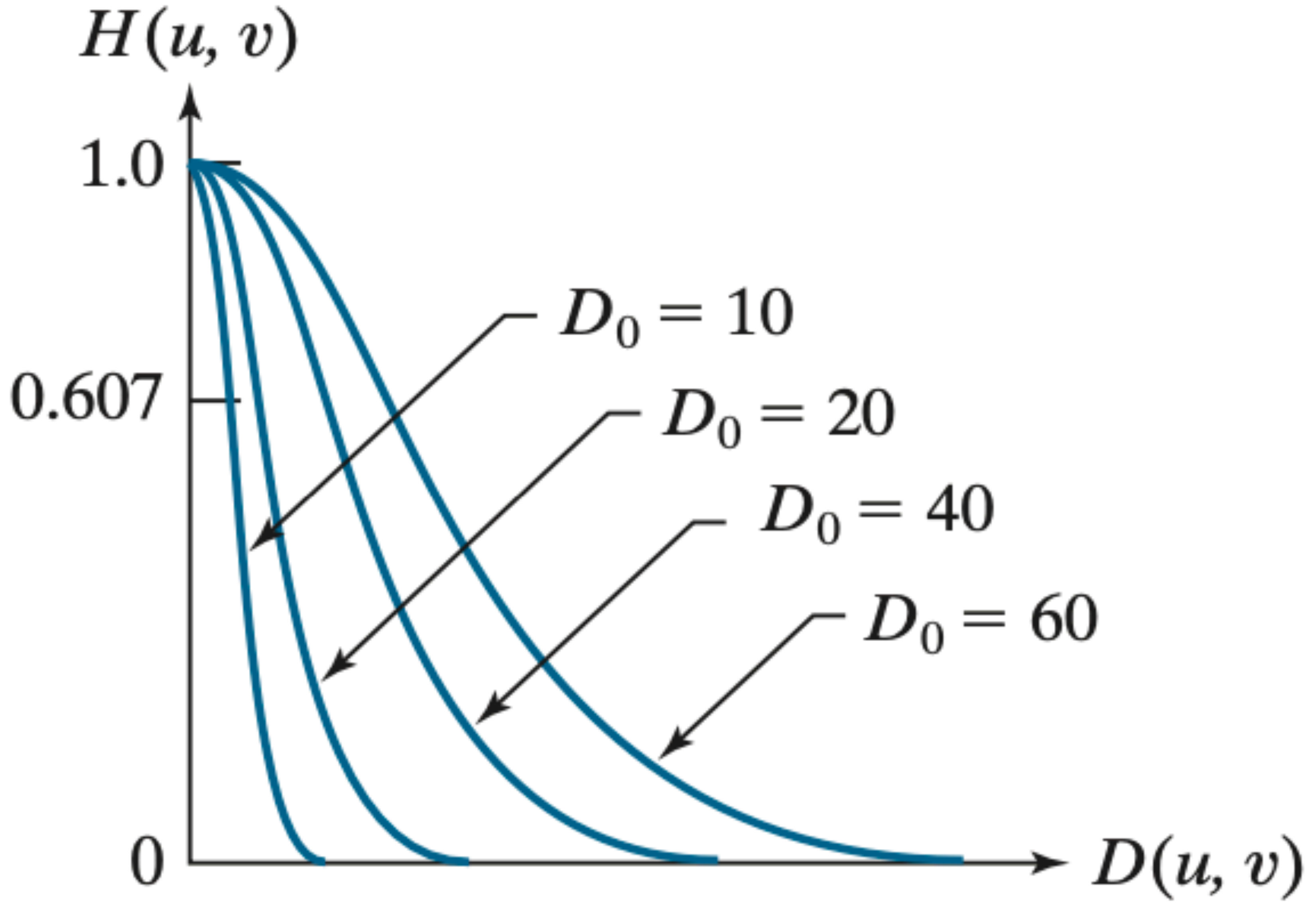
Visualization

Chuyển đổi Fourier 2-D

Visualization : Gaussian Lowpass Filter







Chuyển đổi Fourier 2-D

Làm trơn ảnh: Gaussian Lowpass Filter

Cutoff Freq... 16.00

Original Image

Filtered Image

Gaussian Lowpass Filter

Smoothing comparison with ideal circular lowpass filter

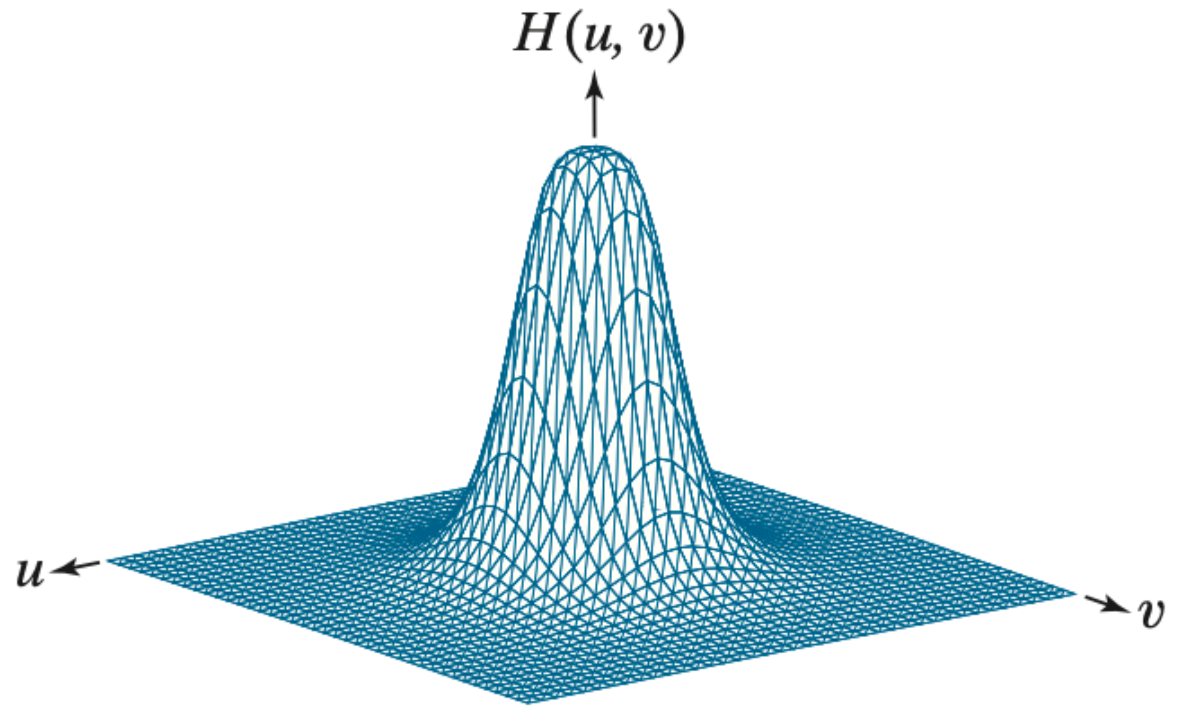
Chuyển đổi Fourier 2-D

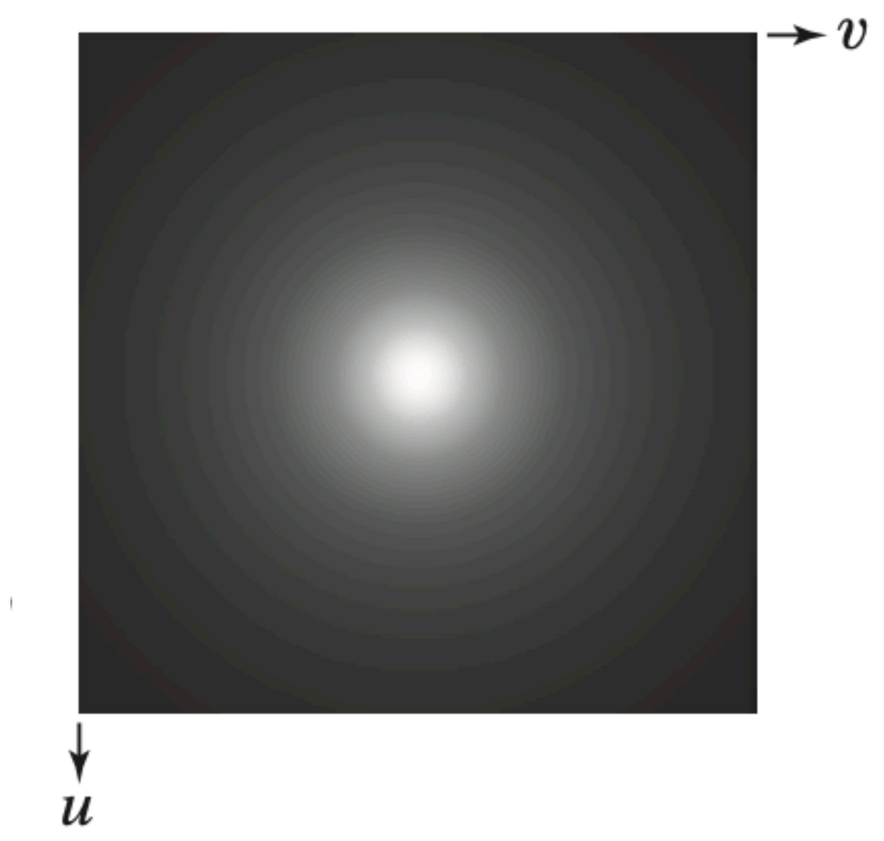
Butter-worth Lowpass Filter

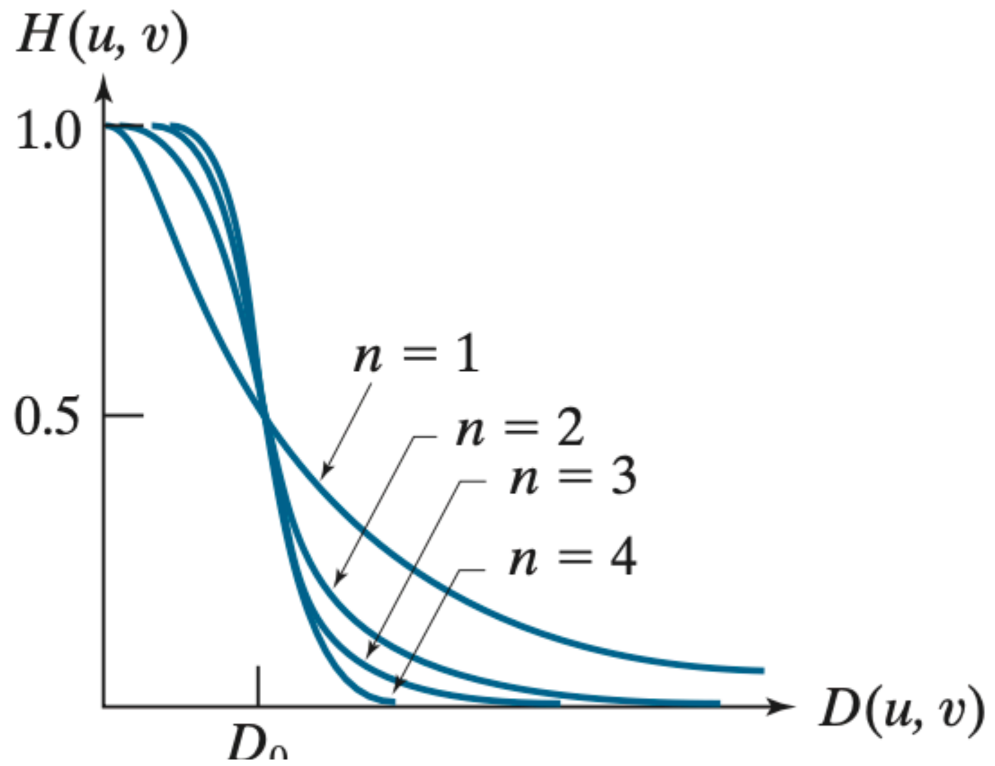
- The Butterworth filter's frequency response

$$H(u,v) = \frac{1}{1 + \left(\frac{D(u,v)}{D_0}\right)^{2n}} \text{ where } D(u,v) = \sqrt{u^2 + v^2}$$

- Frequency Response Characteristics
  - Passband: The area where$D(u,v) < D_0$.
  - Transition Band: The region around D_0. The steepness of this transition depends on the order $n$of the filter.
  - Stopband: The area where $D(u,v) > D_0$.

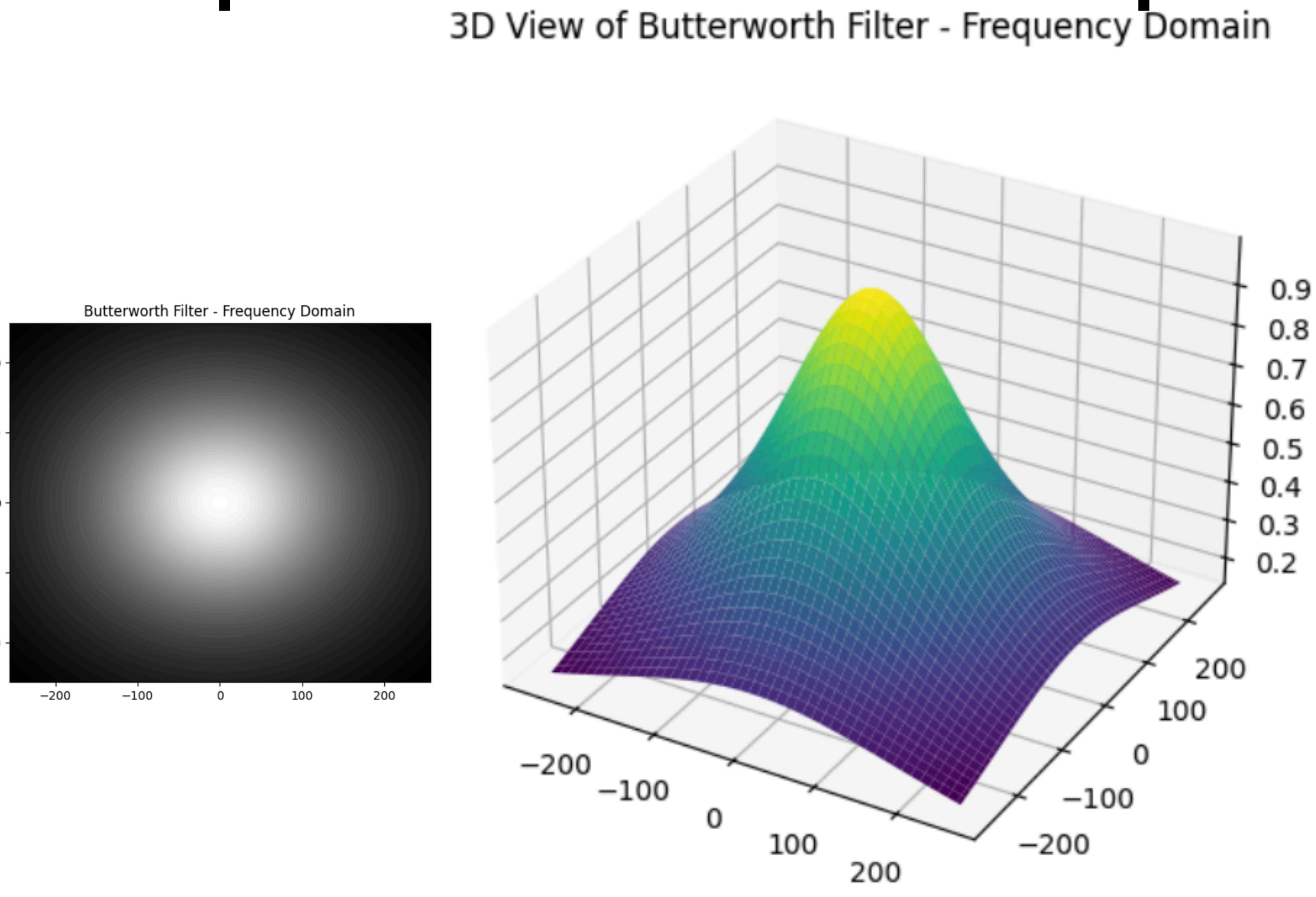




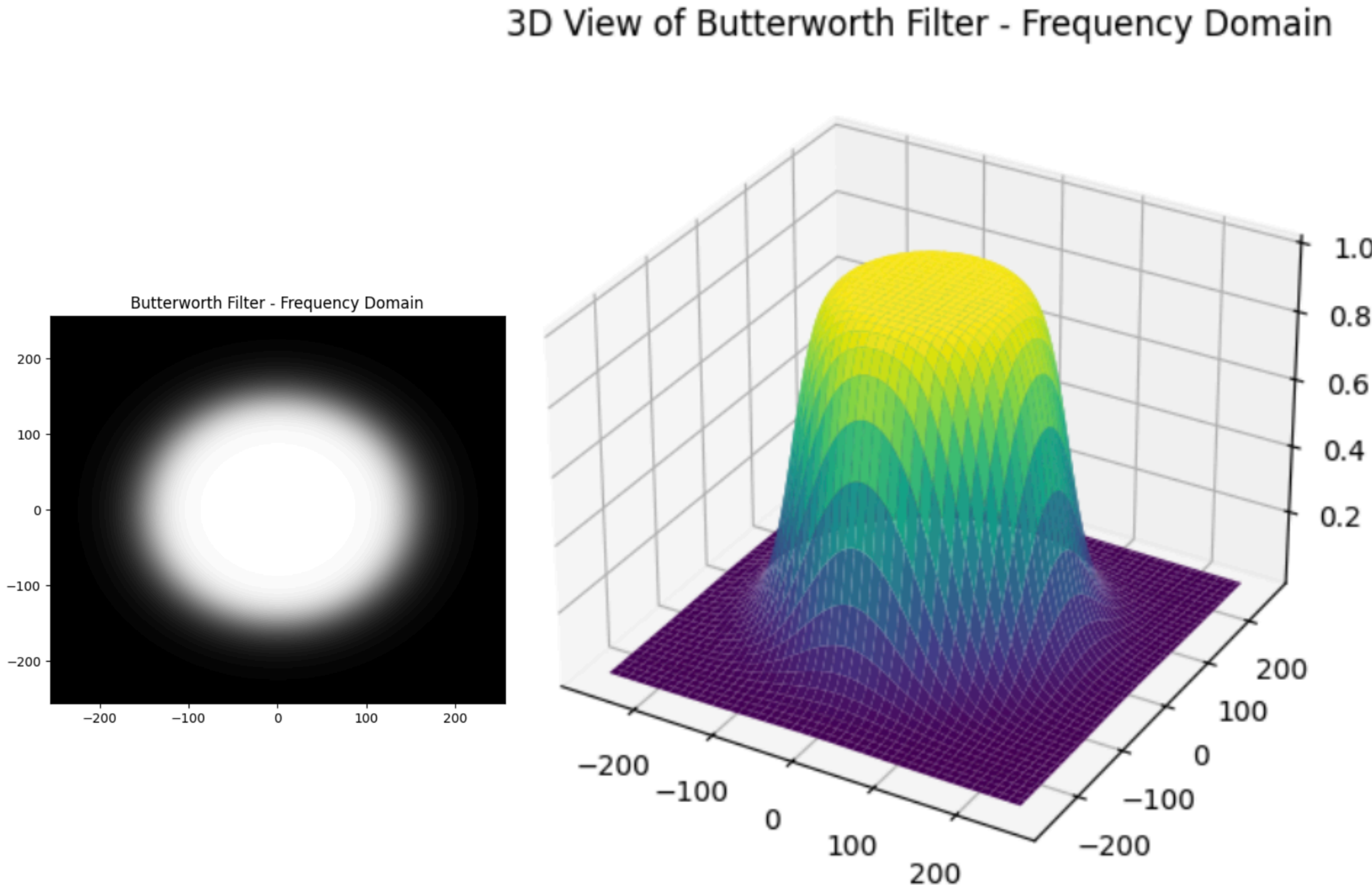


Chuyển đổi Fourier 2-D

Response: Butter-worth Lowpass Filter



$n = 1$



$n = 4$

Chuyển đổi Fourier 2-D

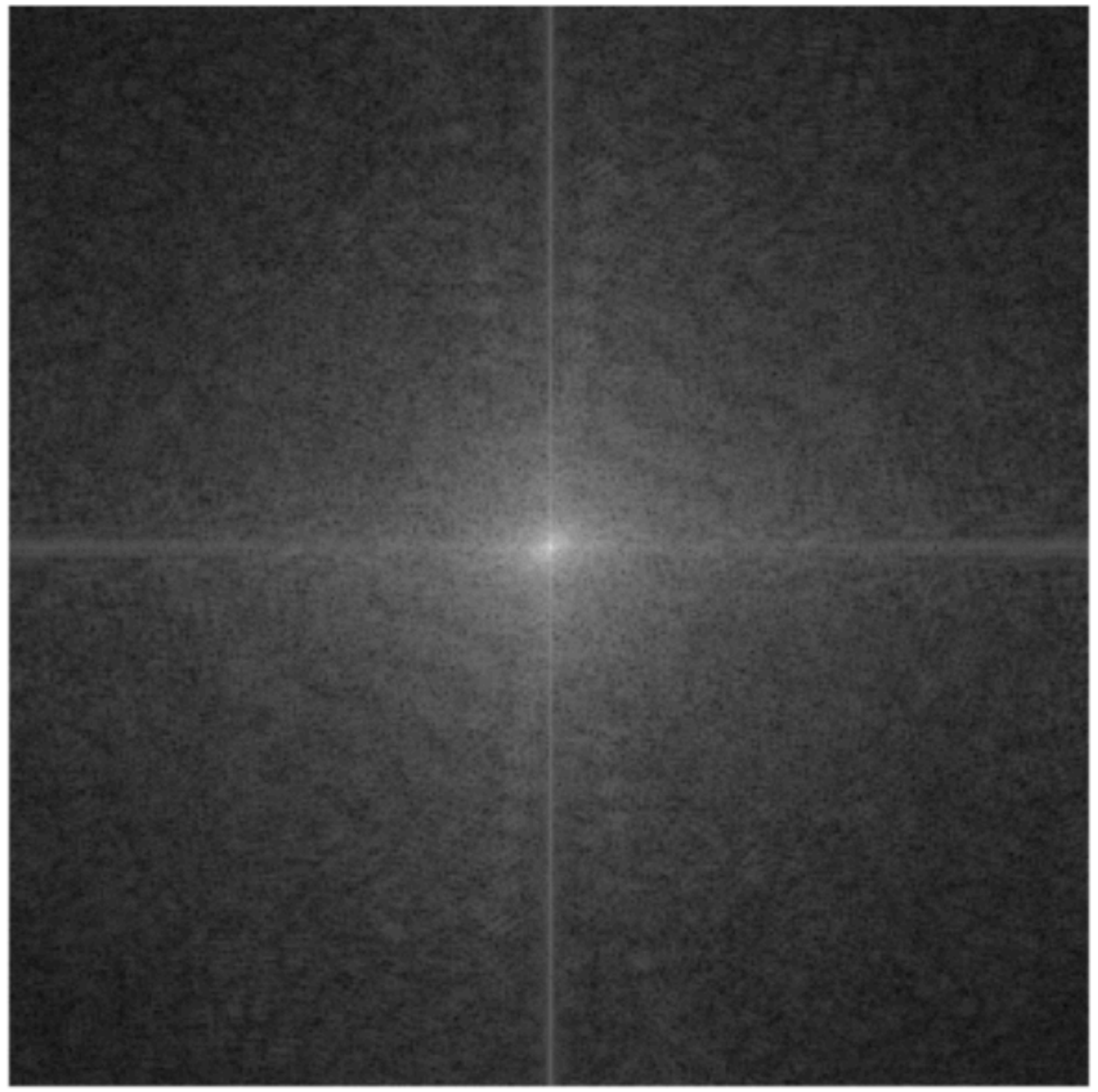
Smoothing application: Butter-worth Lowpass Filter

Cutoff Freq... 13.00

Order n: 6

Original Image

Frequency Spectrum



Filtered Image (D0=13.000000000000002, n=6)

Chuyển đổi Fourier 2-D

Smoothing application: Butter-worth Lowpass Filter

LAB practice: Apply to make person having younger looking

Chuyển đổi Fourier 2-D

Frequency Highpass Filter

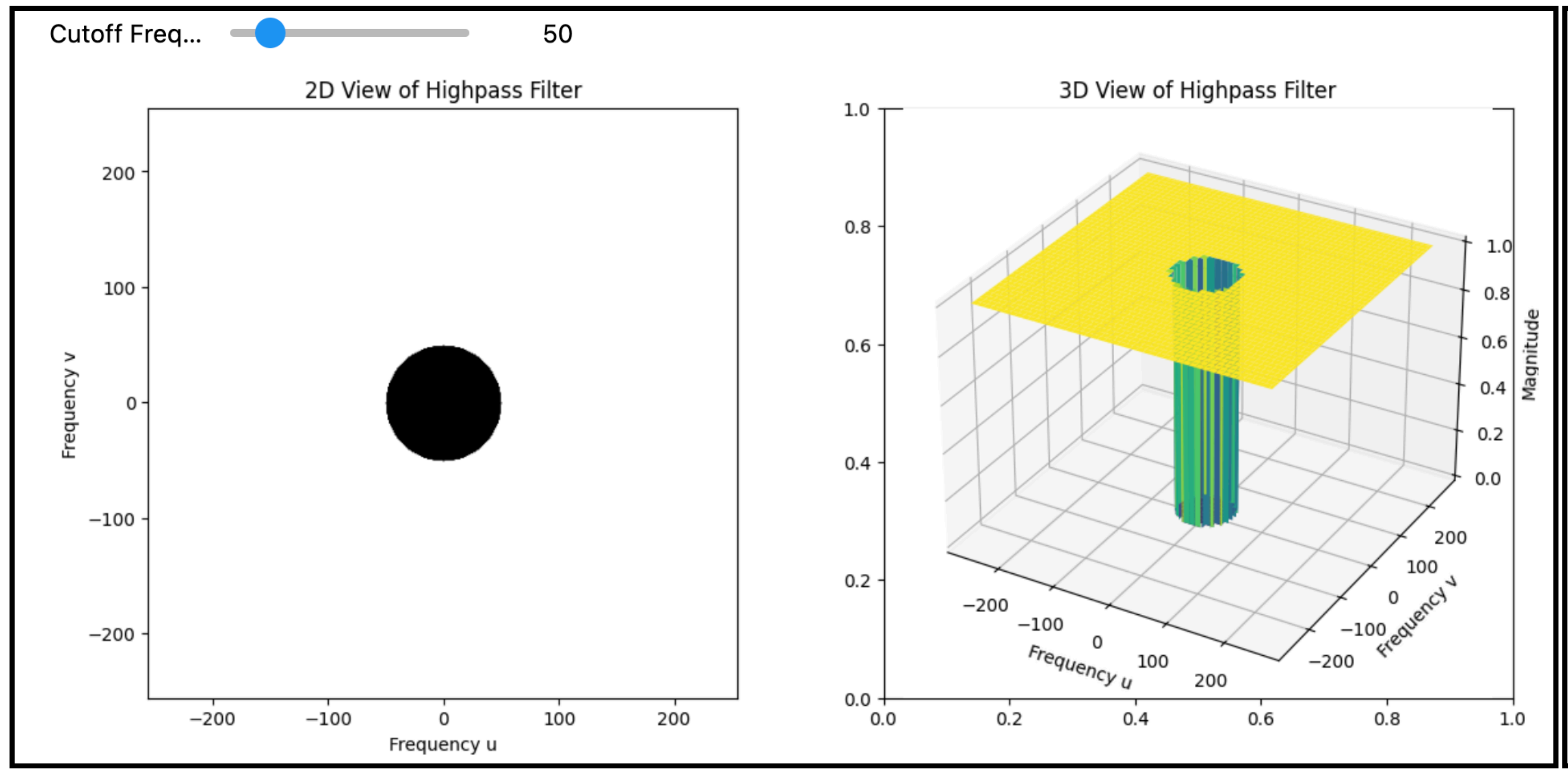
$$\boxed{\text{HPF}(u, v) = 1 - \text{LPF}(u, v) = 1 - H(u, v)}$$

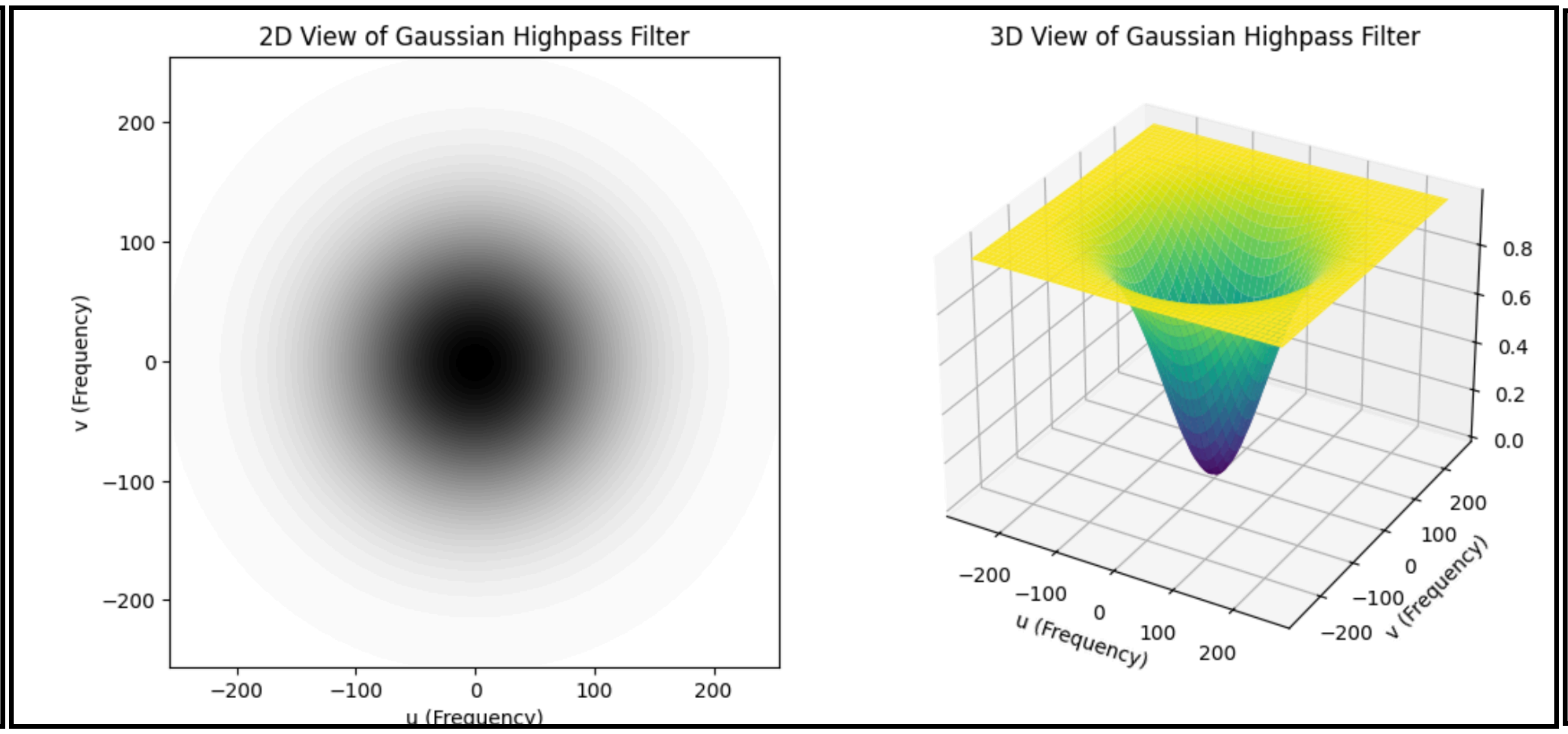
Ideal	Gaussian	Butterworth
$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \leq D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$	$H(u,v) = \dfrac{1}{1 + \left[D_0/D(u,v)\right]^{2n}}$

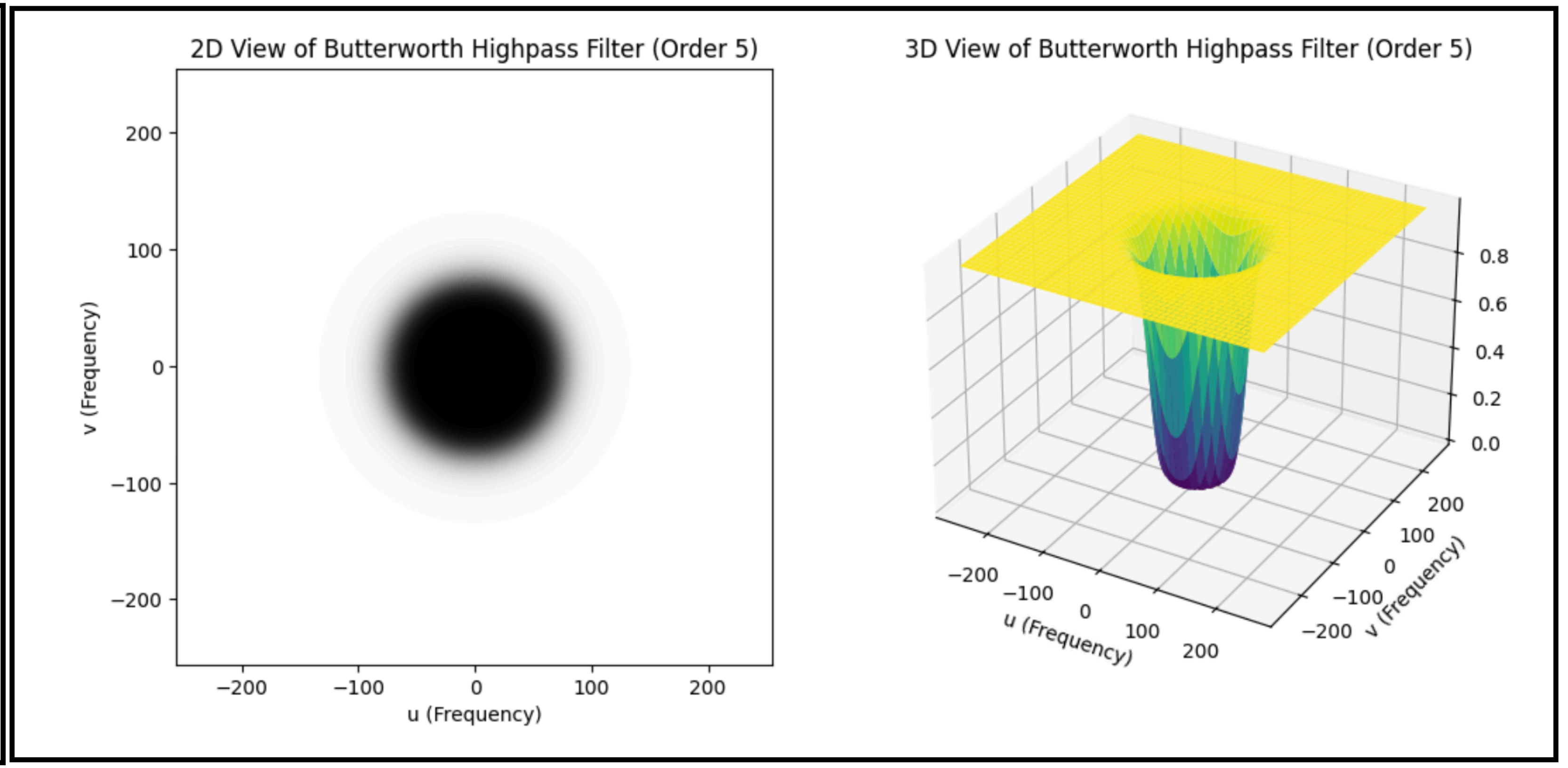
Chuyển đổi Fourier 2-D

Response visualization : Frequency Highpass Filter

Ideal	Gaussian	Butterworth
$H(u,v)=\begin{cases}0 & \text{if } D(u,v)\le D_0\\ 1 & \text{if } D(u,v)>D_0\end{cases}$	$H(u,v)=1-e^{-D^2(u,v)/2D_0^2}$	$H(u,v)=\dfrac{1}{1+\left[D_0/D(u,v)\right]^{2n}}$



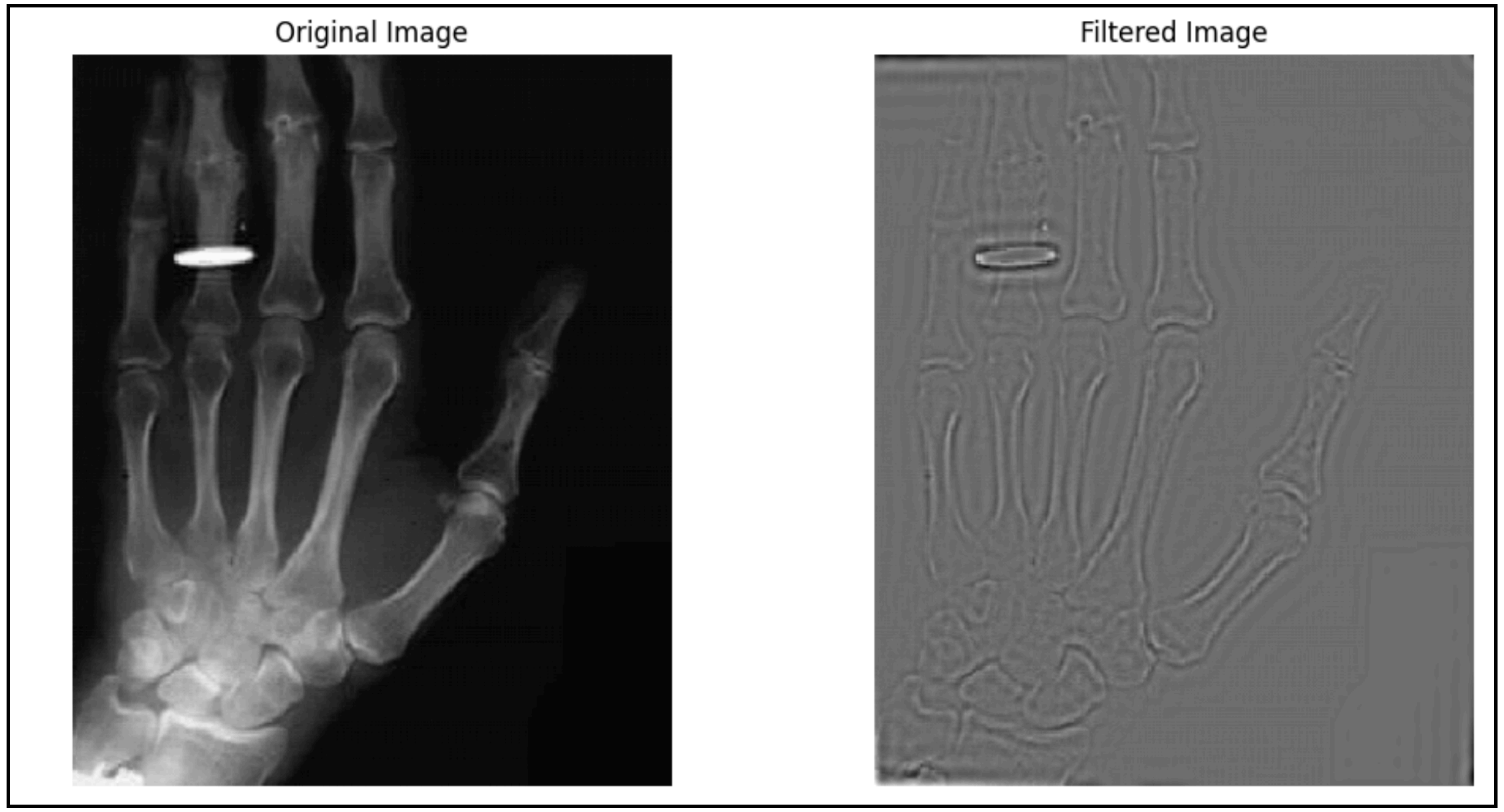


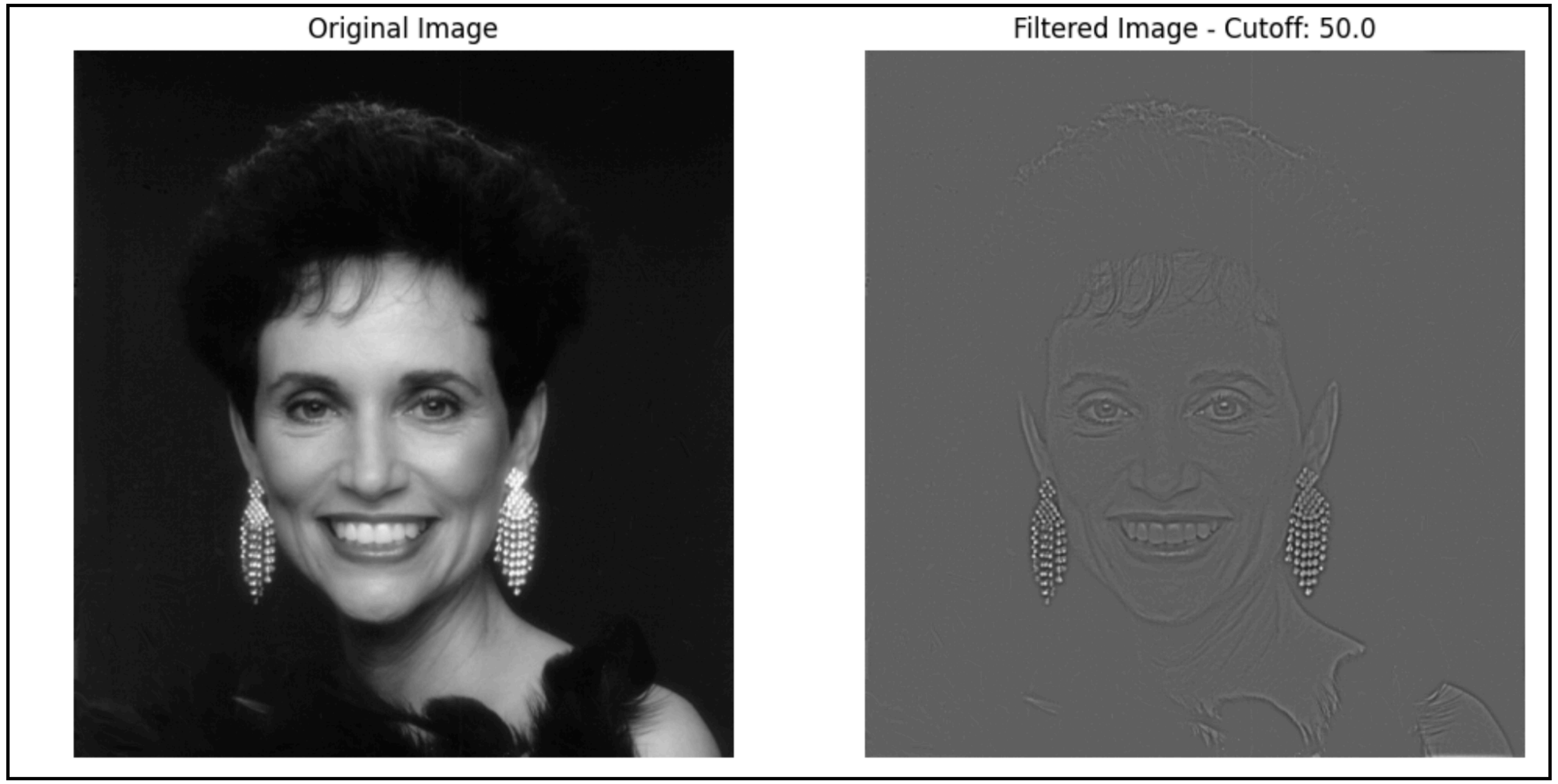


Chuyển đổi Fourier 2-D

Application: Frequency Highpass Filter

LAB practice





Chuyển đổi Fourier 2-D

Laplacian Filter: Frequency Highpass Filter

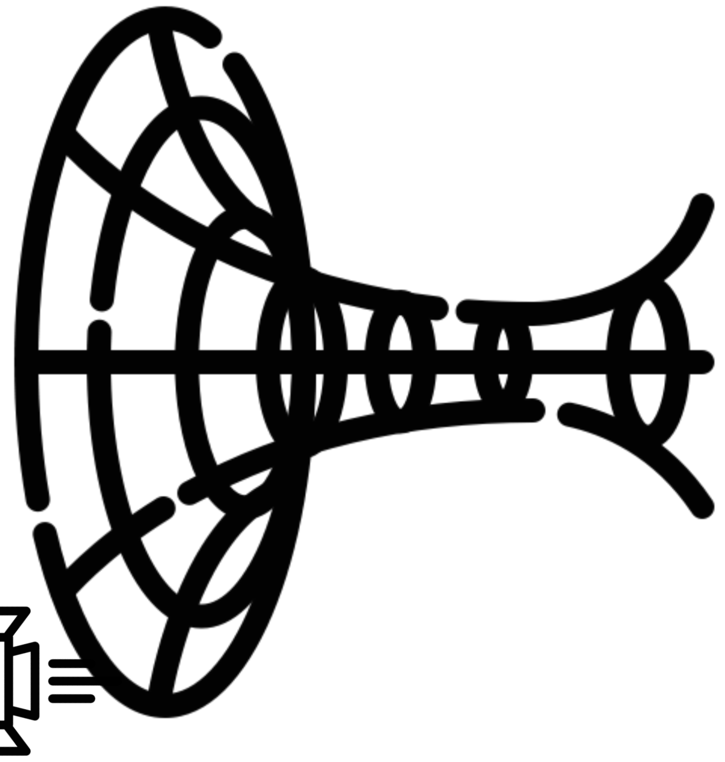
Spatial (time) Domain Universe

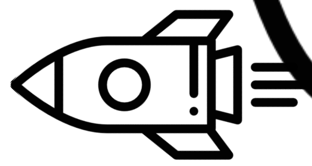
The Laplacian operator in the spatial domain is typically represented by a convolution kernel:

$$\Delta f = \nabla^2 f = \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial z^2}$$

This corresponds to a simple convolution mask such as:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



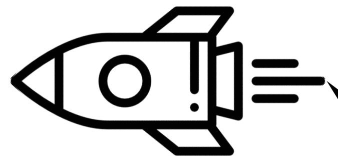


$$\nabla^2 f(t, z) = \mathscr{F}^{-1}\left[F(u, v)H(u, v)\right]$$

Frequency Domain Universe

The frequency response of the Laplacian operator is given by:

$$H(u, v) = -4\pi^2(u^2 + v^2)$$

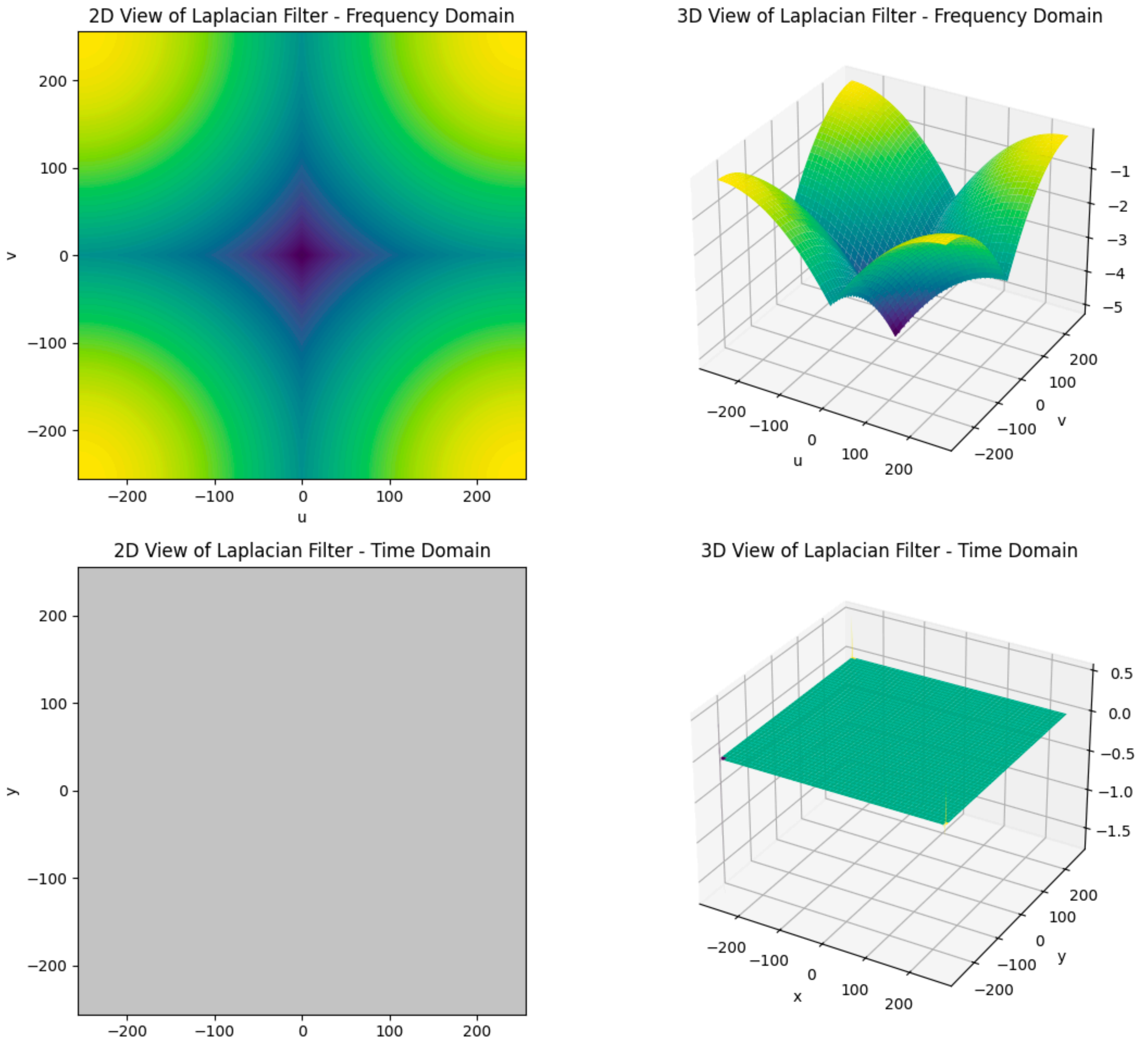


$$F(u, v)H(u, v)$$

Chuyển đổi Fourier 2-D

Laplacian Filter - Frequency Highpass Filter

Laplacian Filter Response

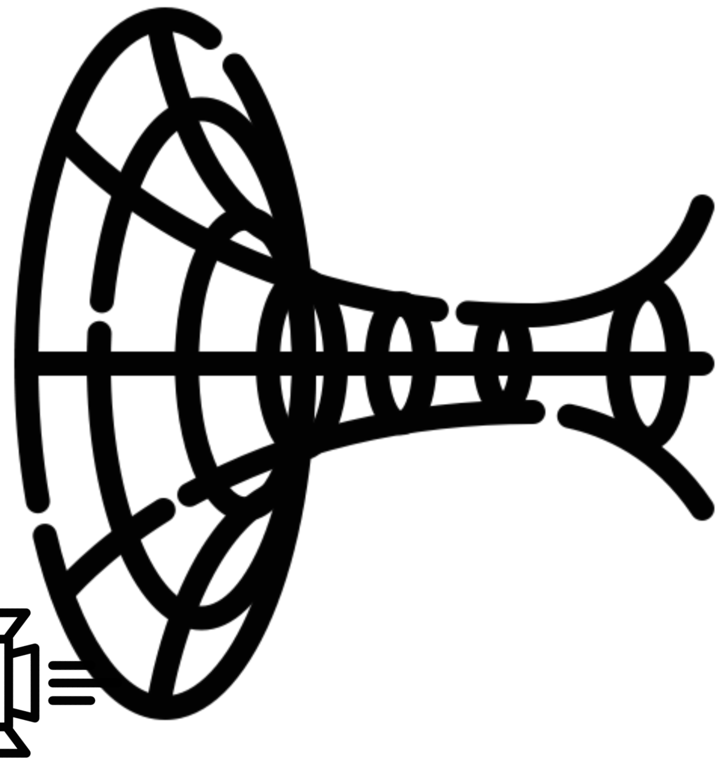


Chuyển đổi Fourier 2-D

Laplacian Filter - Image Sharpening - Frequency Highpass Filter

Miền không gian ảnh

$$g(t,z) = f(t,z) + c\,\nabla^2 f(t,z)$$



Miền tần số 2 chiều

Laplacian Filter:

$$H(u,v) = -\,4\pi^2(u^2+v^2)$$

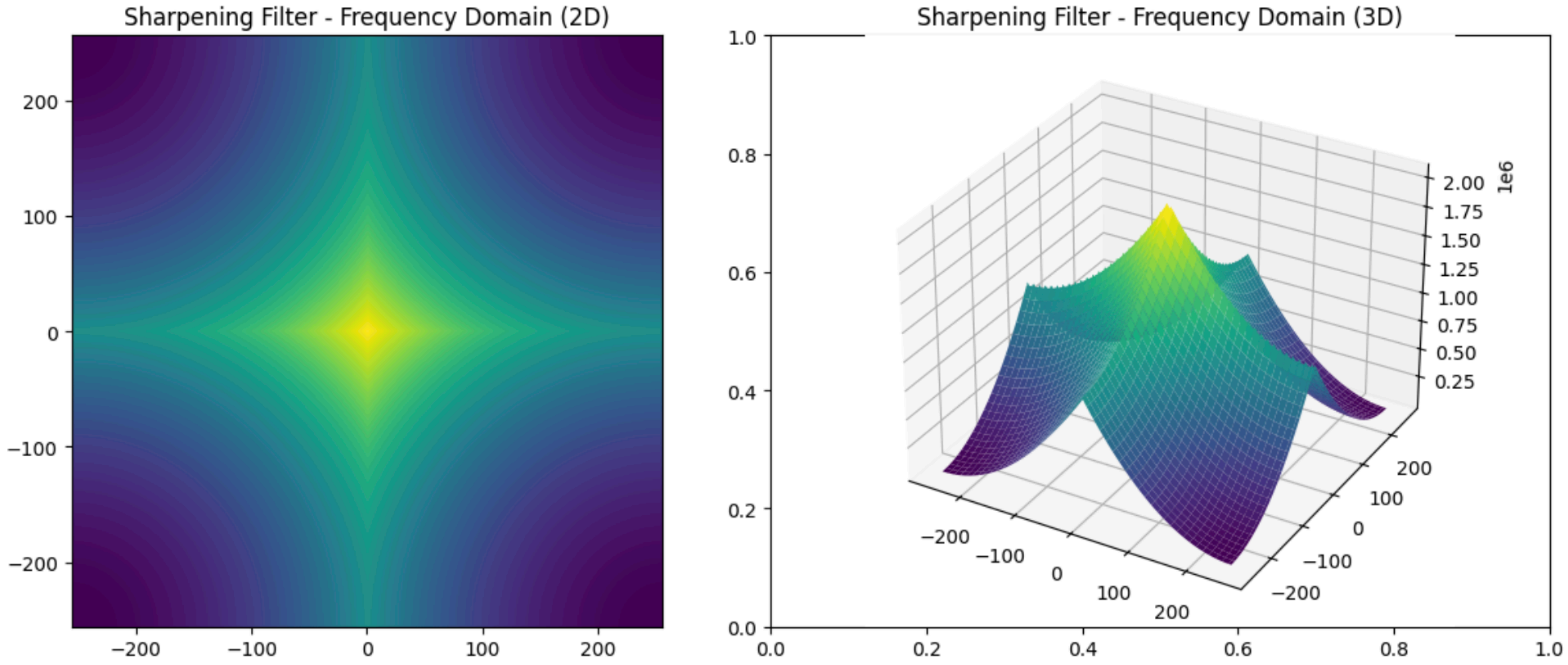
$$F(u,v)\big(1 + cH(u,v)\big)$$

$$g(t,z) = \mathscr{F}^{-1}\left[F(u,v)\big(1+cH(u,v)\big)\right]$$

Chuyển đổi Fourier 2-D

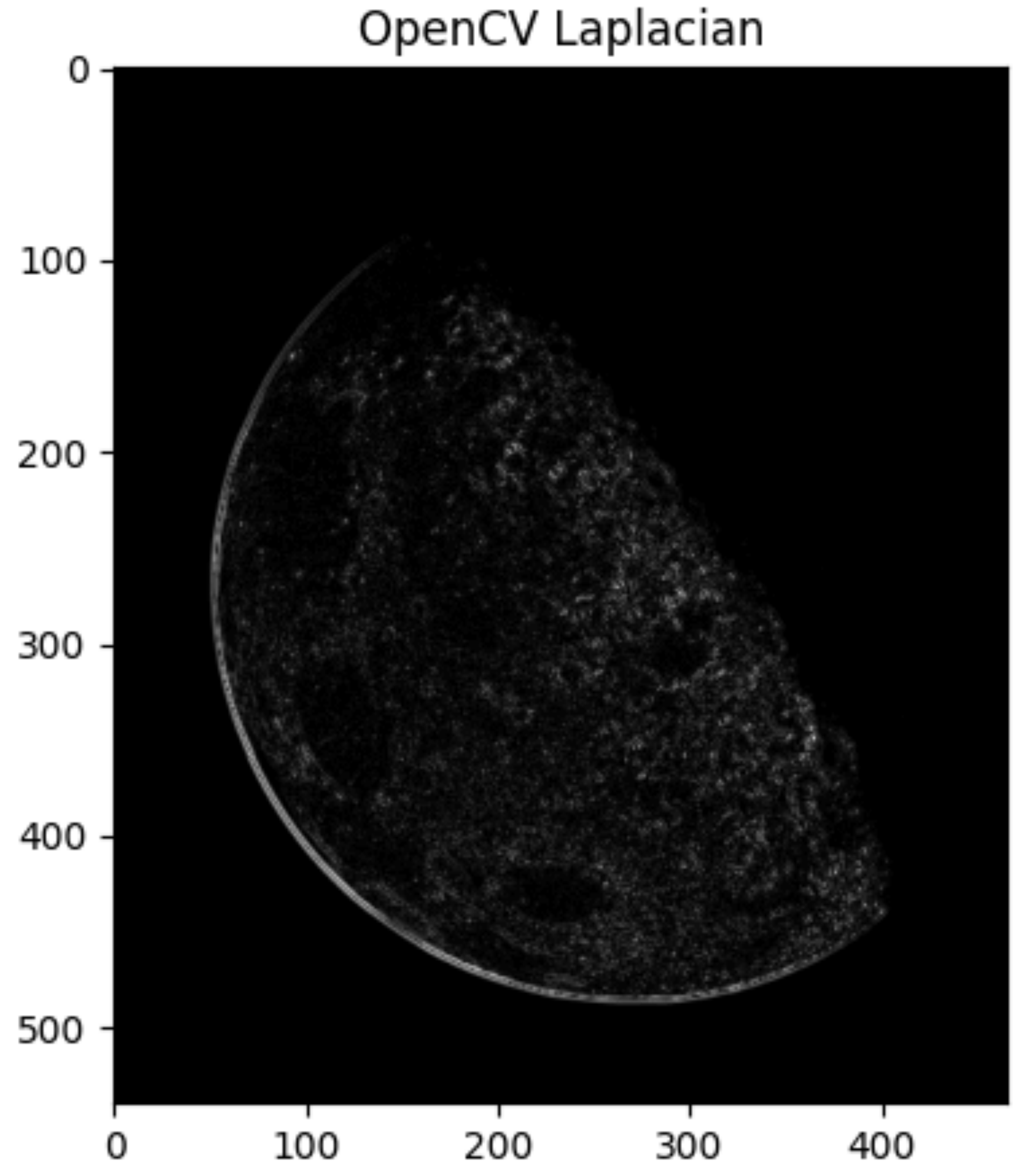
Laplacian Filter - Image Sharpening - Frequency Highpass Filter

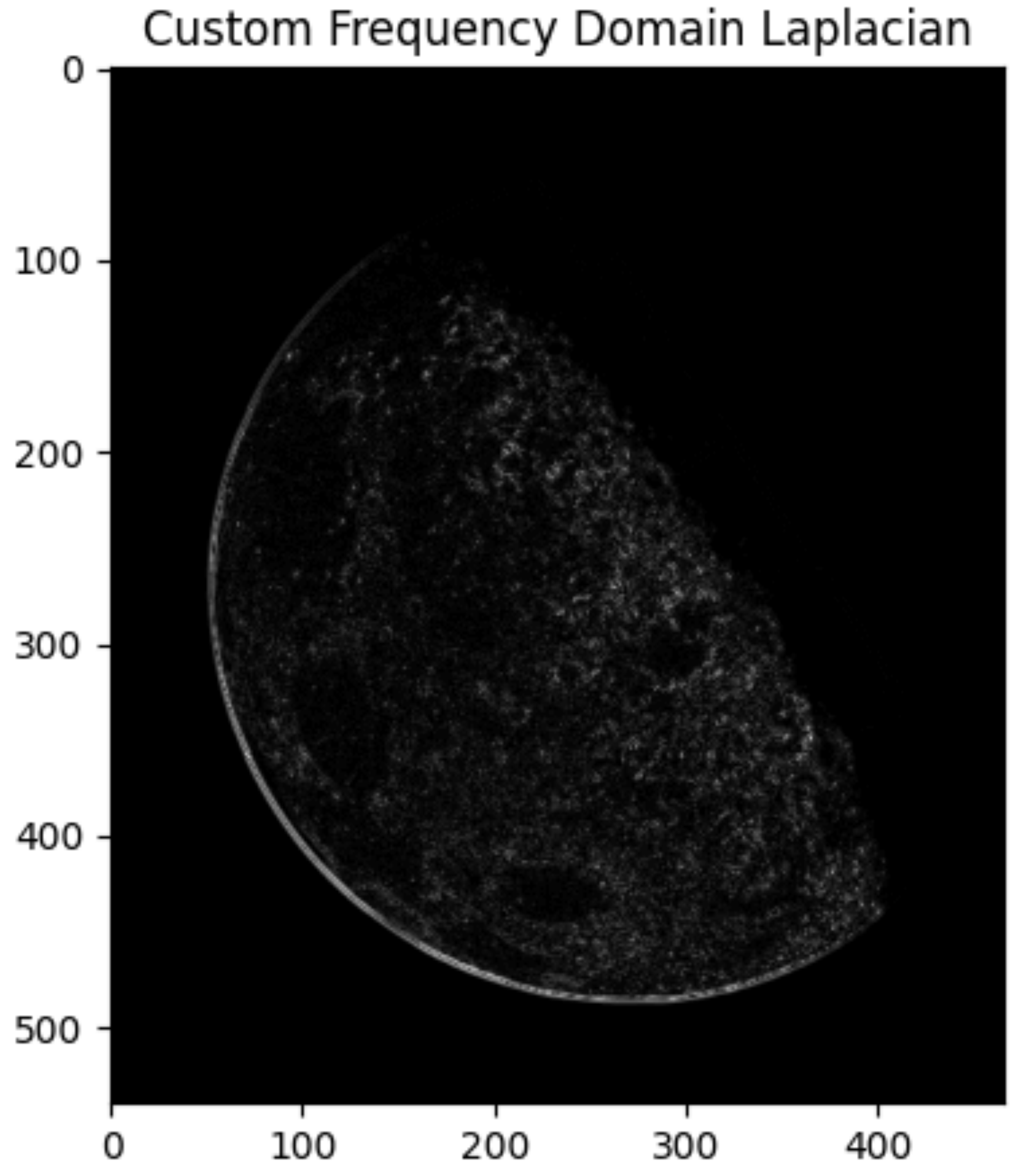
Sharpening Response



Chuyển đổi Fourier 2-D

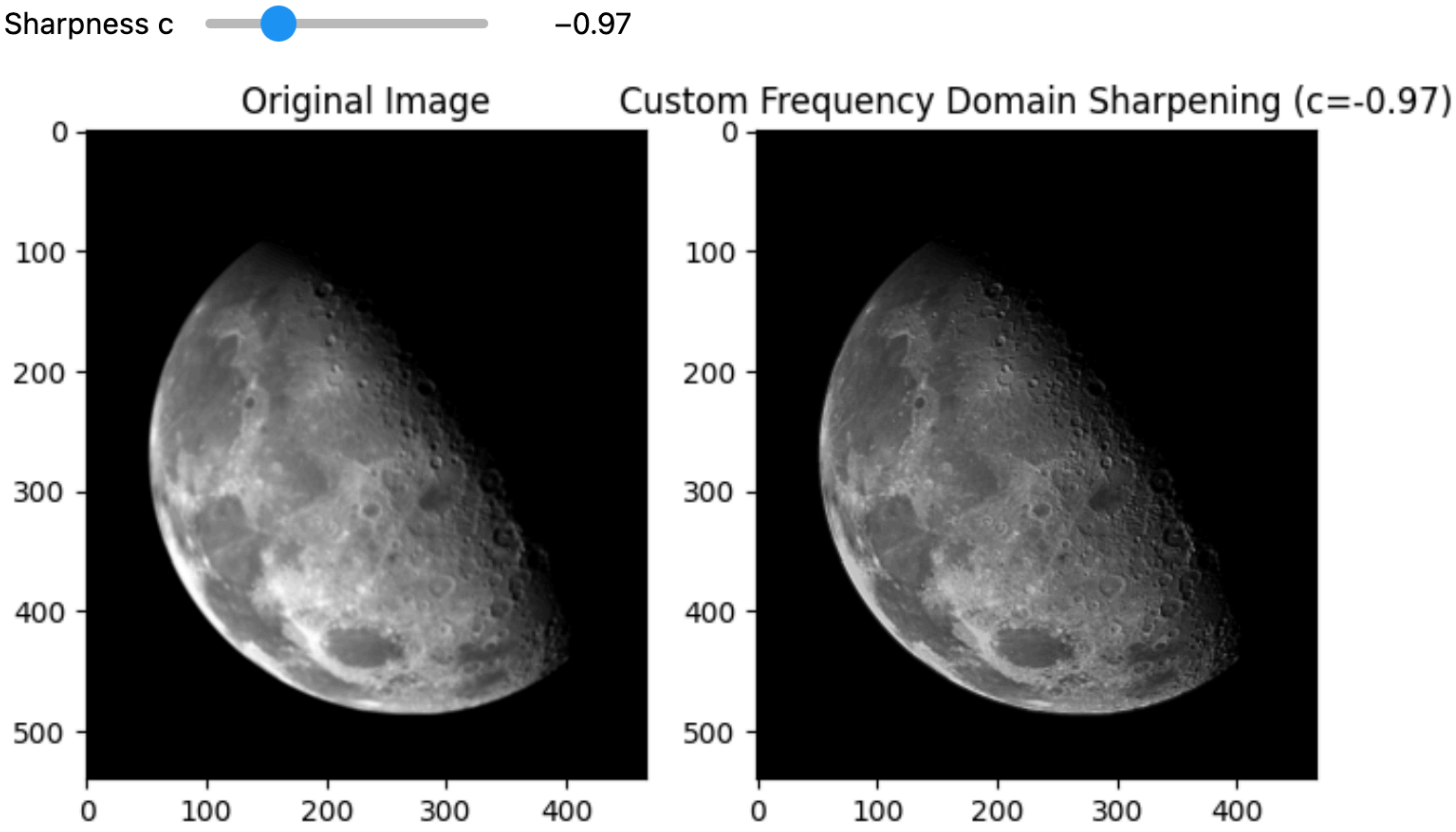
Laplacian Filter Application - Frequency Highpass Filter





Chuyển đổi Fourier 2-D

Laplacian Filter - Application for Sharpening - Frequency Highpass Filter

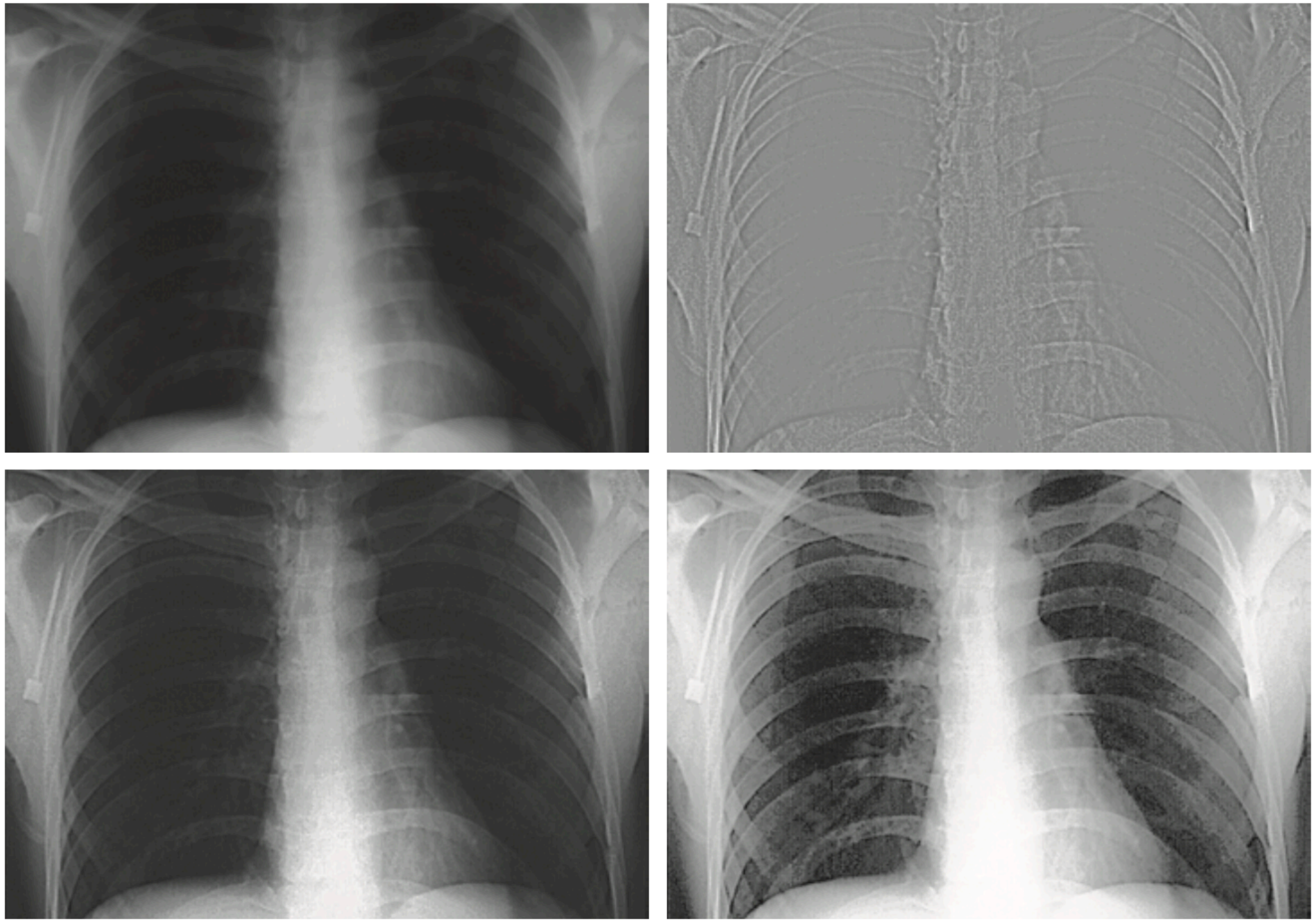


Chuyển đổi Fourier 2-D

Laplacian Filter - Unsharpening and High Frequency Emphasis

Lab practices

(a) A chest X-ray.	(b) Result of filtering with a GHPF function.
(c) Result of high-frequency- emphasis filtering using the same GHPF.	(d) Result of performing histogram equalization on (c)



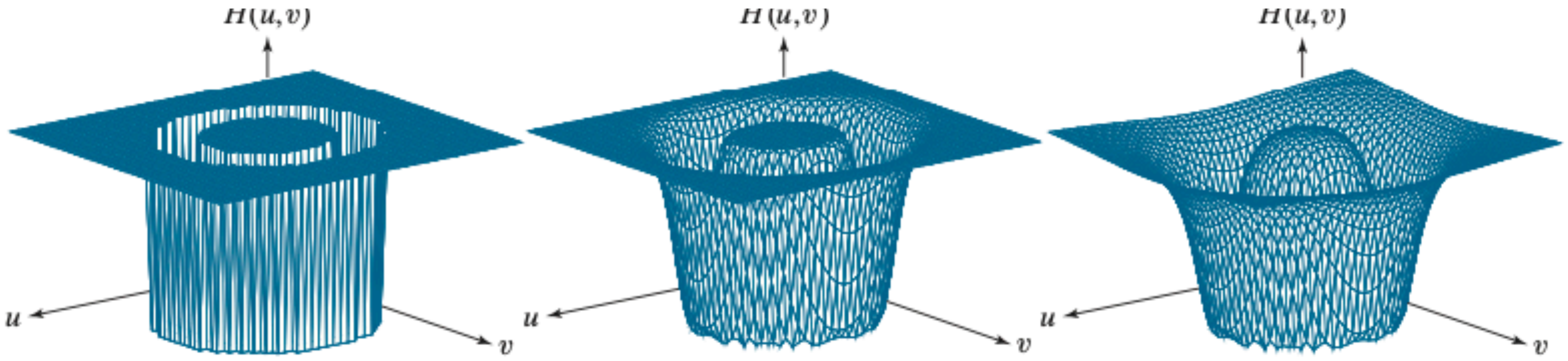
Chuyển đổi Fourier 2-D

Selective Frequency Filter

- **TYPES**
  - **Band-pass filters**: Allow a band of frequencies to pass and attenuate frequencies outside this band.
  - **Band-stop or notch filters**: Attenuate a band of frequencies and allow others to pass.
- **APPLICATIONS**
  - **Noise Reduction**: Removing unwanted frequencies that represent noise.
  - **Image Sharpening and Smoothing**: Enhancing edges (high-pass filtering) or reducing detail for smoothing (low-pass filtering).
  - **Communication Systems**: Filtering out interference or separating channels.
  - **Feature Extraction in Image Processing and Machine Learning**: Enhancing specific features important for analysis or recognition.

BandReject Filter - Selective Frequency Filter

Ideal (IBRF)	Gaussian (GBRF)	Butterworth (BBRF)
$H(u,v)=\begin{cases}0 & \text{if } C_0-\frac{W}{2}\le D(u,v)\le C_0+\frac{W}{2}\\ 1 & \text{otherwise}\end{cases}$	$H(u,v)=1-e^{-\left[\frac{D^2(u,v)-C_0^2}{D(u,v)W}\right]^2}$	$H(u,v)=\dfrac{1}{1+\left[\dfrac{D(u,v)W}{D^2(u,v)-C_0^2}\right]^{2n}}$



a b c

FIGURE 4.62 Perspective plots of (a) ideal, (b) modified Gaussian, and (c) modified Butterworth (of order 1) bandreject filter transfer functions from Table 4.7. All transfer functions are of size 512×512 elements, with $C_0=128$ and $W=60$.

Selective Frequency Filter

BandReject Filter

Inner Radius: 22

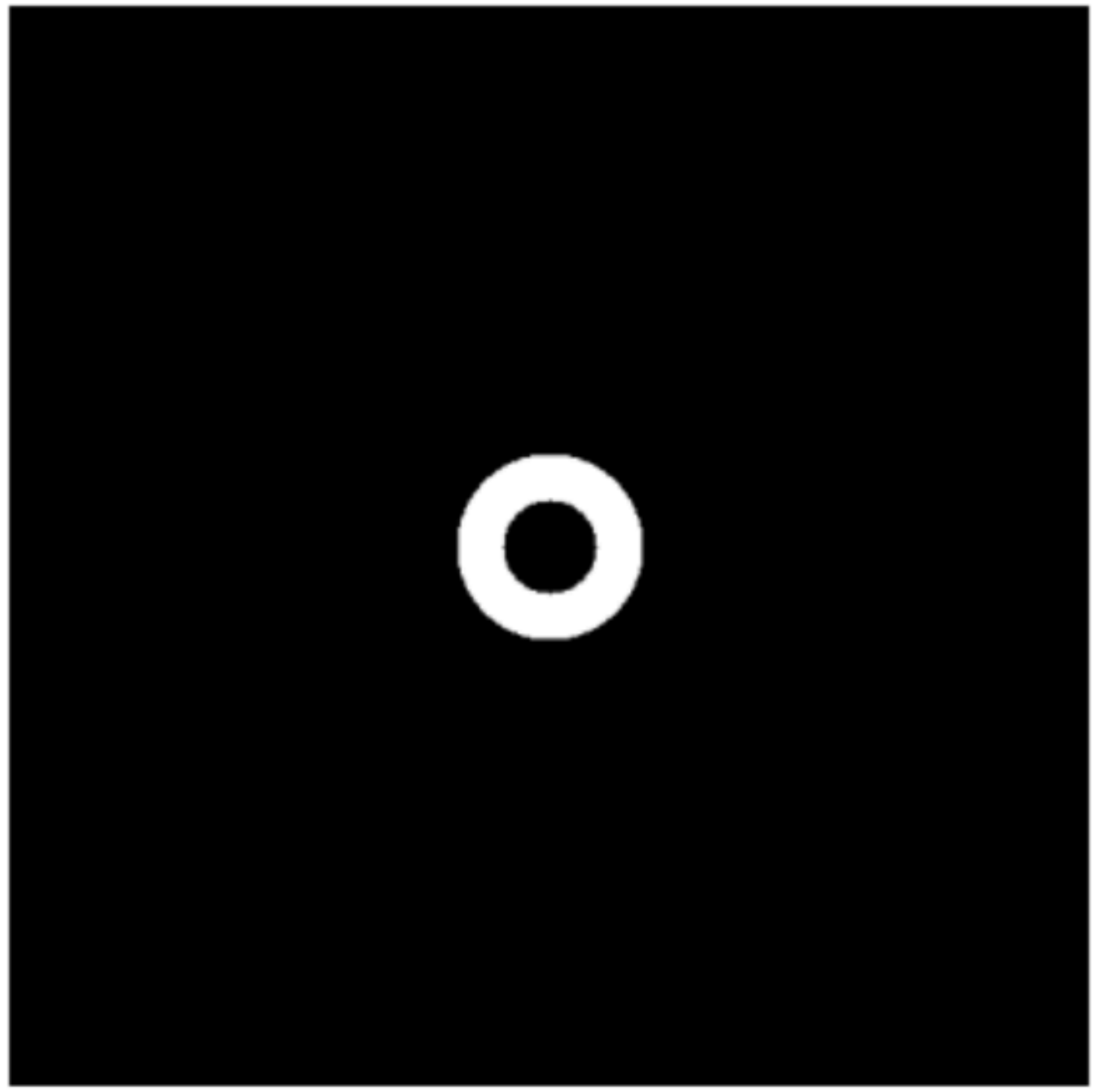
Outer Radi... 44

Ideal (IBRF)

$$H(u,v) = \begin{cases} 0 & \text{if } C_0 - \frac{W}{2} \le D(u,v) \le C_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$$

Original Image

Filter Kernel



Filtered Image

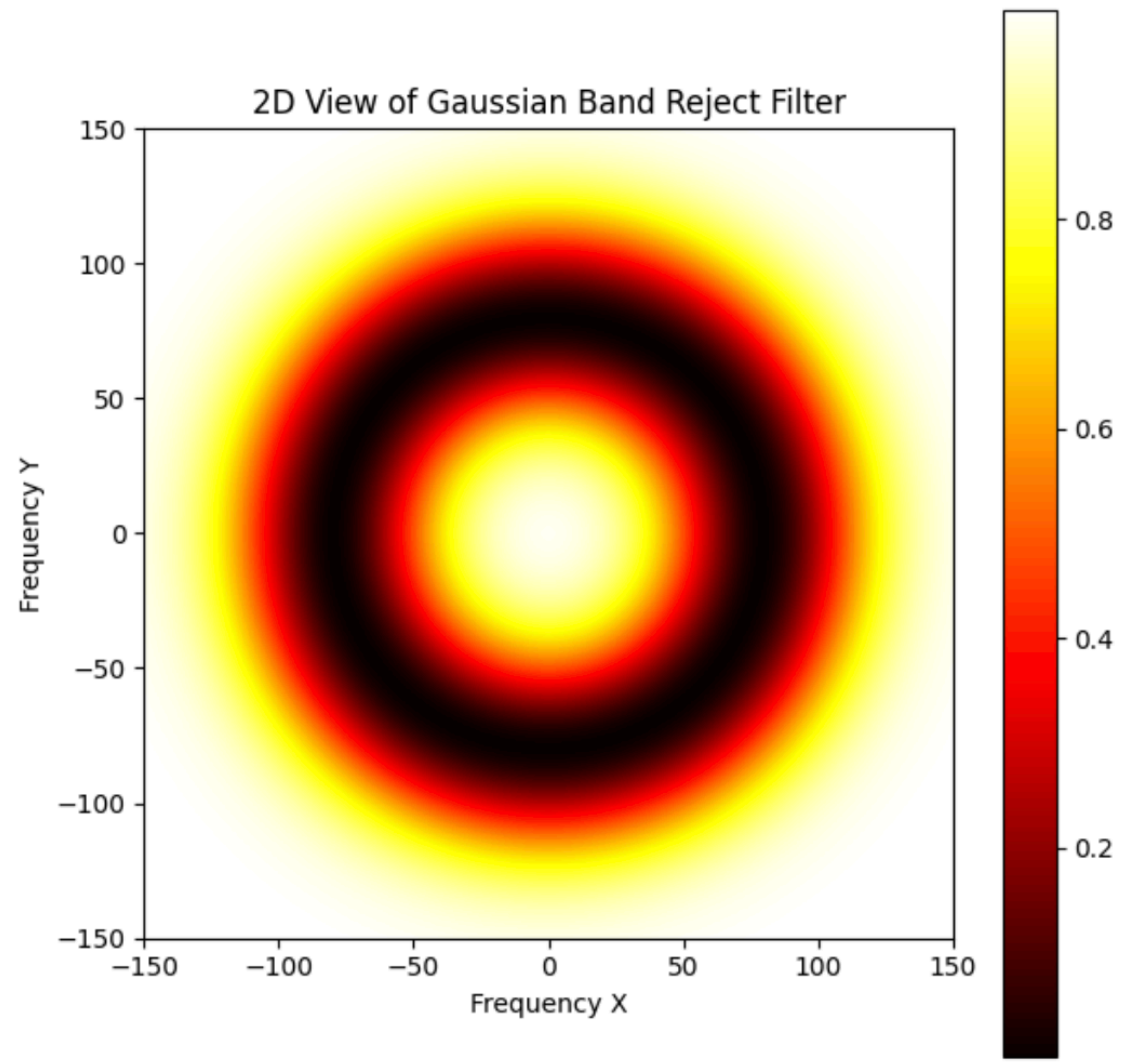
Chuyển đổi Fourier 2-D

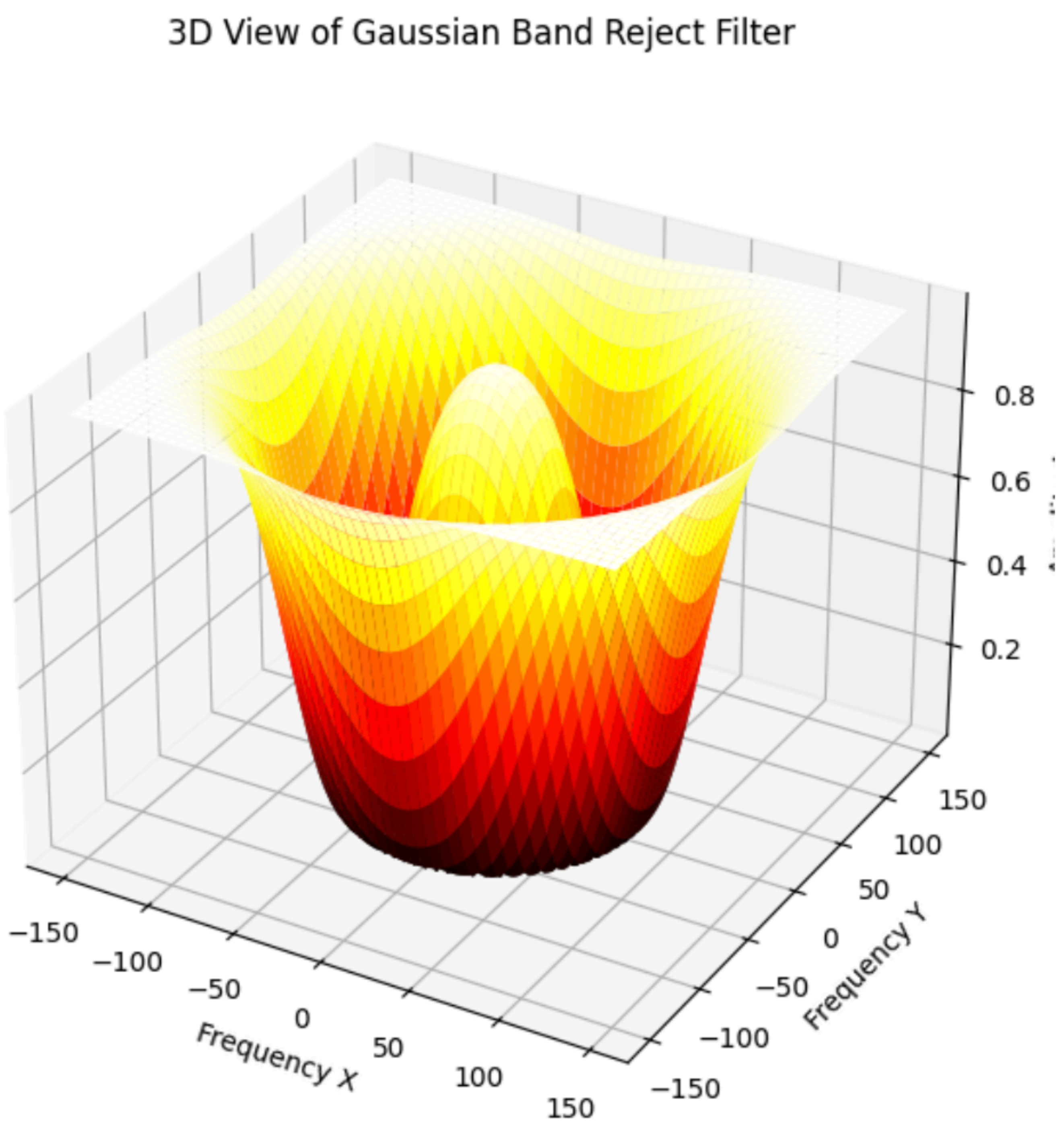
Selective Frequency Filter

Frequency Response

Center Fre... 80.00

Bandwidth 26.00





Gaussian (GBRF)

$$H(u,v) = 1 - e^{-\left[\frac{D^2(u,v) - C_0^2}{D(u,v)W}\right]^2}$$

Chuyển đổi Fourier 2-D

Selective Frequency Filter

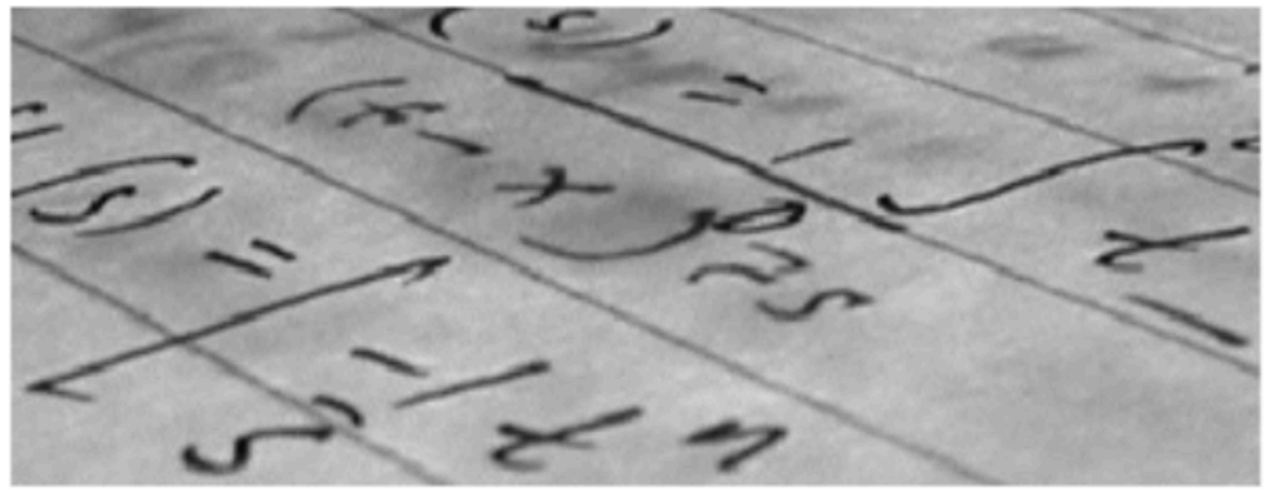
Gaussian (GBRF)

$$H(u,v) = 1 - e^{-\left[\frac{D^2(u,v) - C_0^2}{D(u,v)W}\right]^2}$$

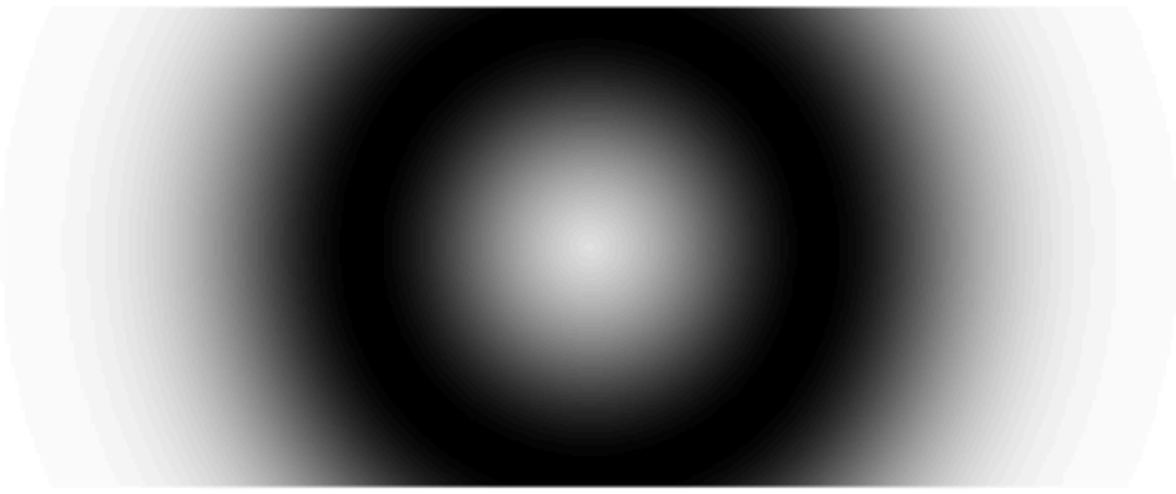
Center Fre... 82.00

Bandwidth: 39.00

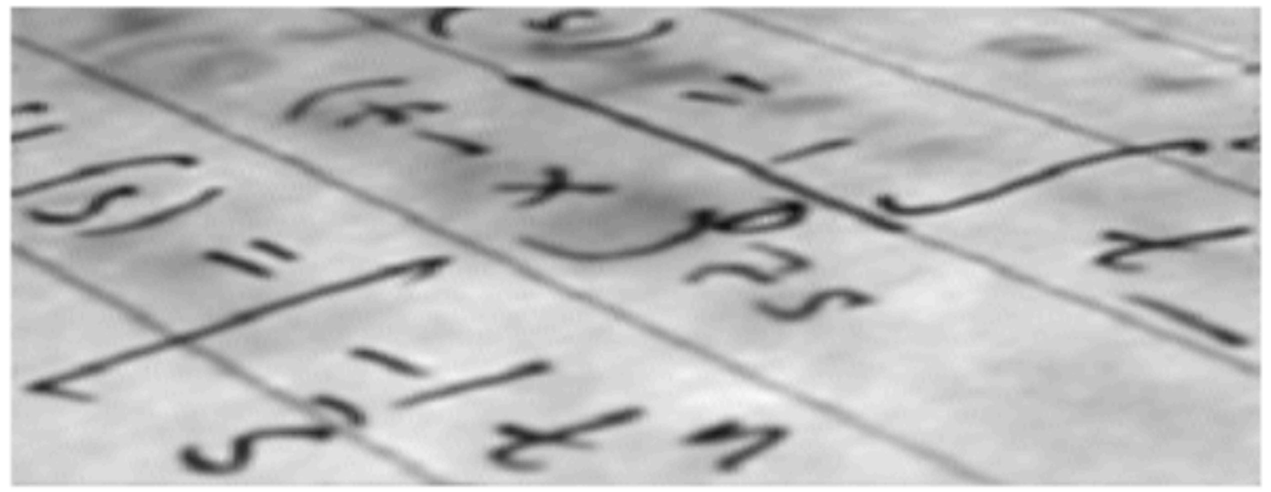
Original Image



Filter Kernel



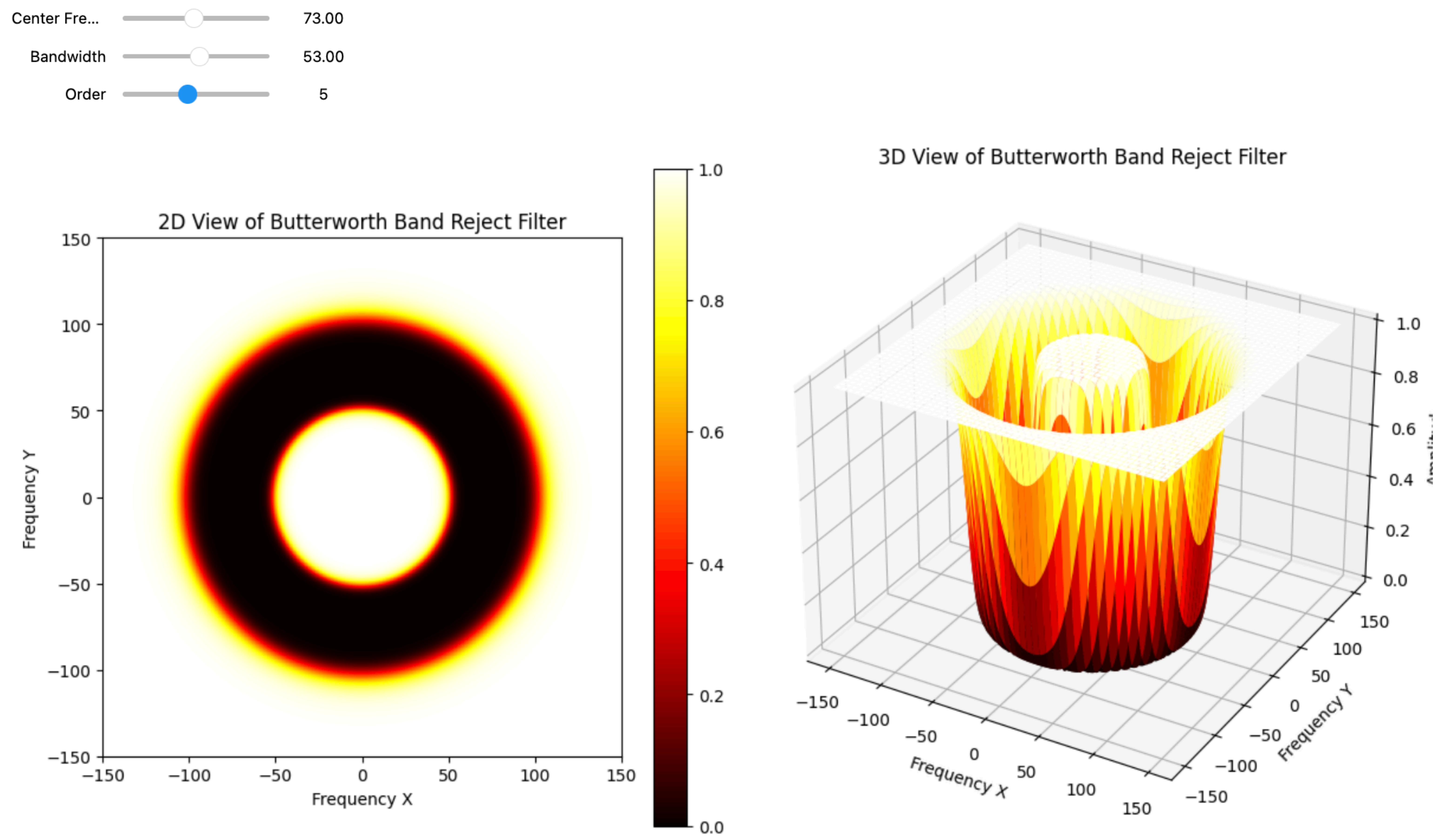
Filtered Image



Chuyển đổi Fourier 2-D

Butterworth BandReject Filter

Frequency Response



Butterworth (BBRF)

$$H(u,v) = \frac{1}{1 + \left[\frac{D(u,v)W}{D^2(u,v) - C_0^2} \right]^{2n}}$$

Chuyển đổi Fourier 2-D

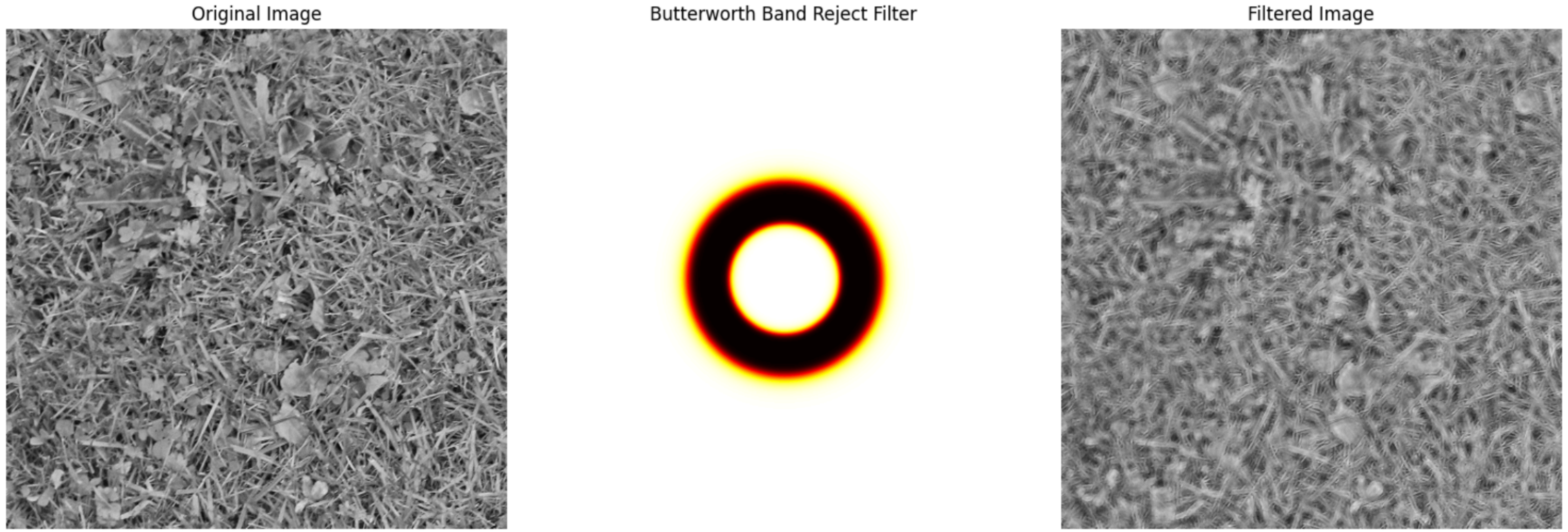
Butterworth BandReject Filter

Center Fre... 75.00

Bandwidth 45.00

Order 4

```
/var/folders/vk/m995pdvx3919575ds09dyx540000gn/T/ipykernel_35557/972265164.py:14: RuntimeWarning: divide by zero encountered in divide
  return 1 / (1 + ((radius * bandwidth) / (radius**2 - center_freq**2))**(2 * order))
```



Butterworth (BBRF)

$$H(u,v) = \frac{1}{1 + \left[\dfrac{D(u,v)W}{D^2(u,v) - C_0^2} \right]^{2n}}$$

Chuyển đổi Fourier 2-D

Notch Filter Principles

- **Introduction to Notch Filtering**
  - Selectively attenuate specific frequency components in an image.
  - Particularly useful in removing periodic noise or artifacts that have a distinct frequency signature.
- **Mathematical Description of Notch Filters**

$$H_{NR}(u, v) = \prod_{k=1}^{Q} H_k(u, v) \cdot H_{-k}(u, v)$$

  - Here, $H_k(u, v)$ and $H_{-k}(u, v)$ are **individual highpass filter** transfer functions centered at frequency coordinates (u_k, v_k) and $(-u_k, -v_k)$ respectively

Chuyển đổi Fourier 2-D

Notch Filter Principles

- **Role of Q in Notch Filters : Multiple Frequency Attenuation**
  - Multiple Notch Locations: Q defines how many distinct pairs of notch filters
  - Symmetrical Attenuation: Each pair includes a filter at (u_k, v_k) and its symmetrical counterpart at $(-u_k, -v_k)$. This symmetry is crucial for maintaining the realness and zero-phase property of the filtered image after applying the inverse Fourier Transform.

Chuyển đổi Fourier 2-D

Notch Filter based on Ideal Highpass Filter

- Creating a Notch Using Ideal Highpass Filters:

$$H_k(u, v) = \begin{cases} 0 & \text{if } \sqrt{(u - u_k)^2 + (v - v_k)^2} < D_0 \text{ or } \sqrt{(u + u_k)^2 + (v + v_k)^2} < D_0 \\ 1 & \text{otherwise} \end{cases}$$

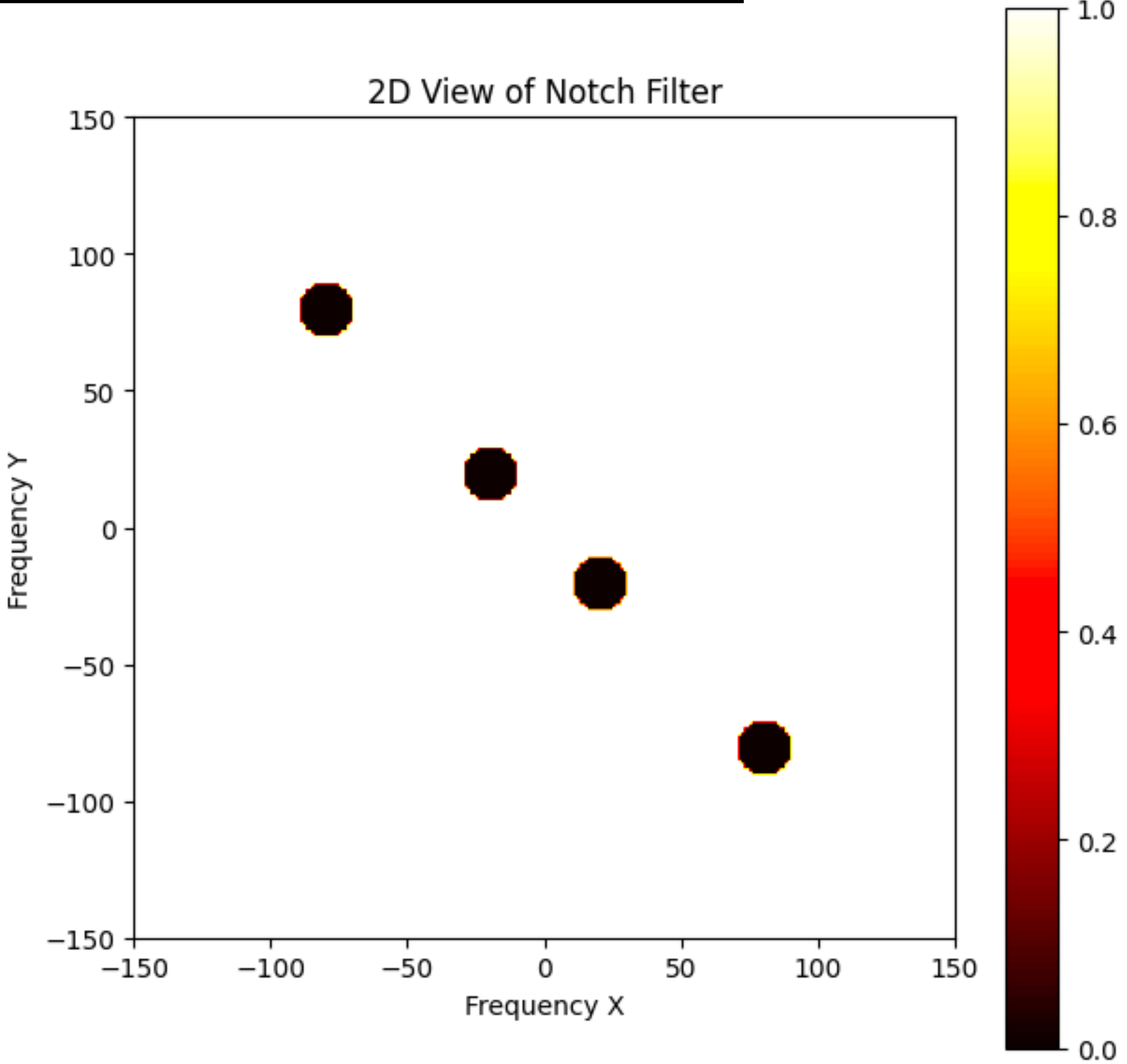
- Combining Multiple Notch Locations

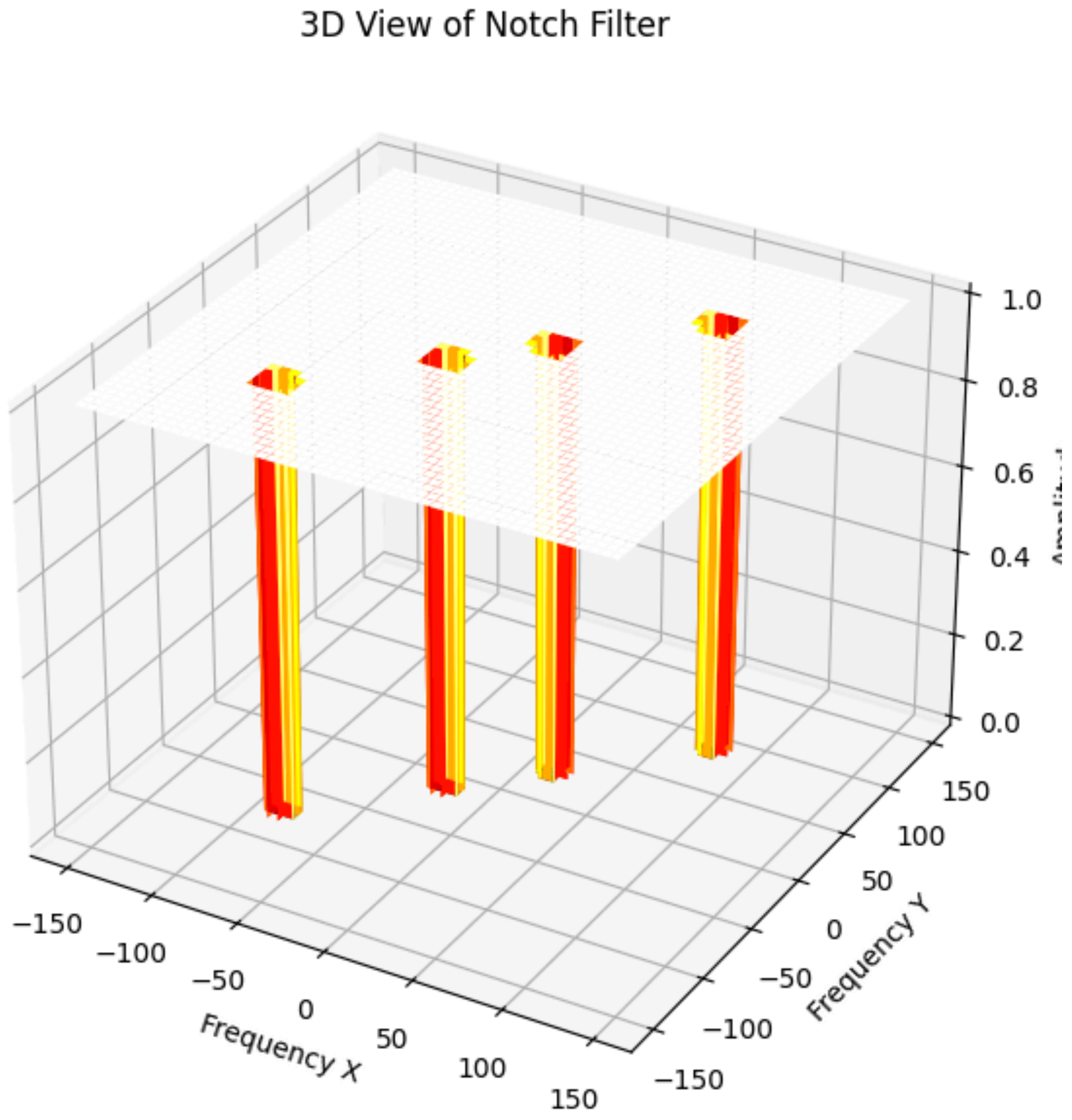
$$H_{NR}(u, v) = \prod_{k=1}^{Q} H_k(u, v) H_{-k}(u, v)$$

Chuyển đổi Fourier 2-D

Notch Filter based on Ideal Highpass Filter

Frequency Response

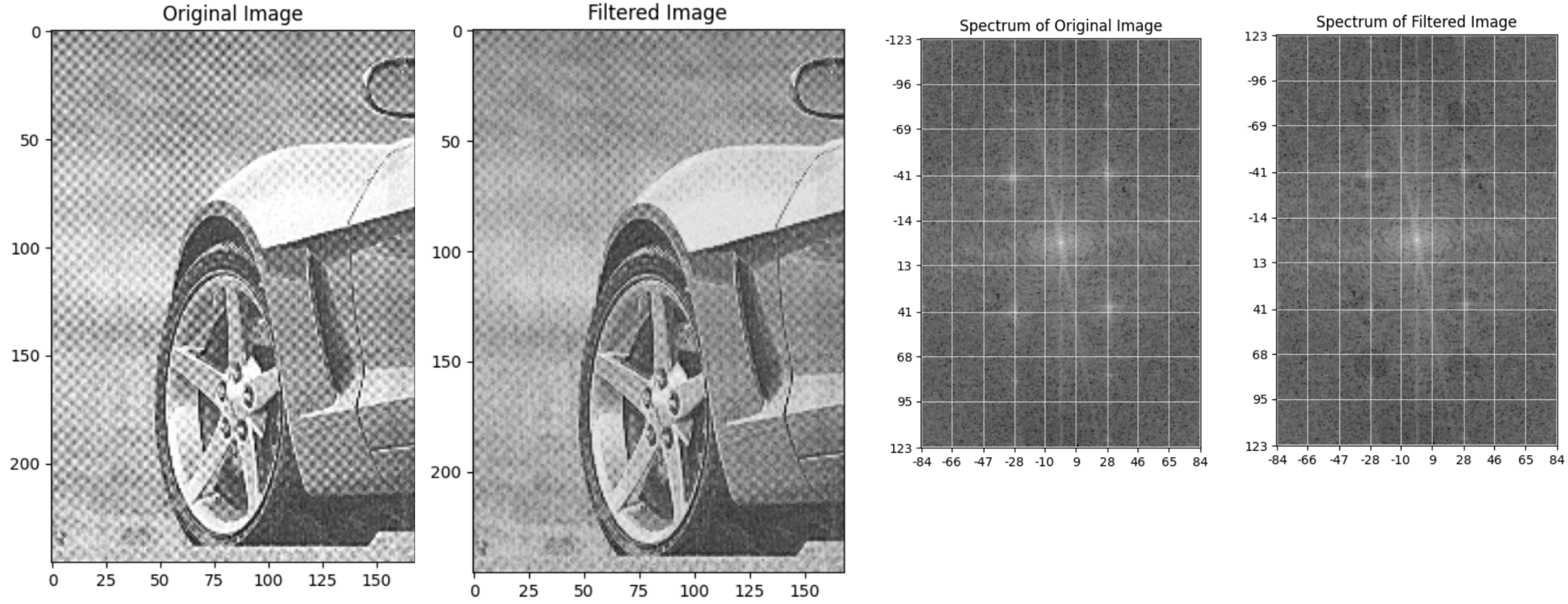




Chuyển đổi Fourier 2-D

Notch Filter based on Ideal Highpass Filter

Application



Quiz và LAB

Quiz và LAB: Chuyển đổi Fourier số 2D

Nội dung

- Quiz: kiểm tra các kiến thức cơ bản về chuyển đổi Fourier
- Lab:
  - Thực hành căn bản về chuyển đổi Fourier 2D
  - Thực hành bộ lọc lowpass, notched pass